Finite Straight Wire

Integral Equation Method

Straight wire of length l and radius a, placed along the y-axis

$$1 = 4\pi\varepsilon_0 \int_0^l \frac{\rho(y')}{R(y,y')} dy'$$
$$0 \le y \le l$$
$$(8\text{-}2)$$

Because we know that uniform charge density will create a scalar electric potential, which has been normalized in this case.

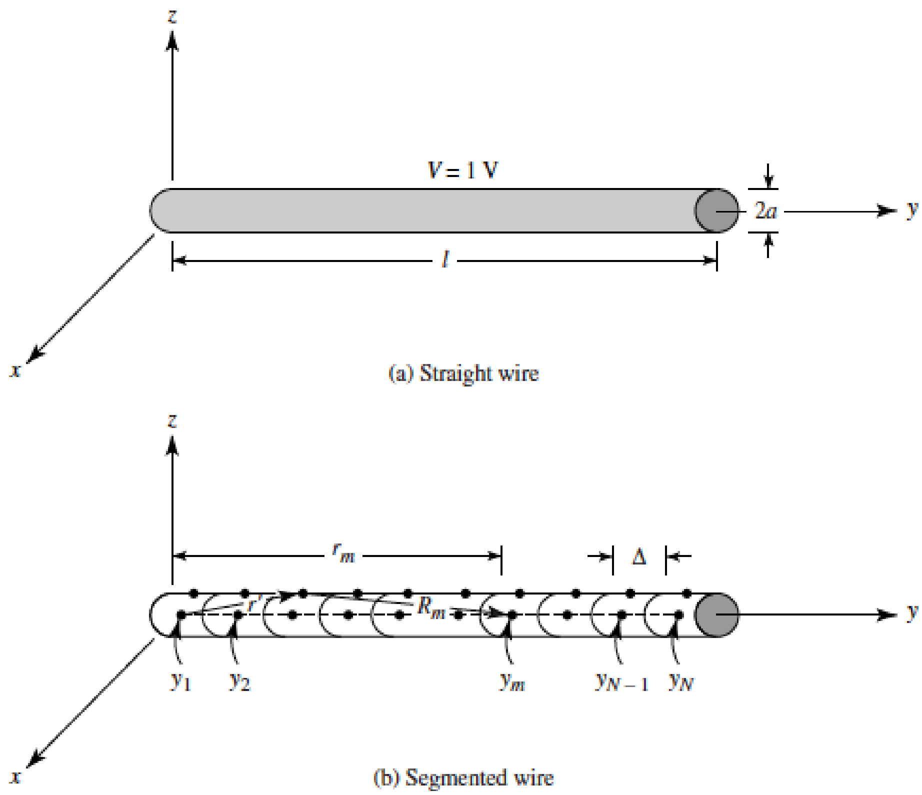

Figure 8.1 Straight wire of constant potential and its segmentation.

$$R(y,y') = R(r,r')|_{x=z=0} = \sqrt{(y-y')^2 + [(x')^2 + (z')^2]}$$
$$= \sqrt{(y-y')^2 + a^2}$$
$$(8\text{-}2a)$$

where

$r'(x', y', z')$ denotes the source coordinate and $r(x, y, z)$ denotes the observation coordinate.

Hence, $R(r, r')$ is the distance from any one point on the source to the observation point.

Basis Function

In this context, we use a pulse function (subdomain piecewise constant function) as the basis function. In analogy, the use of basis function can be thought as 'sampling' in signal processing. There are options for basis function out there though.

$$g_n(y') = \begin{cases} 0 & y' < (n-1)\Delta \\ 1 & (n-1)\Delta \leq y' \leq n\Delta \\ 0 & n\Delta < y' \end{cases}$$

(8-5)

where $g_n(y')$ is the basis function.

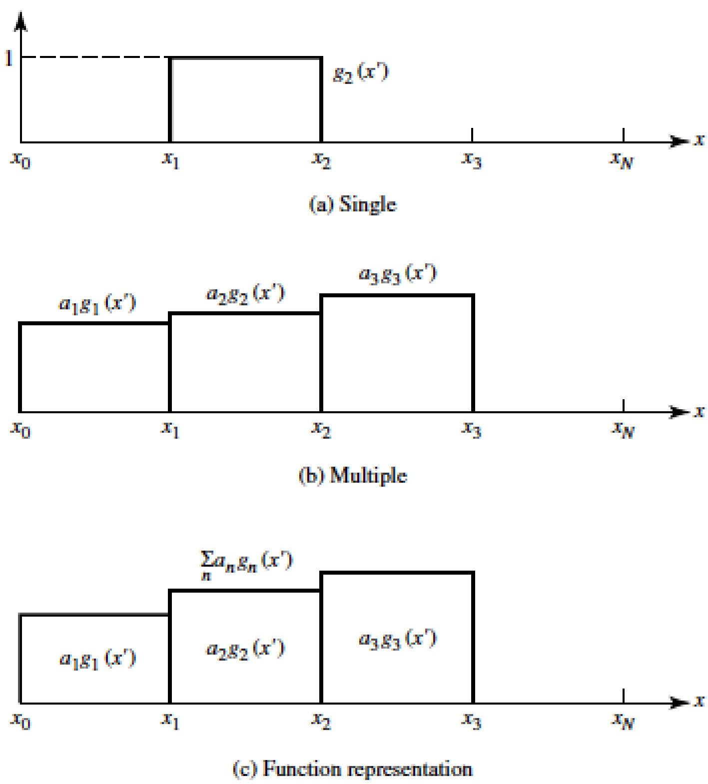

Figure 8.8 Piecewise constant subdomain functions.

The idea is that we need to find the unknown charge density first (by approximating), which will provide N number of linearly independent equations since the solution is unique, I_n. (See [2], Theorem 5 in Chapter 2)

Approximated charge density on wire

$$\rho(y') = \sum_{n=1}^{N} a_n g_n(y')$$

(8-3)

Express (8-2) with the approximated charge distribution.

$$4\pi\varepsilon_0 = \sum_{n=1}^{N} a_n \int_0^l \frac{g_n(y')}{\sqrt{(y-y')^2 + a^2}} dy'$$

(8-4a)

$$\begin{aligned} 4\pi\varepsilon_0 &= a_1 \int_0^{\Delta} \frac{g_1(y')}{R(y_1, y')} dy' + \cdots + a_N \int_{(N-1)\Delta}^{l} \frac{g_N(y')}{R(y_1, y')} dy' \\ &\vdots \\ 4\pi\varepsilon_0 &= a_1 \int_0^{\Delta} \frac{g_1(y')}{R(y_N, y')} dy' + \cdots + a_N \int_{(N-1)\Delta}^{l} \frac{g_N(y')}{R(y_N, y')} dy' \end{aligned} \qquad (8\text{-}6a)$$

where

$$\Delta = l/N$$

The wire now has been divided into uniform N segments.
By pointing y_m to the center of each Δ length element, N linearly independent equations may be produced.

Matrix Equation

$$[V_m] = [Z_{mn}][I_n]$$

where

$$Z_{mn} = \int_{(n-1)\Delta}^{n\Delta} \frac{1}{\sqrt{(y_m - y')^2 + a^2}} dy'$$

and

$$[I_n] = [a_n]$$

$$[V_m] = [4\pi\varepsilon_0]$$

Notice that these are N by 1 size matrix (or a vector).

We may obtain current in explicit form by using the fact that $[Z_{mn}]$ is invertible.

One closed-form evaluation has been introduced in the textbook:

$$Z_{mn} = \begin{cases} 2\ln\left(\dfrac{\frac{\Delta}{2} + \sqrt{a^2 + \left(\frac{\Delta}{2}\right)^2}}{a}\right) & m = n \quad \text{(8-9a)} \\ \ln\left\{\dfrac{d_{mn}^{+} + [(d_{mn}^{+})^2 + a^2]^{1/2}}{d_{mn}^{-} + [(d_{mn}^{-})^2 + a^2]^{1/2}}\right\} & m \neq n \text{ but } |m-n| \leq 2 \quad \text{(8-9b)} \\ \ln\left(\dfrac{d_{mn}^{+}}{d_{mn}^{-}}\right) & |m-n| > 2 \quad \text{(8-9c)} \end{cases}$$

where

$$d_{mn}^{+} = l_m + \frac{\Delta}{2} \quad \text{(8-9d)}$$

$$d_{mn}^{-} = l_m - \frac{\Delta}{2} \quad \text{(8-9e)}$$

l_m is the distance between the m^{th} matching point and the center of the n^{th} source point

Investigate a 1-m long straight wire of radius $a = 0.001m$ is under a constant potential of $1V$

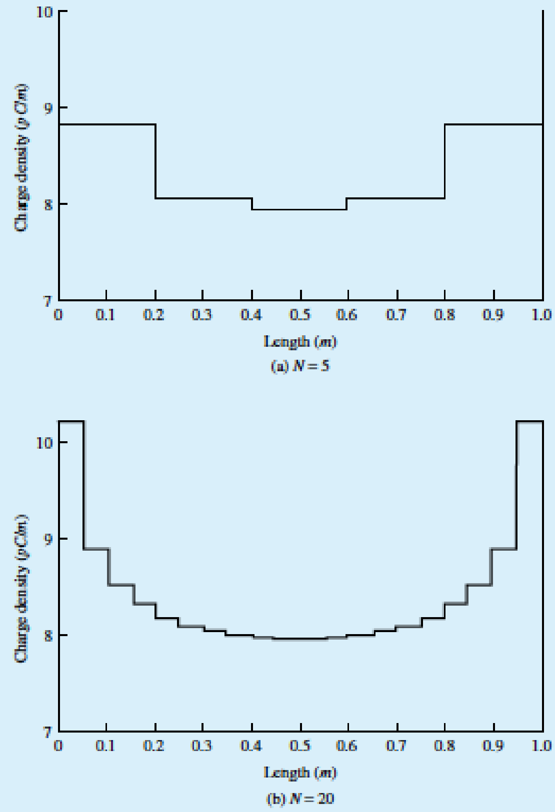

Figure 8.2 Charge distribution on a 1-m straight wire at 1 V.

Reference

[1] Antenna Theory: Analysis and Design, 4th Edition by Constantine A. Balanis

[2] Linear Algebra and its applications, 4th Edition by David C. Lay