Coaxial Cable

I. Introduction

Coaxial cable is one of the famous transmission lines available in market. 50-ohm or 75-ohm coaxial cables are usually used to connect electrical components for power transmission or communication purposes. In an introductory class of EM class, coaxial lines are introduced because of their dominant propagating features (TEM mode) which are very pedagogical to beginners.

One could go into deep analysis of coaxial cable but this document will go over a brief explanation for coaxial cable. I found the analysis was more likely intricate than any other concept because there were too many assumptions and conditions that may not appear as straightforward as they should be.

In this document, *lossless* coaxial cable will be considered first, then move onto more general context of coaxial lines.

There are many ways to transmit signals but the advent of coaxial cable enables the bandwidth of several MHz which in turn allows transmitting multiple (several hundreds) audio signals per cable, for example.

II. Theory of Lossless Coaxial Lines

a. Telegrapher's Equations

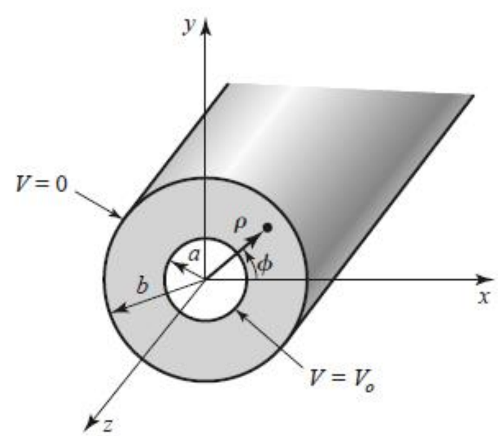

FIGURE 3.15 Coaxial line geometry.

To derive the Telegrapher's equations out of the coaxial geometry, we will look at the following figure:

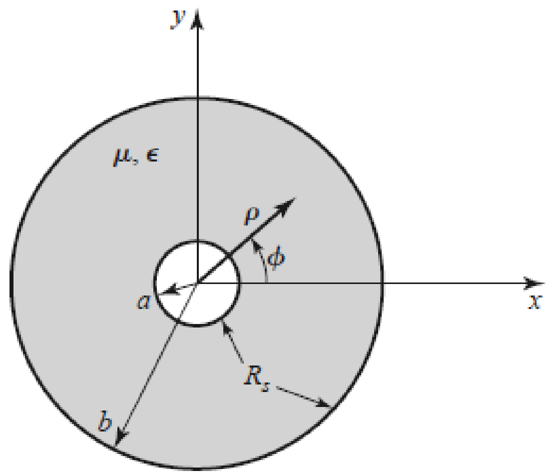

FIGURE 2.3 Geometry of a coaxial line with surface resistance R_S on the inner and outer conductors.

Considering the two Maxwell's equations in phasor:

$$\nabla \times \vec{E} = -j\omega\mu\vec{H}$$

$$\nabla \times \vec{H} = j\omega\epsilon\vec{E}$$

where $\epsilon = \epsilon' - j\epsilon''$ which allows "lossy dielectric filling" in Maxwell's equations. Before expanding the Curl expression, let's investigate some geometrical advantages to clean up vector components.

First, we need to realize that the Coaxial lines are TEM mode line and the propagation will be along the z-direction (i.e. both electric and magnetic vector components cannot be along z-direction). Second, there's no ϕ variation due to *the azimuthal symmetry* (i.e. rotational symmetry along the z-axis). For mathematical formulation,

$$E_z = H_z = 0$$

$$\frac{\partial}{\partial\phi} = 0$$

Expanding the Maxwell's equations <<vector operators in different coordinate systems can be found in another document.>>:

$$-\hat{\rho}\frac{\partial E_\phi}{\partial z} + \hat{\phi}\frac{\partial E_\rho}{\partial z} + \hat{z}\frac{1}{\rho}\frac{\partial(\rho E_\phi)}{\partial \rho} = -j\omega\mu(\hat{\rho}H_\rho + \hat{\phi}H_\phi)$$

$$-\hat{\rho}\frac{\partial H_\phi}{\partial z} + \hat{\phi}\frac{\partial H_\rho}{\partial z} + \hat{z}\frac{1}{\rho}\frac{\partial(\rho H_\phi)}{\partial \rho} = j\omega\epsilon(\hat{\rho}E_\rho + \hat{\phi}E_\phi)$$

We are talking about vector quantities that are in spherical coordinate (that is, a set of orthogonal Basis forms the coordinate) which implies that the terms whose basis vector is $\hat{z}$ must vanish.

To do so, we may use a conjecture that if E_ϕ and H_ϕ look like the following, then there's no violation in our previous assumption:

$$E_\phi = \frac{f(z)}{\rho}$$

$$H_\phi = \frac{g(z)}{\rho}$$

that make $\hat{z}$ components vanish.

Now, if we consider the boundary condition, the tangential component (i.e. normal to the surface of conductor) of the electric field would be the E_ϕ because $\hat{\rho}$ points in the direction that is normal to the surface. Recall the geometrical information of coaxial, we note that the first boundary condition occurs at $\rho = a$ and the second one at $\rho = b$ (i.e. $E_\phi = 0$ at $\rho = a$ and $\rho = b$). However, this just essentially implies that $E_\phi = 0$ everywhere which in turn forces H_ρ term to be zero. <<Applying the right-hand rule, one may note that the ratio of E_ϕ and H_ρ is the wave-impedance factor.>>

To sum up,

$$\hat{\phi}\frac{\partial E_\rho}{\partial z} = -j\omega\mu\left(\hat{\phi}H_\phi\right)$$

$$\hat{\rho}\frac{\partial H_\phi}{\partial z} = -j\omega\epsilon\left(\hat{\rho}E_\rho\right)$$

Recall that the mathematical model for H_ϕ component, E_ρ must also have:

$$E_\rho = \frac{h(z)}{\rho}$$

Plugging these assumed functions into the Maxwell's equation:

$$\hat{\phi}\frac{\partial h(z)}{\partial z} = -j\omega\mu\left(\hat{\phi}g(z)\right)$$

$$\hat{\rho}\frac{\partial g(z)}{\partial z} = -j\omega\epsilon\left(\hat{\rho}h(z)\right)$$

To derive the Telegrapher's equations, let's find out voltage and current equation out of the Maxwell's equations:

$$V(z) = \int_a^b E_\rho d\rho = h(z)\ln\left(\frac{b}{a}\right)$$

<<voltage difference between the two conductors.>>

$$I(z) = \int_0^{2\pi} aH_\phi d\phi = 2\pi g(z)$$

<<Total current on the inner conductor>>

Hence,

$$h(z) = -\ln\left(\frac{b}{a}\right) V(z)$$

$$g(z) = \frac{I(z)}{2\pi}$$

$$\boxed{\begin{gathered} E_\rho = \frac{h(z)}{\rho} = -\frac{\ln\left(\frac{b}{a}\right) V(z)}{\rho} \\ H_\phi = \frac{I(z)}{2\pi\rho} \end{gathered}}$$

Plugging them back into the Maxwell's equations:

$$\hat{\phi}\frac{\partial V(z)\ln\left(\frac{b}{a}\right)}{\partial z} = j\omega\mu\left(\frac{\hat{\phi}I(z)}{2\pi}\right)$$

$$\hat{\rho}\frac{\partial I(z)}{2\pi\partial z} = j\omega\epsilon\left(\hat{\rho}\ln\left(\frac{b}{a}\right)V(z)\right)$$

Clean up a little:

$$\hat{\phi}\frac{\partial V(z)}{\partial z} = -\hat{\phi}\frac{j\omega\mu}{2\pi}\ln\left(\frac{b}{a}\right)I(z)$$

$$\hat{\rho}\frac{\partial I(z)}{\partial z} = \hat{\rho}j2\pi\omega(\epsilon' - j\epsilon'')\ln\left(\frac{b}{a}\right)V(z)$$

$$= -\hat{\rho}\left[\frac{j2\pi\omega\epsilon'}{\ln\left(\frac{b}{a}\right)} + \frac{2\pi\omega\epsilon''}{\ln\left(\frac{b}{a}\right)}\right]V(z)$$

Note that

$$L = \frac{\mu}{2\pi}\ln\left(\frac{b}{a}\right)$$

$$G = \frac{2\pi\omega\epsilon''}{\ln\left(\frac{b}{a}\right)}$$

$$C = \frac{2\pi\epsilon'}{\ln\left(\frac{b}{a}\right)}$$

Finally, we may write the Telegrapher's equations as the following:

$$\hat{\phi}\frac{\partial V(z)}{\partial z} = -\hat{\phi} j\omega L I(z)$$
$$\hat{\rho}\frac{\partial I(z)}{\partial z} = -\hat{\rho}(G + j\omega C)V(z)$$

Because we assumed the conductors are perfect conductor, there's no loss due to resistance. If one includes the loss, then the first Telegrapher's equation should have the resistance term within the equation.

b.Other Transmission Parameters of Lossless Coaxial line

Along with the previous section (where we assumed there's no loss in transmission line), let's find out important parameters of lossless coaxial line such as wave impedance, characteristic impedance, power flow, etc. Specifically, while deriving power flow expression, we emphasize an important concept at the end.

Solving the two Maxwell's equations simultaneously, then we can obtain the (homogenous) Helmholtz's equation [2] (or, wave equation).

$$\hat{\phi}\frac{\partial E_\rho}{\partial z} = -j\omega\mu\left(\hat{\phi} H_\phi\right)$$

$$\hat{\rho}\frac{\partial H_\phi}{\partial z} = -j\omega\epsilon\left(\hat{\rho} E_\rho\right)$$

<<the whole procedure to the wave equation should be dealt in another document. >>

i. Propagation Constant

$$\frac{\partial^2 E_\rho}{\partial z^2} + \omega^2 \mu\epsilon E_\rho = 0$$

which is known as the wave equation. The propagation constant is given as

$$\gamma^2 = -\omega^2\mu\epsilon$$

<<introductory electromagnetic course uses this propagation constant in the wave equation and get the solution because that minus term leads to a *characteristic equation* whose solution is easy to find.>>

Since we assume that the media is lossless, the propagation constant can be reduced to

$$\beta = \omega\sqrt{\mu\epsilon}$$

Recall that this β (propagation constant for lossless media) is the general result for TEM line.

ii. Wave Impedance

Wave impedance (Z_w) is defined as the ratio between E_ρ and H_ϕ.

$$Z_w = \frac{E_\rho}{H_\phi} = \frac{\omega\mu}{\beta} = \sqrt{\frac{\mu}{\epsilon}} = \eta$$

One may verify the above result by taking the Maxwell's equation and solve for E and H. As noting that the solution will be in the form of $e^{-j\beta z}$ dependence, one may derive the above equation. In other word,

$$\frac{\partial E_\rho}{\partial z} = -j\omega\mu(H_\phi)$$

$$-j\beta E_\rho = -j\omega\mu(H_\phi)$$

we obtained the above result.

iii. Characteristic Impedance

The characteristic impedance of TL (transmission line) is defined as

$$Z_0 = \frac{V_0}{I_0}$$

As we already derived earlier, the voltage between the two conductors (e.g. usually the voltage at the inner conductor V_0 and that at the outer conductor is taken as zero) and the current sitting on the inner conductor can be expressed as

$$\frac{V_0}{I_0} = \frac{E_\rho \ln\left(\frac{b}{a}\right)}{H_\phi 2\pi} = \frac{\eta \ln(\frac{b}{a})}{2\pi} = \sqrt{\frac{\mu}{\epsilon}}\frac{\ln(\frac{b}{a})}{2\pi}$$

$$Z_0 = \sqrt{\frac{\mu}{\epsilon}} \frac{\ln(\frac{b}{a})}{2\pi}$$

iv. Power Flow

Since the TL is along z-axis, so is the power flow (or, this can be easily checked by considering the Poynting vector and see where the basis vector is pointing toward)

The power can be derived from the Poynting vector which is defined as

$$P = \frac{1}{2} \int_s \vec{E} \times \vec{H}^* \cdot d\vec{s}$$

(To make the above equation sense mathematically, $d\vec{s}$ must point z-axis which implies $d\vec{s} = \hat{z}\rho d\phi d\rho$ because $\vec{E} \times \vec{H}^*$ will spit out a vector in $\hat{z}$.)

$$P = \frac{1}{2} \int_{\phi=0}^{2\pi} \int_{\rho=a}^{b} \hat{\rho} E_\rho \times \hat{\phi} H_\phi \cdot \hat{z}\rho d\phi d\rho$$

$$= \frac{1}{2} \int_{\phi=0}^{2\pi} \int_{\rho=a}^{b} \hat{z} \frac{V_0 I_0^*}{\rho^2 \ln\left(\frac{b}{a}\right) 2\pi} \cdot \hat{z}\rho d\phi d\rho = \frac{1}{2} V_0 I_0^*$$

$$\boxed{P = \frac{1}{2} V_0 I_0^*}$$

which shows a great harmony with the basic circuit knowledge.

In conclusion, this whole derivation process exhibits an important point: power is not transmitted through the conductors themselves! What does it mean? It means that the flow of power will be caused due to the electric and magnetic field between the two conductors. For finite conductivity (i.e. loss in TL introduced), power may enter the conductors but this power will be converted into “heat”, or “loss” and *won’t be delivered to the load*.

III. Theory of General Coaxial Lines

One starts with the background knowledge that the fields can be induced from a scalar potential function $\Phi(\rho, \phi)$.

With this scalar potential function, one may write the Laplace equation as [3]:

$$\nabla^2 \Phi(\rho, \phi) = 0$$

Considering the geometry of coaxial line, it’s easy to take the Cylindrical coordinate for other mathematical computations. Note that the function is not a function of z.

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\Phi(\rho,\phi)}{\partial\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2\Phi(\rho,\phi)}{\partial\phi^2}=0$$

To solve this equation, we must need boundary conditions and luckily, we already formulated one in the previous section. As seen before, the voltage between the inner and outer conductors was V_0. Here, we set V_0 as the voltage at the inner conductor and zero as the one at the outer one.

Mathematically, this means (i.e. the boundary conditions):

$$\Phi(a,\phi)=V_0$$

$$\Phi(b,\phi)=0$$

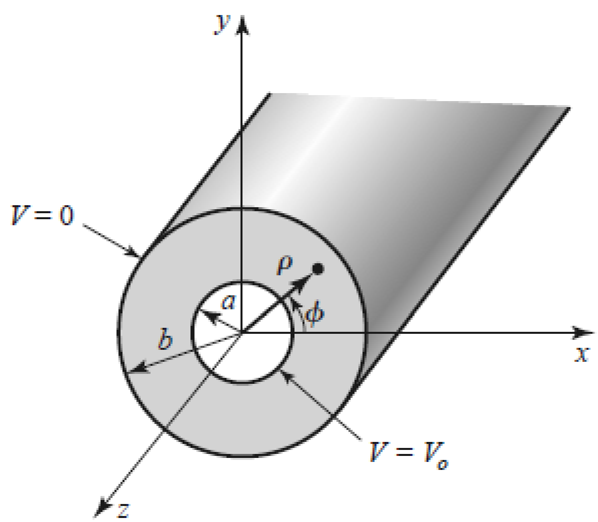

FIGURE 3.15 Coaxial line geometry.

For now, we assume that the scalar potential function can be separated by variable (in the product form) such that <<the method of separation of variable [4]>>

$$\Phi(\rho,\phi)=R(\rho)P(\phi)$$

Plugging this into the original Laplace equation and multiplying by ρ^2:

$$\rho\frac{\partial}{\partial\rho}\left(\rho\frac{\partial R}{\partial\rho}\right)P+\frac{\partial^2 P}{\partial\phi^2}R=0$$

Since $R=P=0$ would yield the trivial solution, we exclude the case and obtain the following by dividing by R and P:

$$\frac{\rho}{R}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial R}{\partial\rho}\right)P+\frac{1}{P}\frac{\partial^2 P}{\partial\phi^2}R=0$$

<<from the separation of variables, the following procedure will be taken in place.>>

$$\frac{\rho}{R}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial R}{\partial\rho}\right)P = -k_\rho^2$$

$$\frac{1}{P}\frac{\partial^2 P}{\partial\phi^2}R = -k_\phi^2$$

$$k_\rho^2 + k_\phi^2 = 0$$

<<I have to re-visit the following section since I forgot how to do the separation of variables method for differential equations.>>

The general solution to

$$\frac{1}{P}\frac{\partial^2 P}{\partial\phi^2}R = -k_\phi^2$$

would be

$$P(\phi) = Acos(n\phi) + Bsin(n\phi)$$

where

$$k_\phi = n$$

must be an integer. Because

$$P(\phi + 2\pi) = Acos\big(n(\phi + 2\pi)\big) + Bsin\big(n(\phi + 2\pi)\big) = P(\phi)$$

Therefore, n must be an integer. In addition, from the boundary condition, we note that

$$\Phi(\rho, \phi)$$

can't be a function of ϕ and to make $P(\phi)$ independent of ϕ, we see n must be zero which brings about

$$P(\phi) = Acos(0) + Bsin(0)$$
$$= A$$

Hence, the scalar potential function can be re-written as:

$$\Phi(\rho, \phi) = \Phi(\rho) = AR(\rho)$$

where A is some constant. Now, we use the fact that

$$k_\phi = n = 0$$

and see $k_\rho = 0$ as well because

$$k_\rho^2 + k_\phi^2 = 0$$

Finally,

$$\frac{\rho}{R}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial R}{\partial\rho}\right)P = -k_\rho^2 = 0$$

Since $R \neq 0$, and $P \neq 0$ and $\rho \neq 0$ (<<this coaxial line analysis excludes the point at $\rho = 0$?>>),

$$\frac{\partial}{\partial\rho}\left(\rho\frac{\partial R}{\partial\rho}\right) = 0$$

$$\rho\frac{\partial R}{\partial\rho} = F$$

where F is some constant. Hence,

$$R = C\ ln(\rho) + D$$

$$\Phi(\rho) = R(\rho) = C\ ln(\rho) + D$$

Because we already formulated the boundary condition, we may find the scalar potential function:

$$\Phi(a,\phi) = V_0 = C\ln(a) + D$$

$$\Phi(b,\phi) = 0 = C\ln(b) + D$$

$$D = -C\ln(b)$$

$$V_0 = C\ln(a) - C\ln(b)$$

$$C = \frac{V_0}{\ln\left(\frac{a}{b}\right)}$$

$$\Phi(\rho,\phi) = \frac{V_0}{\ln\left(\frac{a}{b}\right)}\ln(\rho) - \frac{V_0}{\ln\left(\frac{a}{b}\right)}\ln(b)$$

Note that I got something that is different than the textbook result.

$$\Phi(\rho,\phi) = \frac{V_0}{\ln\left(\frac{a}{b}\right)}\ln\left(\frac{\rho}{b}\right)$$

In conclusion, this is the scalar potential function to find the electric field in the general TEM coaxial line analysis. Another document may expound this relation between the scalar potential function and the electric field.

IV. Higher Modes

Now that we've made it up to this point: the higher modes. Interestingly, the coaxial lines can support not only TEM mode but also TE and TM modes. However, TE and TM modes of coaxial lines are usually cut off (evanescent). It is important to find out the cut off frequency of the lowest order modes to avoid the propagations of these modes (TE and TM). This is because when two or more modes (with different propagation constants) would lead to undesirable effects.

Assuming, still, that waves are propagating along z-axis. We may summaries the basic characteristics of TE and TM modes:

TE	$E_z = 0$ and $H_z \neq 0$
TM	$E_z \neq 0$ and $H_z = 0$

V. HFSS Simulation

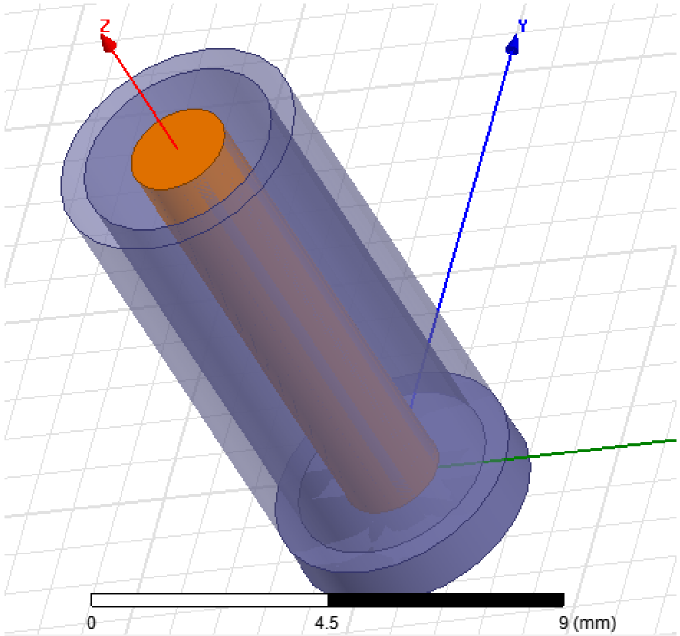

Figure. Coaxial Design on HFSS

Actual coaxial cable looks the above HFSS design simulation. [7] We may observe the cut-plane of RG58C model:

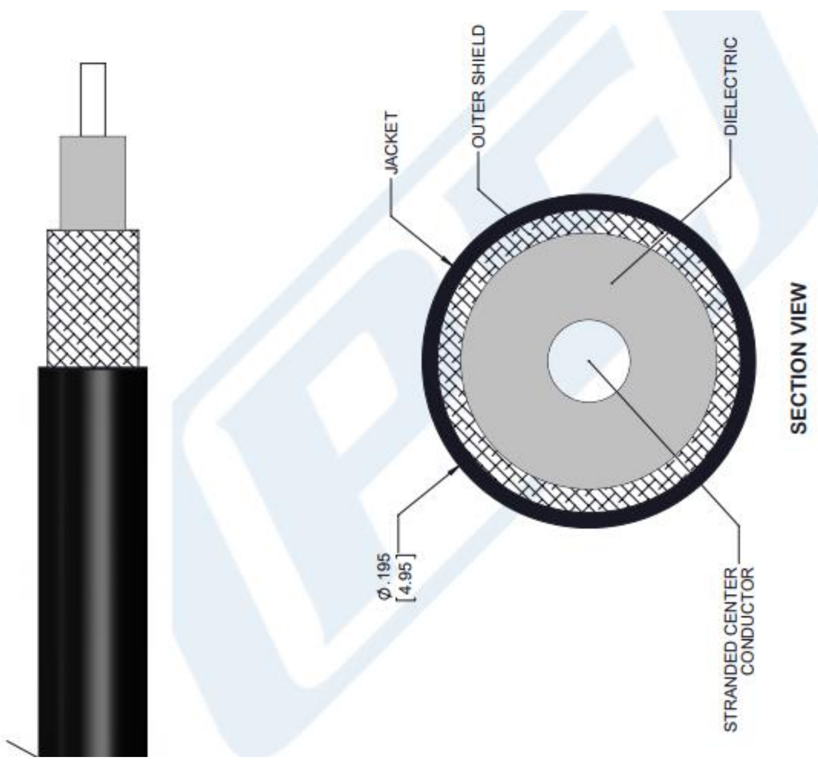

Figure. [7] RG58C model

The following is for the given parameters:

$$a = 1\ mm$$

$$b = 2.5\ mm$$

$$l = 10\ mm$$

The characteristic of this coaxial cable can be found as

$$Z_0 = 41.557\ \Omega$$

The frequency sweep was performed on the interval

$$0.5\ GHz < f < 2\ GHz$$

The above characteristic impedance remains unchanged. However, the input impedance of coaxial cable indeed changed:

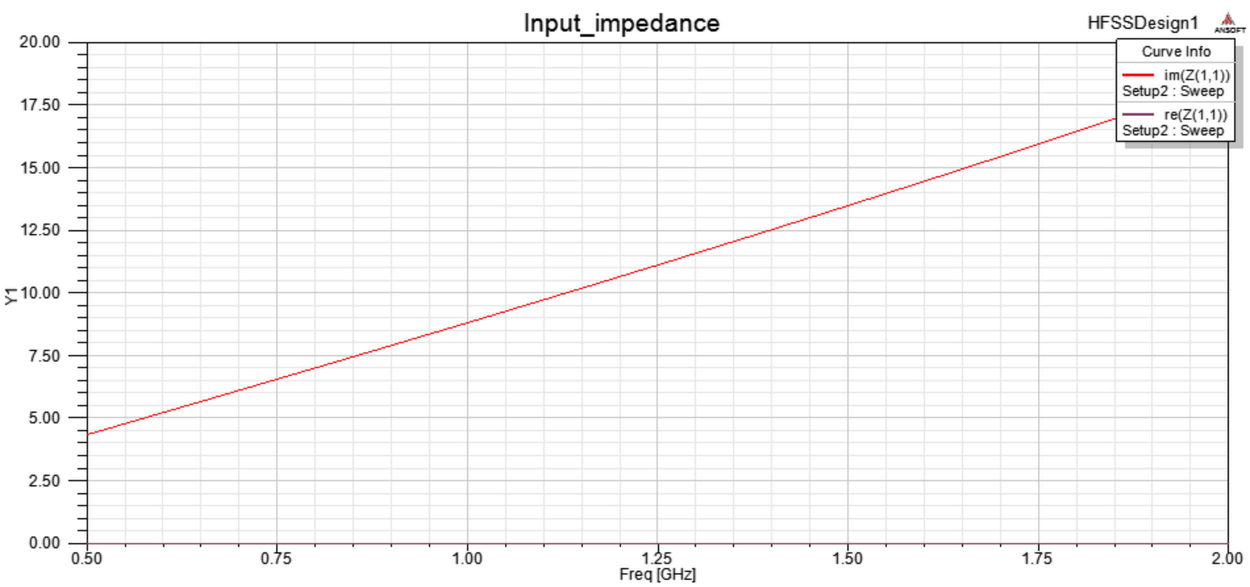

Figure. Input Impedance of Coaxial Cable

Note that input impedance introduced was purely real.

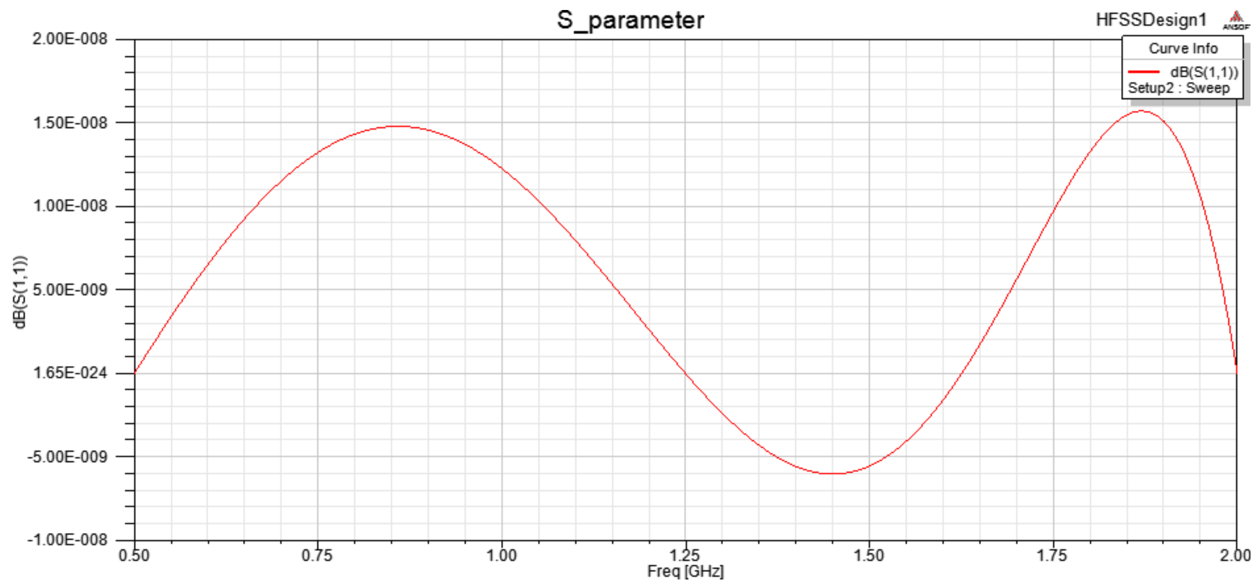

Figure. Reflection Coefficient of Coaxial Cable

VI. LTspice Simulation

a. Example 1 – RG58 with single pulse source

RG58 model can be used to transmit signal at high frequencies; however, the attenuation depends on the frequency. [6] Here, we assume the coaxial line is lossless for investigating theoretical performance of reflected waves.

Before proceeding, let's review the voltage expression at any location on transmission line:

$$\tilde{V}(z) = V_0^+ e^{-j\beta z} + V_0^- e^{+j\beta z}$$

where V_0^+ voltage amplitude toward the load.

The goal of this example is to show how the reflected waves are accounted by investigating the potential model available in market (but the transmission model on spice simulator would be lossless). The single pulse source may help visualizing the process of summing reflected and incident waves at the load.

For LTspice simulation, RG-58 model [5] will be modeled from "tline" spice model with the following characteristics:

$$Z_0 = 50\ ohm$$

$$l = 100\ m$$

$$v_p = 0.659c$$

<<Refer to [8] to find out more information about the propagation velocity.>>

$$c = speed\ of\ light$$

Hence, we can calculate ***the time delay*** caused by this RG58 coaxial cable.

$$t_d = 0.506\ \mu s$$

Let me break down the transmission line by half to measure the voltage at the center of transmission line (the coaxial cable).

The reflection coefficient can be found by

$$\Gamma = \frac{25}{125} = 0.2 = -13.98\ dB$$

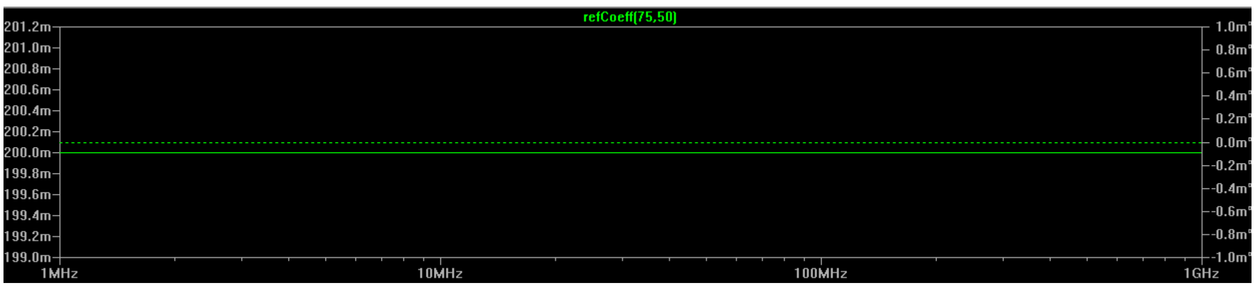

Figure. Reflection Coefficient

<<Note that I used "user defined function" whose function name is ".func" in LTspice. The Wiki section of my github contains how to deal with this.>>

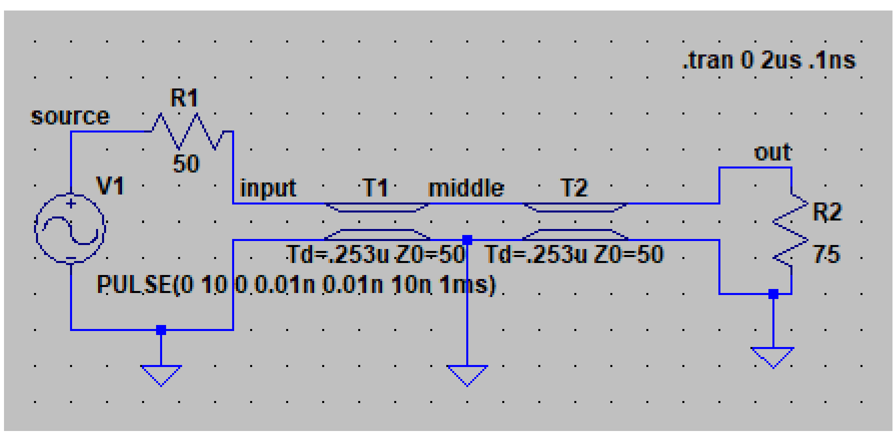

Figure. RG58 Circuit model terminated with 75-ohm

Simulation Tips   Please note that both the rise and fall time are "0.01 ns"; therefore, the maximum step size should be something like 0.1 ns to get a faster result. Any values less than this value might cause a longer time for completing the simulation.

Not specifying the maximum step time will cause distorted square waves.

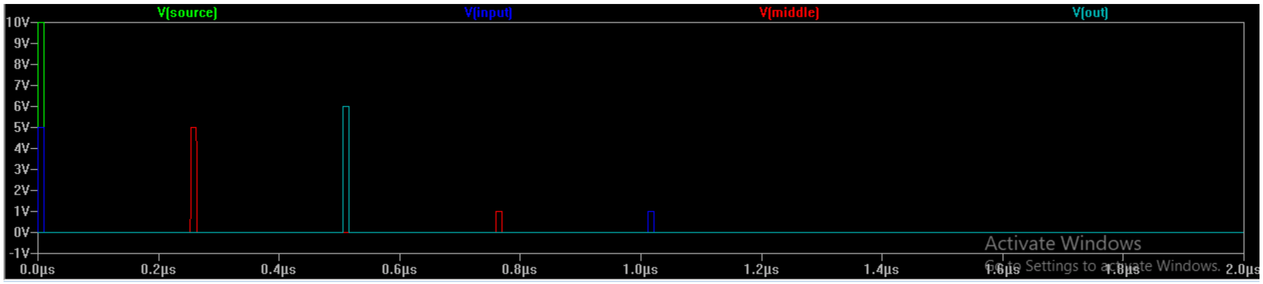

Figure. Transient responses of voltages at different locations

The input excites only once so that we can purely look at the reflected waves without confusing ourselves. The reflected wave can be calculated from

$$V_0^- = \Gamma V_0^+$$

where + sign indicates the wave propagating toward the load and the – sign represents the one toward the generator (the source). <<Note that no phase in the reflection coefficient because the load is purely real for this example.>>

Voltage at the input side of the coaxial cable can be easily found from the voltage division. <<Voltage division between the source impedance Z_s and the characteristic impedance Z_0 because there's GND at the "middle" node of the transmission line. The very first segment of the transmission line will affect the voltage at the "input" node. Regardless of >>

$$\tilde{V}_i = \frac{\tilde{V}_g Z_{in}}{Z_g + Z_{in}} = \frac{10(50)}{50 + 50} = 5\ V$$

or

$$\tilde{V}_i = \tilde{V}(-l) = V_0^+ (e^{j\beta l} + \Gamma e^{-j\beta l})$$

Where $\tilde{V}_i$ voltage (phasor) at "input" node and $\tilde{V}_g$ voltage (phasor) at "source".

As we note that the source voltage provided $10\ V$ but then it became $5\ V$ right at the input side of the coaxial cable. In addition, as we expected, voltage measured at the center of the coaxial cable occurred around $0.253\ \mu s$ with $5\ V$ (no attenuation introduced for transmission line). After $0.506\ \mu s$, we note that voltage at the same location became $1\ V$ which is due to the reflected wave.

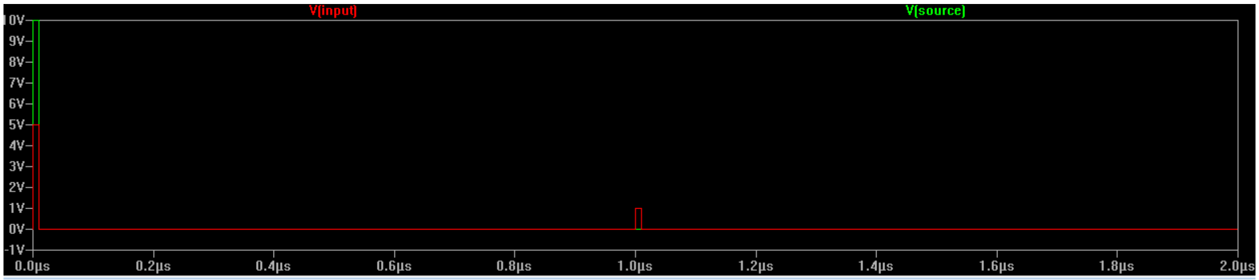

Figure. Voltages at input side of coaxial cable

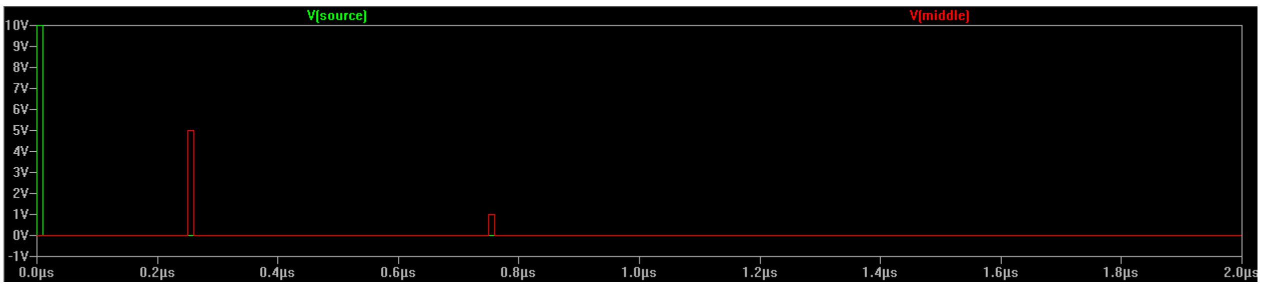

Figure. Voltages at the middle of coaxial cable

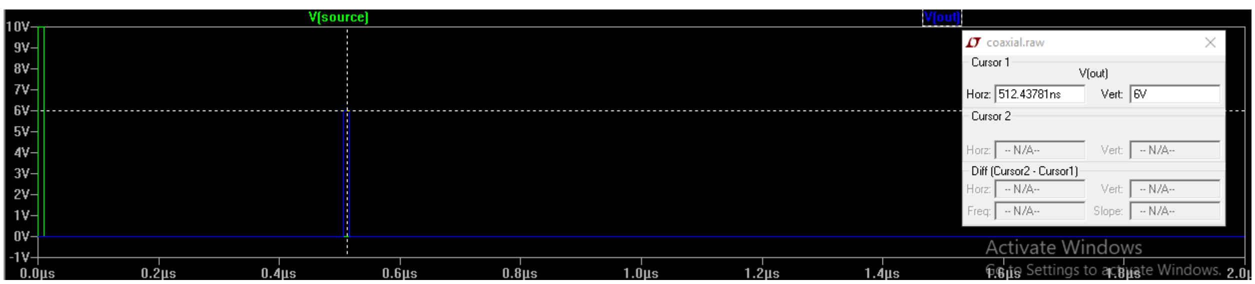

Figure. Voltage at the output

As the figure shows, voltage at the output (75-ohm load) is measured as $6\ V = 5\ V + 1\ V$, which is the sum of the incident and reflected waves. For summary, let us build a table and recall that the source provides $10\ V$. Mathematically,

$$\tilde{V}(z) = V_0^+ e^{-j\beta z} + V_0^- e^{+j\beta z}$$

at the load, we observe: <<to avoid confusion, the location of $z = 0$ is set to be the load whereas $z = -l$ for the source.>>

$$\tilde{V}(0) = V_0^+ e^0 + V_0^- e^0 = V_0^+ + \Gamma V_0^+ = 5 + 1 = 6\ V$$

	Input	Middle	Load
$0\ s$	$5\ V$	$0\ V$	$0\ V$
$0.253\ \mu s$	$0\ V$	$5\ V$	$0\ V$
$0.506\ \mu s$	$0\ V$	$0\ V$	$5\ V + 1V$
$0.759\ \mu s$	$0\ V$	$1\ V$	$0\ V$

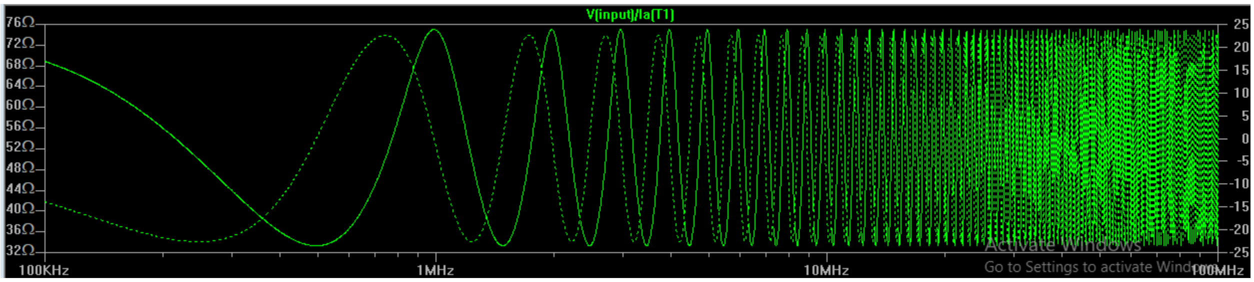

Figure. Input Impedance

Note that input impedance plot's interval was in "decade". The input impedance is bounded in the range between 33 and 75-ohm.

In conclusion, the results that I obtained from the spice well-behaved. Please note that these results are only valid for the "theoretical source" as shown in the schematic. For sources that constantly generate signal, analysis should involve "the small reflection theory", which can be found in the document named "Multisection Matching Transformers", to fully characterize all the reflected waves and their sums.

<<Another point, recall that digital signals (e.g. square-wave pulse) ideally possess infinite bandwidth (or large bandwidth) so I think each harmonic would see a different input impedance whose range is bounded between 33 and 75-ohm, as we saw in the input impedance plot. >>

VII. References

[1] Microwave Engineering, David Pozar, 4th Edition

[2] https://en.wikipedia.org/wiki/Helmholtz_equation

[3] https://en.wikipedia.org/wiki/Laplace%27s_equation

[4] https://en.wikipedia.org/wiki/Separation_of_variables

[5] https://en.wikipedia.org/wiki/Coaxial_cable

[6] https://en.wikipedia.org/wiki/RG-58

[7] https://www.pasternack.com/images/ProductPDF/RG58C-U.pdf

[8] http://www.scte.org/TechnicalColumns/10-03-01%20velocity%20of%20propagation.pdf

[9] Fundamentals of Applied Electromagnetics, Fawwaz T. Ulaby, Eric Michielssen, and Umberto Ravailoi, 6th Edition