Active Filters

I. Introduction

So far, I've worked on passive filters that the maximum gain out of a circuit don't exceed the unity. In addition, passive filters do not require external power source. In active filter circuits, the gain may exceed the unity depends on the feedback loop and the external power supply is required to activate the amplifier. For amplifier, op-amp will be considered for handiness of simulation.

II. RC Filters

a. Introduction to low pass RC filter

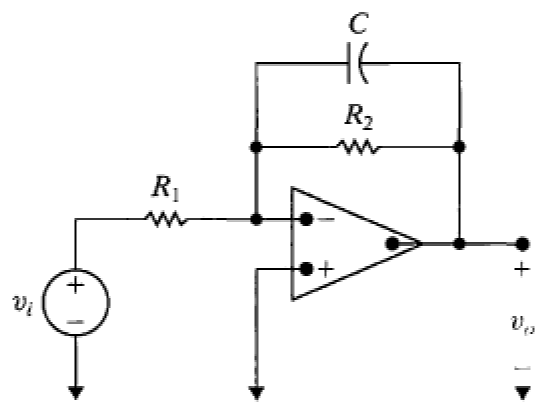


Figure 15.1 ▲ A first-order low-pass filter.

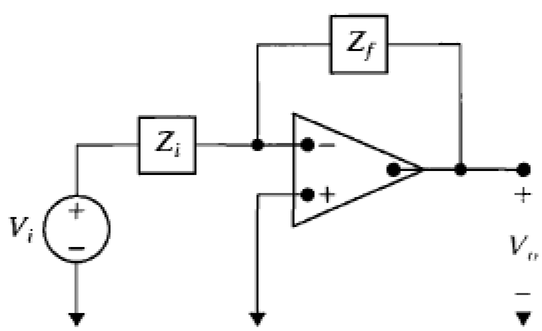


Figure 15.2 ▲ A general op amp circuit.

The basic configuration of RC active filter is now introduced. Let's look at the controlling equations for important parameters. The transfer function can be found:

$$H(s) = -K\frac{\omega_c}{s+\omega_c}$$

It looks the same exact one found in natural response of RC circuit except that the K factor with a negative sign. Negative sign comes from the op-amp circuit. Fortunately, the cutoff frequency equation is the same one in RC circuit.

$$\omega_c = \frac{1}{R_2 C}$$

The gain can be found:

$$K = \frac{R_2}{R_1}$$

Now, for frequency response, passive RC circuit utilized the same frequency axis so that we can stack the magnitude on top of the phase plot. For active filters, the frequency response can be analyzed with Bode Plots. Two important features of Bode plot differ from the frequency response:

(1) Bode plot uses logarithmic axis for frequency range. It covers wider range. Three or four decades will be on the axis, say $1kHz\ to\ 1MHz$.
(2) Bode plot uses decibel axis (dB) for magnitude axis. The conversion can be found: $A_{dB} = 20\log_{10}\lvert H(j\omega)\rvert$

i. Examples - Prototype low pass op-amp filter

Given that $R_1 = 1\ ohm, K = 1, C = 1\ F\ and\ \omega_c = 1\frac{rad}{s}$

These resistor and capacitor values are important because it can be used for a good start point for more complicated filters with more realistic component values.

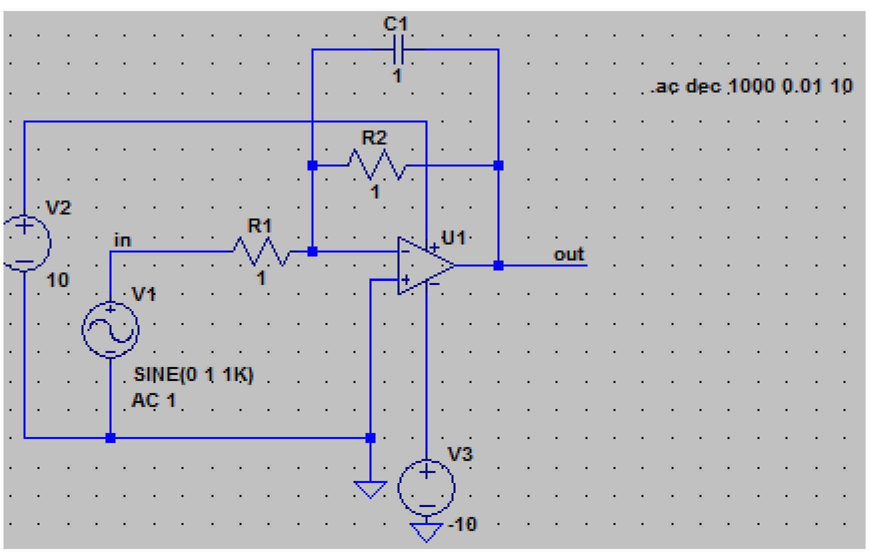


Figure. Prototype Circuit

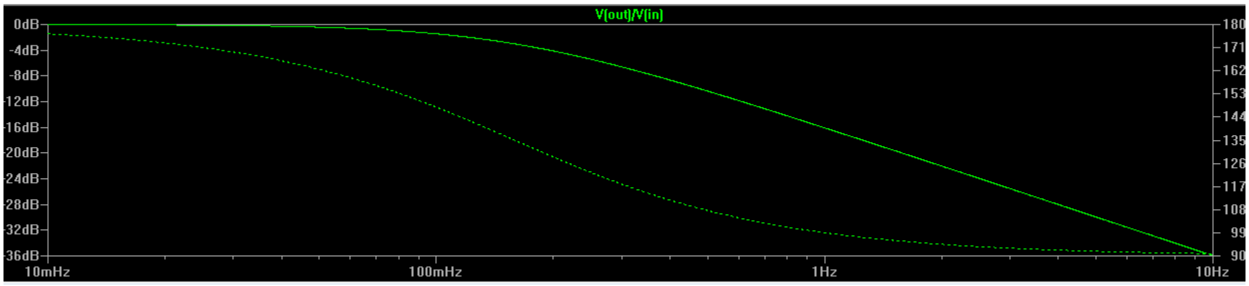


Figure. Bode Plot for RC prototype low pass filter

b.Introduction to RC high pass filters

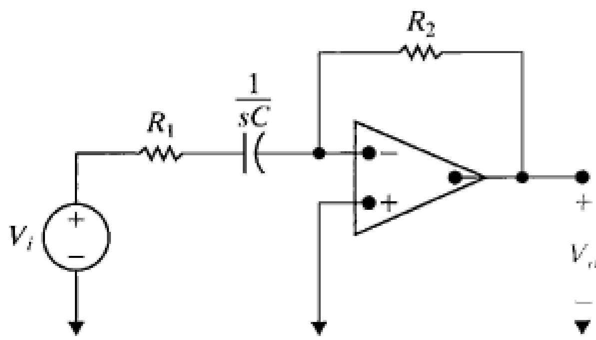


Figure 15.4 ▲ A first-order high-pass filter.

Prototype High Pass Filter Circuit?
We can obtain the prototype high pass filter circuit by setting $R_1 = 1\ ohm, R_2 = 1\ ohm, and\ C = 1F$. This circuit will be considered later when examining the Butterworth High Pass filter design.

The basic configuration for high pass RC circuit is introduced. Note that the location of capacitor is now connected in series with R_1. Let's look at the transfer function for high pass filter.

$$H(s) = -K\frac{s}{s+\omega_c}$$

Note that the form is the one that we can find in the natural response of RC circuit except that the gain (K factor) with the negative sign come from the op-amp.

$$K = \frac{R_2}{R_1}$$

Here, the cutoff frequency can be found and it differs from the low pass one:

$$\omega_c = \frac{1}{R_1 C}$$

i. Example – High pass RC filter

Given that $C = 0.1\mu F$ and the following plot to calculate the rest of the parameters.

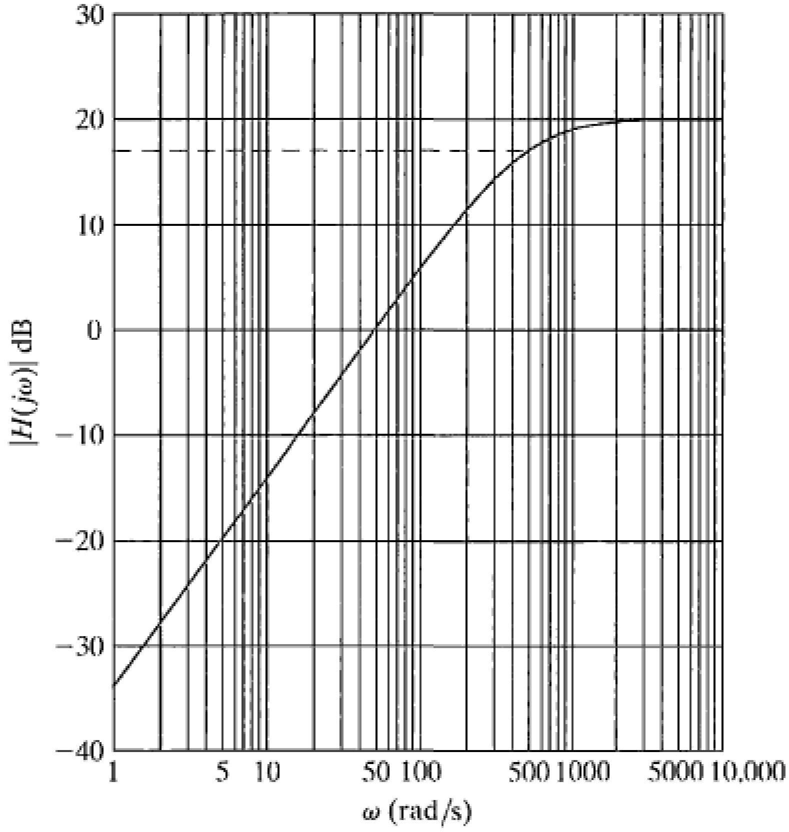


Figure 15.5 ▲ The Bode magnitude plot of the high-pass filter for Example 15.2.

In addition, predict the magnitude plot when $R_L = 10\ khom$ is added.

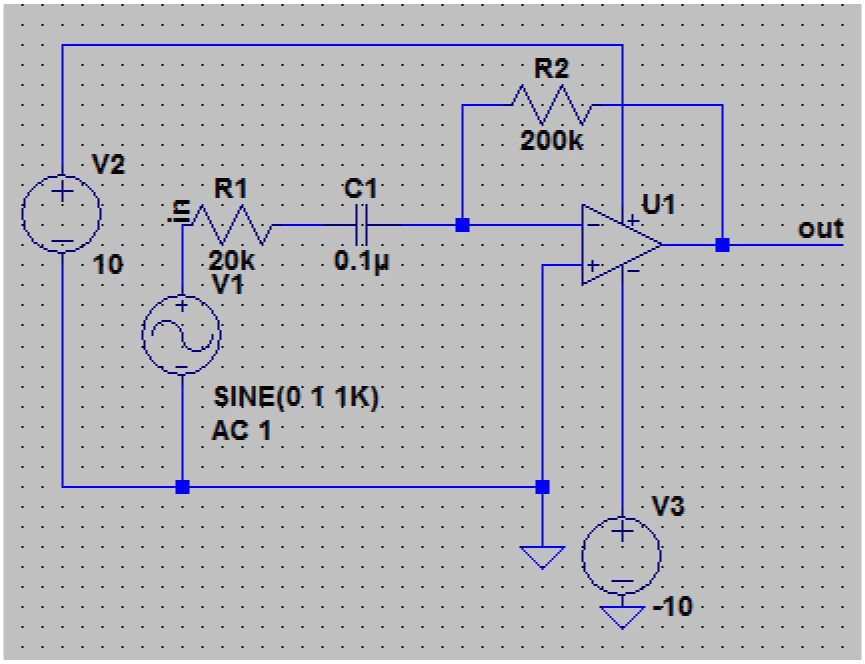


Figure. High pass filter circuit

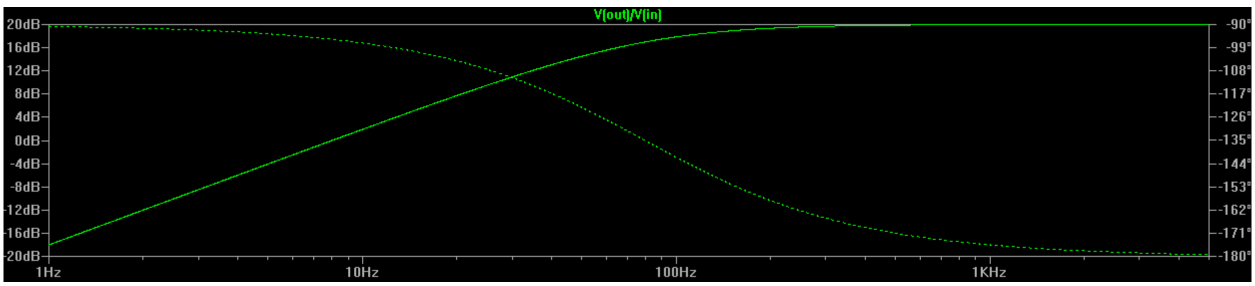


Figure. Bode Plot for High pass Filter

When the load resistor is connected, there would be no change due to that load resistor. There may be a change in the magnitude plot for current but not in voltage response.

III. Scaling

For design purpose, scaling method is one of the convenient ways to compute find the circuit parameters. There are two types of scaling: (1) Magnitude scaling and (2) Frequency Scaling. We assume that resistor has no effect on frequency variations.

$$R' = k_m R,$$

$$L' = \frac{k_m}{k_f} L,$$

$$C' = \frac{1}{k_m k_f} C.$$

Figure. The Scaling Factors

a. Scaling in the Op-amp circuit

Procedure:

First, select the cutoff frequency, $\omega_c = 1\frac{rad}{s}$, if you are designing low or high pass filter, and select the center frequency, $\omega_0 = 1\frac{rad}{s}$, if you are designing bandpass or band reject filter. Then, select $C = 1F$ and compute the rest. Finally, do scaling and it's done!

i. Example – Scaling a series RLC circuit

Given that $Q = 1, f_0 = 500\ Hz$ and $C = 2\mu F$

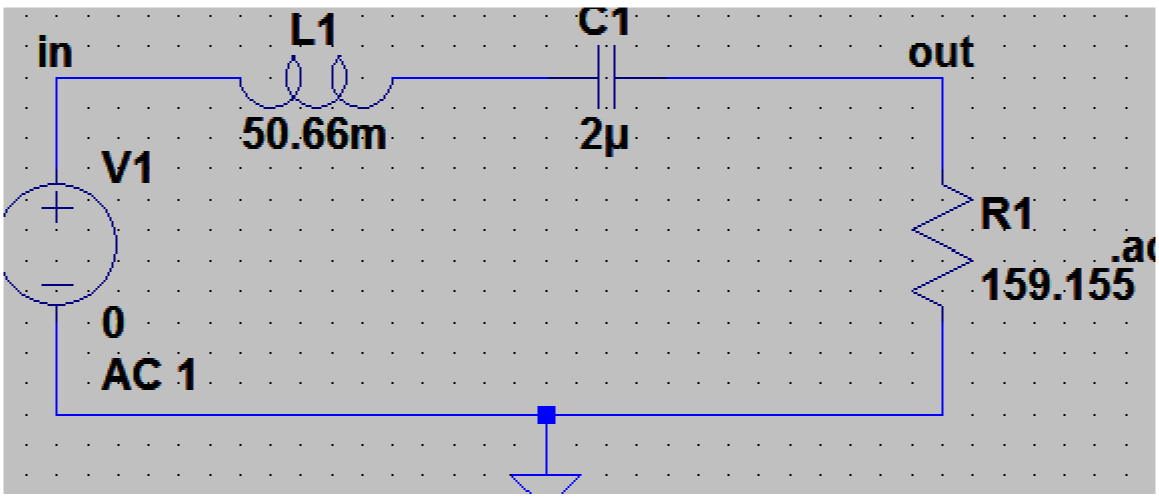


Figure. A series RLC filter from Scaling

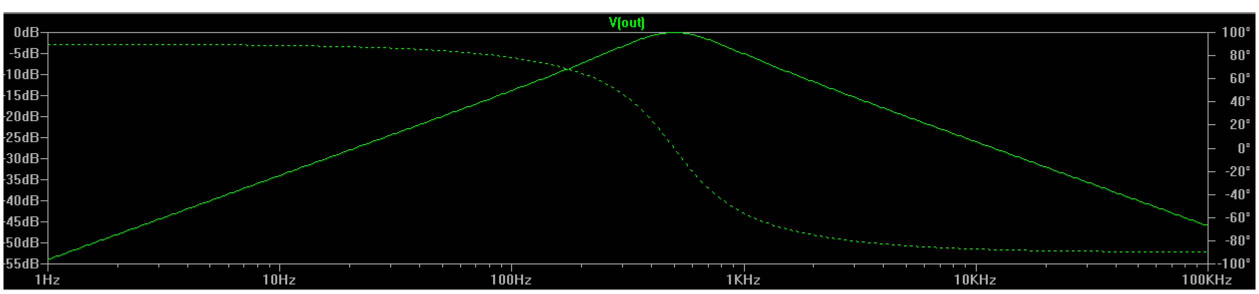


Figure. Bode Plot for the series RLC

IV. Higher Order Op-Amp Filters

Recall that an ideal filter has discontinuity at the cutoff frequency (or the center frequency) which sharply defines the passband and the stopband. Non-ideal filters, rather, has a smooth curve around this point. It's quite impossible to construct an ideal filter, and yet possible to construct a filter that has sharper transition between passband and stopband. Let's look at what methods can be applied.

a. Cascading Identical Filters

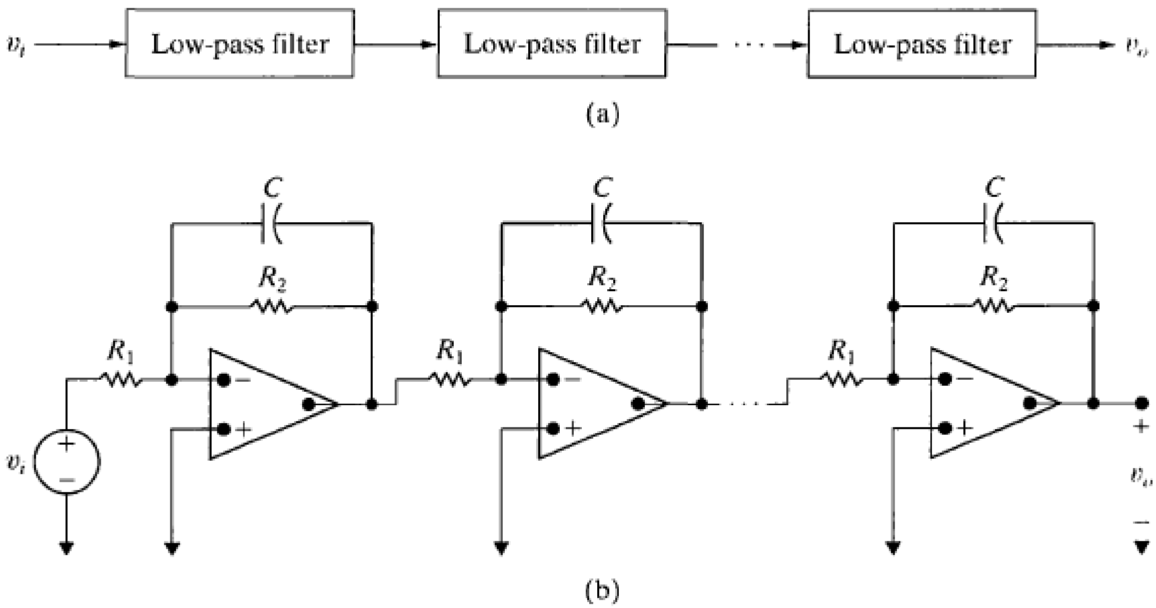


Figure 15.17 ▲ A cascade of identical unity-gain low-pass filters. (a) The block diagram. (b) The circuit.

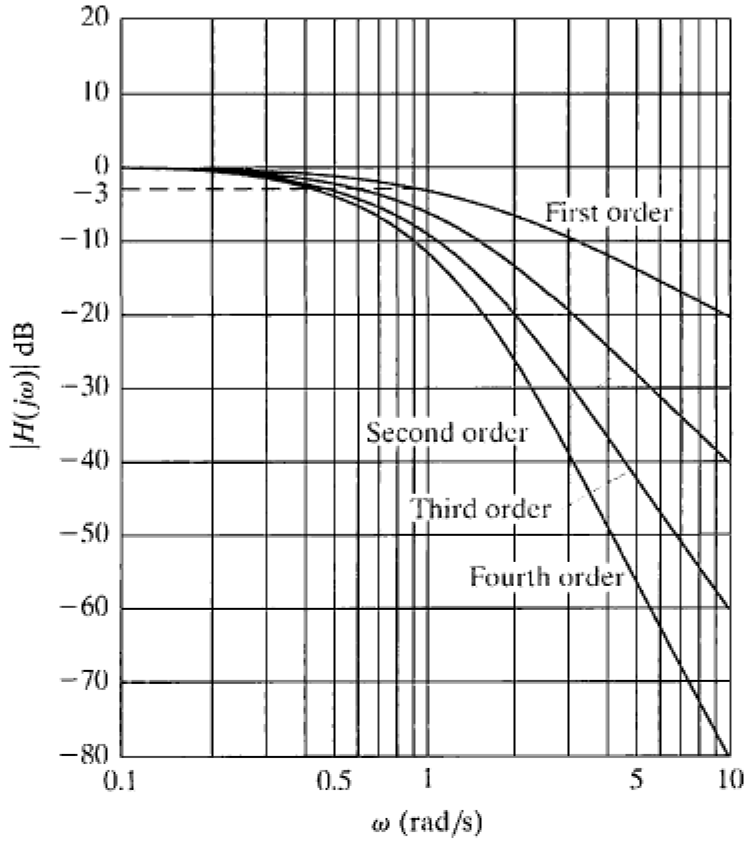


Figure 15.16 ▲ The Bode magnitude plot of a cascade of identical prototype first-order filters.

From this figure, we note that as the order increases, it exposes a sharper transition. The fundamental idea here is that the magnitudes for cascade circuits are additive on Bode Plot.

TABLE E.1 Actual Amplitudes and Their Decibel Values

A_{dB}	A	A_{dB}	A
0	1.00	30	31.62
3	1.41	40	100.00
6	2.00	60	10^3
10	3.16	80	10^4
15	5.62	100	10^5
20	10.00	120	10^6

In general, identical n cascade filters has a slope of

$$20n\frac{dB}{dec}$$

The transfer function for n prototype cascade low pass filter:

$$H(s) = \left(-\frac{1}{s+1}\right)\left(-\frac{1}{s+1}\right)\left(-\frac{1}{s+1}\right)\cdots = \left(-\frac{1}{s+1}\right)^{n}$$

Where $n = the\ order\ of\ H(s) = poles$ in the transfer function

Drawbacks?

The cutoff frequency would change as the order of filter changes. Refer to Figure 15.16.

Solution?
Frequency scaling must be needed. Hence, we must re-calculate the cutoff frequency from the n cascade prototype filter transfer function. Find the magnitude equation for the transfer function and equate it to -3dB point.

$$|H(j\omega_{cn})| = \left|\frac{1}{j\omega_{cn}+1}\right| = \frac{1}{\sqrt{2}}$$

Solving for ω_{cn}:

$$\omega_{cn} = \sqrt{\sqrt[n]{2} - 1}$$

This prototype filter has the maximum unity gain. However, we can obtain non-unity gain by **cascading an inverting amplifier**.

i. Example 15.7 – Design fourth order low pass op-amp filter

Given that $\omega_c' = 2\pi(500Hz), C' = 1\mu F, and\ K = 10$

$$\omega_{cn} = \sqrt{\sqrt[4]{2} - 1} = 0.435$$

Since there’s has been a frequency shift, we must take this into account.

$$k_f = \frac{\omega_c'}{\omega_c} \rightarrow \frac{\omega_c'}{\omega_{cn}} = 7222.39$$

And we can get magnitude scaling factor from:

$$k_m = \frac{1}{k_f C' C} = 138.46$$

We can get the resistor value from:

$$R' = k_m R = 138.46$$

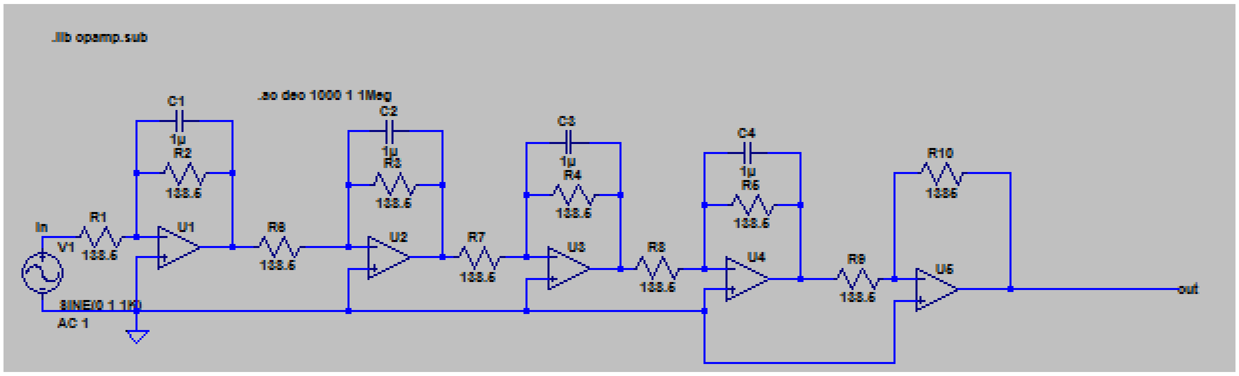


Figure. 4th order low pass filter designed from scaling method

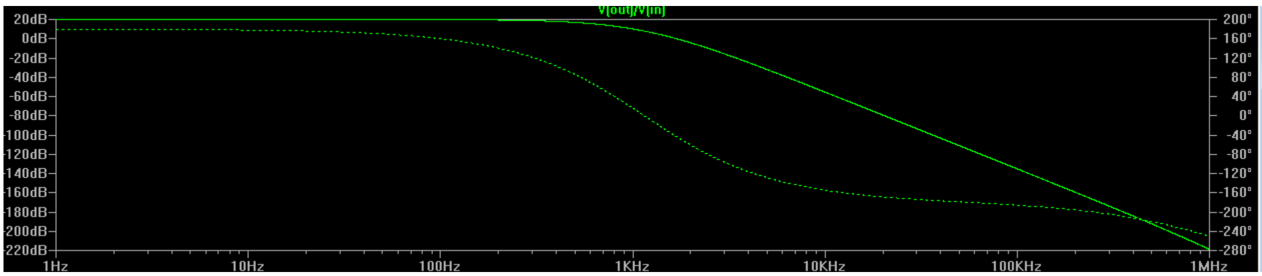


Figure. Magnitude of the transfer function

Simulation Results		
Frequency	$502.09Hz$	$1.684\ kHz$
Magnitude	$16.96dB$	$75.67dB$

Let's check if our simulation results correspond to the calculation results.

$$The\ Gain = 10 = 20dB$$

Since Magnitudes are additive in Bode plot:

$$20dB - 3dB = 17dB \to Cutoff\ Frequency\ occurs = 500Hz$$

We check this is true.

Let's complete the transfer function behavior of cascade low pass filter:

$$H(j\omega) = \left(\frac{\omega_{cn}}{s+\omega_{cn}}\right)^n$$

This form looks similar to the one found in the low pass filter analysis. Hence, the magnitude transfer function should look like:

$$H(j\omega) = \left(\frac{\omega_{cn}}{\sqrt{\omega^2 + \omega_{cn}^2}}\right)^n = \left(\frac{1}{\sqrt{(\omega/\omega_{cn})^2 + 1}}\right)^n = \frac{1}{\left(\sqrt{\left(\frac{\omega}{\omega_{cn}}\right)^2 + 1}\right)^n}$$

From this equation, we notice:

(1) $\omega \ll \omega_{cn} \to |H(j\omega)| \cong 1$

(2) As $\omega \rightarrow \omega_{cn}$, $|H(j\omega)| < 1$ (Recall Ideal low pass filter exhibits unity gain in the passband region. Hence, this is problematic.)

b. Conclusion – Cascade Low Pass Filter

In conclusion, cascade low pass filters are easy to design; however, do not exhibits unity gain in the passband which is problematic. This suggests another consideration for cascading filter design. Next, we will look at the Butterworth design and see if it overcomes the shortcoming of regular cascading low pass filter.

c. Introduction to Butterworth Filters

As introduced earlier, the Butterworth filter exposes unity gain in the passband. The transfer function is given as:

$$|H(j\omega)| = \frac{1}{\sqrt{1 + \left(\frac{\omega}{\omega_c}\right)^{2n}}}$$

Where $n = order\ of\ the\ filter\ (Hence, it\ is\ an\ integer)$

Later on, we need to develop the general s-domain transfer function and find the pattern so that we can begin our design by considering a "prototype circuit", rather than by starting from scratch.

Three important points of this transfer function:

(1) The cutoff frequency is ω_c for all values of n (Meaning that there's no ω_{cn} like cascade low pass filter)
(2) If n is large enough, the denominator is close to unity for $\omega < \omega_c$. (Look at below figure)
(3) The exponent of $\frac{\omega}{\omega_c}$ is always even number.

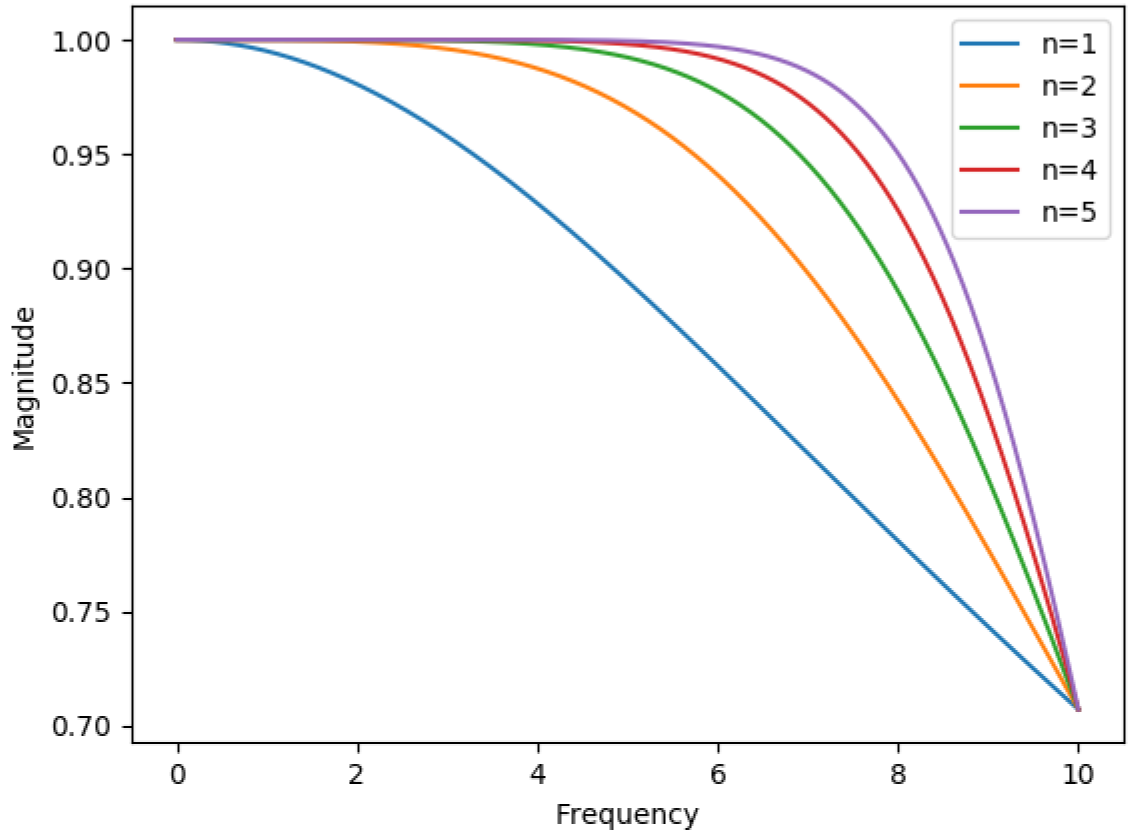
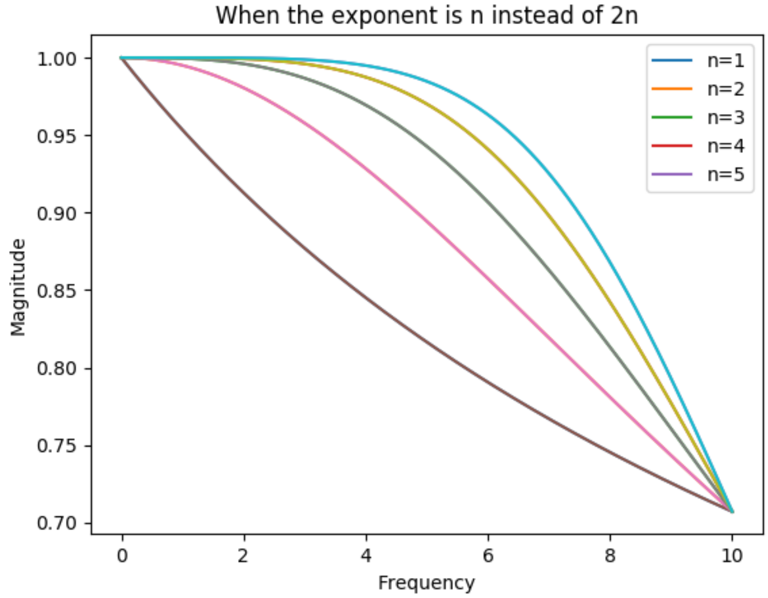


Figure. Python Figure to understand the magnitude of transfer function for Butterworth

This plot is for (2) point which shows that as n increases, graph would tend to sustain its magnitude to unity value longer. In this plot, the cutoff frequency is assumed 10. The plot is merely examined for understanding its behavior.

What happens when the exponent is n instead of 2n?
After looking at the plot, it seems like 2n would yield faster convergence.



Figure

However, an important point of even number of the exponent is not because of the faster convergence characteristic, rather it is for the fact that ***only even number allows realizable circuits***.

Meaning of a realizable circuit? (??)
Although the textbook does not explain directly about it, I interpret it as the follow: it's quite difficult to transform the transfer function that we derive from design procedure into real circuit. Hence, just like the prototype we found in low pass filter, we derive a second order filter (Butterworth type) first and cascade them for the corresponding n. So that we can design and analyze the circuit in more predictable way.

Derivation of s-domain transfer function?
Using the fact that $|H(j\omega)|^2 = H(j\omega)H^*(j\omega) = H(j\omega)H(-j\omega) = H(s)H(-s)$ ($\because s = j\omega$),

$$= \frac{1}{1 + \left(\frac{\omega}{\omega_c}\right)^{2n}}$$

Recall from the prototype low pass filter circuit,

$$\omega_c = 1\frac{rad}{s}$$

We can scale things up later. Now, back to equation,

$$|H(j\omega)|^2 = \frac{1}{1 + (\omega)^{2n}} = H(s)H(-s) = \frac{1}{1 + (-s^2)^n}$$

$$= \frac{1}{1 + (-1)^n s^{2n}}$$

For each n, solving the following:

$$1 + (-1)^n s^{2n} = 0$$

There are always even number of solutions to this equation and they can be expressed in polar coordinate. Now, this step is important which still confuses me a little:

“Assign the left-half plane to $H(s)$ and assign the right-half plane to $H(-s)$”

In s-domain, the $H(s)$ is a reflected image of $H(-s)$ along the y-axis, just simple function property.

For an exercise, let’s do $n = 2$:

$$H(s)H(-s) = \frac{1}{1 + s^4}$$

Hence, we need to consider the following equation to get solution:

$$1 + s^4 = 0$$
$$(s^2 + j)(s^2 - j) = 0$$
$$\left(s + j\sqrt{j}\right)\left(s - j\sqrt{j}\right)\left(s + \sqrt{j}\right)\left(s - \sqrt{j}\right) = 0$$

Now, we obtained the four solutions.

$$s_1 = -j\sqrt{j} = e^{j180°}e^{j90°}e^{j45°} = e^{j315°} = \frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}$$
$$s_2 = +j\sqrt{j} = e^{j90°}e^{j45°} = e^{j135°} = -\frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}}$$
$$s_3 = -\sqrt{j} = e^{j180°}e^{j45°} = e^{j225°} = -\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}$$
$$s_4 = +\sqrt{j} = e^{j45°} = \frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}}$$

I used the following equalities: $j = e^{j180°}\ and\ \sqrt{j} = e^{\frac{j90}{2}°} = e^{j45°}$

Now, consider the solutions that are on the left-half plane, which are $s_2\ and\ s_3$. Hence,

$$Left - half\ plane\ solutions \rightarrow H(s) = \frac{1}{(s - s_2)(s - s_3)}$$
$$= \frac{1}{\left(s + \frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}\right)\left(s + \frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}}\right)}$$
$$H(s) = \frac{1}{s^2 + \sqrt{2}s + 1}$$

TABLE 15.1 Normalized (so that $\omega_c = 1$ rad/s) Butterworth Polynomials up to the Eighth Order

n	nth-Order Butterworth Polynomial
1	$(s + 1)$
2	$(s^2 + \sqrt{2}s + 1)$
3	$(s + 1)(s^2 + s + 1)$
4	$(s^2 + 0.765s + 1)(s^2 + 1.848s + 1)$
5	$(s + 1)(s^2 + 0.618s + 1)(s^2 + 1.618s + 1)$
6	$(s^2 + 0.518s + 1)(s^2 + \sqrt{2} + 1)(s^2 + 1.932s + 1)$
7	$(s + 1)(s^2 + 0.445s + 1)(s^2 + 1.247s + 1)(s^2 + 1.802s + 1)$
8	$(s^2 + 0.390s + 1)(s^2 + 1.111s + 1)(s^2 + 1.6663s + 1)(s^2 + 1.962s + 1)$

Hence, see that we derived the equation for $n = 2$ case correctly.

Note that for every odd number of n, the polynomial contains $H(s) = \frac{1}{s+1}$ term, which is low-pass prototype op-amp filter. The rest of terms can be **always** expressed in terms of some quadratic equations in the form of $H(s) = \frac{1}{s^2+b_1 s+1}$. Let's find the circuit for this second order factor.

Kindly, the textbook just provides the second order factor circuit for us. Now, let's derive the transfer function for the building block for Butterworth filter. The following figure is the circuit for it.

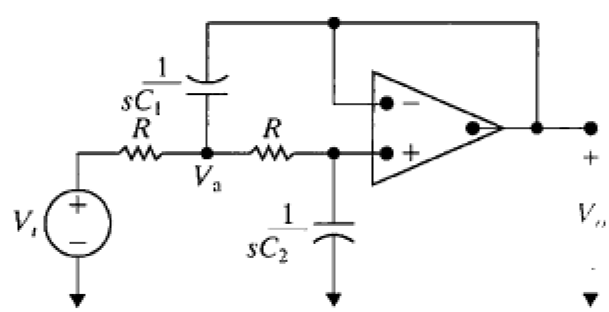


Figure 15.21 ▲ A circuit that provides the second-order transfer function for the Butterworth filter cascade.

Writing the KCL for V_a node and the node at the non-inverting pin of op-amp will give us two simultaneous equations. At the end of day,

$$H(s) = \frac{V_o}{V_i} = \frac{\frac{1}{R^2 C_1 C_2}}{s^2 + \frac{2}{RC_1}s + \frac{1}{R^2 C_1 C_2}}$$

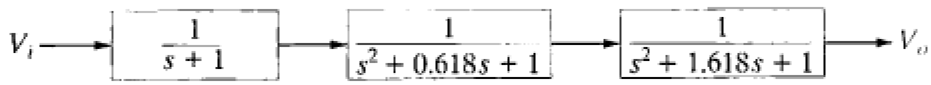


Figure 15.20 ▲ A cascade of first- and second-order circuits with the indicated transfer functions yielding a fifth-order low-pass Butterworth filter with $\omega_c = 1$ rad/s.

Since we have a great tool for design, which is called “Scaling”. We can set $R = 1\ ohm$ and this yield:

$$H(s) = \frac{\frac{1}{C_1C_2}}{s^2 + \frac{2}{C_1}s + \frac{1}{C_1C_2}}$$

Now, let’s compare with the second-order factor:

$$H(s) = \frac{1}{s^2 + b_1 s + 1}$$

This requires:

$$b_1 = \frac{C_1}{2}\ and\ C_1C_2 = 1$$

<u>Don’t forget</u> we came to this expression by assuming $\omega_c = 1\frac{rad}{s}, R = 1\ ohm$, and gain in the passband is equal to the unity. Hence, scaling must be required for an actual design circuit.

i. Example 15.9 – Design fourth-order Butterworth low pass filter

Given: $f_c = 500\ Hz, Gain = 10$ Use $R = 1\ kohm$ as many as possible.

From the Table, we note that the fourth-order Butterworth filter has the following transfer function in the s-domain:

$$H(s) = \frac{1}{(s^2 + 0.765s + 1)(s^2 + 1.848s + 1)}$$

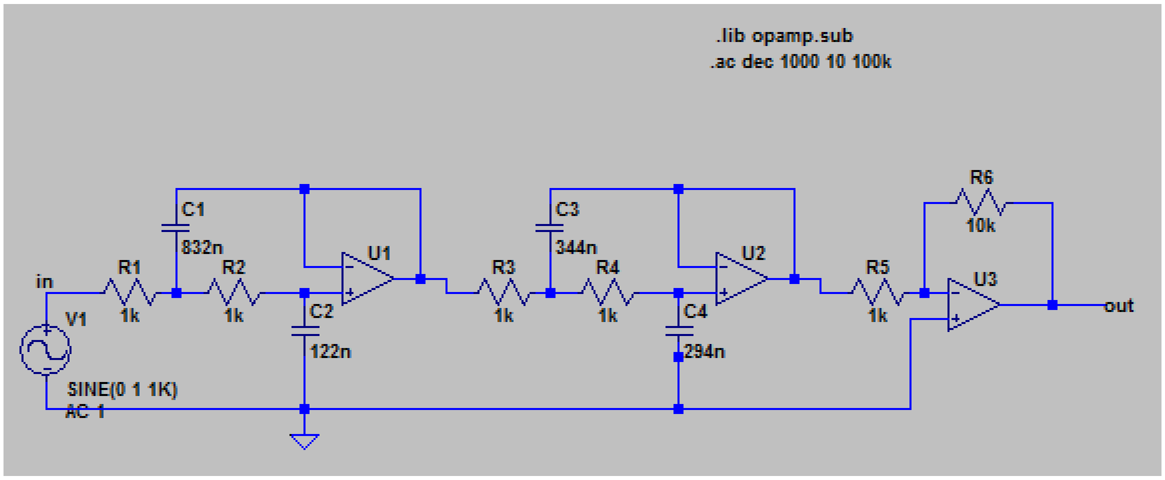


Figure. Constructing the circuit for the example 15.9

Using the fact that

$$k_f = \frac{\omega_c'}{\omega_c} = \frac{2\pi(500\ Hz)}{2\pi(1\ Hz)} = 3141.59$$

$$k_m = \frac{R'}{R} = \frac{1k}{1} = 1000\ (we\ can\ select\ this\ value.)$$

$$C' = \frac{1}{k_m k_f} C$$

Now, let's find out all C's.

Note from the transfer function for the fourth order:

$$b_1 = 0.765\ and\ b_2 = 1.848$$

Hence, we obtain the following results:

$$C_1 = 2.614$$
$$C_2 = 0.3825$$
$$C_3 = 1.082$$
$$C_4 = 0.924$$

After scaling:

$$C'_1 = 832nF$$
$$C'_2 = 122nF$$
$$C'_3 = 344nF$$
$$C'_4 = 294nF$$

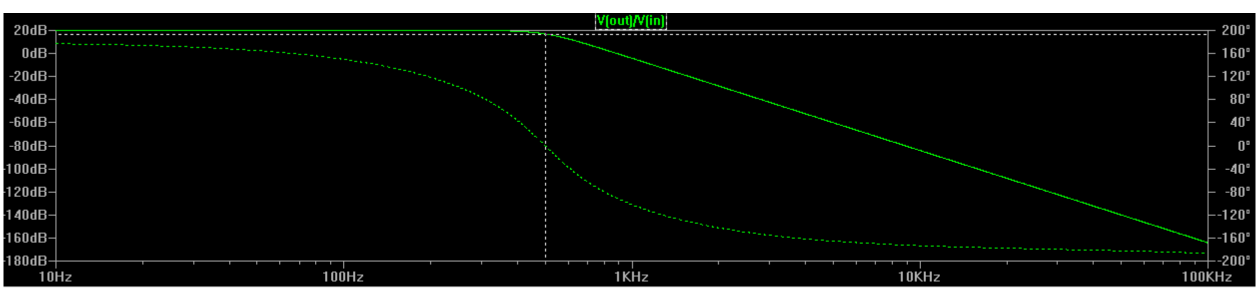


Figure. Frequency Response

From the simulation:

$$H\big(2\pi(501.09\ Hz)\big) = 16.94\ dB \cong 17\ dB$$

Hence, the design was successful. The textbook also provides a plot for comparing the Butterworth and an identical cascade filter:

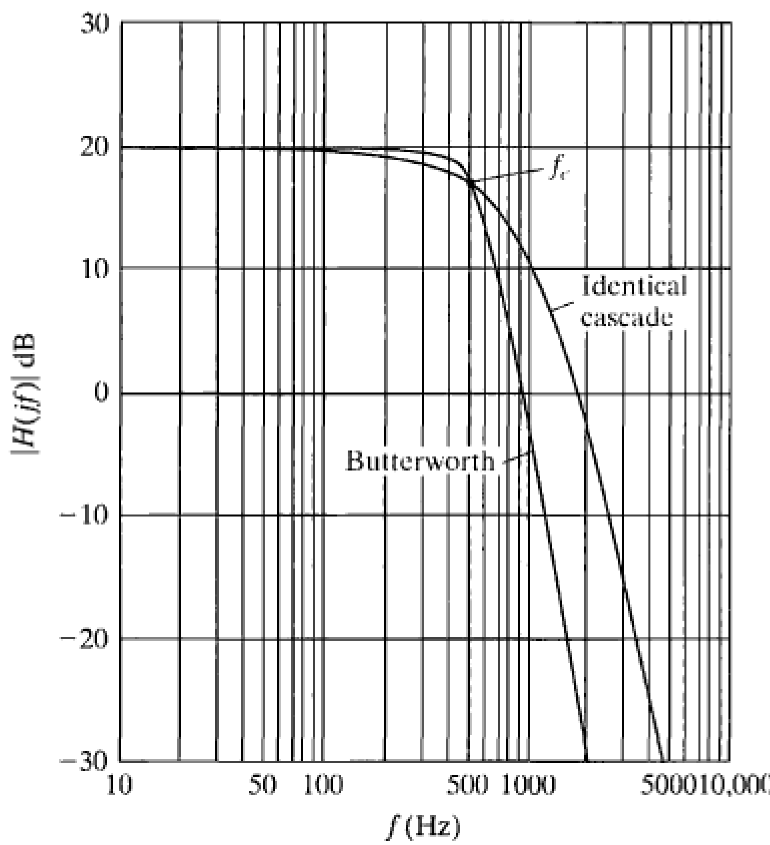


Figure 15.23 ▲ A comparison of the magnitude responses for a fourth-order low-pass filter using the identical cascade and Butterworth designs.

As expected, the Butterworth sustains unity gain longer for the same frequency range than the identical cascade filter.

Determine *<u>the order</u>* of Butterworth Filter?

Now, we've come to a point where we need to determine the order of Butterworth filter. We can't just arbitrarily choose the order. Let's see what are the benefits of higher orders:

(1) Magnitude stays close to unity in the passband
(2) Sharper transition between passband and stopband
(3) Magnitude stays close to zero in the stopband

However, all of these characteristics comes from the cost of circuit, meaning it will be more expensive as it gets higher order. So, how can we determine the smallest n and yet meets the filtering specification?

Filtering specifications usually come from the transition band:

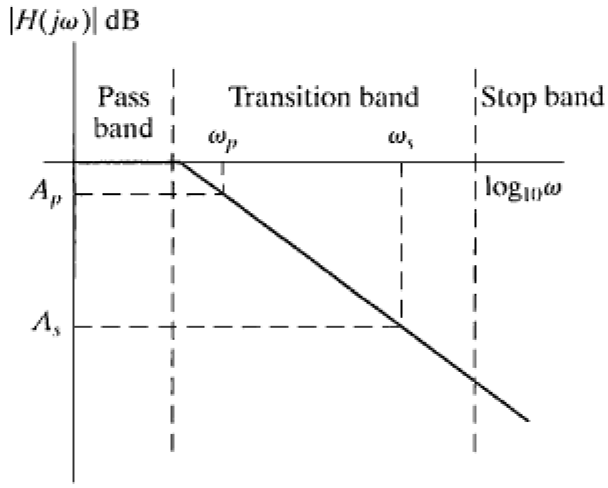


Figure 15.24 ▲ Defining the transition region for a low-pass filter.

$$A_p = 20\log_{10}\left(\frac{1}{\sqrt{1+\omega_p^{2n}}}\right)$$

$$A_s = 20\log_{10}\left(\frac{1}{\sqrt{1+\omega_s^{2n}}}\right)$$

After some math,

$$\left(\frac{\omega_s}{\omega_p}\right)^n = \frac{\sqrt{10^{-0.1A_s}-1}}{\sqrt{10^{-0.1A_p}-1}} = \frac{\sigma_s}{\sigma_p}$$

Where $\sigma_s\ and\ \sigma_p$ assigned for convenience. Note that if this ratio is small, then the length of transition band is also small. Now, let's find n.

$$n = \frac{\log_{10}(\sigma_s)}{\log_{10}\left(\frac{\omega_s}{\omega_p}\right)}\ (\because\ \omega_p = cutoff, A_p = -20\log_{10}(\sqrt{2}) \rightarrow \sigma_p = 1$$

The condition to make the ratio a small value (steep transition) is that:

$$10^{-0.1A_s} \gg 1$$

How I interpret this?
I think since $A_s < 0\ dB$, the steeper transition can be achieved if $|A_s|$ is a large value. A_p is a fixed value in this case. Hence,

$$\sigma_s = \sqrt{10^{-0.1A_s} - 1} \cong 10^{-0.05A_s}$$

Hence, a good approximation for calculating n:

$$n \geq -\frac{0.05A_s}{\log_{10}\left(\frac{\omega_s}{\omega_p}\right)}$$

Because n is an integer.

V. References

[1] Electric Circuits, 9th Edition