The Bode-Fano Criterion

I. Introduction

In this document, the Bode-Fano criterion will be covered. It specifies the upper limit on the reflection for an arbitrary load. Hence, this technique can be used to compare the performances of matching networks. However, the circuit model is limited to the following figure:

FIGURE 5.1 A lossless network matching an arbitrary load impedance to a transmission line.

The whole point of the Bode-Fano Criterion is that it can answer the following three questions:

(1) A perfect match over a specified bandwidth?
(2) If not, how far can we go? ***Trade-off*** between the maximum allowable reflection in the passband, and the bandwidth?
(3) How complex must it be to meet the specification?

Throughout this document, the quality of match may refer to:

$$\Gamma_{\mathrm{m}} = Reflection\ coefficient\ in\ the\ passband$$
$$\Delta\omega = Bandwidth$$

We would like to have smaller reflection in the passband (inverse quantity) and larger bandwidth (directly quantity).

The Bode-Fano Criterion can be summarized by the following figure:

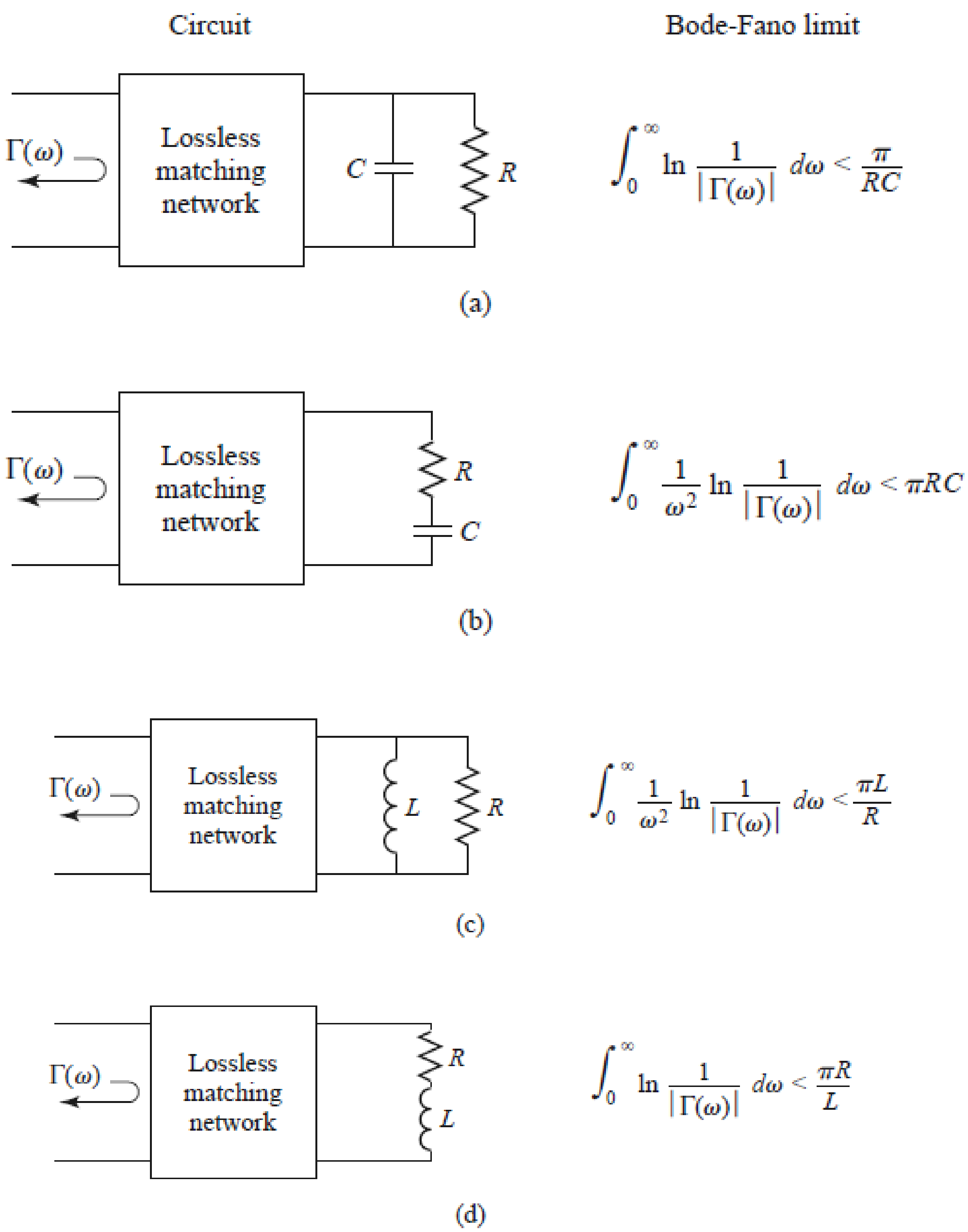


FIGURE 5.22 The Bode–Fano limits for *RC* and *RL* loads matched with passive and lossless networks (ω_0 is the center frequency of the matching bandwidth). (a) Parallel *RC*. (b) Series *RC*. (c) Parallel *RL*. (d) Series *RL*.

In the equation, $\Gamma(\omega) = reflection\ coefficient\ seen\ looking\ into\ the\ arbitrary\ network.$

The following diagram shows what we want for the reflection coefficient response:

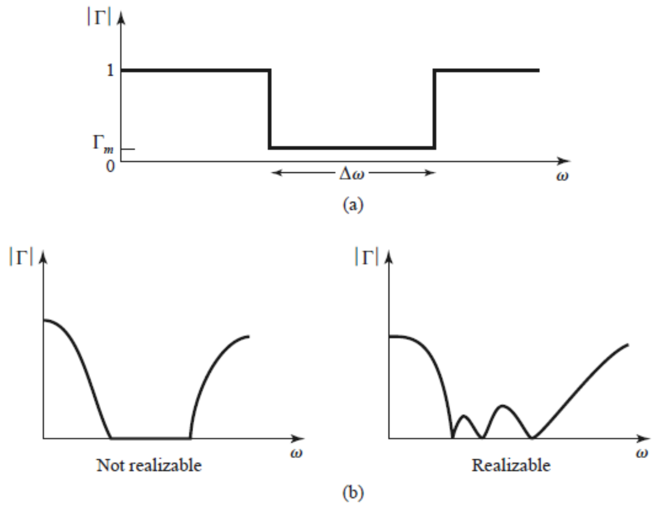


FIGURE 5.23 Illustrating the Bode–Fano criterion. (a) A possible reflection coefficient response. (b) Nonrealizable and realizable reflection coefficient responses.

Figure 5.23 (a) can be realized only with ***infinite number of matching networks***; therefore, it is not achievable in practice.

II. Illustration

Let's first illustrate how this can be done for parallel RC network load.

$$\int_0^\infty \ln \frac{1}{|\Gamma(\omega)|} d\omega$$

Instead of examining the whole region, let's look at the response for a bandwidth with the passband reflection coefficient. Then, we obtain:

$$\int_0^\infty \ln \frac{1}{|\Gamma(\omega)|} d\omega = \int_{\Delta\omega} \ln \frac{1}{\Gamma_m} d\omega$$

$$\boxed{\int \frac{1}{y} dx = x ln\left(\frac{1}{y}\right) + C}$$

Hence,

$$\int_{\Delta\omega} \ln \frac{1}{\Gamma_m} d\omega$$

$$= \Delta\omega \ln \frac{1}{\Gamma_m} \leq \frac{\pi}{RC}$$

We may notice three important observations from this result (for a fixed RC constant):

(1) Bandwidth and the reflection coefficient in the passband is directly related. (which means, increasing bandwidth only happens at the expense of increasing the reflection coefficient.)
(2) The passband reflection can't be zero unless the bandwidth is zero. However, perfect matches may occur at finite frequencies (this point is well illustrated in Figure 5.23 (b): there are 3 distinct finite frequencies where perfect matches occur.)
(3) We saw inverse relation between RC constant and the quality of match. Therefore, higher Q circuit will be intrinsically harder to match than the lower Q circuit. (For better understanding, go to section III. Resonant Circuit Review.)

The quality factor in terms of RC constant is given:

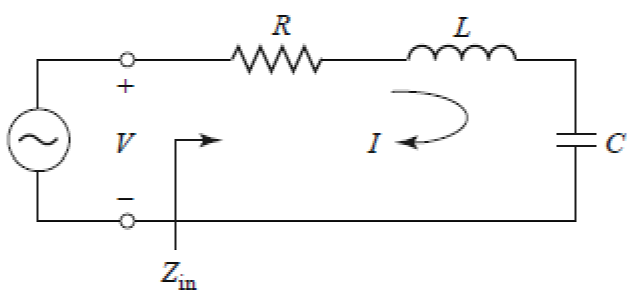
$$Q_0 = \frac{1}{\omega_0 RC} = \frac{\omega_0 L}{R}$$

Seciton III.(a) shows the process.

Chebyshev response (Chebyshev matching transformer) can be approximated when the amplitude is made to equal to Γ_m.

III. Resonant Circuit Review

This section is to review the series RLC circuit to revise the Quality factor. The quality factor can be ***approximated*** in terms of the ratio of the center frequency to the bandwidth.



(a)

The impedance of this series RLC circuit in phasor:

$$Z_{in} = R + j\omega L - \frac{j}{\omega C}$$

Resonance occurs when the imaginary part cancels each other (This point can be checked by investigating if the magnetic stored energy cancels the electric stored energy, $W_m = W_e$).

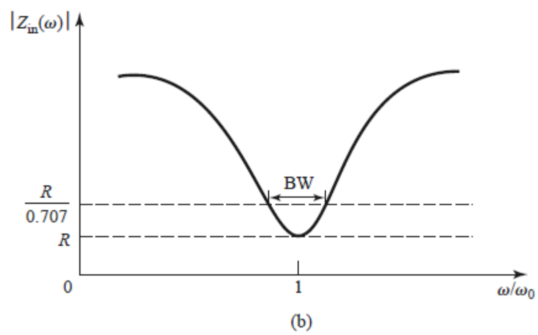


FIGURE 6.1 A series *RLC* resonator and its response. (a) A series *RLC* resonator circuit. (b) Input impedance magnitude versus frequency.

The average magnetic energy stored in inductor can be expressed as:

$$W_m = \frac{1}{4}|I|^2 L$$

The average electric energy stored in capacitor can be expressed as:

$$W_e = \frac{1}{4}|V_C|^2 C$$

The capacitor voltage and currents are related:

$$V_C = -\frac{j}{\omega C} I$$

$$|V_C| = \frac{I}{\omega C}$$

Hence, the average electric energy stored in capacitor can also be expressed in terms of current:

$$W_e = \frac{1}{4}|V_C|^2 C = \frac{1}{4}|I|^2 \frac{1}{\omega^2 C}$$

Resonance occurs when these two-stored energies are equal amount:

$$\omega_0 = \frac{1}{\sqrt{LC}}$$

Now, ***to define*** the quality factor of circuit, we need to know about the power loss of the resonator:

$$P_{loss} = \frac{1}{2}|I|^2 R$$

Because the only lossy unit (assume ideal capacitor and ideal inductor) is the resistor.

The Quality Factor of circuit is defined as:

$$Q = \omega \frac{W_m + W_e}{P_{loss}}$$

$$= \omega \frac{Average\ Stored\ Energy}{Energy\ Loss \big/ second}$$

$$= \omega \frac{Average\ Stored\ Energy}{Power\ Loss}$$

Thus, Q is a measure of loss of a resonant circuit. Higher Q refers to lower loss.

a. Unloaded Resonant Circuit

By adding a load to resonant circuit will cause lowering the Q, which in turn increases the loss. (but increase the bandwidth) Hence, we need to compare $Q's$ of the circuits with/without load. Unloaded Q is denoted by:

$$Q|_{unloaded} = Q_0$$

To derive the equation, recall:

$$Q = \omega \frac{W_m + W_e}{P_{loss}}$$

Using the fact that the two stored energies are equal:

$$Q_0 = \omega_0 \frac{2W_m}{P_{loss}} = \omega_0 \frac{\frac{1}{2}|I|^2 L}{\frac{1}{2}|I|^2 R} = \frac{\omega_0 L}{R}$$

Or

$$Q_0 = \omega_0 \frac{2W_e}{P_{loss}} = \omega_0 \frac{\frac{1}{2}|I|^2 \frac{1}{\omega_0^2 C}}{\frac{1}{2}|I|^2 R} = \frac{1}{\omega_0 RC}$$

Equating the two results:

$$Q_0 = \frac{1}{\omega_0 RC} = \frac{\omega_0 L}{R}$$

Note that both cases show that if R increases, the quality factor will decrease, increasing the loss of the resonant circuit.

b. Finding Q

For most practical resonators, the loss is very small. Hence, ***the perturbation method*** can be used to find the Q.

IV. References

[1] Microwave Engineering, David Pozar, 4th Edition