Matrix Equations

I. Introduction

Many engineering fields utilize matrix equations to describe a system/phenomenon. The general form of a matrix equation is:

$$A\boldsymbol{x} = \boldsymbol{b}$$

The bolded items are called "vector" and A is a matrix. In addition, if $\boldsymbol{b}$ is a non-zero vector, then the equation is categorized as "non-homogenous equation". If it were a zero vector, $\mathbf{0}$, then the equation is called "homogenous equation". This terminology may sound familiar if you have taken differential equation courses.

Given that a matrix has m rows and n columns, there are two ways to look at that matrix. One is called column view and the other one is called row view. Column view would provide a visual aid (e.g. space concept) whereas row view would yield a system of equations view.

I'm not going through how to perform reduced echelon/echelon form of a matrix. Furthermore, python may help throughout this document.

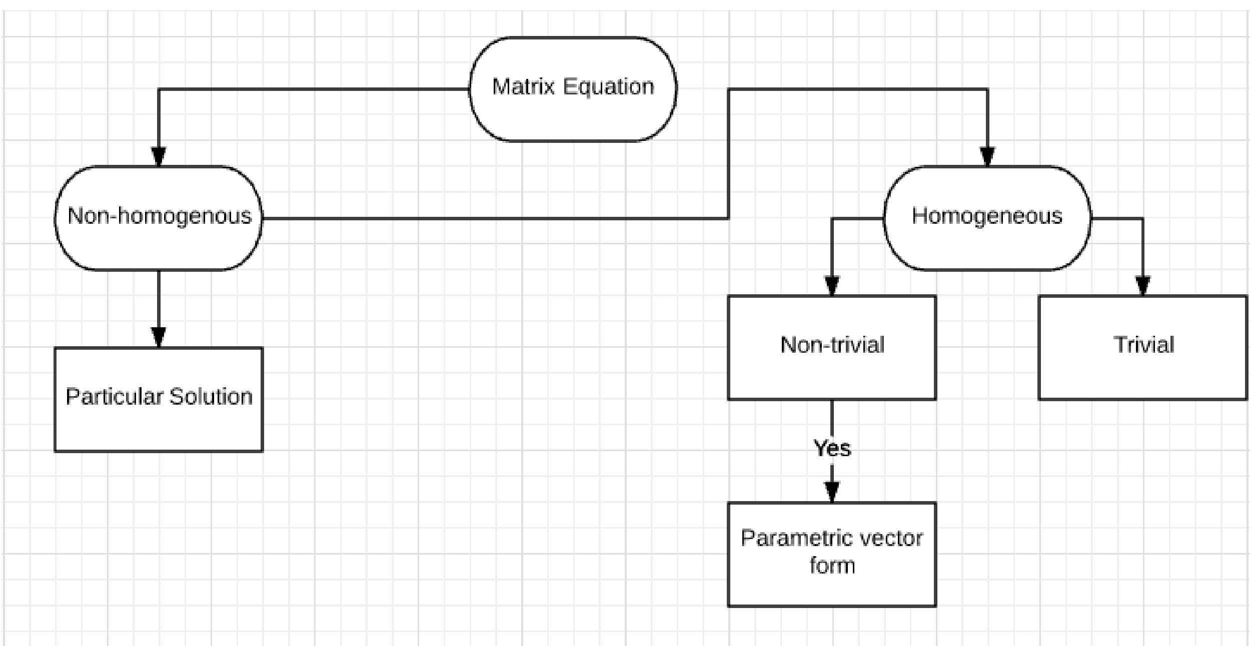

Figure. Matrix Equation Overview

In word, matrix equation can be thought as non-homogenous equation in general. However, if the vector in the right-hand side of equation is the zero vector, then the equation becomes a homogeneous equation. A homogeneous equation always contains zero vector for a solution, which is known as the trivial solution but it could also contain non-trivial solution. If it has non-trivial solution, then the solution can be expressed in the form of parametric vector. Now, non-homogeneous equation's solution has the particular solution along with the non-trivial solution from the homogeneous equation.

In differential equation course, the particular solution arouse along with the initial condition.

II. The Matrix Equation

If you look at the equation closely,

$$A\boldsymbol{x} = [\boldsymbol{a_1}\ \boldsymbol{a_2}\ \boldsymbol{a_3}\ \cdots \boldsymbol{a_n}]\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = x_1\boldsymbol{a_1} + x_2\boldsymbol{a_2} + \cdots x_n\boldsymbol{a_n}$$

In this case, the vector $\boldsymbol{x}$ is in $\mathbb{R}^n$

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

The elements in this vector are like coefficients for corresponding vector elements in the matrix.

The matrix is an $m \times n$

$$A = [\boldsymbol{a_1}\ \boldsymbol{a_2}\ \boldsymbol{a_3}\ \cdots \boldsymbol{a_n}] = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & a_{ii} & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}$$

The vector $\boldsymbol{b}$ is in $\mathbb{R}^m$

$$\boldsymbol{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$$

Note that the number of rows n (for vector $\boldsymbol{x}$) determines the dimension of space $\mathbb{R}^n$. This makes sense if you consider a vector in the Euclidian space, say,

$$s = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Then this means

$$x = 1, y = 2, and\ z = 3$$

We are looking at a vector in 3-dimension space.

Augmented Matrix form?

$$A\boldsymbol{x} = \boldsymbol{b}$$

The equation can also be expressed as the follow:

$$[\boldsymbol{a_1}\ \boldsymbol{a_2}\ \boldsymbol{a_3}\ \cdots \boldsymbol{a_n}\ \boldsymbol{b}]$$

Existence of Solutions?
I think this question makes what makes linear algebra "linear algebra". It's that fundamental question.

In short, the matrix equation has ***a solution*** if:

"$\boldsymbol{b}$ is in Span of $[\boldsymbol{a_1}\ \boldsymbol{a_2}\ \boldsymbol{a_3}\ \cdots \boldsymbol{a_n}]$"

or

"The matrix equation is ***consistent***"

General view of Matrix Equation?
I think if the following statement makes sense to you and you can somehow visualize it, then you get the idea of it correctly.

$$\begin{gathered} if\ a\ set\ of\ vectors\ [\boldsymbol{v_1}\ \boldsymbol{v_2}\ \cdots \boldsymbol{v_p}]\ in\ \mathbb{R}^m\ spans\ \mathbb{R}^m, \\ then\ every\ vector\ b\ in\ \mathbb{R}^m\ is\ a\ linear\ combination\ of \\ the\ set\ of\ vectors\ \boldsymbol{v_1}, \boldsymbol{v_2}, \cdots, \boldsymbol{v_p} \end{gathered}$$

or

$$Span[\boldsymbol{v_1}\ \boldsymbol{v_2}\ \cdots \boldsymbol{v_p}] = \mathbb{R}^m$$

The following theorem is very useful in linear algebra:

THEOREM 4

Let A be an $m \times n$ matrix. Then the following statements are logically equivalent. That is, for a particular A, either they are all true statements or they are all false.

a. For each $\mathbf{b}$ in $\mathbb{R}^m$, the equation $A\mathbf{x} = \mathbf{b}$ has a solution.
b. Each $\mathbf{b}$ in $\mathbb{R}^m$ is a linear combination of the columns of A.
c. The columns of A span $\mathbb{R}^m$.
d. A has a pivot position in every row.

Note that this theorem is for a "coefficient matrix", not for an "augmented matrix".

III. Solution Sets of Linear systems

There are two: (1) homogeneous and (2) non-homogeneous.

a. Homogenous Equations

A equation

$$A\boldsymbol{x} = \boldsymbol{0}$$

It always has a solution, which is known as the trivial solution,

$$\boldsymbol{x} = \boldsymbol{0}$$

However, it's obvious and un-interesting. The real question is:

"Is there non-trivial solution(s) that satisfy the homogenous equation?"

The condition for non-trivial solution in a homogeneous equation?

"If and only if there is at least one free variable."

This can be checked by solving the augmented matrix form and ***if you get anything except***:

$$A\boldsymbol{x} \sim [I\ \boldsymbol{0}]$$

Where I represents the identity matrix. In other words, the matrix A must be singular, or its determinant is equal to zero.

i. Example 1 – Determine if a homogeneous system has a non-trivial solution.

A homogenous system is given:

$$3x_1 + 5x_2 - 4x_3 = 0$$

$$-3x_1 - 2x_2 + 4x_3 = 0$$

$$6x_1 + x_2 - 8x_3 = 0$$

After taking $rref([A\ \boldsymbol{b}])$

$$rref([A\ \boldsymbol{b}]) = \begin{bmatrix} 1 & 0 & -1.33 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Immediately, we notice x_3 is a free variable in this case.

$$\begin{bmatrix} x_1 - 1.33x_3 \\ x_2 \\ 0 \end{bmatrix} = \boldsymbol{0}$$

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 1.33 \\ 0 \\ 1 \end{bmatrix} = x_3\boldsymbol{v}$$

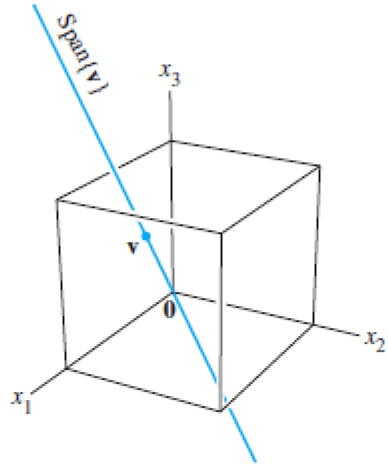

FIGURE 1

Geometrically, the solution set is ***a line through the origin (zero vector)*** in $\mathbb{R}^3$

j. Example 2 – A single linear equation as a simple system of equations

Given that

$$10x_1 - 3x_2 - 2x_3 = 0$$

Hence, the solution

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0.3x_2 + 0.2x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} 0.3 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0.2 \\ 0 \\ 1 \end{bmatrix} = x_2\boldsymbol{u} + x_3\boldsymbol{v}$$

In this case, x_2 and x_3 are free variables. Graphically, the solutions are represented ***as a plane***:

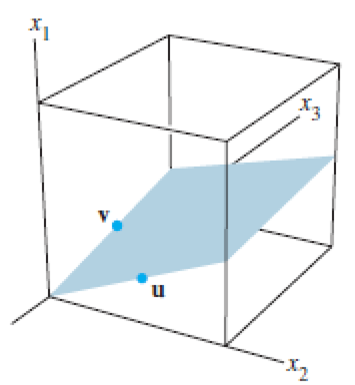

FIGURE 2

b.Non-homogeneous Equations

THEOREM 6

> Suppose the equation $A\mathbf{x} = \mathbf{b}$ is consistent for some given $\mathbf{b}$, and let $\mathbf{p}$ be a solution. Then the solution set of $A\mathbf{x} = \mathbf{b}$ is the set of all vectors of the form $\mathbf{w} = \mathbf{p} + \mathbf{v}_h$, where $\mathbf{v}_h$ is any solution of the homogeneous equation $A\mathbf{x} = \mathbf{0}$.

Recall the Example 1, let's do the following example to understand the theorem 6.

i. Example 3 – Non-homogeneous equation

$$A\boldsymbol{x} = \boldsymbol{b}$$

$$\begin{bmatrix} 3 & 5 & -4 \\ -3 & -2 & 4 \\ 6 & 1 & -8 \end{bmatrix} \boldsymbol{x} = \begin{bmatrix} 7 \\ -1 \\ -4 \end{bmatrix}$$

Note that the matrix is identical to the one in the Example 1. However, the vector $\boldsymbol{b}$ is now different.

$$rref([A\ \boldsymbol{b}]) = \begin{bmatrix} 1 & 0 & -1.33 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} x_1 - 1.33x_3 \\ x_2 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$$

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 1.33 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} = x_3\boldsymbol{v} + \boldsymbol{p}$$

We notice that this result and the theorem 6 shows a great accordance. In more general form, the solution to non-homogeneous equation can be expressed:

$$\boldsymbol{x} = t\boldsymbol{v} + \boldsymbol{p}$$

"Parametric vector form"

> Note that the solution $\boldsymbol{x}$ is parallel to the particular solution $\boldsymbol{p}$**.** If two vectors are parallel to each other, then two vectors are linearly dependent to each other.

> Translation in the homogeneous equation solution?
> How can we graphically interpret this particular solution along with the non-trivial solution from the homogeneous equation? This is merely a "translation" in vector.

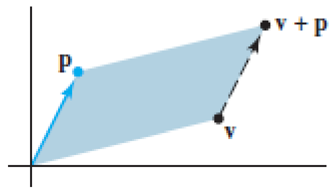

FIGURE 3
Adding $\mathbf{p}$ to $\mathbf{v}$ translates $\mathbf{v}$ to $\mathbf{v}+\mathbf{p}$.

"We say $\boldsymbol{v}$ is translated by $\boldsymbol{p}$ to $\boldsymbol{v}+\boldsymbol{p}$"

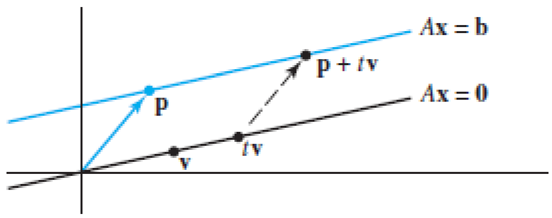

FIGURE 5 Parallel solution sets of $A\mathbf{x}=\mathbf{b}$ and $A\mathbf{x}=\mathbf{0}$.

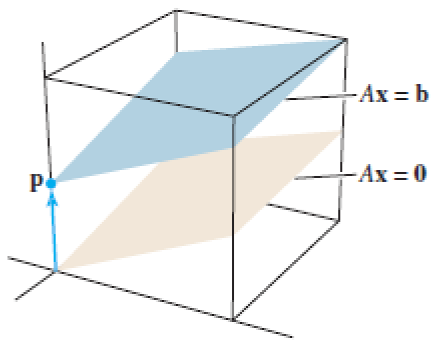

FIGURE 6 Parallel solution sets of $A\mathbf{x}=\mathbf{b}$ and $A\mathbf{x}=\mathbf{0}$.

"The solution set of $A\boldsymbol{x}=\boldsymbol{b}$ is a line through $\boldsymbol{p}$ parallel to the solution set of $A\boldsymbol{x}=\boldsymbol{0}$"
Theorem 6 and Figure 5 and 6 are valid only when there is at least one nonzero solution to nonhomogeneous equation.

IV. References

[1] Linear Algebra and its Applications, David C. Lay, 4^{th} Edition