Waveguides Analysis

I. Introduction

In this document, TEM, TE, and TM modes will be explained. In general, the geometry of arbitrary transmission lines or waveguides look like:

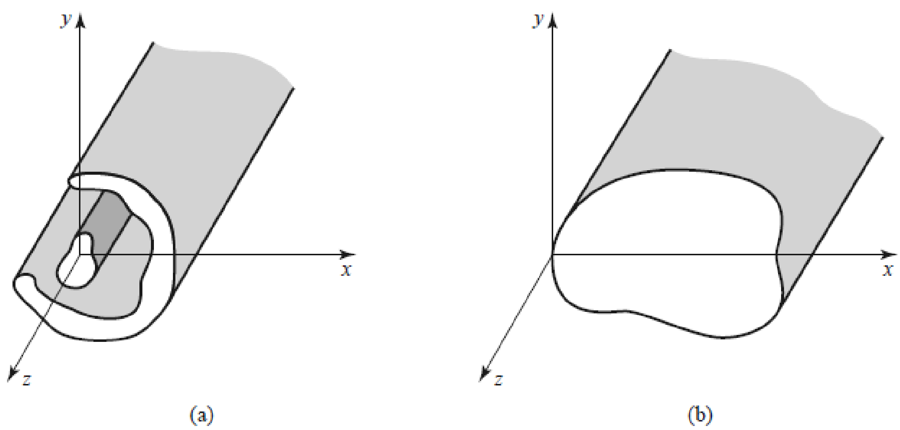

FIGURE 3.1 (a) General two-conductor transmission line and (b) closed waveguide

For simplicity, conductors for analysis are assumed to be perfect conductor; however, the attenuation can be even found from ***the perturbation method***.

Throughout this document, we assume that all fields are time-harmonic, that is $e^{j\omega t}$ dependency.

<<The general idea of approaching to solutions (i.e. E- and H- fields) would be as the following:

The ***Maxwell's equations*** help identifying wave parameters such as the cutoff wavenumber, the wavenumber of the filling material of the transmission line/ the wavenumber of the region in the waveguide, and the propagation constant, depending on the mode of operation (i.e. TEM, TE, or TM). To find out the solutions of field expression, ***the Helmholtz equations*** must be used. For the nature of the Helmholtz equation, the boundary condition must be used to complete the solution set that will be uniquely defined for the particular design/geometry under concern. I had read about the uniqueness theorem in EM fields somewhere but I couldn't find any relevant theoretical paper on online. I will come back for this point later though.>>

Due to the fact that there many types of waveguide out there, the only introductory argument about waveguide will be covered in this document. Specific types of waveguide (e.g. rectangular waveguide, etc.) will be covered in another document in different names.

II. Theory

a. Plane Wave Propagation in Lossless Media

In this section, we are going to examine whether guided waves can have the component (i.e. vector component) in which the waves are propagating. To be specific, if we assume that the wave is propagating in the z-direction (in the Cartesian coordinate), then the fields (waves) can't have z-variation in their vector components.

b. General Solutions for TEM, TE, and TM modes

We start by writing the electric field and magnetic field in the cartesian coordinate as

$$\vec{E}(x,y,z) = [\vec{e}(x,y) + \hat{a}_z e_z(x,y)]e^{-j\beta z}$$

$$\vec{H}(x,y,z) = [\vec{h}(x,y) + \hat{a}_z h_z(x,y)]e^{-j\beta z}$$

As usual, $e^{-j\beta z}$ account for wave propagation in the $+z$ direction. If conductor loss or dielectric loss is introduced, then $e^{-j\beta z}$ should be converted into $e^{-\gamma z}$ where $\gamma = \alpha + j\beta$. In addition, $\vec{e}(x,y)$ and $\vec{h}(x,y)$ are the transverse fields for each E and H.

To find these fields, of course, we start with the Maxwell's equations (Faraday's and Ampere's):

$$\nabla \times \vec{E} = -j\omega\mu\vec{H}$$

$$\nabla \times \vec{H} = j\omega\epsilon\vec{E}$$

Assuming the Cartesian coordinate for the basis coordinate, we can generate six different equations from the Maxwell's equations. A general curl can be expanded in the following format:

$$\hat{a}_x\left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}\right) - \hat{a}_y\left(\frac{\partial E_z}{\partial x} - \frac{\partial E_x}{\partial z}\right) + \hat{a}_z\left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right) = -j\omega\mu(\hat{a}_x H_x + \hat{a}_y H_y + \hat{a}_z H_z)$$

$$\hat{a}_x\left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}\right) - \hat{a}_y\left(\frac{\partial H_z}{\partial x} - \frac{\partial H_x}{\partial z}\right) + \hat{a}_z\left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right) = j\omega\epsilon(\hat{a}_x E_x + \hat{a}_y E_y + \hat{a}_z E_z)$$

The vector components that are in the same axis can be re-written as (Hint: $E_x, E_y, H_x, and H_y$ do not contain z dependency but they are multiplied to the factor $e^{-j\beta z}$, which has the information about the propagation and location.):

The following <u>six</u> *equations* are useful in TEM mode waves analysis:

$$\frac{\partial E_z}{\partial y} + j\beta E_y = -j\omega\mu H_x$$

$$-\frac{\partial E_z}{\partial x} - j\beta E_x = -j\omega\mu H_y$$

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = -j\omega\mu H_z$$
$$\frac{\partial H_z}{\partial y} + j\beta H_y = j\omega\epsilon E_x$$
$$-\frac{\partial H_z}{\partial x} - j\beta H_x = j\omega\epsilon E_y$$
$$\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = j\omega\epsilon E_z$$

Note: For TE and TM, we should concern modified version of these six equations which will be reduced to four equations.

<<This might be the first time solving something like the above system. I'm not sure is there a systematic way to solve such system though. The textbook insisted that the transverse components (i.e. x and y components) The textbook approach is as the following: solve for x and y components in terms of z component (direction of the propagation for TEM mode).>>

$$\frac{\partial E_z}{\partial y} + \frac{j\beta}{j\omega\epsilon}\left(-\frac{\partial H_z}{\partial x} - j\beta H_x\right) = -j\omega\mu H_x$$

$$\frac{\partial E_z}{\partial y} - \frac{\beta}{\omega\epsilon}\frac{\partial H_z}{\partial x} = H_x\left(\frac{j\beta^2}{\omega\epsilon} - j\omega\mu\right) = H_x\left(\frac{j\beta^2 - j\omega^2\mu\epsilon}{\omega\epsilon}\right)$$

$$H_x = \frac{\omega\epsilon}{j\beta^2 - j\omega^2\mu\epsilon}\frac{\partial E_z}{\partial y} - \frac{\beta}{j\beta^2 - j\omega^2\mu\epsilon}\frac{\partial H_z}{\partial x}$$

$$= \frac{j\omega\epsilon}{k_c^2}\frac{\partial E_z}{\partial y} - \frac{j\beta}{k_c^2}\frac{\partial H_z}{\partial x}$$

The following *__four__ equations* are governing a variety of waveguiding structures:

$$H_x = \frac{j}{k_c^2}\left(\omega\epsilon\frac{\partial E_z}{\partial y} - \beta\frac{\partial H_z}{\partial x}\right)$$

The same procedure to find out the rest of the equations:

$$H_y = \frac{-j}{k_c^2}\left(\omega\epsilon\frac{\partial E_z}{\partial x} + \beta\frac{\partial H_z}{\partial y}\right)$$
$$E_x = \frac{-j}{k_c^2}\left(\beta\frac{\partial E_z}{\partial x} + \omega\mu\frac{\partial H_z}{\partial y}\right)$$
$$E_y = \frac{j}{k_c^2}\left(-\beta\frac{\partial E_z}{\partial y} + \omega\mu\frac{\partial H_z}{\partial x}\right)$$

where

$$k_c^2 = k^2 - \beta^2$$

"***The cutoff wavenumber***" which will play an important role later when we examine waveguide structure with different modes.

$$k = \omega\sqrt{\mu\epsilon} = \frac{2\pi}{\lambda}$$

which is the wavenumber of the waveguide region (or the material filling the transmission line).

If dielectric loss is introduced, then one may consider the complex permittivity as:

$$\epsilon = \epsilon_0\epsilon_r(1 - j\tan\delta)$$

$$\tan\delta = loss\ tangent\ of\ material$$

Note: For TEM mode, these four equations may not seem a strong candidate for analysis. We may use the previous six equations.

In addition, if the cutoff wavenumber were zero (i.e. $k_c^2 = 0$), then the system of equations has an indeterminate solution.

c. Conditions for Modes

TEM	$E_z = 0$ and $H_z = 0$
TE	$E_z = 0$ and $H_z \neq 0$
TM	$E_z \neq 0$ and $H_z = 0$

d. TEM mode

For TEM waves, we note that the following condition should be present:

$$E_z = 0 \text{ and } H_z = 0$$

which implies that if we use this condition into the four equations, we would not obtain a meaningful observation out of it. However, the six equations will spit out something meaningful:

$$\beta E_y = -\omega\mu H_x$$

$$-\beta E_x = -\omega\mu H_y$$

$$\frac{\partial E_y}{\partial x} = 0$$

$$\beta H_y = \omega\epsilon E_x$$

$$-\beta H_x = \omega\epsilon E_y$$

$$\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = 0$$

From these equations, we observe a useful relation:

$$\omega\epsilon E_y = \beta \frac{\beta E_y}{\omega\mu}$$

$$\omega^2\mu\epsilon E_y = \beta^2 E_y$$

$$\beta^2 = \omega^2\mu\epsilon$$
$$\beta = \omega\sqrt{\mu\epsilon} = k$$

we note that $\beta = k$ in TEM mode! In addition, this implies that the cutoff wavenumber $k_c = 0$

Now that we develop useful relation between the wave parameters, let’s find out the actual solutions. To do so, we need to consider the Maxwell’s equations (source-free version as before) once again to find the Helmholtz equation. The Faraday’s and Ampere’s equations in Phasor form can be written:

$$\nabla \times \vec{E} = -j\omega\mu\vec{H}$$

$$\nabla \times \vec{H} = j\omega\epsilon\vec{E}$$

Plugging $\vec{H} = \frac{1}{-j\omega\mu}\nabla \times \vec{E}$ into the other equation yield the Helmholtz equation for electric field.

$$\nabla \times \left(\frac{1}{-j\omega\mu}\nabla \times \vec{E}\right) = j\omega\epsilon\vec{E}$$

$$\nabla \times \nabla \times \vec{E} = \omega^2\epsilon\mu\vec{E}$$

Now, we use useful vector identity:

$$\nabla \times \nabla \times \vec{E} = \nabla\left(\nabla \cdot \vec{E}\right) - \nabla^2\vec{E}$$

Because we consider the source-free Maxwell’s equation, the Gauss equation $\nabla \cdot \vec{E} = 0$

$$\nabla^2\vec{E} + \omega^2\epsilon\mu\vec{E} = 0$$

The same procedure led us to

$$\nabla^2\vec{H} + \omega^2\epsilon\mu\vec{H} = 0$$

First, let’s consider the electric field <<and use the fact that Helmholtz operator is a linear operator. >>

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + k^2\right)E_x = 0$$

To be consistent with our initial assumption on the form of electric field,

$$E_x = f(x,y)e^{-j\beta z}$$

which implies that

$$\frac{\partial^2}{\partial z^2} E_x = (-j\beta)^2 f(x,y) e^{-j\beta z} = -\beta^2 E_x$$

and as we already note that $\beta = k$

$$\frac{\partial^2}{\partial z^2} E_x = -k^2 E_x$$

Hence, the Helmholtz equation is reduced to

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) E_x = 0$$

The same idea goes for the y-component:

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) E_y = 0$$

Hence, we may re-write those in more compact form:

$$\hat{a}_x \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) E_x + \hat{a}_y \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) E_y$$

$$\boxed{= \nabla_t^2 \vec{e}(x,y) = 0}$$

where ∇_t^2 represents the Laplacian operator acting on the "transverse" components. In short, the above equation suggests us that the Laplace's equation is satisfied for the transverse field of TEM mode. Likewise, the same argument can help identifying the magnetic field:

$$\nabla_t^2 \vec{h}(x,y) = 0$$

If you recall the electrostatics, electric field that is generated by a scalar potential (between two conductors) also satisfies the Laplace's equation. In other words, the transverse fields are the same as the static fields that may have been generated by the scalar potential between two conductors. <<I think the uniqueness of EM fields also support this point. I will add reference if I find a relevant paper or something.>>

To verify that

$$\nabla_t^2 \vec{e}(x,y) = 0$$

let's examine the electrostatics. As we know, the relation between electric field and a scalar potential as:

$$\vec{e}(x,y) = -\nabla_{\mathrm{t}} \Phi(x,y)$$

The curl of the electric field:

$$\nabla_{\mathrm{t}} \times \vec{e}(x,y) = \nabla \times \left(-\nabla_{\mathrm{t}} \Phi(x,y)\right) = 0$$

<u>or</u> if we expand the curl: ($E_z = 0$)

$$\nabla_t \times \vec{e}(x,y) = \hat{a}_x \frac{\partial E_y}{\partial z} + \hat{a}_y \frac{\partial E_x}{\partial z} + \hat{a}_z \left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} \right) = \hat{a}_z \left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} \right)$$

I shouldn't write $\frac{\partial}{\partial z}$ because the operator is acting on the transverse field ∇_t but I did it for showing mathematical flow. Recall from the very beginning section of this document,

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = -j\omega\mu H_z$$

For TEM waves, $H_z = 0$. Hence,

$$\nabla_t \times \vec{e}(x,y) = 0$$

again. We verify it.

Finally, we may show that the scalar potential also satisfies the Laplace's equation by using the source free Maxwell's equation (the Gauss equation):

$$\nabla \cdot D = \epsilon \nabla \cdot E = 0$$

$$= \nabla \cdot \left(-\nabla_t \Phi(x,y) \right)$$

$$= -\nabla_t^2 \, \Phi(x,y)$$

Hence, the scalar potential also satisfies the Laplace's equation!

$$\nabla_t^2 \, \Phi(x,y) = 0$$

i. Voltage

We may find the voltage expression using

$$V_{12} = \Phi_1 - \Phi_2 = \int_1^2 \vec{E} \cdot d\vec{l}$$

where Φ_1 is the potential at the conductor 1 and Φ_2 for conductor 2.

ii. Current

Using the Ampere's law, we may find the current expression as

$$I = \oint_C \vec{H} \cdot d\vec{l}$$

where C is the cross-sectional contour of the conductor (i.e. it must be Gaussian region).

iii. The Characteristic Impedance

The characteristic impedance is defined as

$$Z_0 = \frac{V}{I} = \frac{\int_1^2 \vec{E} \cdot d\vec{l}}{\oint_C \vec{H} \cdot d\vec{l}}$$

iv. Wave Impedance for TEM mode

Recall from the general six equations,

$$\frac{\partial H_z}{\partial y} + j\beta H_y = j\omega\epsilon E_x$$

$$-\frac{\partial E_z}{\partial x} - j\beta E_x = -j\omega\mu H_y$$

Now $H_z = 0$ and $E_z = 0$ for TEM wave:

$$Z_{TEM} = \frac{E_x}{H_y} = \frac{\beta}{\omega\epsilon} = \frac{2\pi}{\lambda}\frac{1}{2\pi f\epsilon} = \frac{\sqrt{\mu\epsilon}}{\epsilon} = \sqrt{\frac{\mu}{\epsilon}}$$

$$Z_{TEM} = \frac{E_x}{H_y} = \frac{\omega\mu}{\beta} = \frac{2\pi f\mu}{2\pi/\lambda} = \frac{\mu}{\sqrt{\mu\epsilon}} = \sqrt{\frac{\mu}{\epsilon}}$$

Either case, we end up with same expression for wave impedance.

In addition, note that this wave impedance is the same impedance of the plane wave in a lossless medium. Another characteristic of the wave impedance would be the fact that it connects the relation between the transverse electric and magnetic fields.

$$\vec{h}(x,y) = \frac{1}{Z_{TEM}}\hat{a}_z \times \vec{e}(x,y)$$

e. TE Waves

Unlike TEM wave, we approach to the solutions of transverse fields indirectly as we first look for either E_z or H_z (depending on the type of wave, either TM or TE, respectively) and then use the four equations that lead to the transverse fields.

TE waves are also known as H-waves because $E_z = 0$ and $H_z \neq 0$ are the two conditions for TE waves. Recall that we derived the four equations for TE and TM waves already. Let's re-write them:

$$H_x = \frac{j}{k_c^2}\left(\omega\epsilon\frac{\partial E_z}{\partial y} - \beta\frac{\partial H_z}{\partial x}\right) = \frac{-j}{k_c^2}\left(\beta\frac{\partial H_z}{\partial x}\right)$$

$$H_y = \frac{-j}{k_c^2}\left(\omega\epsilon\frac{\partial E_z}{\partial x} + \beta\frac{\partial H_z}{\partial y}\right) = \frac{-j}{k_c^2}\left(\beta\frac{\partial H_z}{\partial y}\right)$$

$$E_x = \frac{-j}{k_c^2}\left(\beta\frac{\partial E_z}{\partial x} + \omega\mu\frac{\partial H_z}{\partial y}\right) = \frac{-j}{k_c^2}\left(\omega\mu\frac{\partial H_z}{\partial y}\right)$$

$$E_y = \frac{j}{k_c^2}\left(-\beta\frac{\partial E_z}{\partial y} + \omega\mu\frac{\partial H_z}{\partial x}\right) = \frac{j}{k_c^2}\left(\omega\mu\frac{\partial H_z}{\partial x}\right)$$

where

$$\beta = \sqrt{k^2 - k_c^2}$$

$$k_c \neq 0$$

which are usually functions of frequency and geometric information about the waveguide.

As discussed earlier, to find the transverse fields, we may first need to find out H_z from the Helmholtz equation: (note that $E_z = 0$ so we can only consider H_z Helmholtz equation)

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + k^2\right)H_z = 0$$

where

$$H_z = h_z(x, y)e^{-j\beta z}$$

Hence, the Helmholtz equation can be reduced to:

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + k_c^2\right)H_z = 0$$

If there's a boundary condition for the geometry of waveguide, one can solve the Helmholtz equation for TE waves.

i. Wave impedance for TE

$$Z_{TE} = \frac{E_x}{H_y} = -\frac{E_y}{H_x} = \frac{\omega\mu}{\beta} = \frac{kZ_{TEM}}{\beta}$$

In conclusion, unlike TEM waves, closed conductor (e.g. rectangular waveguide) can support TE waves.

f. TM Waves

The same procedure as we did for TE waves but this time, with different condition:

$$E_z \neq 0 \text{ and } H_z = 0$$

which is why TM waves are also known as E-waves.

$$H_x = \frac{j}{k_c^2}\left(\omega\epsilon\frac{\partial E_z}{\partial y} - \beta\frac{\partial H_z}{\partial x}\right) = \frac{j}{k_c^2}\left(\omega\epsilon\frac{\partial E_z}{\partial y}\right)$$

$$H_y = \frac{-j}{k_c^2}\left(\omega\epsilon\frac{\partial E_z}{\partial x} + \beta\frac{\partial H_z}{\partial y}\right) = \frac{-j}{k_c^2}\left(\omega\epsilon\frac{\partial E_z}{\partial x}\right)$$

$$E_x = \frac{-j}{k_c^2}\left(\beta\frac{\partial E_z}{\partial x} + \omega\mu\frac{\partial H_z}{\partial y}\right) = \frac{-j}{k_c^2}\left(\beta\frac{\partial E_z}{\partial x}\right)$$

$$E_y = \frac{j}{k_c^2}\left(-\beta\frac{\partial E_z}{\partial y} + \omega\mu\frac{\partial H_z}{\partial x}\right) = \frac{j}{k_c^2}\left(-\beta\frac{\partial E_z}{\partial y}\right)$$

where

$$\beta = \sqrt{k^2 - k_c^2}$$

$$k_c \neq 0$$

To find the transverse fields, we need to solve the Helmholtz equation for TM wave condition.

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + k^2\right)E_z = 0$$

where

$$E_z = e_z(x,y)e^{-j\beta z}$$

Hence, the Helmholtz equation can be reduced to:

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + k_c^2\right)E_z = 0$$

If there's a boundary condition for the geometry of waveguide, one can solve the Helmholtz equation for TM waves.

i. Wave Impedance for TM Waves

$$Z_{TM} = \frac{E_x}{H_y} = -\frac{E_y}{H_x} = \frac{\beta}{\omega\epsilon} = \frac{\beta Z_{TEM}}{k}$$

g. Attenuation Due to Dielectric Loss

As you know, attenuation in a transmission line or waveguide caused by two types of loss: dielectric loss and conductor loss. In this section, we develop dielectric loss expression to account for the attenuation in waveguide. The conductor loss can be calculated by the perturbation method. In general, the total loss can be expressed as the sum of dielectric and conductor loss:

$$\alpha = \alpha_c + \alpha_d$$

The propagation constant can derive the dielectric loss; however, the material filling the waveguide must be homogenous throughout the structure obviously. If a waveguide consists of two different material or more, then the propagation constant will be changed for different filling material which in turn change the propagation constant too.

So, our assumption is that the waveguide under concern is made of homogenous material to ease the analysis for dielectric loss.

The propagation constant can be expressed as:

$$\gamma = \alpha_d + j\beta$$

III. References

[1] Microwave Engineering, David Pozar, 4th Edition

[2] http://web.mit.edu/sahughes/www/8.022/lec20.pdf