Introduction to MM

I. Introduction

In this document, I will introduce one of basic examples for MM (Method of Moment), which will be used in the analysis of finite-diameter wire dipole antenna. The code was realized in Python program for plotting purpose and the corresponding code can be found in the folder named “Code”.

Throughout this document, the primed coordinate represents the source coordinate and the unprimed coordinate represents the observation (basis coordinate).

II. Electrostatic Charge Distribution

Charge distribution of a wire will be considered along with MM. From basic physics point of view, charge distribution and voltage are related to each other. In real world, one can apply voltage across the wire and find the charge distribution whereas the opposite process might be more difficult to do.

One of the Maxwell’s equations (the Coulomb’s law) can be modified a little and be shown as

$$V(r) = \frac{1}{4\pi\varepsilon_0}\int_{source}\frac{\rho(r')}{R}dl'$$

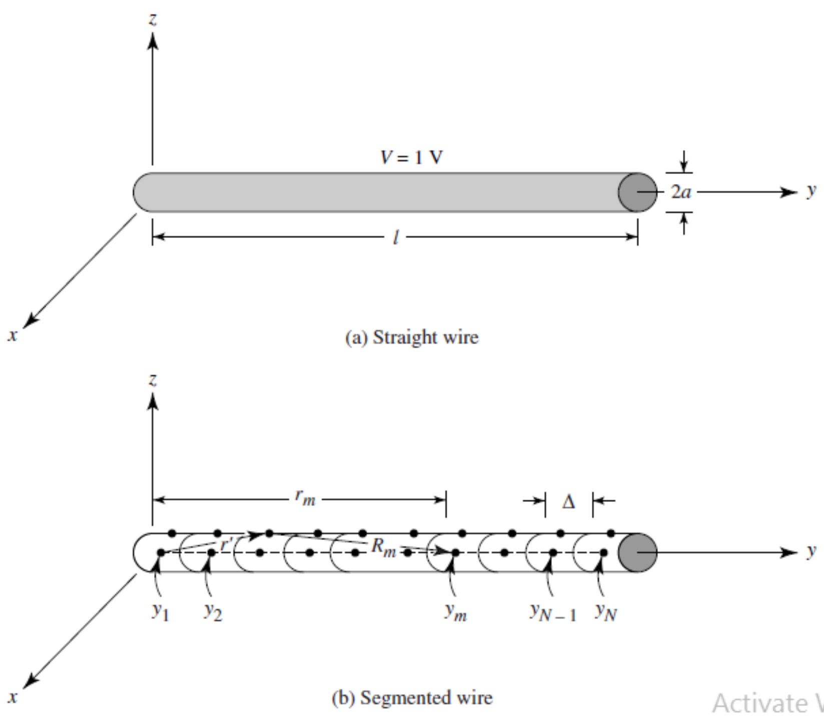

Figure 8.1 Straight wire of constant potential and its segmentation.

If the geometry of structure becomes more convoluted, then one might not be able to find the charge distribution even when they know about the voltage across the structure which means, in mathematics, the problem becomes non-trivial problem. (i.e. more variables than equations)

<<In my opinion about MM, it is the process of forcing the unknown variables to be reduced so that the problem become consistent. The matrix would be Toeplitz matrix which always has the inverse due to its matrix shape (i.e. triangular diagonal matrix) [2]

In addition, I feel like MM is similar to the Fourier series where the coefficients are determined for each basis function.>>

Hence, it is evident that we need a special way to calculate the charge distribution. MM allows one to approximate such charge distribution even when the structure is complicated. Luckily, we are examining rather simple structure for practice.

Let's first express R:

$$R(r, r') = |\vec{r} - \vec{r}'| = \sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2}$$

<<In my opinion, finding the charge distribution would be analogous process of finding the impulse response of a system. [2] Toeplitz matrix is related to the convolution operator! Check this point later.>>

The shape of structure is simple (i.e. wire); therefore, we should take advantage of it. By letting the observation point

$$x = y = 0$$

$$R = \sqrt{(x')^2 + (z')^2 + (y - y')^2}$$

$$R = \sqrt{a^2 + (y - y')^2}$$

where

$$a = radius\ of\ wire$$

Now, we can re-write the voltage equation:

$$V(y) = \frac{1}{4\pi\varepsilon_0}\int_0^l \frac{\rho(y')}{\sqrt{a^2 + (y - y')^2}} dy'$$

where

$$\varepsilon_0 = permittivity\ of\ free\ space$$

$$\rho(y') = charge\ density$$

Note that we are missing the information about the charge density. Let's approximate this first in terms of a set of basis functions (known) with coefficients (unknown).

$$\rho(y') \approx \sum_{n=1}^{N} a_n g_n\,(y')$$

where

$$g_n = basis\ function$$

$$V(y) = \frac{1}{4\pi\varepsilon_0}\int_0^l \frac{1}{\sqrt{a^2 + (y - y')^2}}\left[\sum_{n=1}^{N} a_n g_n\,(y')\right] dy'$$

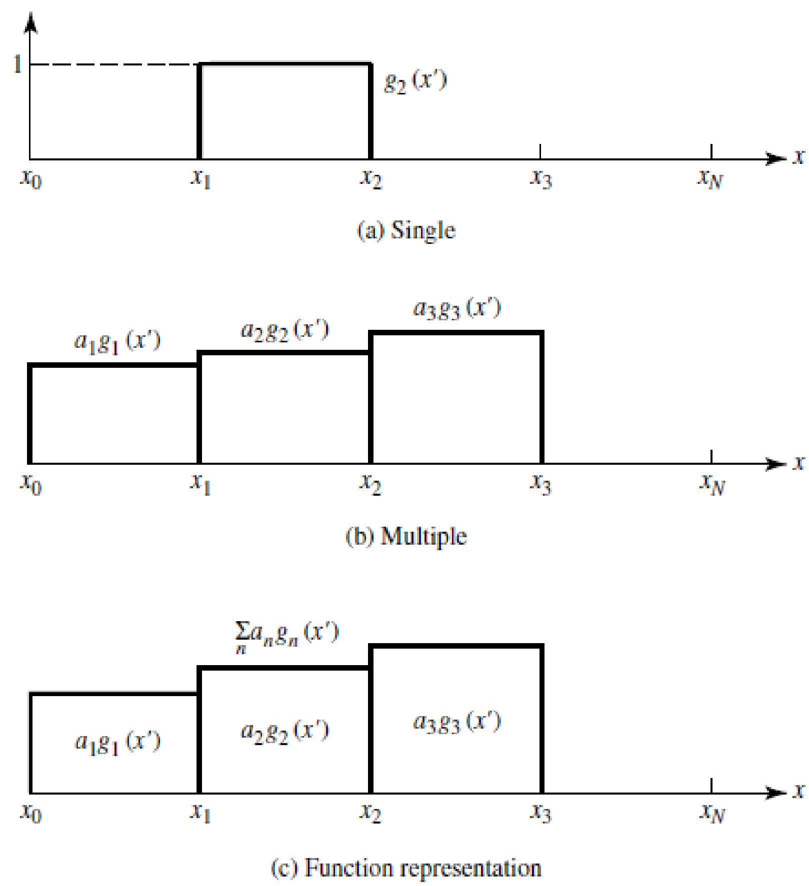

Figure 8.8 Piecewise constant subdomain functions.

The basis function is piecewise subdomain function that is

$$g_n(y') = \begin{cases} 0, & y' < (n-1)\Delta \\ 1, & (n-1)\Delta \leq y' \leq n\Delta \\ 0, & y' > n\Delta \end{cases}$$

There are many types of piecewise subdomain functions that can be used to represent the charge density function. Here we use the unit pulse function for basis function. Note this basis function will modify the integral equation:

$$V(y) = \frac{1}{4\pi\varepsilon_0}\sum_{n=1}^{N} a_n \int_{(n-1)\Delta}^{n\Delta} \frac{g_n(y')}{\sqrt{a^2 + (y - y')^2}}\,dy'$$

$$= \frac{1}{4\pi\varepsilon_0}\left[a_1 \int_0^{\Delta} \frac{g_1(y')}{\sqrt{a^2 + (y - y')^2}}\,dy' + a_2 \int_{\Delta}^{2\Delta} \frac{g_2(y')}{\sqrt{a^2 + (y - y')^2}}\,dy' + \cdots \right.$$
$$\left. + a_N \int_{(N-1)\Delta}^{N\Delta} \frac{g_N(y')}{\sqrt{a^2 + (y - y')^2}}\,dy'\right]$$

Note that the integral itself doesn't have any problem (i.e. it will spit out another function in terms of y) but what we did for coefficients is problematic. (i.e. generating N variables in one equation) To form an $N \times N$ matrix, let us consider the inner product of weighing function and the voltage function (i.e. the point matching technique with the Dirac-delta function)

$$\langle V(y), w_m \rangle = \int_{observation} w_m^* V(y)\,dy$$

where

$$w_m = \delta_m \ and \ m = 1,2,\cdots$$

Which will generate N different equations.

$$V(y_1) = \frac{1}{4\pi\varepsilon_0}\left[a_1 \int_0^{\Delta} \frac{g_1(y')}{\sqrt{a^2 + (y_1 - y')^2}}\,dy' + a_2 \int_{\Delta}^{2\Delta} \frac{g_2(y')}{\sqrt{a^2 + (y_1 - y')^2}}\,dy' + \cdots \right.$$
$$\left. + a_N \int_{(N-1)\Delta}^{N\Delta} \frac{g_N(y')}{\sqrt{a^2 + (y_1 - y')^2}}\,dy'\right]$$

$$V(y_2) = \frac{1}{4\pi\varepsilon_0}\left[a_1 \int_0^{\Delta} \frac{g_1(y')}{\sqrt{a^2 + (y_2 - y')^2}}\,dy' + a_2 \int_{\Delta}^{2\Delta} \frac{g_2(y')}{\sqrt{a^2 + (y_2 - y')^2}}\,dy' + \cdots \right.$$
$$\left. + a_N \int_{(N-1)\Delta}^{N\Delta} \frac{g_N(y')}{\sqrt{a^2 + (y_2 - y')^2}}\,dy'\right]$$

$$\vdots$$

$$V(y_N) = \frac{1}{4\pi\varepsilon_0}\left[a_1 \int_0^{\Delta} \frac{g_1(y')}{\sqrt{a^2 + (y_N - y')^2}}\,dy' + a_2 \int_{\Delta}^{2\Delta} \frac{g_2(y')}{\sqrt{a^2 + (y_N - y')^2}}\,dy' + \cdots \right.$$
$$\left. + a_N \int_{(N-1)\Delta}^{N\Delta} \frac{g_N(y')}{\sqrt{a^2 + (y_N - y')^2}}\,dy'\right]$$

In a compact form,

$$\sum_{m=1}^{N} V(y_m) = \frac{1}{4\pi\varepsilon_0}\sum_{m=1}^{N}\sum_{n=1}^{N} a_n \int_{(n-1)\Delta}^{n\Delta} \frac{g_n(y')}{\sqrt{a^2 + (y_m - y')^2}}\,dy'$$

$$= \frac{1}{4\pi\varepsilon_0}\sum_{m=1}^{N}\sum_{n=1}^{N}\int_{(n-1)\Delta}^{n\Delta}\frac{g_n(y')}{\sqrt{a^2+(y_m-y')^2}}dy' \cdot \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$$

$$[V] = [Z][a] = [Z][I]$$

We simply change the notation.

The coefficients can be found from finding the inverse matrix of $[Z]$.

$$[I] = [Z]^{-1}[V]$$

Here, we can set up the voltage matrix as the follow:

$$[V] = [4\pi\varepsilon_0]$$

At this point, let's check this equation and see if we understand this idea correctly.

III. Examples

Consider

$$a = radius\ of\ wire = 0.001\ m$$
$$N = number\ of\ elements\ for\ approximation = 5$$

For piece-wise pulse function, one can refer to [2].

The following items are the results from computation.

```
('I = ', array([  8.81086801e-12,   8.08998704e-12,   7.97285998e-12,
         8.08998704e-12,   8.81086801e-12]))
('Z = ', array([[ 10.59668473,   1.09859007,   0.51082385,   0.33647175,
          0.25131423],
       [  1.09859007,  10.59668473,   1.09859007,   0.51082385,
          0.33647175],
       [  0.51082385,   1.09859007,  10.59668473,   1.09859007,
          0.51082385],
       [  0.33647175,   0.51082385,   1.09859007,  10.59668473,
          1.09859007],
       [  0.25131423,   0.33647175,   0.51082385,   1.09859007,
         10.59668473]]))
```

Just as before,

$$Z = impedance\ Matrix$$

$$I = coefficient\ Matrix$$

If we plot the result,

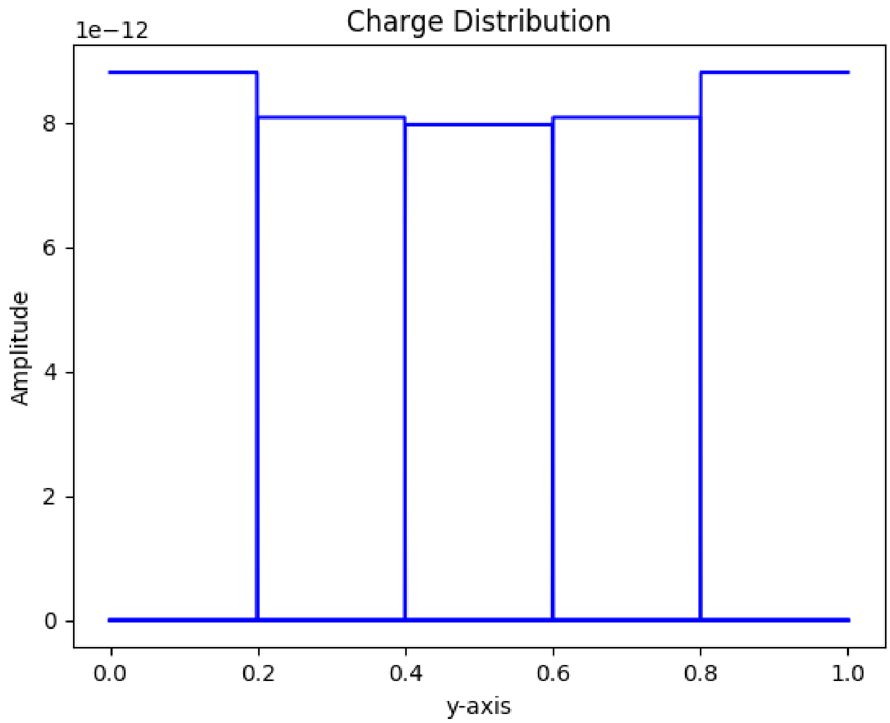

Figure.

The above result can be easily verified by looking at the textbook example result and indeed it came out correctly. (although the textbook approached to the problem slightly differently.)

$$\begin{bmatrix} 10.60 & 1.10 & 0.51 & 0.34 & 0.25 \\ 1.10 & 10.60 & 1.10 & 0.51 & 0.34 \\ 0.51 & 1.10 & 10.60 & 1.10 & 0.51 \\ 0.34 & 0.51 & 1.10 & 10.60 & 1.10 \\ 0.25 & 0.34 & 0.51 & 1.10 & 10.60 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix} = \begin{bmatrix} 1.11 \times 10^{-10} \\ 1.11 \times 10^{-10} \\ \vdots \\ 1.11 \times 10^{-10} \end{bmatrix}$$

Figure. Textbook solution result

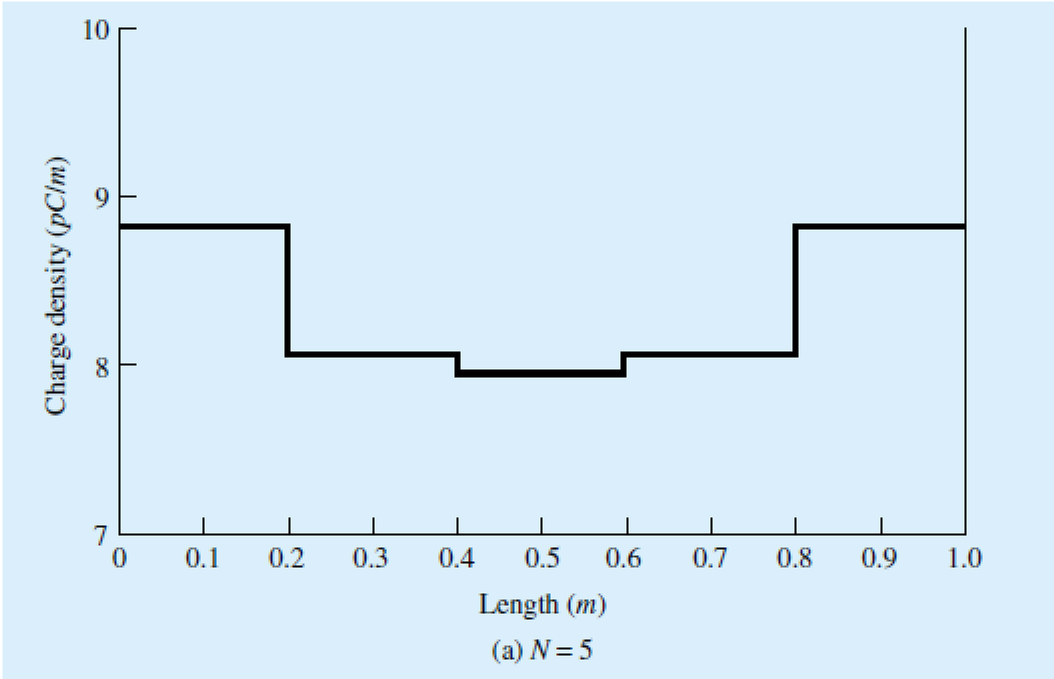

Figure. Textbook solution plot

I think the textbook used stair function or something. Plotting method can be varied as long as the idea is there.

The Python code that realizes the above matrix and plot can be found in the folder named “Code” and whose name is “IntroMM.py”.

IV. References

[1] Antenna Theory Analysis and Design, Constantine Balanis, 4th Edition

[2] https://books.google.com/books?id=C_2zCgAAQBAJ&pg=PA53&lpg=PA53&dq=piecewise+pulse+python&source=bl&ots=rGV74hZjfO&sig=rC3dhRGkS-eSBlFxb0O5zTmBepM&hl=en&sa=X&ved=0ahUKEwigo86IgbLWAhXHq1QKHfNSARgQ6AEIQDAE#v=onepage&q=piecewise%20pulse%20python&f=false