Green’s Function

I. Introduction

Green’s function is the impulse response of inhomogeneous linear differential equation on a domain with a boundary condition (BC).

This idea is useful in wave equations, Helmholtz equation, and other differential equations, etc.

II. Theory

a. Definition

$$LG(x,s) = \delta(s-x)$$

where

L = Linear operator

$G(x,s)$ =Green’s function

$\delta(s-x)$ = the Dirac-delta function

In other words, we are looking for the Green’s function such that when the linear operator (e.g. Helmholtz operator) acted on it, the Dirac-delta function is the solution.

This green’s function is used to solve the following differential equation:

$$Lu(x) = f(x)$$

b. General Idea

Consider the following integral

$$\int LG(x,s)f(s)ds = \int \delta(s-x)f(s)ds = f(x)$$

Then,

$$Lu(x) = \int LG(x,s)f(s)ds$$

Note that the linear operator is acted on the variable x. Hence, we may extract the linear operator out of the integral:

$$= L\left[\int G(x,s)f(s)ds\right]$$

Which leads to the following relation:

$$u(x) = \int G(x, s) f(s) ds$$

$$u(x) = \int G(s - x) f(s) ds$$

From Laplace Transformation, one may be familiar with this form. In other words, the solution to the differential equation can be obtained from considering the convolution between the Green's function of the linear operator and <<the particular solution of the inhomogeneous differential equation. >>

III. Examples

IV. References

[1] https://en.wikipedia.org/wiki/Green%27s_function