# Vector Identities

## I. Introduction

In this document, useful vector identities will be derived.

The right-hand rule might be a good tool to be consistent which always occur in Cross product between two vectors to determine the orientation of the resulted vector direction.

### a. Einstein Summation Convention

A vector can be written using the Einstein Summation Convention by the follow:

$$\vec{r} = a_1\hat{x}_1 + a_2\hat{x}_2 + a_3\hat{x}_3 = \sum_{i=1}^{3} a_i\hat{x}_i = a_i\hat{x}_i$$

### b. Levi-Civita Symbol

It is also known as the permutation symbol.

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|---|
| $\epsilon_{ijk} = 1$ when $(ijk)$ is even permutation i.e. $\epsilon_{123} = \epsilon_{231} = \epsilon_{312} = 1$ |
| $\epsilon_{ijk} = -1$ when $(ijk)$ is odd permutation i.e. $\epsilon_{213} = \epsilon_{321} = \epsilon_{132} = -1$ |
| $\epsilon_{ijk} = 0$ when two or more indices are the same i.e. $\epsilon_{112} = \epsilon_{121} = \epsilon_{222} = 0$ |

One can obtain the same value from the following formula:

$$\epsilon_{ijk} = \frac{1}{2}(i-j)(j-k)(k-i)$$

where $i, j, k = 1,2,3$

Another important property occurs in the matrix form as we concern the product of two Levi-Civita symbols:

$$\epsilon_{ijk}\epsilon_{lmh} = \begin{bmatrix} \delta_{il} & \delta_{im} & \delta_{ih} \\ \delta_{jl} & \delta_{jm} & \delta_{jh} \\ \delta_{kl} & \delta_{km} & \delta_{kh} \end{bmatrix}$$

let $h = k$ (because other combination will yield 0.)

$$\epsilon_{ijk}\epsilon_{lmk} = \begin{bmatrix} \delta_{il} & \delta_{im} & \delta_{ik} \\ \delta_{jl} & \delta_{jm} & \delta_{jk} \\ \delta_{kl} & \delta_{km} & \delta_{kk} \end{bmatrix}$$

where the Kronecker Delta symbol for the elements was used:

| |
|---|
| $\delta_{ij} = 1$ if $i = j$ |
| $\delta_{ij} = 0$ if $i \neq j$ |

The determinant of this $3 \times 3$ matrix:

$$= \delta_{il}\delta_{jm}\delta_{kk} + \delta_{im}\delta_{jk}\delta_{kl} + \delta_{ik}\delta_{jl}\delta_{km} - \delta_{ik}\delta_{jm}\delta_{kl} - \delta_{il}\delta_{jk}\delta_{km} - \delta_{im}\delta_{jl}\delta_{kk}$$

Since $\delta_{kk} = \sum_{k=1}^{3} \delta_{kk} = 3$

$$= 3\delta_{il}\delta_{jm} + \delta_{im}\delta_{jk}\delta_{kl} + \delta_{ik}\delta_{jl}\delta_{km} - \delta_{ik}\delta_{jm}\delta_{kl} - \delta_{il}\delta_{jk}\delta_{km} - 3\delta_{im}\delta_{jl}$$

$$= 3\delta_{il}\delta_{jm} - 3\delta_{im}\delta_{jl} + \delta_{im}\delta_{jl} + \delta_{im}\delta_{jl} - \delta_{il}\delta_{jm} - \delta_{il}\delta_{jm}$$

$$= \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$$

$$\boxed{\epsilon_{ijk}\epsilon_{lmk} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}}$$

Note that the following useful relation was used for simplifying:

$$\sum_k \delta_{ik}\,\delta_{kj} = \delta_{ij}$$

The above relation can be verified by equating $k = j$.

## c. Cross Product

Recall that unit vectors in the Cartesian coordinate are related by the right-hand rule.

$$\hat{x} \times \hat{y} = \hat{z} \; and \; \hat{y} \times \hat{z} = \hat{x} \; and \; \hat{z} \times \hat{x} = \hat{y}$$

However, other combination such as

$$\hat{y} \times \hat{x} = -\hat{z}$$

Note that the minus sign. This leads to an important relation in general unit vector notation:

$$\boxed{\hat{e}_i \times \hat{e}_j = \epsilon_{ijk}\hat{e}_k}$$

Cross product can be expressed in the same way:

$$\boxed{\vec{a} \times \vec{b} = (a_i\hat{e}_i) \times (b_j\hat{e}_j) = a_i b_j(\hat{e}_i \times \hat{e}_j) = \epsilon_{ijk} a_i b_j \hat{e}_k}$$

## d. Dot Product

Unit vector dot product can be shown:

$$\boxed{\hat{e}_i \cdot \hat{e}_j = \delta_{ij}}$$

The dot product of two vectors can be shown:

$$\boxed{\vec{a} \cdot \vec{b} = \sum_i \sum_j a_i b_j \, \delta_{ij} = \sum_i a_i b_i = a_i b_i}$$

# II. Vector Identities

## a. The Scalar Triple Product

$$\vec{a} \cdot \left(\vec{b} \times \vec{c}\right) = \vec{c} \cdot \left(\vec{a} \times \vec{b}\right) = \vec{b} \cdot (\vec{c} \times \vec{a})$$

See if this identity works.

$$\vec{a} \cdot \left(\vec{b} \times \vec{c}\right) = a_i \hat{e}_i \cdot \left(b_j c_k \epsilon_{jkl} \hat{e}_l\right)$$

$$= a_i b_j c_k \epsilon_{jkl} (\hat{e}_i \cdot \hat{e}_l)$$

$$= a_i b_j c_k \epsilon_{jkl} \delta_{il}$$

Note that the above term is only meaningful when $l = i$, and

$$= a_i b_j c_k \epsilon_{jki}$$

Useful relation can be detected:

$$\vec{a} \cdot \left(\vec{b} \times \vec{c}\right) = a_i b_j c_k \epsilon_{ijk}$$

Note that the permutation rule has been applied to the Levi-Civita symbol.

Re-arrange terms:

$$c_k a_i b_j \epsilon_{kij} = \vec{c} \cdot \left(\vec{a} \times \vec{b}\right)$$

Likewise,

$$b_j c_k a_i \epsilon_{jki} = \vec{b} \cdot (\vec{c} \times \vec{a})$$

## b. Vector Triple Product

It is also known as “BAC CAB” rule.

$$\vec{a} \times \left(\vec{b} \times \vec{c}\right) = \vec{b}(\vec{a} \cdot \vec{c}) - \vec{c}\left(\vec{a} \cdot \vec{b}\right)$$

Proof goes here.

# III. References


[1] https://en.wikipedia.org/wiki/Levi-Civita_symbol

[2] https://en.wikipedia.org/wiki/Kronecker_delta

[3] https://en.wikipedia.org/wiki/Triple_product