Finite Dimension Dipole Antenna

I. Introduction

In this document, rigorous approaches to analyze finite radius and length of dipole antennas will be covered. Many approximation methods can be done for analysis of such dipole. The method of Moment (MM) will be considered in this document with EFIE equations.

Self-impedance can also be used along with MM to find the current distribution along dipole. The closed form solutions of self-impedance and driving-point impedance can be determined by a method named *"the induced emf method"*.

For an exercise, a half-wavelength dipole will be considered. Python will be used to compute the Pocklington's integrodifferential equation with MM (Method of Moment). In addition, I will add the HFSS simulation to compare the results.

Throughout this document, ***the primed coordinate will represent the source*** where the unprimed coordinate notation will be used for the Euclidean coordinates.

II. Background Idea

Unlike the analysis of dipole analysis where we assume that the radius of the wire is negligible, we solve for unknown current density that is developed on the radiator/scatterer. For thin wire dipole analysis, the current distribution can be modeled as sinusoidal function; however, the current distribution would no longer be sinusoidal as the radius of wire become non-negligible.

The general idea for IE (integral equation) is that we set up an integral equation to solve for unknown current density that is induced on the surface of radiator/scatterer. MM (Method of Moment) then can be done for approximating method. Interestingly, the unknown current density is part of the integrand.

Before proceeding analysis, we first set up the excitation source, voltage matrix. This excitation source will be injected through the gap at the center of the wire. There are two famous voltage source models for MM: (1) Delta-gap model (2) Magnetic-Frill Generator

For integral equation, there are two famous models: (1) Pocklington's integral equation (2) Hallen's integral equation

III. Integral Equations

In this section, we are going to examining the two IE's, the Pocklington's and Hallen's IE. In terms of voltage source model, the Hallen's IE method is restricted to use the Delta gap model whereas many types of sources are possible for the Pocklington's IE.

a. Pocklington's Integral Equations

To derive the Pocklington's IE, we first assume that the total electric field can be modeled as

$$\vec{E}^t(\vec{r}) = \vec{E}^i(\vec{r}) + \vec{E}^s(\vec{r})$$

where the total field is expressed as the sum of the incident and scattered field vectors. Note that the scattered field was induced by the scattered current density, which had been induced by the incident field $\vec{E}^i(\vec{r})$ for the wire being "antenna". ("Scatter" case would be exactly the opposite the "antenna" case.)

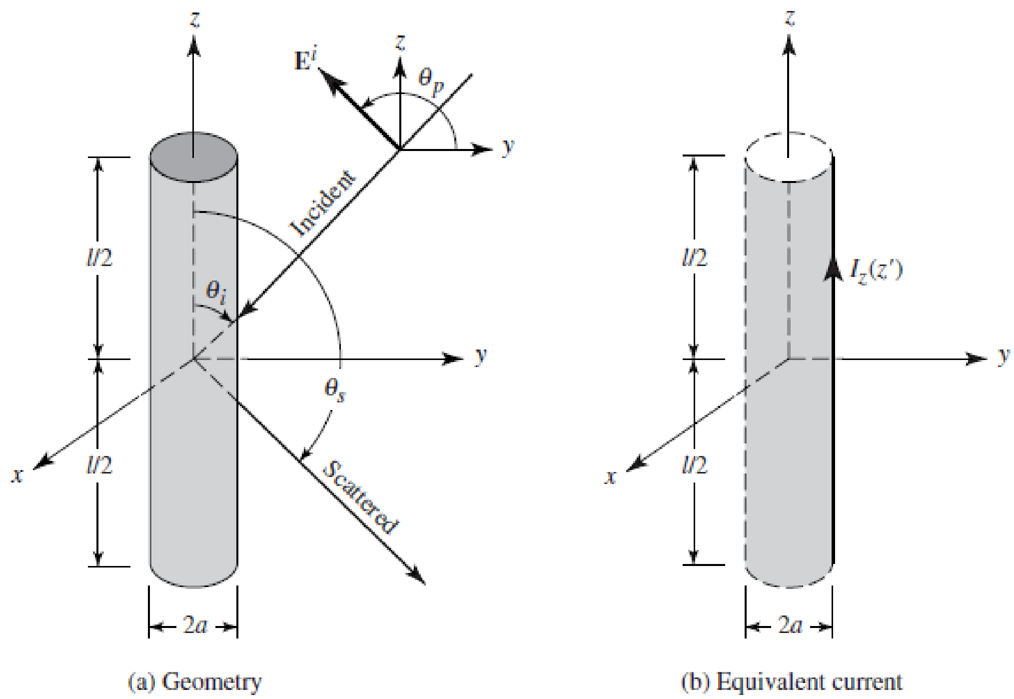


Figure 8.5 Uniform plane wave obliquely incident on a conducting wire.

$$E_\rho = E_y = j\frac{\eta I_0}{4\pi y}\left[\left(z - \frac{l}{2}\right)\frac{e^{-jkR_1}}{R_1} + \left(z + \frac{l}{2}\right)\frac{e^{-jkR_2}}{R_2} - 2z\cos\left(\frac{kl}{2}\right)\frac{e^{-jkr}}{r}\right]$$

$$E_z = -j\frac{\eta I_0}{4\pi}\left[\frac{e^{-jkR_1}}{R_1} + \frac{e^{-jkR_2}}{R_2} - 2z\cos\left(\frac{kl}{2}\right)\frac{e^{-jkr}}{r}\right]$$

The above expressions are the radiated fields by the dipole whose derivation process is not shown here. This point may be covered later. Note that these radiated fields by dipole contain two electric field components which are the radial (E_ρ) and tangential (E_z) components.

Using the boundary condition, if the observation point is located on the surface of the wire, then the total tangential component of electric field should be zero. That is,

$$\vec{E}_z^t(\vec{r} = \vec{r}_s) = \vec{E}_z^i(\vec{r} = \vec{r}_s) + \vec{E}_z^s(\vec{r} = \vec{r}_s)$$

Hence,

$$\vec{E}_z^s(\vec{r}=\vec{r}_s) = -\vec{E}_z^i(\vec{r}=\vec{r}_s)$$

To find the Scattered electric field expression, which had been induced by the scattered current density, we may recall that the vector potential induced by the current density can be used.

$$\vec{E}_A = -\vec{\nabla}\phi - j\omega\vec{A} = -j\omega\vec{A} - \frac{j}{\omega\mu\varepsilon}\vec{\nabla}(\vec{\nabla}\cdot\vec{A})$$

$$= -\frac{j}{\omega\mu\varepsilon}\left(k^2\vec{A} + \vec{\nabla}(\vec{\nabla}\cdot\vec{A})\right)$$

$$= E_x\hat{e}_x + E_y\hat{e}_y + E_z\hat{e}_z$$

Since the set $\{\hat{e}_x, \hat{e}_y, \hat{e}_z\}$ forms an orthonormal set, we may consider each component at a time. The tangential component E_z is in our primary concern.

$$E_z = -\frac{j}{\omega\mu\varepsilon}\left(k^2A_z + \frac{\partial^2 A_z}{\partial z^2}\right)$$

or equivalently, we could think of this as an operator acting on the vector potential:

$$E_z = -\frac{j}{\omega\mu\varepsilon}\left(k^2 + \frac{\partial^2}{\partial z^2}\right)A_z$$

Note that this operator is known as the Helmholtz operator in the general context (i.e. Laplacian involved). [3]

Recall the vector potential can be derived by the current density as

$$\vec{A} = \frac{\mu}{4\pi}\iint_s \vec{J}_s\,(x',y',z')\,\frac{e^{-jkR}}{R}\,d\vec{s}'$$

$$A_z = \frac{\mu}{4\pi}\iint_s J_z\,\frac{e^{-jkR}}{R}\,ds'$$

If we consider the surface integral done to be in the cylindrical coordinate, then

$$= \frac{\mu}{4\pi}\int_{-l/2}^{l/2}\int_0^{2\pi} J_z\,\frac{e^{-jkR}}{R}\,\rho\,d\varphi'dz'$$

For the sake of simplicity, let us assume that the current does not vary along φ' direction. That is,

$$I_z(z') = 2\pi a J_z(z')$$

in turn,

$$J_z(z') = \frac{I_z(z')}{2\pi a}$$

where a = radius of the wire

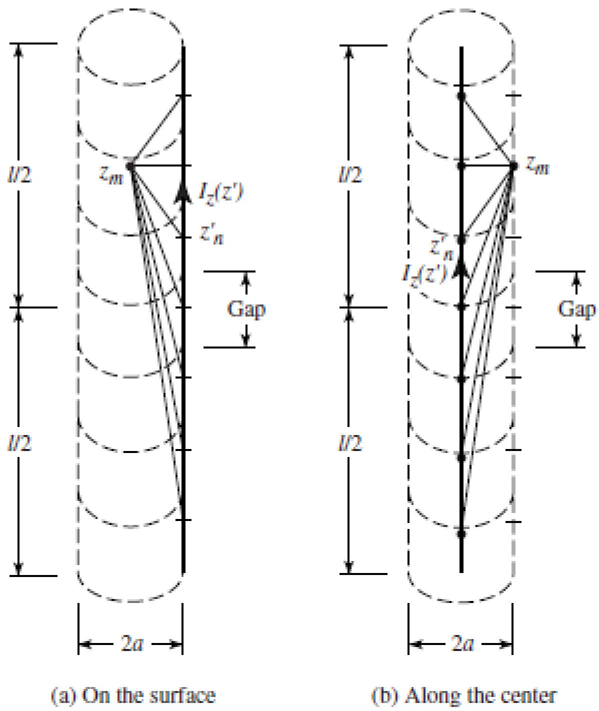


Figure 8.6 Dipole segmentation and its equivalent current.

From the figure, current is only on the surface of the wire (i.e. the skin effect for high frequencies [4]) which leads to

$$\rho = a$$

This further reduces the equation

$$A_z = \frac{\mu}{4\pi}\int_{-l/2}^{l/2}\int_0^{2\pi}\frac{I_z}{2\pi a}\frac{e^{-jkR}}{R}\,a\,d\varphi' dz'$$

$$= \mu\int_{-l/2}^{l/2}\left(I_z(z')\frac{1}{2\pi}\int_0^{2\pi}\frac{e^{-jkR}}{4\pi R}d\varphi'\right)dz'$$

$$= \mu\int_{-l/2}^{l/2} I_z(z')G(z,z')dz'$$

This may look like abrupt transition in equation but it is not if we consider the Green's function in free-space: [2]

$$G(z,z') = \frac{1}{2\pi}\int_0^{2\pi} \frac{e^{-jkR}}{4\pi R} d\varphi'$$

where

$$R = \sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}$$

In cylindrical coordinate, this expression can be re-written:

$$R = \sqrt{\rho^2 + a^2 - 2\rho\, a\cos(\varphi - \varphi') + (z-z')^2}$$

$$\rho = radial\ distance\ to\ the\ observation\ point$$

$$a = radius\ of\ the\ wire$$

On the observation point, the equations become

$$A_z(\rho = a) = \mu \int_{-l/2}^{l/2} I_z(z') G(z,z') dz'$$

$$R = \sqrt{a^2 + a^2 - 2a\, a\cos(\varphi - \varphi') + (z-z')^2}$$

$$= \sqrt{2a^2\big(1 - cos(\varphi - \varphi')\big) + (z-z')^2}$$

since the structure is symmetric along φ direction, let $\varphi = 0$ for simplicity in calculation.

$$= \sqrt{2a^2\left(2\sin^2\frac{\varphi'}{2}\right) + (z-z')^2}$$

$$R(\rho = a) = \sqrt{4a^2\, sin^2\frac{\varphi'}{2} + (z-z')^2}$$

Now, we collected all the pieces we need for the integrodifferential equation.

Re-writing the equation for electric field:

$$E_z^s = -\frac{j}{\omega\varepsilon}\left(k^2 + \frac{\partial^2}{\partial z^2}\right)\int_{-l/2}^{l/2} I_z(z') G(z,z') dz' = -E_z^i$$

Hence,

$$-j\omega\varepsilon E_z^i(\rho = a) = \int_{-l/2}^{l/2} I_z(z')\left[\left(k^2 + \frac{\partial^2}{\partial z^2}\right) G(z,z')\right] dz'$$

Pocklington's Integral equation.

The above Pocklington's IE can be further simplified by assuming the wire is very thin such that

$$a \ll \lambda$$

which will lead to

$$G(z,z') = \frac{1}{2\pi}\int_0^{2\pi} \frac{e^{-jkR}}{4\pi R} d\varphi' \rightarrow G(R) = \frac{e^{-jkR}}{4\pi R}$$

because there's no φ' variation. For more rigorous analysis, one may account for the φ' variation in their calculations. In addition, the expression for R can be further simplified in more convenient way:

$$R = \sqrt{a^2 + (z-z')^2}$$

The way I think of the rationale that we could do this is fixing $\varphi' = {}^{\pi}/_{3}$ since there's no φ' variation.

To simplify further, let's consider the Helmholtz operator on the Green's function, particularly the partial derivatives.

$$\frac{\partial}{\partial z}\left[\frac{\partial}{\partial z}\left(\frac{e^{-jkR}}{R}\right)\right] = \frac{\partial}{\partial z}\left[\frac{\partial}{\partial R}\left(\frac{e^{-jkR}}{R}\right)\frac{\partial\left(\sqrt{a^2+(z-z')^2}\right)}{\partial z}\right]$$

<<Note that we take $\frac{1}{4\pi}$ out of the equation. >>

$$= \frac{\partial}{\partial z}\left[-\frac{e^{-jkR}}{R}(z-z')\left(\frac{1+jkR}{R^3}\right)\right]$$

$$= \frac{e^{-jkR}}{R^5}\{-R^2(1+jkR) - [jkR(z-z')^2 - (1+jkR)(3+jkR)(z-z')^2]\}$$

<<Usually, I would like to derive every single equation I encounter but I would rather have a faith on the author and believe this partial derivative is correct this time around than verify it for myself.>>

Using the following relation, we could express the Pocklington's IE in more simplified (convenient for Code) form:

$$(z-z')^2 = R^2 - a^2$$

$$\boxed{-j\omega\varepsilon E_z^i(\rho = a) = \int_{-l/2}^{l/2} I_z(z')\frac{e^{-jkR}}{4\pi R^5}[(1+jkR)(2R^2-3a^2) + (kaR)^2]dz'}$$

IV. The Method of Moment (MM)

To approximate using MM, the unknown function must be expressed in terms of a linear combination of Basis functions. Famous basis functions are (1) unit pulse (2) piece-wise sinusoidal function.

a. General Idea of MM

Consider the following equation model:

$$F(g) = h$$

where

$$F = linear\ operator$$

$$g = g(z') = unknown\ response\ function$$

$$h = known\ excitation\ function$$

From basic circuit theory, we can use similar analogy of a transfer function for a system. For example, let's say we have a simple RC circuit where the transfer function can be easily derived. When the input is a simple sinusoidal in time-domain (or equivalently, unitary in frequency-domain), the natural response would be the system response. In other words, the system response equals to the natural response such that the transfer function of the system fully describes the characteristic of the system.

Now that the unknown function is expressed in terms of a linearly independent set with corresponding amplitudes (coefficients).

$$g(z') \approx a_1 g_1(z') + a_2 g_2(z') + \cdots + a_N g_N(z') = \sum_{n=1}^{N} a_n g_n(z')$$

Which leads to

$$h = \sum_{n=1}^{N} a_n F\big(g_n(z')\big)$$

However, what we just did here is that we generated N unknowns (***variables***) with a single equation which is an *inconsistent system* in Linear algebra language. To make such system to be consistent, we need another N ***equations***. This can be done by considering *"the inner product"* between the function on which the linear operator is acted and appropriate *"weighing functions"*.

<<Here comes the idea of boundary condition. The point-matching technique allows one to use the Dirac-delta function for the weighing function to satisfy the boundary condition (i.e. vanishing the tangential component of electric field on the surface of electric conductor) on *each element* surface of wire. >>

b.Inner Product

$$\langle \vec{w}, \vec{h} \rangle = \iint_S \vec{w}^* \cdot \vec{h}\, dS$$

where

$$\vec{w}^* = complex\ conjugate\ of\ \vec{w}$$

$$S = the\ surface\ of\ the\ strucuture\ being\ analyzed$$

c. Impedance Matrix

For simplicity, let's choose the unit pulse function for the basis function. Then, the current can be expressed in the following way:

$$I_z(z') = \sum_{n=1}^{N} a_n g_n(z')$$

where a_n the coefficients for current amplitude that we need to find, and the basis function is:

$$g_n(z') = \begin{cases} 1, & (n-1)\Delta z' \le z' \le n\Delta z' \\ 0, & elsewhere \end{cases}$$

$$E_z^i = -\frac{j}{\omega\varepsilon} \sum_{n=1}^{N} \int_{(n-1)\Delta z\prime}^{n\Delta z\prime} a_n \left[\left(k^2 + \frac{\partial^2}{\partial z^2} \right) G(z, z') \right] dz'$$

Now, to form an impedance matrix, we need the inner product

$$\langle w_m, E_z^i \rangle = \int w_m \left[-\frac{j}{\omega\varepsilon} \sum_{n=1}^{N} \int_{(n-1)\Delta z\prime}^{n\Delta z\prime} a_n \left[\left(k^2 + \frac{\partial^2}{\partial z^2} \right) G(z, z') \right] dz' \right] dz$$

$$= \sum_{m=1}^{N} \left[-\frac{j}{\omega\varepsilon} \sum_{n=1}^{N} \int_{(n-1)\Delta z\prime}^{n\Delta z\prime} a_n \left[\left(k^2 + \frac{\partial^2}{\partial z^2} \right) G(z_m, z') \right] dz' \right]$$

$$= Z_{mn}$$

If we were to choose the Dirac-delta function for the weighing function, then the point-matching technique can be used to generate N different equations. In that case, the Dirac-delta function can be defined as the follow:

$$\delta_m(z - z_m) = \begin{cases} 1, & z = z_m \\ 0, & elsewhere \end{cases}$$

Note that if we use piece-wise pulse function for basis and the Dirac-delta function for weighing function, then one might want to locate each z_m at the center of the corresponding pulse function.

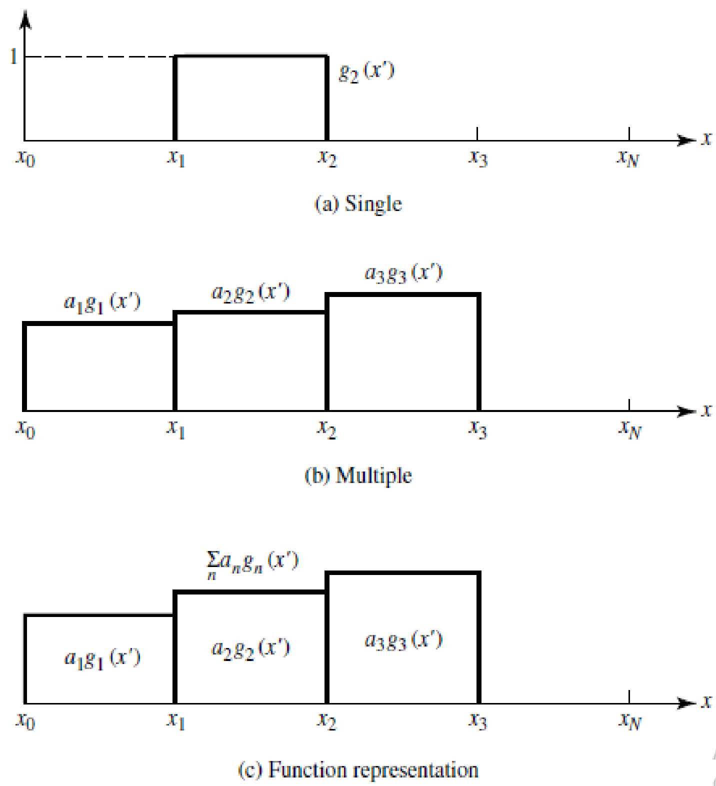


Figure 8.8 Piecewise constant subdomain functions.

d.Code Realization in Python

$$[Z_{mn}][I] = -\frac{j}{\omega\varepsilon}\sum_{m=1}^{N}\sum_{n=1}^{N}\int_{(n-1)\Delta z'}^{n\Delta z'} a_n\left[\frac{e^{-jkR}}{4\pi R^5}[(1+jkR)(2R^2-3a^2)+(kaR)^2]\right]dz'$$

where

$$R = \sqrt{a^2+(z_m-z')^2}$$

$$[I] = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$$

Note that the above impedance matrix has been derived from the point-matching technique.

<<I realized the above equation doesn't yield the correct output. After researching, it turned out that Pocklington's IE would yield slower convergence rate than that of Hallen's because of the inverse of R. As the power of the inverse of R increases, the convergence even goes slower. This point made me try out a new way to approach.>>

Let’s start with the Pocklington’s IE:

$$-j\omega\varepsilon E_z^i(\rho = a) = \int_{-l/2}^{l/2} I_z(z') \left[\left(k^2 + \frac{\partial^2}{\partial z^2}\right) G(z, z')\right] dz'$$

From [3], we know that the Helmholtz operator is a linear operator. In addition, as introduced in the general idea of MM section,

$$F = k^2 + \frac{\partial^2}{\partial z^2}$$

$$g = I_z(z')$$

$$h = E_z^i$$

Choosing basis function for the pulse function and considering the Galerkin method where basis function and weighing function are the same.

Using the pulse function for the basis function, the current can be re-modeled as:

$$I_z = \sum_{n=1}^{N} a_n b_n$$

$$\sum_{n=1}^{N} a_n \int_{b_n} b_n(z') \left[\left(k^2 + \frac{\partial^2}{\partial z^2}\right) G(z, z')\right] dz' = -j\omega\varepsilon E_z^i(z)$$

To apply the weighing function, we need the inner product:

$$\langle b_m(z), \sum_{n=1}^{N} a_n \int_{b_n} b_n(z') \left[\left(k^2 + \frac{\partial^2}{\partial z^2}\right) G(z, z')\right] dz' \rangle = \langle b_m(z), -j\omega\varepsilon E_z^i(z) \rangle$$

The inner product can be expressed using integral,

$$\sum_{n=1}^{N} a_n \int_{b_m} b_m(z) \int_{b_n} b_n(z') \left[\left(k^2 + \frac{\partial^2}{\partial z^2}\right) G(z, z')\right] dz'\, dz = -j\omega\varepsilon \int_{b_m} b_m(z)\, E_z^i(z) dz$$

where

b_m *and* b_n are subintervals for the weighing and basis function, respectively. Now, we’ve constructed the following matrix

$$[Z_{mn}][I_n] = [V_m]$$

Recall that the Green’s function is

$$G(R) = \frac{e^{-jkR}}{4\pi R}$$

where

$$R = \sqrt{a^2 + (z - z')^2}$$

Utilize the fact that the operator is linear, the impedance matrix can be re-written:

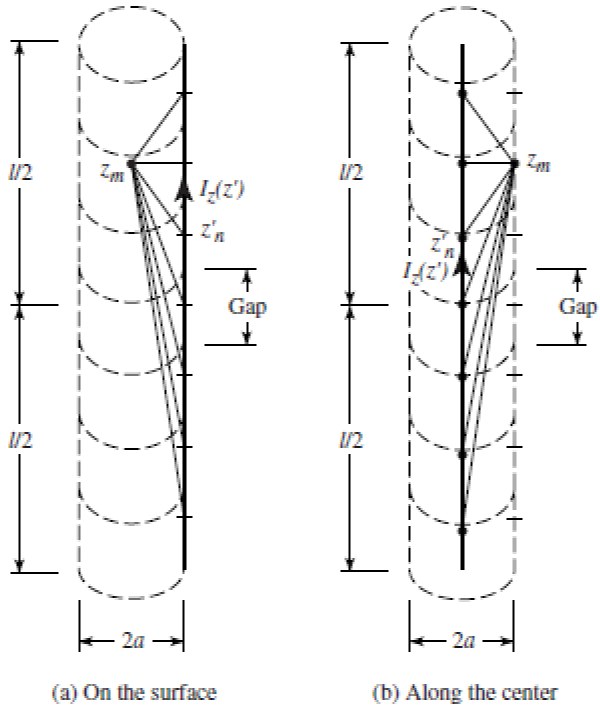


Figure 8.6 Dipole segmentation and its equivalent current.

z_m is at the center of each subsection $\Delta z = {}^{l}/_{N}$ and the integration would be performed over the subinterval. Now, if we were able to find the closed form solution for the linear operator acting on the Green's function term, then we may successfully derive the Pocklington's IE.

Using the Wolfram alpha, we can easily compute the partial derivative:

$$\frac{\partial}{\partial x}\left(\frac{\exp\left(b\sqrt{x^2+a^2}\right)}{\sqrt{x^2+a^2}}\right) = \frac{b\,x\,e^{b\sqrt{a^2+x^2}}}{a^2+x^2} - \frac{x\,e^{b\sqrt{a^2+x^2}}}{(a^2+x^2)^{3/2}}$$

which is equivalent to

$$x\,e^{b\sqrt{a^2+x^2}}\left(\frac{b}{a^2+x^2} - \frac{1}{(a^2+x^2)^{3/2}}\right)$$

$$[Z_{mn}] = k^2\int_{b_n}\frac{e^{-jkR}}{4\pi R}dz' + \int_{b_n}(z_m - z')\frac{e^{-jkR}}{4\pi}\left(\frac{-jk}{R^2} - \frac{1}{R^3}\right)dz'$$

$$= k^2 \int_{b_n} \frac{e^{-jkR}}{4\pi R} dz' + \int_{b_n} (z_m - z') e^{-jkR} \left(\frac{-jkR - 1}{4\pi R^3} \right) dz'$$

Now, let's think about the interval in which the integral is performed. As aforementioned, z_m is located at the center of each subinterval Δz, and the range over which integral is performed is also Δz. The variable dz' is for the integral over z_n coordinate.

$$Z_{mn} = k^2 \int_{z_n - \frac{\Delta z}{2}}^{z_n + \frac{\Delta z}{2}} \frac{e^{-jkR}}{4\pi R} dz' + \left[(z_m - z') e^{-jkR} \left(\frac{jkR + 1}{4\pi R^3} \right) \right]_{z' = z_n - \frac{\Delta z}{2}}^{z' = z_n + \frac{\Delta z}{2}}$$

where

$$R = \sqrt{a^2 + (z_m - z')^2}$$

It might be easier thinking of z_m only for point-matching technique. For verification of the above equation, one may find Python code in "Code" folder.

For source model,

$$h_m = -j\omega\varepsilon \int_{b_m} b_m(z)\, E_z^i(z) dz$$

$$= -j\omega\varepsilon E_z^i(z_m)$$

For source modeling, we are going to use the delta-gap source model. A reminder of this electric field: it's the incident electric field, only the z-component. In addition, to make the electric field vector into "voltage" vector, we use the following simple relation:

$$E_z^i(z_m) = \frac{V_m}{\Delta z}$$

Hence, we may find the impedance scaling factor as

$$z_{scale} = \frac{\Delta z}{-j\omega\varepsilon} = \frac{j\Delta z}{\omega\varepsilon}$$

such that the true voltage expression

$$[v_m] = \frac{j\Delta z}{\omega\varepsilon} [Z_{mn}][i_n]$$

where scaled impedance is expressed:

$$Z = \frac{j\Delta z}{\omega\varepsilon} [Z_{mn}]$$

For checking the electric field pattern derived from the current distribution, we may recall from the pattern multiplication:

$$SF = \int_{-l/2}^{l/2} I_e(x', y', z') e^{jkz\cos\theta} dz'$$

$$EF = j\eta \frac{ke^{-jkr}}{4\pi r} \sin\theta$$

where

$$I_e(x', y', z') = I_0 \sin\left(k\left(\frac{l}{2} - z_n\right)\right)$$

The total field is the product of SF (space factor) and EF (element factor). Note that the element factor would be equal to the field of unit length infinitesimal dipole antenna located at the origin. For field pattern analysis, we're going to look at the space factor for this dipole.

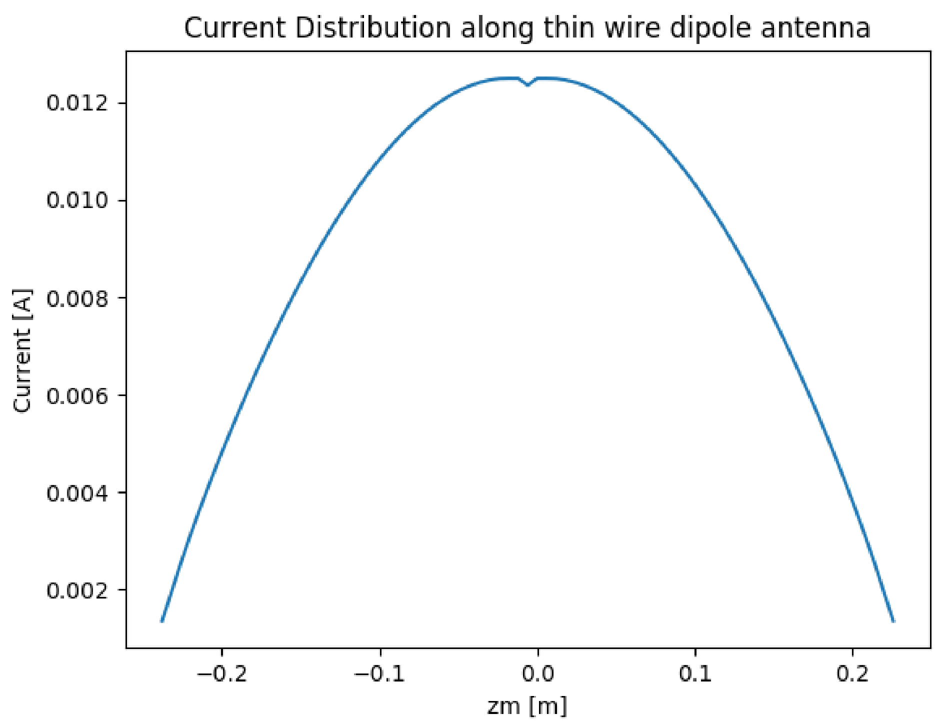


Figure. Z-directed dipole antenna current distribution ($N = 79$)

N	Input Impedance
39	$75.5 - j3.52$
79	$80.5 + j9.16$
101	$82.02 + j8.41$

From the textbook, we note that the input impedance of half-wavelength dipole has

$$Z_{in} = 73 + j42.5$$

which leads to a discrepancy between what we just found and the textbook result. This has to do with the convergence of method. We could obtain more accurate results by applying different IE (i.e. Hallen’s IE) or by applying different source model (i.e. Magnetic Frill). As a matter of fact, <<Pocklington’s IE and delta gap source model are the slowest convergence model. Let me find this reference.>>

Let’s look at the polarization for E_θ component in different cut planes.

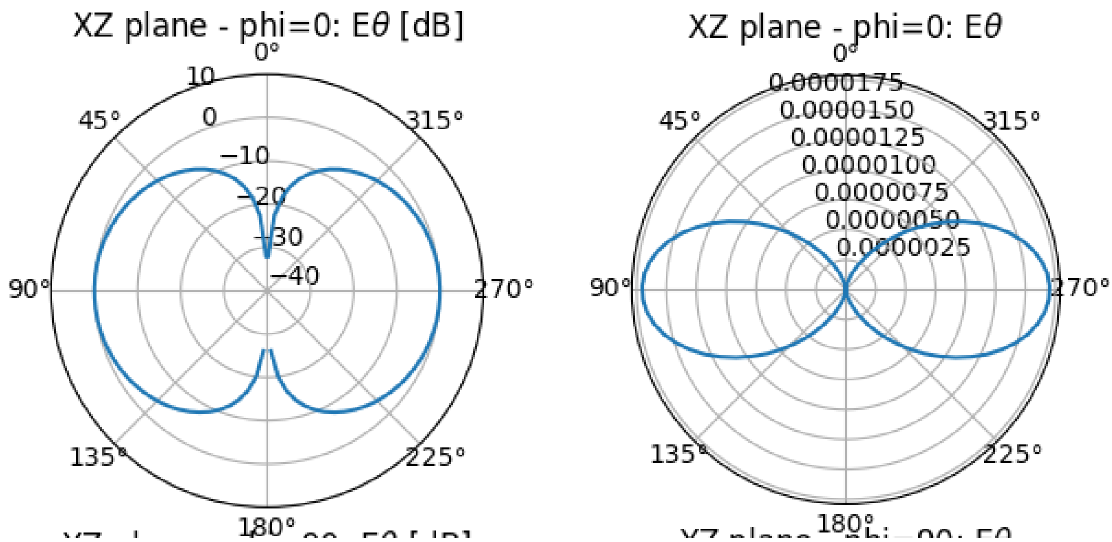


Figure. XZ cut plane view ($N = 79$)

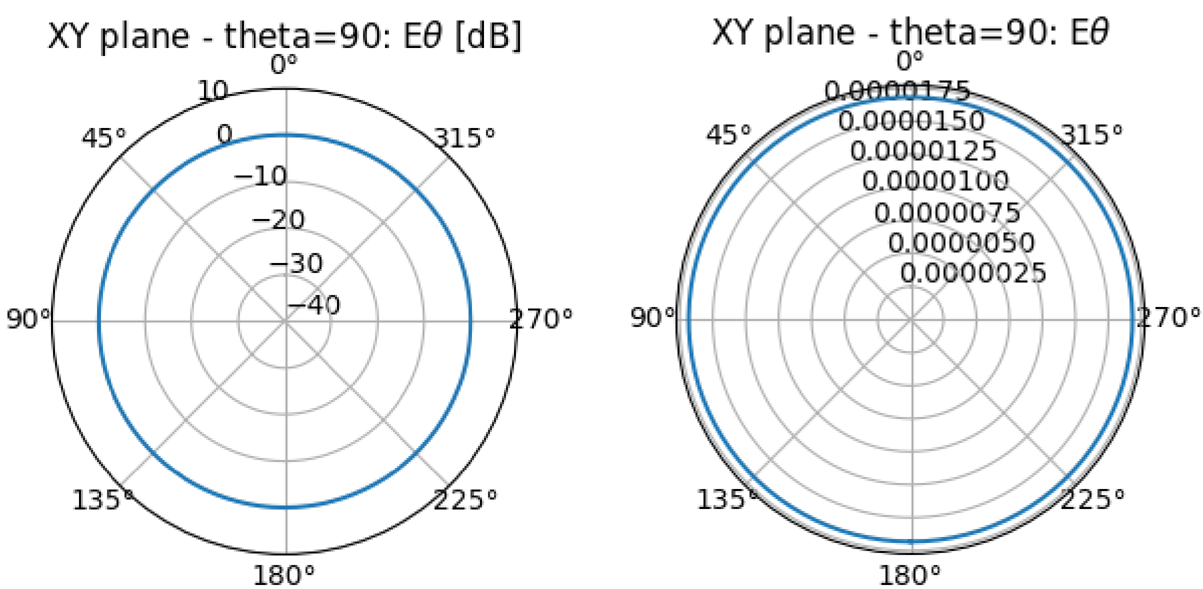


Figure. XY cut plane view ($N = 79$)

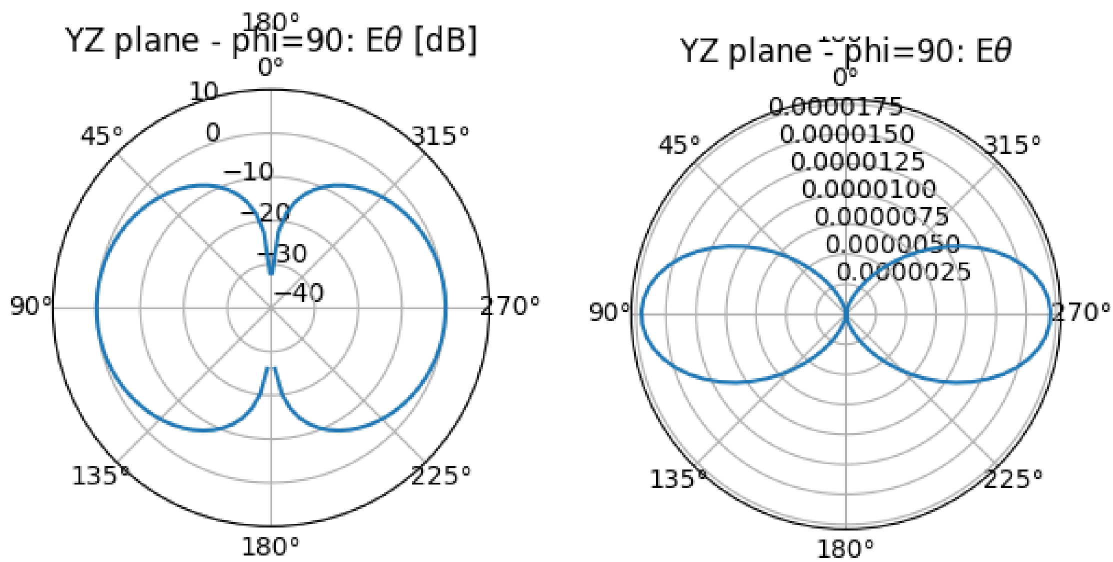


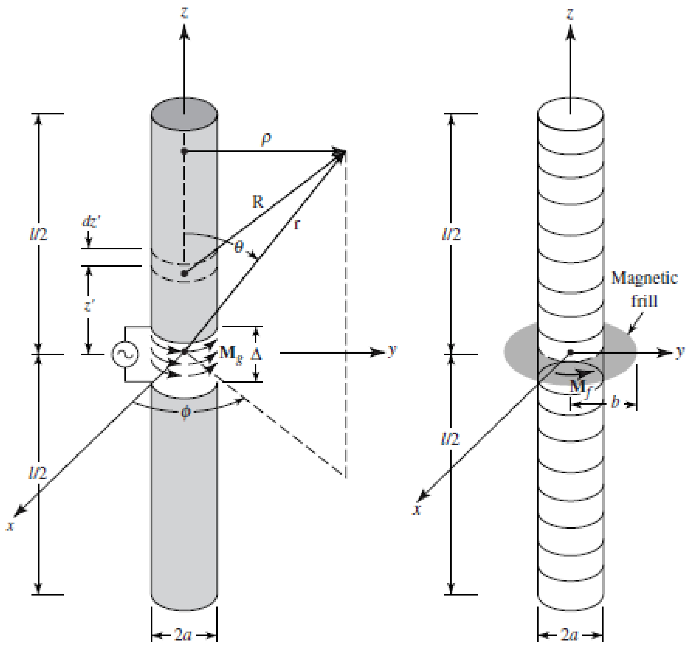
Figure. YZ cut plane view ($N = 79$)

The python code that generated above results can be found in the fold named “Code”.

V. Source Modeling

There are two types of sources for MM introduced in textbook [1]: (1) Delta-gap (2) Magnetic-Frill Generator

a. Delta-Gap



(a) Dipole

(b) Segmentation and gap modeling

Figure 8.7 Cylindrical dipole, its segmentation, and gap modeling.

VI. HFSS Simulation

Along with the MM, let's design a $300MHz$ half-wavelength dipole Antenna on HFSS.

$$f = 300\ MHz$$

$$\lambda = 1m$$

The length of dipole, therefore,

$$l = 0.5\ m$$

As before, the radius of wire will be

$$a = 0.005\ m$$

which is far less than the wavelength.

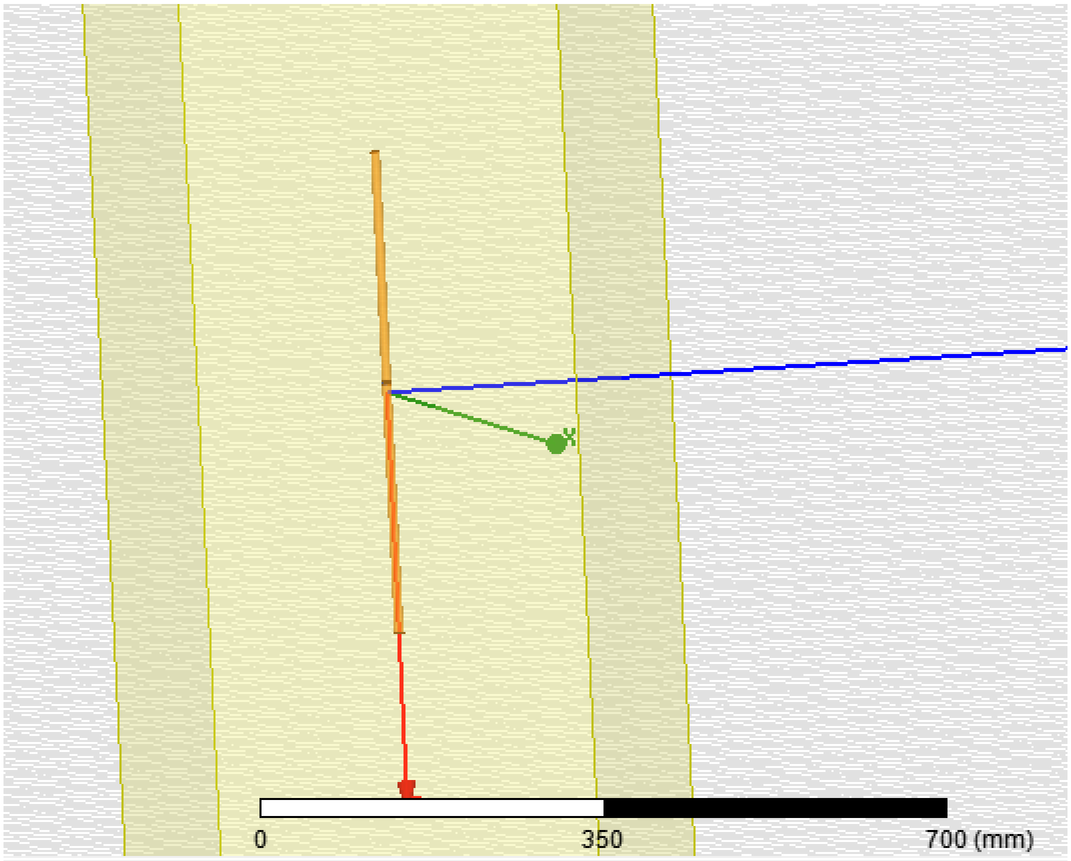


Figure. HFSS Design of Half-Wavelength Dipole

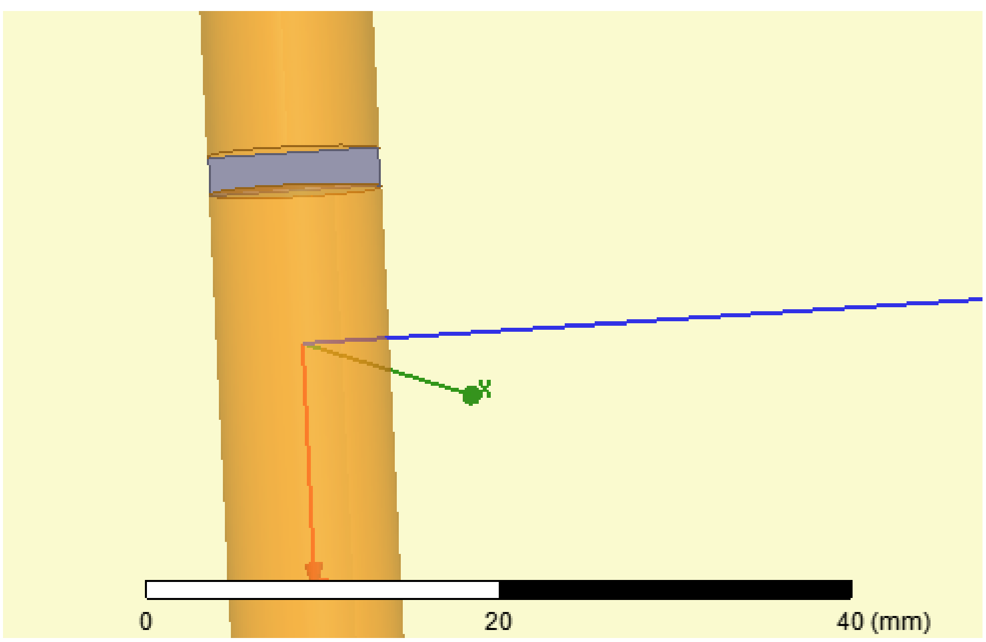


Figure. Input port by Delta-gap Model

$\Delta = 2.2\ mm$

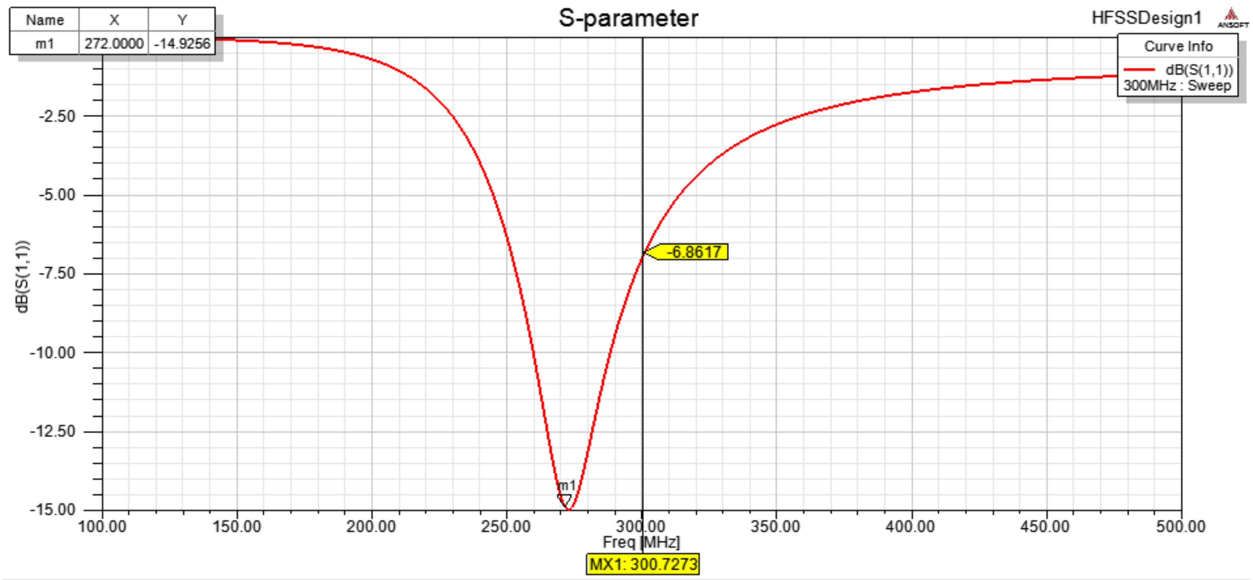


Figure. S_{11}

In the S-parameter plot, we note that the antenna should be operating around

$$272\ MHz$$

It might be a good exercise identifying what other parameters control this reflection. I forgot the reason why the author used 0.47λ instead of half-wavelength.

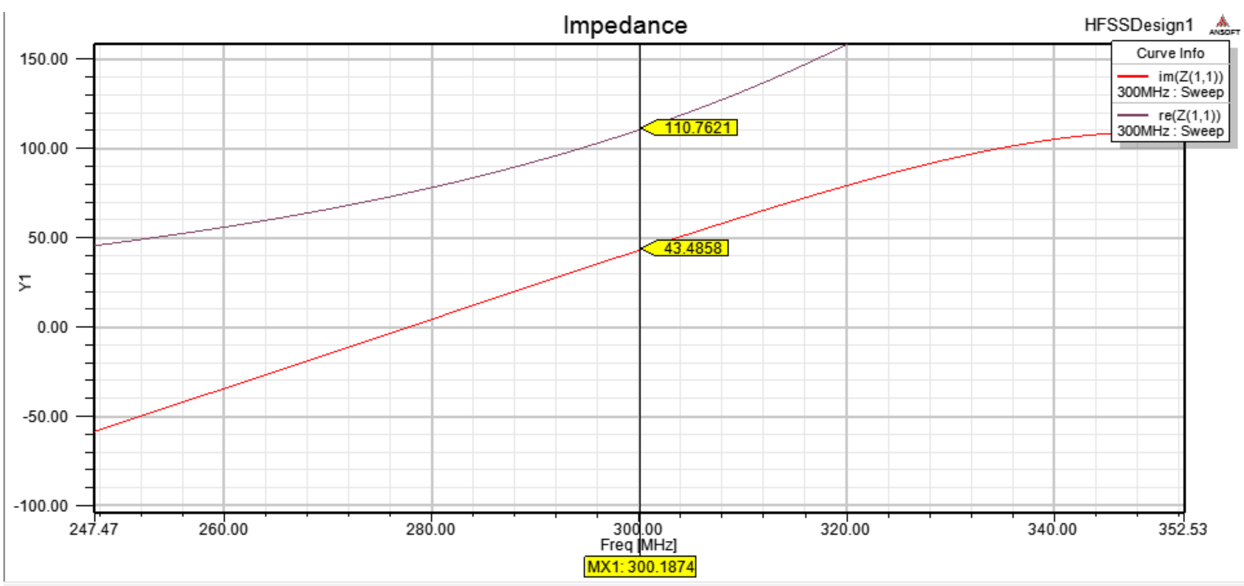


Figure. Input Impedance of Dipole

$$Z_{in} = 43.48 + j110.76$$

From the impedance plot, a well-designed matching network is required for coaxial cable if needed.

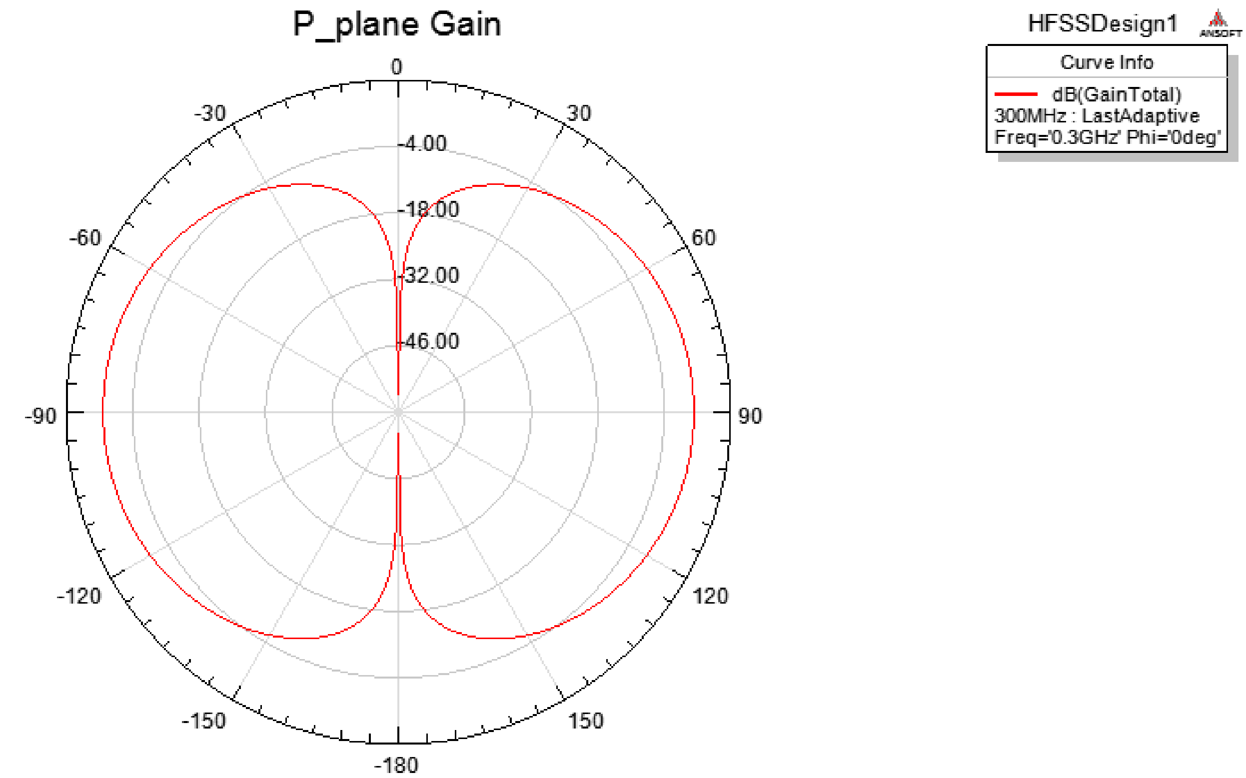


Figure. Radiation Pattern of Dipole

$$G_{max} = 2.2457$$

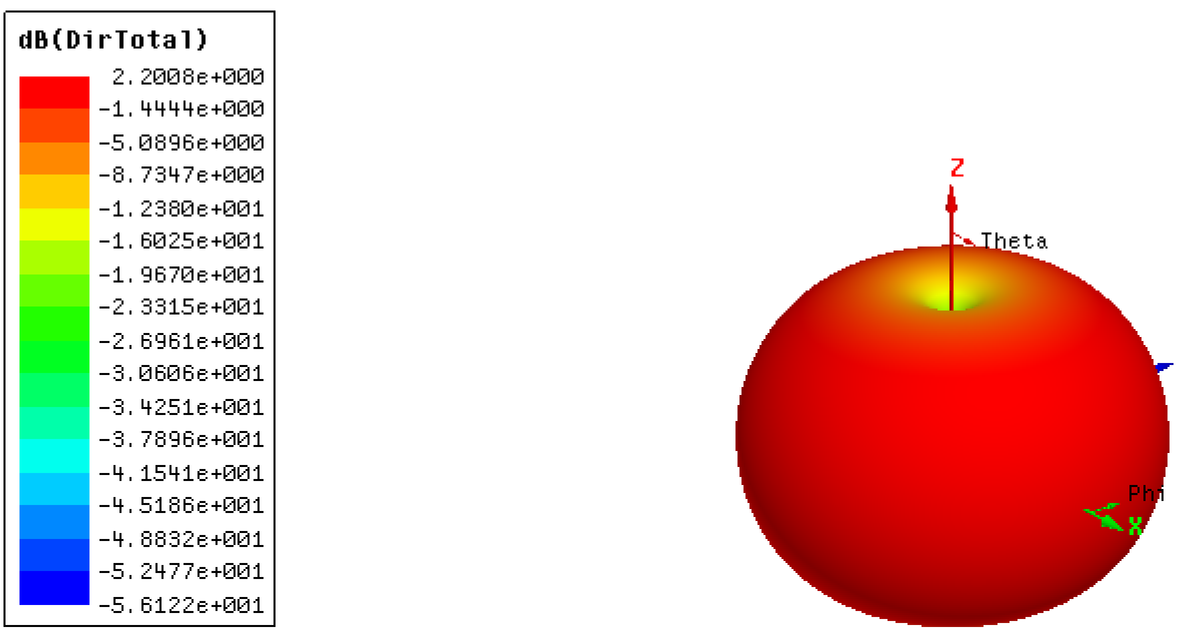


Figure. 3D Directivity

VII. References

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[4] https://en.wikipedia.org/wiki/Skin_effect

[5] https://www.wolframalpha.com/input/?i=d(exp(b*sqrt(x%5E2%2Ba%5E2))%2Fsqrt(x%5E2%2Ba%5E2))%2Fdx

[6] http://emlab.utep.edu/ee5390cem/Lecture%2028%20--%20Method%20of%20Moments%20for%20Thin%20Wire%20Antennas.pdf