Isomorphism

I. Introduction

In this document, simple ideas of isomorphism will be covered. I hope this document would alleviate the confusions arose from all the mathematical terminologies.

II. Terminologies

Automorphism
Homomorphism
Homeomorphism
Diffeomorphism

(1) Isomorphism

In mathematics, an isomorphism is a homomorphism or morphism that admits an inverse.

(2) Homomorphism

For algebraic structure (ring or group), a homomorphism is an isomorphism iff it is bijective.

(3) Automorphism

An isomorphism whose target and source coincide. In other words, it is an isomorphism from a mathematical object to itself. In set theory, this is known as “permutation”.

(4) Homeomorphism

In topology, where morphisms are continuous functions, isomorphisms are called homeomorphism.

(5) Diffeomorphism

In mathematical analysis, where the morphisms are differential functions, isomorphisms are called diffeomorphism.

III. Examples

a. Exponentials and logarithms

Let $\mathbb{R}^+$be a multiplicative group of positive real numbers and $\mathbb{R}$ be an additive group of real numbers. If the followings are met, then the functions (from A to B and from B to A) between the groups are homomorphism each.

Group?

It's an algebraic structure consisting of sets and elements. [2]

(1) $f(a+b) = f(a) \ddagger f(b)$
(2) $f(a \cdot b) = f(a) * f(b)$
(3) $f(1_R) = 1_S$

Where the third condition may drop if the algebraic structure doesn't have to be unital.

The function "Logarithm function" $log: \mathbb{R}^+ \rightarrow \mathbb{R}$ satisfies $\log(xy) = \log(x) + \log(y)\ \ for\ all\ x, y \in \mathbb{R}^+$. Hence, logarithm function is a group homomorphism.

The function "Exponential function" $exp: \mathbb{R} \rightarrow \mathbb{R}^+$satisfies $\exp(x+y) = \exp(x)\exp(y)\ \ for\ all\ x, y\ \in \mathbb{R}$. Hence, exponential function is a homomorphism.

Now, the two identities:

$$\log(\exp(x)) = x$$

$$\exp(\log(y)) = y$$

Allow us to determine that they are inverse to each other.

Since **log** is a homomorphism and its inverse, **exp**, is also a homomorphism, **log** is an isomorphism of groups. Because **log** is an isomorphism, it translates multiplications of positive real numbers into addition of real numbers.

This might be a great tool for simplifying complicated formula or some equation but a few functional properties of the function should be nailed before identifying whether the function possess isomorphic properties.

b.Relation between $\mathbb{P}^2$ and $\mathbb{R}^3$ [3]

In $\mathbb{P}^2$ space, a polynomial function can be defined as the follow:

$$a_0 + a_1 x + a_2 x^2$$

Meanwhile in $\mathbb{R}^3$, a vector can be expressed in the following way:

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix}$$

The two vector spaces are isomorphic to each other if

(1) One-to-one correspondence can be established between them
(2) Addition while preserving structures
(3) Scalar Multiplication while preserving structures

Let's see what these means.

(1) One-to-one correspondence

To find out if there's one-to-one correspondence,

$$a_0 + a_1 x + a_2 x^2 \overset{f}{\longmapsto} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix}$$

For f to be one-to-one,

$$f(a_0 + a_1 x + a_2 x^2) = f(b_0 + b_1 x + b_2 x^2)$$

is valid only when

The two sets are equal. That is,

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} b_0 \\ b_1 \\ b_2 \end{bmatrix}$$

(2) Additive

$$a_0 + a_1 x + a_2 x^2 + b_0 + b_1 x + b_2 x^2$$
$$= (a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2$$

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} + \begin{bmatrix} b_0 \\ b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} a_0 + b_0 \\ a_1 + b_1 \\ a_2 + b_2 \end{bmatrix}$$

Hence, their structures are preserved under addition.

(3) Scalar Multiplication

For some $c \in \mathbb{C}$,

$$c(a_0 + a_1 x + a_2 x^2) = (ca_0) + (ca_1)x + (ca_2)x^2$$

$$c\begin{bmatrix}a_0\\a_1\\a_2\end{bmatrix}=\begin{bmatrix}ca_0\\ca_1\\ca_2\end{bmatrix}$$

Hence, their structures are preserved under scalar multiplication.

Why is the idea of "Isomorphism" is important? My understanding of its importance is that we may find simpler approaches to the result by using their structures are "similar". It might be tedious if one had to perform some mathematical operations on a series of polynomials, assuming the order is relatively high, then the "math/calculation" may be lengthy. In this case, we may consider a set of vectors such that we can reduce the burdens of calculation while maintaining the mathematical operations that should have acted on the polynomials. In addition, if two vectors are isomorphic to each other, then the transformation (or, function) that relate two vector spaces does have the inverse matrix (or, the inverse function).

c. Example 3 [3]

Let a vector space $\mathbb{G}=\{c_1\cos\theta+c_2\sin\theta\,|c_1,c_2\in\mathbb{R}\}$. Now, we are interested in the relation between a vector space $\mathbb{R}^2$ and $\mathbb{G}$. (i.e. isomorphism can be established between them?)

(1) One-to-one correspondence

One-to-one correspondence is formed between the two spaces you may check this point by investigating the same procedure used in Example (b) in this document.

(2) Addition

$$f(c_1\cos\theta+c_2\sin\theta+d_1\cos\theta+d_2\sin\theta)$$

where

$$d_1,d_2\in\mathbb{R}$$

Then,

$$f\big((c_1+d_1)\cos\theta+(c_2+d_2)\sin\theta\big)$$

$$=\begin{bmatrix}c_1+d_1\\c_2+d_2\end{bmatrix}$$

$$=\begin{bmatrix}c_1\\c_2\end{bmatrix}+\begin{bmatrix}d_1\\d_2\end{bmatrix}$$

(3) Scalar Multiplication

For some scalar

$$t \in \mathbb{R}$$

$$tf = f(tc_1 \cos\theta + tc_2 \sin\theta)$$

$$= \begin{bmatrix} tc_1 \\ tc_2 \end{bmatrix} = t \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

Therefore, the two vectors space are isomorphic to each other.

d.Coordinates (between Cartesian and Cylindrical)

For an exercise, let’s do a simple example. Let’s say a vector in Cartesian coordinate

$$\vec{u} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

The same vector in Cylindrical coordinate

$$\vec{u} = \begin{bmatrix} \rho \\ \varphi \\ z \end{bmatrix} = \begin{bmatrix} \sqrt{x^2 + y^2} \\ \tan^{-1}\dfrac{y}{x} \\ z \end{bmatrix}$$

The transformation matrix is given as (from Cylindrical to Cartesian)

$$\frac{\partial(x, y, z)}{\partial(r, \varphi, z)} = \mathcal{M} = \begin{bmatrix} \cos\varphi & -r\sin\varphi & 0 \\ \sin\varphi & r\cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Let’s do a scalar multiplication and see the two coordinates form an isomorphism. (although we need examine whether the other two conditions are also met.)

IV. References

[1] https://en.wikipedia.org/wiki/Isomorphism

[2] https://en.wikipedia.org/wiki/Group_(mathematics)

[3] https://en.wikibooks.org/wiki/Linear_Algebra/Definition_and_Examples_of_Isomorphisms