Tensor

I. Introduction

Tensor will provide the fundamental view of vector notations that are usually taken for granted. To understand why the idea of Tensor is important, one should have experienced with some problems concerning anisotropic properties of material. (i.e. wave propagation through anisotropic material, or fluid velocity in general)

In this document, my goal is identifying what are contravariant, covariant, mixed, and physical vectors, and how they are related. For example, a velocity of an object in basic physics course may have been defined as:

$$\vec{v} = (\dot{r}, \dot{\varphi}, \dot{z})$$

Which is said to be a contravariant vector. Note that "over-dot" notation is for derivative with respect to time.

Let's start this document with a claim:

"Tensors are objects in multilinear algebra that can have aspects of both co-variance and contra-variance."

Let's see what are co-variance and contra-variance.

a. Background

To identify if a vector is either co-variant or contra-variant vector, we need to look at its component. Categorizing co- or contra- variant vector results from the process when one wants the vector (or the co-vector) to be basis-independent. Because, for us, basis-independent vectors (or their components) are what we measure in real-life.

b. What are Contra-variant components?

Examples of vectors with contra-variant components:

(1) Position relative to an observer (not a length. Lengths are scalars.)
(2) Any derivative of position with respect to time (e.g. velocity, acceleration, and jerk)

In Einstein notation, contra-variant components are indicated with upper indices:

$$\vec{x} = v^i \hat{e}_i$$

To sum up, the idea of contra-variant components appears as we concern about *a vector* to be basis-independent. In contrast, that of co-variant components appears as *a co-vector* to be basis-independent.

For contra-variant components, one must transform them with *the inverse matrix* that transforms the basis vectors. To understand this, as an analogy, we would multiply the magnitude (vector component) by 100 when we convert from meter to centimeter but the dimension (the basis; meter to centimeter) has been divided by 100 to compensate.

c. What are Co-variant components?

Examples of co-vectors with co-variant components:

(1) They generally appear by taking a gradient of a function

Likewise, for a co-vector to be a basis-independent, its component must co-vary with the change of basis.

For co-variant components, one must transform them with *the same matrix* that transforms the basis vectors.

In Einstein notation, co-variant components are indicated with lower indices:

$$\vec{x} = v_i \hat{e}^i$$

d. Mathematically

A coordinate system is described by a coordinate vector $\vec{x}$ and it is transformed by a matrix $\mathcal{M}$. Then, the transformed can be done:

$$\vec{x}' = \mathcal{M}\vec{x}$$

To compensate this, the contra-variant vector, $\vec{v}$, written in the coordinate system should be transformed as the follow:

$$\vec{v}' = \mathcal{M}^{-1}\vec{v}$$

II. Basis Vectors

Vectors and matrices can be considered as Tensor only when they are coordinate invariant. Cartesian coordinate vectors are familiar to us as it appears may times in class and to be easy to picture them. Here, we represent the Cartesian coordinate as

$$X = (x^1, x^2, x^3)$$

and any arbitrary curvilinear coordinate system as

$$U = (u^1, u^2, u^3)$$

Then, general coordinate transformation can be done by

$$u^1 = u^1(x^1, x^2, x^3), u^2 = u^2(x^1, x^2, x^3), and\ u^3 = u^3(x^1, x^2, x^3)$$

In addition, if we want to express a vector with respect to each coordinate:

$$\vec{r} = \vec{r}(u^1, u^2, u^3) = P(x^1, x^2, x^3) - O(x^1, x^2, x^3)$$

Note that to express a vector, we use a position vector for any arbitrary curvilinear coordinate system. On the other hand, the difference between two points will be used to express a vector in the Cartesian coordinate where O represents the origin in the Cartesian.

As we saw before, there are two types of basis vectors: covariant vector and contravariant vector.

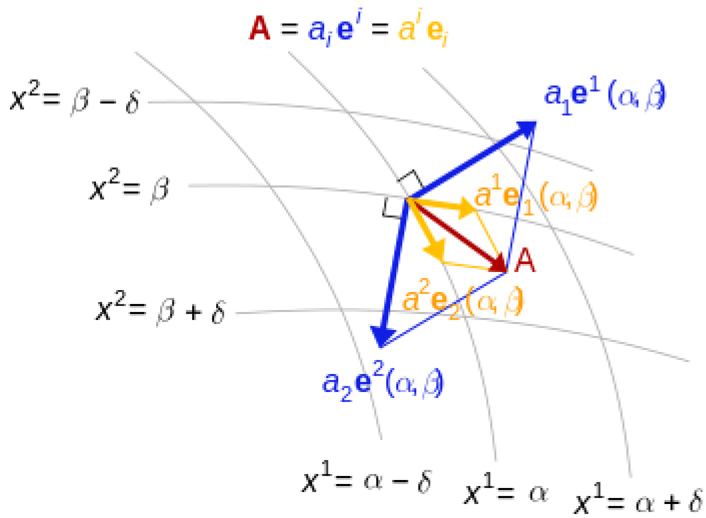


Figure. [1]

Now, let's see how covariant and contravariant are defined in mathematics.

The covariant vector can be defined as

$$\vec{a}_i = \frac{\partial \vec{r}}{\partial u^i} = \frac{\partial x^1}{\partial u^i}\hat{x}_1 + \frac{\partial x^2}{\partial u^i}\hat{x}_2 + \frac{\partial x^3}{\partial u^i}\hat{x}_3$$

What does it mean by $\frac{\partial \vec{r}}{\partial u^i}$?
It means that we are taking partial derivative of $\vec{r}$ (i.e. we will obtain a vector that varies in the direction of u^i varying.)

The contravariant vector can be defined as

$$\vec{a}^i = \nabla u^i = \frac{\partial u^i}{\partial x^1}\hat{x}_1 + \frac{\partial u^i}{\partial x^2}\hat{x}_2 + \frac{\partial u^i}{\partial x^3}\hat{x}_3$$

What does it mean by ∇u^i?
It means that we take a gradient of u^i (i.e. a gradient will point to the maximum value it varies. Hence, we get to know the direction to the maximum of u^i)

In Cartesian coordinate, the contravariant vector and the covariant vector coincide.

$$\vec{a}_1 = \frac{\partial x^1}{\partial x^1}\hat{x}_1 + \frac{\partial x^2}{\partial x^1}\hat{x}_2 + \frac{\partial x^3}{\partial x^1}\hat{x}_3 = \hat{x}_1$$

Likewise, $\vec{a}_2 = \hat{x}_2$ and $\vec{a}_3 = \hat{x}_3$

For contravariant vector,

$$\vec{a}^1 = \frac{\partial x^1}{\partial x^1}\hat{x}_1 + \frac{\partial x^1}{\partial x^2}\hat{x}_2 + \frac{\partial x^1}{\partial x^3}\hat{x}_3 = \hat{x}_1$$

Likewise, $\vec{a}^2 = \hat{x}_2$ and $\vec{a}^3 = \hat{x}_3$

Now, let’s relate the covariant and contravariant vectors by taking the inner product.

$$\vec{a}^i \cdot \vec{a}_j = \frac{\partial u^i}{\partial x^1}\hat{x}_1 \cdot \frac{\partial \vec{r}}{\partial u^j} + \frac{\partial u^i}{\partial x^2}\hat{x}_2 \cdot \frac{\partial \vec{r}}{\partial u^j} + \frac{\partial u^i}{\partial x^3}\hat{x}_3 \cdot \frac{\partial \vec{r}}{\partial u^j}$$

$$= \frac{\partial u^i}{\partial x^1}\frac{\partial x^1}{\partial u^j} + \frac{\partial u^i}{\partial x^2}\frac{\partial x^2}{\partial u^j} + \frac{\partial u^i}{\partial x^3}\frac{\partial x^3}{\partial u^j}$$

$$= \frac{\partial u^i}{\partial u^j} = \delta^i_j$$

<<The above relation can be verified by plugging actual index. Realize that the above terms are in the form of “*exact differential*”.>>

Hence, we verify that

$$\boxed{\vec{a}^i \cdot \vec{a}_j = \delta^i_j}$$

a. Contravariant and Covariant components

To be consistent, let’s set up some terminologies:

(1) Contravariant components u^i
(2) Covariant components u_i
(3) Contravariant base vectors a^i
(4) Covariant base vectors a_i

Contravariant base vectors and covariant base vectors (reciprocal to each other) are related by the Kronecker Delta:

$$a^i \cdot a_j = \delta^i_j$$

which holds even when bases do not form orthonormal sets. <<It seems like one determines the other so that uniqueness of one kind can be guaranteed. In other word, for example, if covariant vectors were given, then contravariant vectors will be determined from the Kronecker delta. This is true!>> [1] In general, a vector can be expressed in the following ways:

$$\vec{u} = u^i a_i$$

$$\vec{u} = u_i a^i$$

From this, we can extract components (i.e. either contravariant or covariant) by applying appropriate base vector, for example, to find the covariant component of a vector, we apply dot product with covariant base vectors:

$$\vec{u} \cdot a^i = \left(u^j a_j\right) \cdot a^i = u^j \delta_j^i$$

To obtain u^i from $u^j \delta_j^i$

$$j = i$$

since the Kronecker Delta symbol is

$$\delta_j^i = 1$$

when the indexes are the same.

Hence, in a compact form, we can obtain the following relation:

$$\boxed{\begin{array}{c} \vec{u} \cdot a^i = u^i \\ \vec{u} \cdot a_i = u_i \end{array}}$$

However, what would happen if a vector were written in terms of, for example, contravariant components with covariant base vectors and dot product-ed with another covariant base vector? In turn, this would introduce a new mathematical object, named “metric tensor”.

$$\vec{u} \cdot a^i = \left(u^j a^j\right) \cdot a^i = u^j \left(a^j \cdot a^i\right)$$

where the metric tensor in general

$$a^i \cdot a^j = g^{ij}$$

$$a_i \cdot a_j = g_{ij}$$

Please note that g^{ij} forms the inverse of g_{ij}. (this can be easily checked by expanding the entries to see their matrices. What happens when we consider the dot product of two vectors? i.e. transpose) Is there relation between covariant and contravariant components? Yes, there is.

Let’s examine the following case:

$$u_j g^{ij} = u_j\left(a^i \cdot a^j\right) = a^i \cdot \left(u_j a^j\right) = \vec{u} \cdot a^i = u^i$$

$$u^j g_{ij} = u^j\left(a_i \cdot a_j\right) = a_i \cdot \left(u^j a_j\right) = \vec{u} \cdot a_i = u_i$$

Hence,

$$u_j g^{ij} = u^i$$
$$u^j g_{ij} = u_i$$

"Relation between contravariant and covariant components"

Likewise,

$$a_i = g_{ij} a^j$$
$$a^i = g^{ij} a_j$$

"Relation between contravariant and covariant vectors"

b.Calculation of Metric Tensor

We've seen a number of simple relations between contravariant and covariant components and vectors. The metric tensor plays like a bridge for connecting them. Now, let's see how we can calculate the metric tensor.

First, we will look at the definition of covariant vector.

$$a_i = \frac{\partial \vec{r}}{\partial u^i}$$

This definition makes sense since any coordinate system represents displacement which implies components should be contra-variant with respect to the change of basis. For example, if the position vector $\vec{r}$ was written in the rectangular Cartesian coordinate (note that contravariant and covariant vectors are coincide in the Cartesian coordinate), and u^i represents some curvilinear coordinate system:

$$\vec{r} = x^1 a_1 + x^2 a_2 + x^3 a_3$$

Hence,

$$\frac{\partial \vec{r}}{\partial u^i} = \frac{\partial x^1}{\partial u^i} a_1 + \frac{\partial x^2}{\partial u^i} a_2 + \frac{\partial x^3}{\partial u^i} a_3$$

If we consider u^j that is some other curvilinear coordinate system, then

$$a_j = \frac{\partial \vec{r}}{\partial u^j} = \frac{\partial x^1}{\partial u^j} a_1 + \frac{\partial x^2}{\partial u^j} a_2 + \frac{\partial x^3}{\partial u^j} a_3$$

The dot product of two covariant vectors that have different change of basis:

$$a_i \cdot a_j = \frac{\partial x^1}{\partial u^i}\frac{\partial x^1}{\partial u^j} + \frac{\partial x^2}{\partial u^i}\frac{\partial x^2}{\partial u^j} + \frac{\partial x^3}{\partial u^i}\frac{\partial x^3}{\partial u^j}$$

In Einstein notation convention,

$$a_i \cdot a_j = \frac{\partial x^k}{\partial u^i}\frac{\partial x^k}{\partial u^j} = g_{ij}$$

The matric tensor can be expressed in the partial differential form as

$$g_{ij} = \frac{\partial x^k}{\partial u^i}\frac{\partial x^k}{\partial u^j}$$

The transformation from covariant vector to contravariant vector in terms of the set of given covariant vector basis may be an obvious task. However, their relation was given <<I should look for the proof later. For now, I can only provide why it would be so.>>

$$a^1 = \frac{a_2 \times a_3}{a_1 \cdot (a_2 \times a_3)}$$

$$a^2 = \frac{a_3 \times a_1}{a_2 \cdot (a_3 \times a_1)}$$

$$a^3 = \frac{a_1 \times a_2}{a_3 \cdot (a_1 \times a_2)}$$

Let's see what this would mean as examining the case of a^1. We know $a_2 \times a_3$ will produce a third vector that is orthogonal to both a_2 and a_3. In three-dimensional Euclidean space, this third vector will be a_1. From the Kronecker Delta relation between contravariant and covariant vectors, we can sense that they do not form 90 degrees (nor 270 degrees) because their dot product doesn't yield zero. <<I'm not saying that they are parallel but they just have components that are parallel. >> My trial for verification may look like:

$$\vec{a}^1 = \frac{\vec{a}_2 \times \vec{a}_3}{\vec{a}_1 \cdot (\vec{a}_2 \times \vec{a}_3)}$$

$$= \frac{g_{2j}\vec{a}^j \times g_{3k}\vec{a}^k}{g_{1i}\vec{a}^i \cdot \left(g_{2j}\vec{a}^j \times g_{3k}\vec{a}^k\right)} = \frac{1}{g_{1i}}\frac{\vec{a}^j \times \vec{a}^k}{\vec{a}^i \cdot (\vec{a}^j \times \vec{a}^k)} = \frac{1}{g_{1i}}\frac{\vec{a}^i}{g^{ii}}$$

$$= \vec{a}^1\frac{\partial u^1}{\partial u^1}\frac{\partial u^1}{\partial u^1} + \vec{a}^1\frac{\partial u^1}{\partial u^1}\frac{\partial u^2}{\partial u^1} + \vec{a}^1\frac{\partial u^1}{\partial u^1}\frac{\partial u^3}{\partial u^1} = \vec{a}^1$$

where

$$g_{ij} = \frac{\partial x^k}{\partial u^i}\frac{\partial x^k}{\partial u^j} = \frac{1}{g^{ij}}$$

This inverse relation is valid because the metric tensor forms Jacobian matrix. Having Jacobian matrix automatically ensures that the matrix is invertible. <<? Check for this point later.>>

I couldn’t find proof or derivation related to it but here’s my trial.

If the Jacobian matrix for the contravariant metric tensor represents something like

$$J = \begin{bmatrix} \frac{\partial u^1}{\partial x^1} & \frac{\partial u^1}{\partial x^2} & \frac{\partial u^1}{\partial x^3} \\ \frac{\partial u^2}{\partial x^1} & \frac{\partial u^2}{\partial x^2} & \frac{\partial u^2}{\partial x^3} \\ \frac{\partial u^3}{\partial x^1} & \frac{\partial u^3}{\partial x^2} & \frac{\partial u^3}{\partial x^3} \end{bmatrix}$$

and let the determinant $\sqrt{g}$

c. Differentiation of Tensor

First, let us consider the derivative of vector valued function with respect to general coordinate system, U.

$$\frac{\partial \vec{F}}{\partial u^j} = \frac{\partial (f^i \vec{a}_i)}{\partial u^j}$$

Applying the chain rule,

$$= \frac{\partial f^i}{\partial u^j} \vec{a}_i + f^i \frac{\partial \vec{a}_i}{\partial u^j}$$

where the covariant vector $\vec{a}_i = \frac{\partial \vec{r}}{\partial u^i}$

$$= \frac{\partial f^i}{\partial u^j} \vec{a}_i + f^i \frac{\partial^2 \vec{r}}{\partial u^j \partial u^i}$$

Note that the derivative of a covariant vector with respect to contravariant component seems heavy for calculation. (i.e. forms a matrix) Let’s define a new mathematical symbol, namely “Christoffel Symbol of the second kind”.

$$\frac{\partial \vec{a}_i}{\partial u^j} = \frac{\partial^2 \vec{r}}{\partial u^j \partial u^i} = \Gamma^k_{ij} \vec{a}_k = \begin{Bmatrix} k \\ ij \end{Bmatrix} \vec{a}_k = \Gamma^k_{ji} \vec{a}_k = \begin{Bmatrix} k \\ ji \end{Bmatrix} \vec{a}_k$$

<<$\frac{\partial \vec{a}_i}{\partial u^j}$ → change in vector is still a vector which can be expanded in another basis with “connection coefficients” that are dependent on the position. This connection coefficients are the Christoffel symbol. [3]>>

Note that

$$\Gamma^k_{ij} \vec{a}_k = \Gamma^k_{ji} \vec{a}_k$$

Christoffel symbol of the second kind is symmetric.

Indexing the Christoffel symbol would be the same as we did for contravariant and covariant vectors and components but note that Christoffel symbols are not “tensor”. [3]

$$\frac{\partial \vec{F}}{\partial u^j} = \frac{\partial f^i}{\partial u^j}\vec{a}_i + f^i \Gamma_{ij}^k \vec{a}_k$$

$$= \frac{\partial f^i}{\partial u^j}\vec{a}_i + f^i \Gamma_{ij}^k \vec{a}_k$$

<<I was not fully convinced that we can *swap the index*. The following would work.>>

$$= \frac{\partial f^i}{\partial u^j}\vec{a}_i + f^k \Gamma_{jk}^i \vec{a}_i$$

$$\frac{\partial \vec{F}}{\partial u^j} = \left(\frac{\partial f^i}{\partial u^j} + f^k \Gamma_{jk}^i\right)\vec{a}_i$$

d. Christoffel Symbols

There are two kinds. Christoffel symbol of the first kind and that of the second kind.

i. The Second kind

It sounds little odd but we need to know about the second kind first to understand the first kind easily. In fact, we already study the second kind.

$$\frac{\partial \vec{a}_i}{\partial u^j} = \Gamma_{ij}^k \vec{a}_k = \begin{Bmatrix} k \\ ij \end{Bmatrix} \vec{a}_k$$

Which is in the form of the partial derivative of *covariant* basis vectors with respect to a generalized coordinate system. The Christoffel symbol of the second kind:

$$\Gamma_{ij}^k = \frac{\partial \vec{a}_i}{\partial u^j} \cdot \vec{a}^k = \Gamma_{ij}^l \vec{a}_l \cdot \vec{a}^k$$

It’s a bit tricky but the following relation also holds for *contravariant* basis vector case:

$$\frac{\partial \vec{a}^i}{\partial u^j} = -\Gamma_{kj}^i \vec{a}^k$$

In addition to this,

$$\frac{\partial \vec{a}^i}{\partial u^j} \cdot \vec{a}^k = -\Gamma_{kj}^i$$

ii. The First kind

$$\Gamma_{ij\,k} = [ij, k]$$

The first kind is related to the second kind as:

$$\Gamma_{ij\,k} = g_{kl}\Gamma_{ij}^{l}$$

From the inverse relation between metric tensors, we may come up with

$$\Gamma_{ij}^{l} = g^{kl}\Gamma_{ij\,k}$$

In the form of partial differential:

$$\Gamma_{ij\,k} = g_{kl}\frac{\partial \vec{a}_i}{\partial u^j}\cdot \vec{a}^l = \frac{\partial \vec{a}_i}{\partial u^j}\cdot \vec{a}_k$$

e. Covariant Derivative

One may think of covariant derivative as generalized version of "*directional derivative*". (i.e. gradient operator)

As you may sense, there are two types of covariant derivative: (1) Covariant derivative of contravariant vector (2) Covariant derivative of covariant vector

Covariant derivative of contravariant vector	$\nabla_j f^i$
Covariant derivative of covariant vector	$\nabla_j f_i$

i. Covariant derivative of covariant vector

$$\nabla_j f^i \equiv f^i_{,j} = \frac{\partial f^i}{\partial u^j} - f_k\Gamma_{ij}^{k}$$

This will lead to the covariant of a vector valued function we study before:

$$\frac{\partial \vec{F}}{\partial u^j} = \nabla_j f^i \vec{a}_i$$

Please note that <u>covariant derivative of contravariant vector</u> is a *tensor*.

ii. Covariant derivative of contravariant vector

$$\nabla_j f_i \equiv f_{i,j} = \frac{\partial f_i}{\partial u^j} + f^k\Gamma_{kj}^{i}$$

Hence,

$$\frac{\partial \vec{F}}{\partial u^j} = \nabla_j f_i \vec{a}^i$$

Likewise, note that covariant derivative of covariant vector is a *tensor*.

iii. Covariant derivative Examples

Please try to show:

$$\nabla_j \vec{a}^i = 0$$
$$\nabla_j \vec{a}_i = 0$$

iv. Del and Nabla

Del in the Cartesian coordinate is defined as

$$\vec{\nabla} \equiv \frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y} + \frac{\partial}{\partial z}\hat{z}$$

My goal is to define the same operator in any general coordinate U to find the general equations for divergence, curl, gradient, etc.

Define the Del operator in generalized coordinate U as

$$\vec{\nabla} \equiv \nabla_i \vec{a}^i = \nabla_1 \vec{a}^1 + \nabla_2 \vec{a}^2 + \nabla_3 \vec{a}^3$$
$$\nabla_i \vec{a}^i = \frac{\partial}{\partial u^i}\vec{a}^i + \Gamma^i_{ij}\vec{a}^i$$

which is useful to define the ***divergence and gradient*** but not the Curl. We need to define the Del operator in another way to define the Curl.

where we used the idea of the covariant derivative of a vector valued function and contravariant vector

$$\frac{\partial \vec{F}}{\partial u^j} = \frac{\partial\left(f_i \vec{a}^i\right)}{\partial u^j}$$

$$\nabla_j f_i \equiv f_{i,j} = \frac{\partial f_i}{\partial u^j} + f^k \Gamma^i_{kj}$$

Its dual:

$$\vec{\nabla} \equiv \vec{a}_1 \nabla_1 + \vec{a}_2 \nabla_2 + \vec{a}_3 \nabla_3$$

which is useful to define the ***Curl*** operator on a vector valued function.

v. Differential line in Tensor

“Differential length” is one of the building blocks that forms objects in mathematics. We are familiar with this differential length idea from rudimentary vector calculus course (i.e. introduction to Electromagnetics) Now, it’s time to re-define differential length that is independent of coordinate system.

$$d\vec{l}_1 = \vec{a}_1 du^1$$

$$d\vec{l}_2 = \vec{a}_2 du^2$$

$$d\vec{l}_3 = \vec{a}_3 du^3$$

As before, $\vec{a}_i$ represents covariant basis vectors and du^j represents differential contravariant components so that the differential line $d\vec{s}_k$ exhibits the characteristics of tensor.

The dot product of two differential line will appear later so let’s see what are those.

$$d\vec{l}_a \cdot d\vec{l}_b = du^a du^b (\vec{a}_a \cdot \vec{a}_b) = du^a du^b g_{ab}$$

Note that I used different index to avoid confusion arouse from Einstein notation convention, that is this is ***not*** in the Einstein notation.

vi. Differential Area in Tensor

From rudimentary vector calculus course, we know that the direction of a differential surface vector (or any surface vector) will always be perpendicular to the surface.

$$d\vec{s}_1 = d\vec{l}_2 \times d\vec{l}_3 = du^2 du^3 (\vec{a}_2 \times \vec{a}_3) = du^2 du^3 \big(a_1 \cdot (a_2 \times a_3)\big)\vec{a}^1 = V du^2 du^3 \vec{a}^1$$

$$d\vec{s}_2 = V du^1 du^3 \vec{a}^2$$

$$d\vec{s}_3 = V du^1 du^2 \vec{a}^3$$

where

$$V = a_1 \cdot (a_2 \times a_3)$$

$$\vec{a}^1 = V(\vec{a}_2 \times \vec{a}_3)$$

<<Covariant and Contravariant basis vector transformation can be found in the document named “Vector Identities”.>>

The magnitude of the differential area vector can be found by: [5]

$$ds_1 = |d\vec{s}_1| = \left|d\vec{l}_2 \times d\vec{l}_3\right| = \left|d\vec{l}_2\right|\left|d\vec{l}_3\right| sin\theta$$

$$= \sqrt{d\vec{l}_2 \cdot d\vec{l}_2}\sqrt{d\vec{l}_3 \cdot d\vec{l}_3}\, sin\theta$$

since

$$cos\theta = \frac{d\vec{l}_2 \cdot d\vec{l}_3}{\sqrt{d\vec{l}_2 \cdot d\vec{l}_2}\sqrt{d\vec{l}_3 \cdot d\vec{l}_3}}$$

$$1 - \cos^2\theta = \sin^2\theta = 1 - \frac{\left(d\vec{l}_2 \cdot d\vec{l}_3\right)^2}{\left(d\vec{l}_2 \cdot d\vec{l}_2\right)\left(d\vec{l}_3 \cdot d\vec{l}_3\right)}$$

$$(ds_1)^2 = \left(d\vec{l}_2 \cdot d\vec{l}_2\right)\left(d\vec{l}_3 \cdot d\vec{l}_3\right)\left[1 - \frac{\left(d\vec{l}_2 \cdot d\vec{l}_3\right)^2}{\left(d\vec{l}_2 \cdot d\vec{l}_2\right)\left(d\vec{l}_3 \cdot d\vec{l}_3\right)}\right]$$

$$ds_1 = \sqrt{\left(d\vec{l}_2 \cdot d\vec{l}_2\right)\left(d\vec{l}_3 \cdot d\vec{l}_3\right) - \left(d\vec{l}_2 \cdot d\vec{l}_3\right)^2}$$

$$ds_2 = \sqrt{\left(d\vec{l}_1 \cdot d\vec{l}_1\right)\left(d\vec{l}_3 \cdot d\vec{l}_3\right) - \left(d\vec{l}_1 \cdot d\vec{l}_3\right)^2}$$

$$ds_3 = \sqrt{\left(d\vec{l}_1 \cdot d\vec{l}_1\right)\left(d\vec{l}_2 \cdot d\vec{l}_2\right) - \left(d\vec{l}_1 \cdot d\vec{l}_2\right)^2}$$

Hence,

$$\boxed{\begin{gathered} ds_1 = \sqrt{(du^2du^2(g_{22})^2)(du^3du^3(g_{33})^2) - (du^2du^3g_{23})^2} \\ = \sqrt{(g_{22}g_{33})^2 - (g_{23})^2}du^2du^3 \\ ds_2 = \sqrt{(g_{11}g_{33})^2 - (g_{13})^2}du^1du^3 \\ ds_3 = \sqrt{(g_{11}g_{22})^2 - (g_{12})^2}du^1du^2 \end{gathered}}$$

Similarly, for basis vectors:

$$\boxed{\begin{gathered} |\vec{a}_2 \times \vec{a}_3| = \sqrt{(g_{22}g_{33})^2 - (g_{23})^2} \\ |\vec{a}_1 \times \vec{a}_3| = \sqrt{(g_{11}g_{33})^2 - (g_{13})^2} \\ |\vec{a}_1 \times \vec{a}_2| = \sqrt{(g_{11}g_{22})^2 - (g_{12})^2} \end{gathered}}$$

vii. Differential Volume in Tensor

Recall the transformation between covariant and contravariant vector formula:

$$\vec{a}^i = \frac{\epsilon_{ijk}\vec{a}_j \times \vec{a}_k}{\vec{a}_i \cdot \left(\vec{a}_j \times \vec{a}_k\right)} = \frac{\epsilon_{ijk}\vec{a}_j \times \vec{a}_k}{V}$$

One can sense that $\vec{a}_i \cdot \left(\vec{a}_j \times \vec{a}_k\right)$ represents the volume generated by the three vectors. You might be familiar with the term "Triple scalar Product" form introductory Linear Algebra course.

$$dv_1 = d\vec{l}_1 \cdot \left(d\vec{l}_2 \times d\vec{l}_3\right) = (\vec{a}_1 du^1) \cdot [du^2 du^3 (\vec{a}_2 \times \vec{a}_3)]$$

$$= du^1 du^2 du^3 (\vec{a}_1 \cdot (\vec{a}_2 \times \vec{a}_3)\ ) = V du^1 du^2 du^3$$

Although there's a lengthy derivation process behind, I will skip it in this document and claim that (it's lengthy because we are working with differential volume element in curvilinear coordinate system and the definitions of covariant and contravariant vectors are referenced to the rectangular Cartesian coordinate)

$$dv = \sqrt{\mathcal{G}}du^1 du^2 du^3$$

where

$$\mathcal{G} = \left|g_{ij}\right|$$

which is the determinant of the metric tensor matrix.

$$\boxed{V = \vec{a}_1 \cdot (\vec{a}_2 \times \vec{a}_3) = \sqrt{\mathcal{G}}}$$

viii. Curl

In general, the curl of a vector valued function can be written as

$$\vec{\nabla} \times \vec{F}$$

What we are going to do is that using the covariant derivative of both contravariant- and covariant-vector to find the Curl in curvilinear coordinate.

Recall that coordinate system basis vectors are covariant vectors.

$$= \nabla_{\mathrm{i}}\left(\vec{a}^i \times \vec{F}\right) = \nabla_{\mathrm{i}}\left(\vec{a}^i \times f_j \vec{a}^j\right)$$

$$= \nabla_{\mathrm{i}}\left(f_j \frac{\epsilon_{ijk}\vec{a}_k}{\vec{a}^k \cdot (\vec{a}^i \times \vec{a}^j)}\right) = \epsilon_{ijk}\nabla_{\mathrm{i}}\left(\frac{f_j \vec{a}_k}{\sqrt{\mathcal{G}}}\right)$$

$$= \epsilon_{ijk}\frac{\partial}{\partial u^i}\left(\frac{f_j \vec{a}_k}{\sqrt{\mathcal{G}}}\right)$$

$$\boxed{\vec{\nabla} \times \vec{F} = \frac{1}{\sqrt{\mathcal{G}}} \epsilon_{ijk} \frac{\partial f_j}{\partial u^i} \vec{a}_k}$$

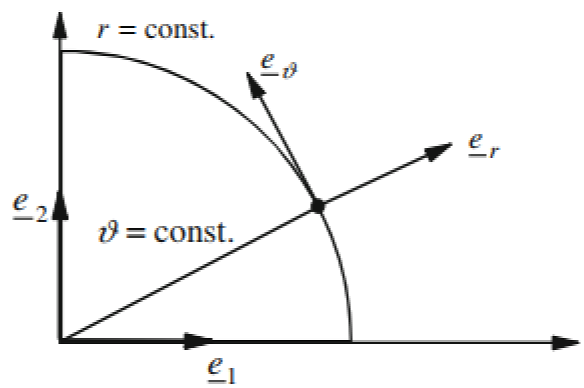
<<we used the general formula for conversion between contravariant and covariant vector. ϵ_{ijk} is known as the Levi-Civita symbol. I thought I wrote about it in this document but it was not. A document named “Vector Identities” will explain this symbol.>>

Now, covariant derivative of contravariant vector version:

$$\vec{\nabla} \times \vec{F} =$$

III. Examples

a. 2D Cartesian to Polar Coordinate



x_i be cartesian coordinate and z^k be curvilinear coordinate.

Then, the metric tensor

$$g_{kj} = \frac{\partial x_i \partial x_i}{\partial z^k \partial z^j}$$

Now, considering z^k be Cylindrical coordinate, then

$$z^1 = r = \sqrt{x_1^2 + x_2^2}$$

$$z^2 = \varphi = \arctan\left(\frac{x_2}{x_1}\right)$$

$$z^3 = z = x_3$$

Now, the reverse relation is obtained:

$$x_1 = z^1 \cos(z^2)$$

$$x_2 = z^1 \sin(z^2)$$

$$x_3 = z^3$$

In other words,

$$z^k(x_1, x_2, x_3) \; and \; x_k(z^1, z^2, z^3)$$

Isomorphism between two coordinates can be shown; hence, the transformation between them should have an invertible matrix. <<hmm.. I may not have a strong reasoning for this statement..>> Let's refer to the following figure.

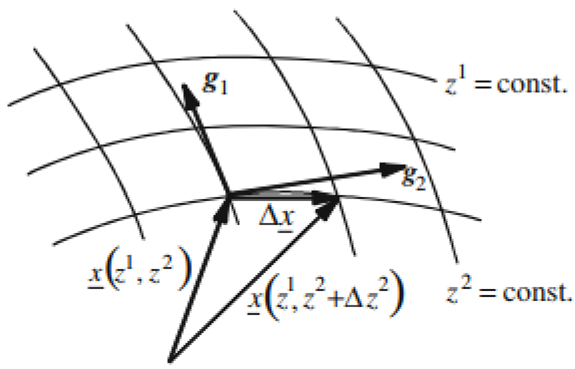


Figure. Tangent vectors to the curvilinear coordinate system

If we define the infinitesimal length (distance) $d\vec{x}$ from $\Delta\vec{x}$

The reason why we concern about this infinitesimal length (between position vectors) is because if the origins of two coordinates do not coincide, their position vectors will be different from each other. We would like to reduce the distance between the position vectors infinitely to coincide them at a point as we consider the differential quantity.

$$d\vec{x} = \lim_{\Delta\vec{x}\to 0} \Delta\vec{x} = \lim_{\Delta x_i \to 0} \Delta x_i \, \hat{e}_i$$

$$= \lim_{\Delta\vec{x}\to 0} \frac{x_i(z^1, (z^2 + \Delta z^2), z^3) - x_i(z^1, z^2, z^3)}{\Delta z^2} \Delta z^2 \hat{e}_i$$

$$= \frac{\partial x_i}{\partial z^2} dz^2 \hat{e}_i$$

where $\hat{e}_i$ are unit vectors in Cartesian. In turn,

$$\frac{d\vec{x}}{dz^2} = \frac{\partial x_i}{\partial z^2} \hat{e}_i \equiv \vec{g}_2$$

Now, let's generalize this: $$d\vec{x} = dz^k \vec{g}_k$$ And the tangent vector to the constant line, $z^j = constant$, is denoted by: $$\vec{g}_k = \frac{\partial x_i}{\partial z^k} \hat{e}_i$$ where Einstein summation rule is applied.

Back to our original problem: relation between the Cartesian (2-D) and the Polar coordinates:

$$g_1 = \frac{\partial x_1}{\partial z^1} \hat{e}_1 + \frac{\partial x_2}{\partial z^1} \hat{e}_2$$

$$= \frac{\partial (z^1 \cos(z^2))}{\partial z^1} \hat{e}_1 + \frac{\partial (z^1 \sin(z^2))}{\partial z^1} \hat{e}_2$$

$$= \cos(z^2)\, \hat{e}_1 + \sin(z^2)\, \hat{e}_2$$

$$\therefore g_1 = \cos(\varphi)\, \hat{e}_1 + \sin(\varphi)\, \hat{e}_2 \equiv \vec{e}_r$$

Likewise,

$$g_2 = \frac{\partial x_1}{\partial z^2} \hat{e}_1 + \frac{\partial x_2}{\partial z^2} \hat{e}_2$$

$$\therefore g_2 = -r \sin(\varphi)\, \hat{e}_1 + r \sin(\varphi)\, \hat{e}_2 \equiv r\vec{e}_\varphi$$

To sum up,

Cartesian	Polar
$\hat{e}_1 = \hat{e}_x = \hat{x}$	$g_1 = \hat{e}_r = \hat{r}$
$\hat{e}_2 = \hat{e}_y = \hat{y}$	$g_2 = r\hat{e}_\varphi = r\hat{\varphi}$

b.Cartesian to Spherical Coordinate

i. Unit Vector Transformation

The Cartesian $(e_1, e_2, e_3) \rightarrow (x, y, z)$

The Spherical $(z^1, z^2, z^3) \rightarrow (r, \theta, \varphi)$

Consider z^k be the Spherical coordinate vector, then

$$x_1 = rsin\theta cos\varphi = z^1 \sin(z^2) \cos(z^3)$$

$$x_2 = rsin\theta sin\varphi = z^1 \sin(z^2) \sin(z^3)$$

$$x_3 = rcos\theta = z^1 \cos(z^2)$$

Recall the tangent vector to the $z^j = constant$:

$$\vec{g}_k = \frac{\partial x_i}{\partial z^k}\hat{e}_i$$

Hence,

$$\vec{g}_1 = \frac{\partial x_1}{\partial z^1}\hat{e}_1 + \frac{\partial x_2}{\partial z^1}\hat{e}_2 + \frac{\partial x_3}{\partial z^1}\hat{e}_3$$

$$= \sin(z^2)\cos(z^3)\,\hat{e}_1 + \sin(z^2)\sin(z^3)\,\hat{e}_2 + \cos(z^2)\,\hat{e}_3$$

$$\therefore \vec{g}_1 = sin\theta cos\varphi\hat{e}_1 + sin\theta sin\varphi\hat{e}_2 + \cos\theta\,\hat{e}_3$$

$$\vec{g}_2 = \frac{\partial x_1}{\partial z^2}\hat{e}_1 + \frac{\partial x_2}{\partial z^2}\hat{e}_2 + \frac{\partial x_3}{\partial z^2}\hat{e}_3$$

$$= z^1\cos(z^2)\cos(z^3)\,\hat{e}_1 + z^1\cos(z^2)\sin(z^3)\,\hat{e}_2 - z^1\sin(z^2)\,\hat{e}_3$$

$$\therefore \vec{g}_2 = r\cos(\theta)\cos(\varphi)\,\hat{e}_1 + r\cos(\theta)\sin(\varphi)\,\hat{e}_2 - r\sin(\theta)\,\hat{e}_3$$

$$\vec{g}_3 = \frac{\partial x_1}{\partial z^3}\hat{e}_1 + \frac{\partial x_2}{\partial z^3}\hat{e}_2 + \frac{\partial x_3}{\partial z^3}\hat{e}_3$$

$$= -z^1\sin(z^2)\sin(z^3)\,\hat{e}_1 + z^1\sin(z^2)\cos(z^3)\,\hat{e}_2$$

$$\therefore \vec{g}_3 = -r\sin(\theta)\sin(\varphi)\,\hat{e}_1 + r\sin(\theta)\cos(\varphi)\,\hat{e}_2$$

To see identify the unit vectors in the Spherical coordinate, we need to find the magnitude of the vectors that we just found:

$$|\vec{g}_1| = \sqrt{(sin\theta cos\varphi)^2 + (sin\theta sin\varphi)^2 + (\cos\theta)^2}$$

$$= \sqrt{\sin^2\theta + \cos^2\theta} = 1$$

$$|\vec{g}_2| = \sqrt{(r\cos(\theta)\cos(\varphi))^2 + (r\cos(\theta)\sin(\varphi))^2 + (-r\sin(\theta))^2}$$

$$= r$$

$$|\vec{g}_3| = \sqrt{(-r\sin(\theta)\sin(\varphi))^2 + (r\sin(\theta)\cos(\varphi))^2}$$

$$= rsin(\theta)$$

Hence,

$$\vec{g}_1 = |\vec{g}_1|\hat{e}_r = \hat{e}_r$$

$$\vec{g}_2 = |\vec{g}_2|\hat{e}_\varphi = r\hat{e}_\varphi$$

$$\vec{g}_3 = |\vec{g}_3|\hat{e}_\theta = rsin(\theta)\hat{e}_\theta$$

In conclusion, we end up with the relation between the Cartesian and the Spherical coordinate vectors as the follow:

$$\hat{e}_r = sin\theta cos\varphi \hat{e}_x + sin\theta sin\varphi \hat{e}_y + \cos\theta\, \hat{e}_z$$

$$r\hat{e}_\varphi = r\cos(\theta)\cos(\varphi)\, \hat{e}_x + r\cos(\theta)\sin(\varphi)\, \hat{e}_y - r\sin(\theta)\, \hat{e}_z$$

$$rsin(\theta)\hat{e}_\theta = -r\sin(\theta)\sin(\varphi)\, \hat{e}_x + r\sin(\theta)\cos(\varphi)\, \hat{e}_y$$

In addition,

IV. References

[1] https://en.wikipedia.org/wiki/Covariance_and_contravariance_of_vectors

[2] https://en.wikipedia.org/wiki/Jacobian_matrix_and_determinant

[3] https://en.wikipedia.org/wiki/Christoffel_symbols

[4] https://ghebook.blogspot.com/2011/06/tensor-coordinate-transformation.html?showComment=1504995811419#c5545146236458364 20

[5] https://en.wikipedia.org/wiki/Cross_product