Vector Spaces

I. Introduction

The concept of Vector spaces and Subspaces will be covered as well as null spaces, column spaces, and linear transformations. In addition, more importantly, the concept of Bases will be covered.

II. Vector spaces and Subspaces

a. Vector space

DEFINITION

A **vector space** is a nonempty set V of objects, called *vectors*, on which are defined two operations, called *addition* and *multiplication by scalars* (real numbers), subject to the ten axioms (or rules) listed below.[1] The axioms must hold for all vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in V and for all scalars c and d.

1. The sum of $\mathbf{u}$ and $\mathbf{v}$, denoted by $\mathbf{u}+\mathbf{v}$, is in V.
2. $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$.
3. $(\mathbf{u}+\mathbf{v})+\mathbf{w}=\mathbf{u}+(\mathbf{v}+\mathbf{w})$.
4. There is a **zero** vector $\mathbf{0}$ in V such that $\mathbf{u}+\mathbf{0}=\mathbf{u}$.
5. For each $\mathbf{u}$ in V, there is a vector $-\mathbf{u}$ in V such that $\mathbf{u}+(-\mathbf{u})=\mathbf{0}$.
6. The scalar multiple of $\mathbf{u}$ by c, denoted by $c\mathbf{u}$, is in V.
7. $c(\mathbf{u}+\mathbf{v})=c\mathbf{u}+c\mathbf{v}$.
8. $(c+d)\mathbf{u}=c\mathbf{u}+d\mathbf{u}$.
9. $c(d\mathbf{u})=(cd)\mathbf{u}$.
10. $1\mathbf{u}=\mathbf{u}$.

The definition of vector space is here straight from the textbook. In most cases, we don't need to define/test if the space is a vector space or not; it is already examined by other mathematicians. The concept of Vector space is like for generalization which is what mathematicians love.

By virtue of Vector space concept, engineers/scientists can work with arrows (geometric figure in vector analysis), polynomials, or even series and do even more complicated work such as calculations. For example, we know we can add two arrows and the result would sit on the tip of a parallelogram. We know we can stretch/dilate arrows by multiplying a constant. More importantly, we know that there is zero vector defined. For relatively easy calculations, the concept of Vector space may not be appreciated; however, when calculation becomes bizarre, say working on some vector operation (e.g. addition) with vectors that have 5 rows; we can't visualize them simply because we don't know how 5-dimension space would look like. We can only rely on the fact that the arrows (vectors) with 5 rows still in vector space; therefore, they should obey rules constituted by it.

What are vector spaces? Let's go through some examples.

i. Examples

1. Example 4 – polynomials

Let's say the set $\mathbb{P}_n$ of polynomials (of degree at most n consists of all polynomials) of the form:

$$\boldsymbol{P}(t) = a_0 + a_1 t + a_2 t^2 + \cdots + a_n t^n$$

Where all the coefficients and the variable are real number.

Let's examine a case where polynomial is defined:

$$\boldsymbol{P}(t) = a_0$$

And if $a_0 \neq 0$, then the degree of $\boldsymbol{P}$ is zero. This may be a confusing part: what if all the coefficients are all zeros? Then, $\boldsymbol{P}(t) = 0$, which is known as "zero polynomial". Zero polynomial is included in $\mathbb{P}_n$ but its degree is not defined for technical reason. (Definition, 4) Now, let us define another polynomial in $\mathbb{P}_n$. (and we can do this as long as the forms are the same.)

$$\boldsymbol{Q}(t) = b_0 + b_1 t + b_2 t^2 + \cdots + b_n t^n$$

If we add two polynomials:

$$\boldsymbol{P}(t) + \boldsymbol{Q}(t) = (a_0 + b_0) + (a_1 + b_1)t + (a_2 + b_2)t^2 + \cdots + (a_n + b_n)t^n$$

Hence this result also is in $\mathbb{P}_n$. In addition, if $b_i = -a_i$, then we see (Definition, 5) holds.

Now, let's see a scalar multiplication would also works.

$$(c\boldsymbol{P})(t) = c\boldsymbol{P}(t) = ca_0 + ca_1 t + ca_2 t^2 + \cdots + ca_n t^n$$

Hence this result is also in $\mathbb{P}_n$.

As a matter of fact, after going through some tedious process (I've already performed the main things, the rest can be easily tested) all the axioms would be met. Hence, $\mathbb{P}_n$ is a vector space.

b. Subspaces

Subspaces are alike vector space except, fortunately, they only require 3 out of 10 axioms from the vector space definition, which are (1) zero vector (2) addition, and (3) scalar multiplication.

DEFINITION

A **subspace** of a vector space V is a subset H of V that has three properties:

a. The zero vector of V is in H.[2]

b. H is closed under vector addition. That is, for each $\mathbf{u}$ and $\mathbf{v}$ in H, the sum $\mathbf{u} + \mathbf{v}$ is in H.

c. H is closed under multiplication by scalars. That is, for each $\mathbf{u}$ in H and each scalar c, the vector $c\mathbf{u}$ is in H.

The formal definition is straight from the textbook. For notational matter, we indicate

$$H = Subspace$$

$$V = Vector\ Space$$

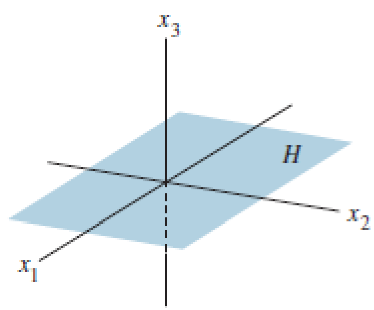

FIGURE 7
The x_1x_2-plane as a subspace of $\mathbb{R}^3$.

i. Examples

1. Example 8 – Be careful about subspace

$\mathbb{R}^2$ is not a subspace of $\mathbb{R}^3$ because it is not even subset of $\mathbb{R}^3$.

Note that $H = \left\{ \begin{bmatrix} s \\ t \\ 0 \end{bmatrix} : s\ and\ t\ are\ real \right\}$ is a subspace of $\mathbb{R}^3$ but it "looks" like $\mathbb{R}^2$. To be a subspace of $\mathbb{R}^3$, it must have all three entries (rows).

2. Example 11 – Subspace spanned by a set

THEOREM 1 If $\mathbf{v}_1, \ldots, \mathbf{v}_p$ are in a vector space V, then Span $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is a subspace of V.

Let H be the set of all vectors of form $(a-3b, b-a, a, b)$:

$$H = \left\{ \begin{bmatrix} a-3b \\ b-a \\ a \\ b \end{bmatrix} : a, b \in \mathbb{R} \right\}$$

Show that H is a subspace of $\mathbb{R}^4$.

Since

$$\begin{bmatrix} a-3b \\ b-a \\ a \\ b \end{bmatrix} = a \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + b \begin{bmatrix} -3 \\ 1 \\ 0 \\ 1 \end{bmatrix} = a\boldsymbol{v} + b\boldsymbol{u}$$

And these two vectors are closed under vector addition and scalar multiplication.

$$(a\boldsymbol{v} + b\boldsymbol{u}) + (c\boldsymbol{v} + d\boldsymbol{u}) = (a+c)\boldsymbol{v} + (b+d)\boldsymbol{u}$$

$(a+c)\boldsymbol{v} + (b+d)\boldsymbol{u}$ are still in $\mathbb{R}^4$. Scalar multiplication can also be examined in a similar way. There's zero vector which can be shown:

$$(a+c)\boldsymbol{v} + (b+d)\boldsymbol{u} = (a-a)\boldsymbol{v} + (b-b)\boldsymbol{u} = \mathbf{0}$$

If $a = -c \; and \; b = -d$, then we obtain zero vector. Hence, all the axioms are examined and we see they work. Therefore, H is a subspace of $\mathbb{R}^4$.

3. Example 12

$$\boldsymbol{v_1} = \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}, \boldsymbol{v_2} = \begin{bmatrix} 5 \\ -4 \\ -7 \end{bmatrix}, and \; \boldsymbol{v_3} = \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}$$

$$\boldsymbol{y} = \begin{bmatrix} -4 \\ 3 \\ h \end{bmatrix}$$

Determine h such that the vector $\boldsymbol{y}$ will be a subspace of $\mathbb{R}^3$ spanned by $\boldsymbol{v_1}, \boldsymbol{v_2}, and \; \boldsymbol{v_3}$

The procedure contains row operations to find suitable h value. Instead of solving it, let's take the answer value and compare it with non-solution value. (I was lazy doing the math….)

There's only one value $h = 5$ such that vector $\boldsymbol{y}$ be a subspace of $\mathbb{R}^3$ spanned by the $\boldsymbol{v}$ vectors.

Let

$$A = [\boldsymbol{v_1}\ \boldsymbol{v_2}\ \boldsymbol{v_3}]$$

Then,

$$rref([A\ \boldsymbol{y}]) = \begin{bmatrix} 1 & 0 & 7 & 1 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The last row shows that the augmented matrix is consistent.

Now, let's examine any value $h \neq 5$, let's choose $h = 7$

$$rref\left(\left[A\ \begin{bmatrix} -4 \\ 3 \\ 7 \end{bmatrix}\right]\right) = \begin{bmatrix} 1 & 0 & 7 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Note that the last row gives:

$$0x_1 + 0x_2 + 0x_3 = 1$$

Which implies that the augmented matrix is inconsistent. Hence, we know that $h = 5$ works.

III. Null Spaces

We know that a homogeneous equation has always the trivial solution, which is basically the zero vector that obviously satisfies the equation. However, what if there are some solution beside the zero vector? The set of all solutions to the homogeneous equation is known as the Null space and it's a vector space.

THEOREM 2

The null space of an $m \times n$ matrix A is a subspace of $\mathbb{R}^n$. Equivalently, the set of all solutions to a system $A\mathbf{x} = \mathbf{0}$ of m homogeneous linear equations in n unknowns is a subspace of $\mathbb{R}^n$.

IV. Coordinate Systems

"Main reason why we look for a basis $\mathcal{B}$ for a vector space V is because it imposes a coordinate system on V."

DEFINITION

Suppose $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ is a basis for V and $\mathbf{x}$ is in V. The **coordinates of x relative to the basis $\mathcal{B}$** (or the **$\mathcal{B}$-coordinates of x**) are the weights $c_1, \dots, c_n$ such that $\mathbf{x} = c_1\mathbf{b}_1 + \cdots + c_n\mathbf{b}_n$.

If $c_1, \ldots, c_n$ are the $\mathcal{B}$-coordinates of $\mathbf{x}$, then the vector in $\mathbb{R}^n$

$$[\,\mathbf{x}\,]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$

is the **coordinate vector of x (relative to $\mathcal{B}$)**, or the **$\mathcal{B}$-coordinate vector of x**. The mapping $\mathbf{x} \mapsto [\,\mathbf{x}\,]_{\mathcal{B}}$ is the **coordinate mapping (determined by $\mathcal{B}$)**.[1]

a. Examples

i. Example 1

Given

$$\boldsymbol{b_1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \; and \; \boldsymbol{b_2} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

If a vector x in $\mathbb{R}^2$ has a coordinate vector $[\boldsymbol{x}]_{\mathcal{B}} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$. Find x.

$$\boldsymbol{x} = \mathcal{B}[\boldsymbol{x}]_{\mathcal{B}} = -2\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 3\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \end{bmatrix}$$

Note that this vector x can be thought as being relative to the standard basis. That is,

$$\boldsymbol{x} = \begin{bmatrix} 1 \\ 6 \end{bmatrix} = 1\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 6\begin{bmatrix} 0 \\ 1 \end{bmatrix} = 1\boldsymbol{e_1} + 6\boldsymbol{e_2}$$

ii. Example 4

Given that

$$\boldsymbol{b_1} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \; and \; \boldsymbol{b_2} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$\boldsymbol{x} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$$

Find the Coordinate vector $[\boldsymbol{x}]_{\mathcal{B}}$. Again, we start with

$$\boldsymbol{x} = \mathcal{B}[\boldsymbol{x}]_{\mathcal{B}}$$

Then,

$$\mathcal{B}^{-1}\mathcal{B}[\boldsymbol{x}]_{\mathcal{B}} = \boldsymbol{x}\mathcal{B}^{-1}$$

$$[\boldsymbol{x}]_{\mathcal{B}} = \boldsymbol{x}\mathcal{B}^{-1}$$

$$[\boldsymbol{x}]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

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Python script:
>>> np.dot(x,B_inv)
matrix([[ 3.,  2.]])
```

iii. Example 6

Use coordinate vectors to identify whether the following polynomials are linearly dependent in $\mathbb{P}^2$: $1 + 2t^2, 4 + t + 5t^2, and\ 3 + 2t.$

The following theorem is the key idea to "Isomorphism" and to the solution procedure to this example:

THEOREM 8 Let $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ be a basis for a vector space V. Then the coordinate mapping $\mathbf{x} \mapsto [\mathbf{x}]_{\mathcal{B}}$ is a one-to-one linear transformation from V onto $\mathbb{R}^n$.

Let $\mathcal{B}$ the Basis of the space $\mathbb{P}^2$: $\mathcal{B} = \{1, t, t^2\}$. Typical element of $\mathbb{P}^2$ take the following form:

$$\boldsymbol{P}(t) = a_0 + a_1 t + a_2 t^2$$

Then, the coordinate vector of $\boldsymbol{P}$:

$$[\boldsymbol{P}]_{\mathcal{B}} = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix}$$

Because of the Theorem 8, we know that there is one to one transformation from $\mathbb{P}^2$ onto $\mathbb{R}^3$.

Now, to check whether they are linearly dependent, it's the same question asking whether a homogeneous equation has a non-trivial solution, or has at least one free variable:

$$A\boldsymbol{x} = \boldsymbol{0} \to \begin{bmatrix} 1 & 4 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 2 & 5 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -5 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

We see that x_3 is the free variable in this case. Hence, they are linearly dependent. As a matter of fact, we already know what are the x_1 and x_2 that make equation $3 + 2t$ in terms of $1 + 2t^2$ and $4 + t + 5t^2$. They are $x_1 = -5$ and $x_2 = 2$.

$$3 + 2t = -5(1 + 2t^2) + 2(4 + t + 5t^2)$$

Note that to check this point, we need to solve:

$$\begin{bmatrix} 1 & 4 & 3 \\ 0 & 1 & 2 \\ 2 & 5 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -5 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

Likewise,
If we were curious about the solution to express $4 + t + 5t^2$ in terms of the other two equations can be found by solving:

$$\begin{bmatrix} 1 & 3 & 4 \\ 0 & 2 & 1 \\ 2 & 0 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2.5 \\ 0 & 1 & 0.5 \\ 0 & 0 & 0 \end{bmatrix}$$

And we check it works:

$$4 + t + 5t^2 = 2.5(1 + 2t^2) + 0.5(3 + 2t)$$

iv. Example 7

EXAMPLE 7 Let

$$\mathbf{v}_1 = \begin{bmatrix} 3 \\ 6 \\ 2 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} 3 \\ 12 \\ 7 \end{bmatrix},$$

and $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2\}$. Then $\mathcal{B}$ is a basis for $H = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$. Determine if $\mathbf{x}$ is in H, and if it is, find the coordinate vector of $\mathbf{x}$ relative to $\mathcal{B}$.

If vector $\boldsymbol{x}$ is in H, then the following augmented matrix equation should be consistent:

$$\begin{bmatrix} 3 & -1 & 3 \\ 6 & 0 & 12 \\ 2 & 1 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

Therefore, it is consistent. Furthermore, it even gives the coordinate vector if you recall the augmented matrix's matrix equation:

$$\mathcal{B}[\boldsymbol{x}]_{\mathcal{B}} = [\boldsymbol{v_1}\ \boldsymbol{v_2}][\boldsymbol{x}]_{\mathcal{B}} = \boldsymbol{x}$$

Hence,

$$[\boldsymbol{x}]_{\mathcal{B}} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

The textbook provides graphical figure to help understanding:

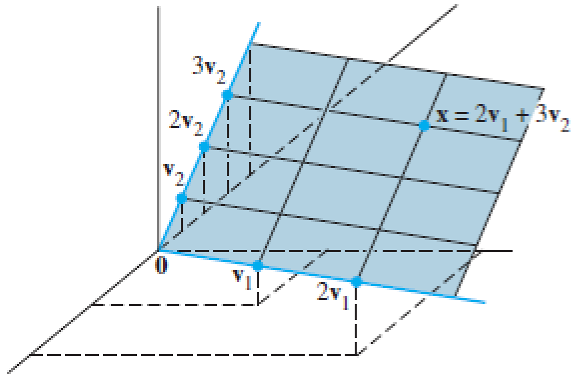

FIGURE 7 A coordinate system on a plane H in $\mathbb{R}^3$.

V. Change of Basis

This section is for studying the relation between two bases $[\boldsymbol{x}]_{\mathcal{B}}$ and $[\boldsymbol{x}]_{\mathcal{C}}$ for each $\boldsymbol{x}$ in V.

VI. References

[1] Linear Algebra and its Applications, David C. Lay, 4th Edition