Natural Response Circuit

I. Introduction

There are three general types of natural responses (or also known as “transient response”):

RC
RL
Parallel RLC
Series RLC

Natural response circuit assumes that either capacitor or inductor initially contains some energy and it release the energy at a time. Natural responses of parallel RLC circuits are also known as the step response. (?)

Transient response?
It’s the response of a system to a change from the equilibrium or the steady-state point. Transient response occurs whenever the equilibrium of the system is affected. It contains two types of response: the impulse response and the step response. Impulse response occurs when the input type is an impulse input whereas the step response occurs when the input type is a step input.

The basic configurations are below:

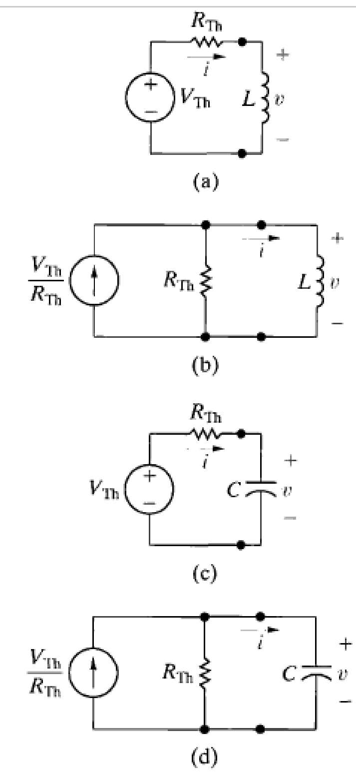

Figure. Equivalent Circuits for Natural Response

(a) Inductor connected to Thevenin
(b) Inductor connected to Norton
(c) Capacitor connected to Thevenin
(d) Capacitor connected to Norton

II. RC Natural Response

Let's investigate RC natural response first.

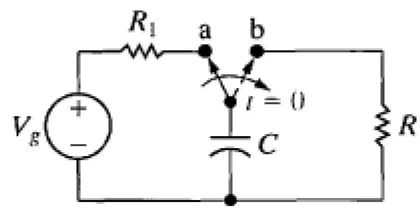

Figure 7.10 ▲ An *RC* circuit.

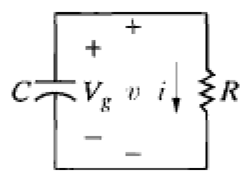

Figure 7.11 ▲ The circuit shown in Fig. 7.10, after switching.

Figure. Basic Models of RC natural Response

a. Develop Equations

Applying the KCL around the loop of Figure 7.11,

$$C\frac{dv}{dt} + \frac{v}{R} = 0$$

$$\int \frac{dv}{v} = -\frac{1}{RC}\int dt$$

$$\ln(v) = -\frac{t}{RC} + V_1$$

Where V_1 is the constant from the indefinite integral

$$v(t) = \exp\left(-\frac{t}{RC} + V_1\right)$$

$$\therefore v(t) = \mathrm{V}_0 \exp\left(-\frac{t}{RC}\right)$$

If $R = 1kohm$ and $C = 1\mu F$

b.Simulation

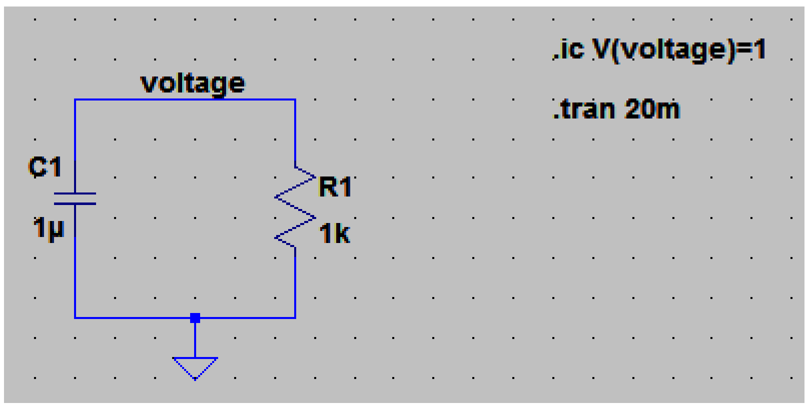

Figure. LTspice model

Note that the spice comment ".ic V(voltage)=1" is for the initial condition of the capacitor voltage at $t = 0$

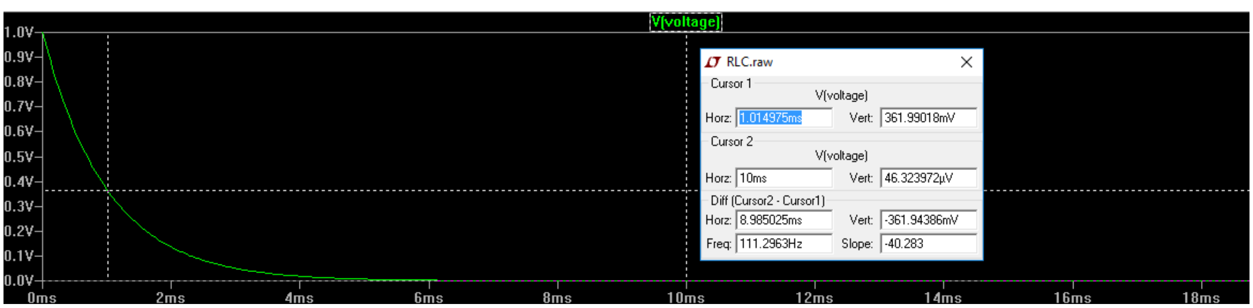

Figure. Transient response of RC circuit (voltage)

We note that the RC time constant in this case,

$$\tau = RC = 0.001$$

Hence, the time at which yields $e^{-1} \cong 0.368$ of the initial voltage is,

$$t = 1ms$$

As it can be shown in the transient response, at $t = 1ms$, the voltage reaches around

$$V(1ms) = 0.36\ V$$

c. Real application

For digital communication, the bus capacitance is usually can be found from either data sheet. In this case, one must find appropriate pull-up resistor to make the following options:

(1) Increase the size of pull-up resistor and reduce the power consumption introduced by the component. In this case, larger pull-up resistor will yield larger RC time constant which in turn take longer time to approach the logic ‘1’. (In digital communication, there are only two ways to represent things, 0 or 1) For the record, by the definition of IC2, the logic ‘1’ means $0.7V_{DD}$ (0.7 times the power supply). The logic ‘0’ means $0.3V_{DD}$.
(2) Decrease the size of pull-up resistor and this will increase the power consumption of the component; however, it will reduce the RC time constant. Thus, engineers might want to reduce the pull-up resistor value when they want to obtain a sharp transition/accuracy in digital reading.

III. RL Natural Response

The procedure for RL circuit is very similar to the one in RC circuit.

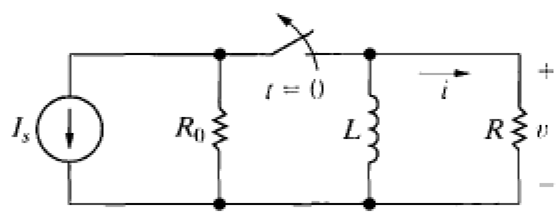

Figure 7.3 ▲ An *RL* circuit.

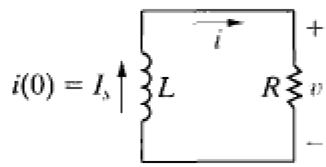

Figure 7.4 ▲ The circuit shown in Fig. 7.3, for $t \geq 0$.

Figure. Basic configuration for RL circuit

a. Develop Equations

Applying the KVL this time,

$$L\frac{di}{dt} + Ri = 0$$

At the end of the day, we will be ended up with:

$$i(t) = I_0 \exp\left(-\frac{R}{L}t\right)$$

$$\tau = \frac{L}{R} = time\ constant$$

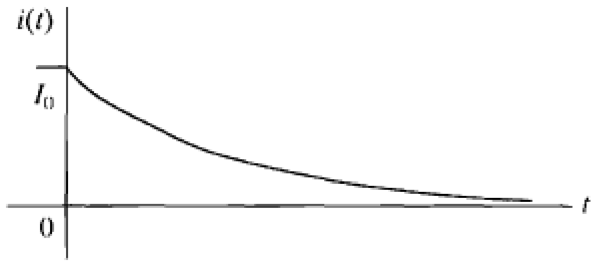

Figure 7.5 ▲ The current response for the circuit shown in Fig. 7.4.

b.Simulation

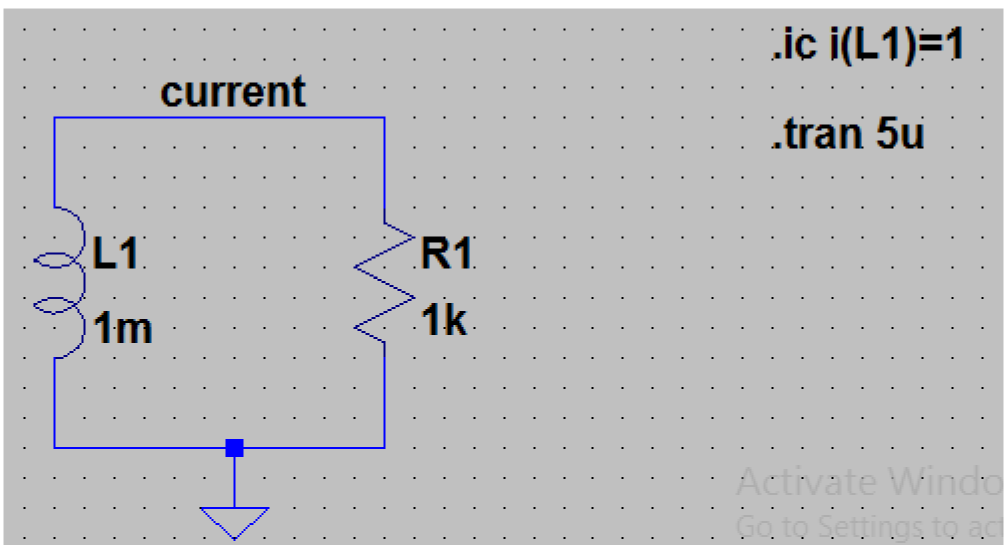

Figure. LTspice Model for RL circuit

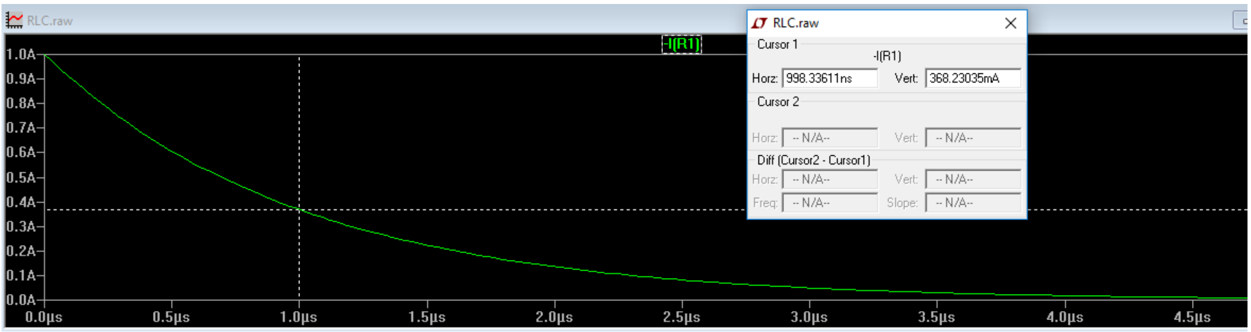

Figure. Transient Response of RL circuit (Current)

Note the polarity of the current in the plot.

The time at which the inductor hits 0.37 of the initial current can be calculated from the time constant and it turned out to be:

$$\tau = \frac{1mH}{1kohm} = 1\mu s$$

Which can be found from the above simulation result.

IV. Natural Response of Parallel RLC circuit

Unlike the RL and RC, the RLC circuit can be analyzed with the second order differential equations; therefore, the solutions will be different from RL and RC natural responses. For RLC, there are parallel and series RLC natural responses. Let’s see how they differ.

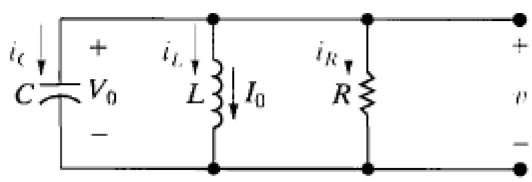

Figure 8.1 ▲ A circuit used to illustrate the natural response of a parallel *RLC* circuit.

The voltage will be the same for each component; therefore, applying the KCL:

$$C\frac{dv}{dt} + \frac{1}{L}\int_0^t vdt + I_0 + \frac{v}{R} = 0$$

$$\frac{d^2v}{dt^2} + \frac{1}{RC}\frac{dv}{dt} + \frac{1}{LC}v = 0$$

Applying the method of the characteristic equation for the solution of the second order differential equation:

$$B = \frac{1}{RC} \; and \; C = \frac{1}{LC}$$

Then we will obtain the two roots:

$$m_1 = -\frac{B}{2} + \sqrt{\frac{B^2}{4} - C} = -\frac{1}{2RC} + \sqrt{\frac{1}{(2RC)^2} - \frac{1}{LC}}$$

$$m_2 = -\frac{B}{2} - \sqrt{\frac{B^2}{4} - C} = -\frac{1}{2RC} - \sqrt{\frac{1}{(2RC)^2} - \frac{1}{LC}}$$

Now, the final solution is in the form of linear combination:

$$v = v_1 + v_2 = A_1 e^{m_1 t} + A_2 e^{m_2 t}$$

Note that the two roots, $m_!$ And m_2 are determined by the circuit components, R,L, and C.

Now, let's express the two roots in the famous circuit parameters:

$$m_1 = Complex\ Frequency\ [rad/s] = -\alpha + \sqrt{\alpha^2 - \omega_0^2}$$

$$m_2 = Complex\ Frequency\ [rad/s] = -\alpha - \sqrt{\alpha^2 - \omega_0^2}$$

Where

$$\alpha = Neper\ Frequency = \frac{1}{2RC}\ [rad/s]$$

$$\omega_0 = Resonant\ Radian\ Frequency = \frac{1}{\sqrt{LC}}\ [rad/s]$$

a. Damping Responses

There are three types of damping responses:

Underdamped	$\omega_0^2 > \alpha^2$	$v(t) = B_1 e^{-\alpha t} \cos(\omega_d t) + B_2 e^{-\alpha t} \sin(\omega_d t)$
Overdamped	$\omega_0^2 < \alpha^2$	$v(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$ $(s_1\ and\ s_2\ are\ real\ and\ distinct)$
Critically Damped	$\omega_0^2 = \alpha^2$	$v(t) = D_1 t e^{-\alpha t} + D_2 e^{-\alpha t}$
$\omega_d = \sqrt{\omega_0^2 - \alpha^2}$		

Note that whenever $\alpha \neq 0$, we say that the response will have the "damping"; therefore, the amplitude will go to the steady-state value (for the underdamped response) at some point eventually. In addition to this feature, the oscillating frequency (ω_d) will be lower than the resonant radian frequency when $\alpha \neq 0$.

b. Mechanical Analogy

Parallel RLC model take similar model for the mass attached to a spring. The motions can be described in the same manner; hence, the same formula to describe the effect. It is possible because the energy storing elements (inductor and capacitor) behave the same as the spring and mass. How a simple mechanical energy model works? From the law of conservation of energy, ALL the kinetic energy will be transformed into the potential energy. Once the kinetic energy is fully transformed, the potential energy now releases its energy to transform back to the kinetic energy and this process sustains if there's no energy loss in the process. Parallel RLC circuit will do the same exact thing except the energy store elements are inductor and capacitor and energy types can be described in the form of voltage and current.

c. LTspice examples

i. Example 8.2 (Overdamped Response)

Initial conditions: (1) $v(0^+) = 12V$ (2) $i(0^+) = 30mA$

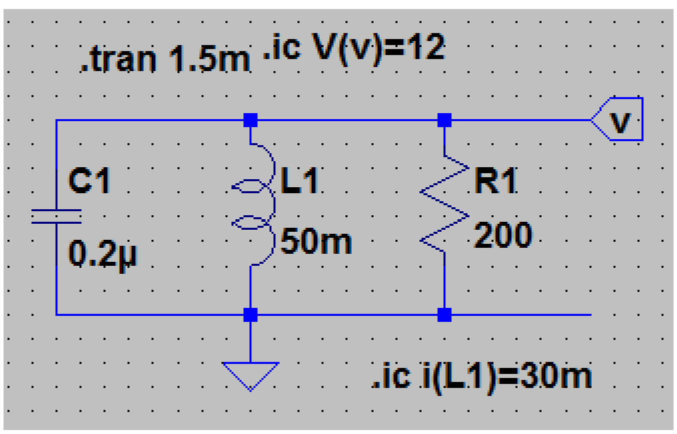

Figure. Circuit

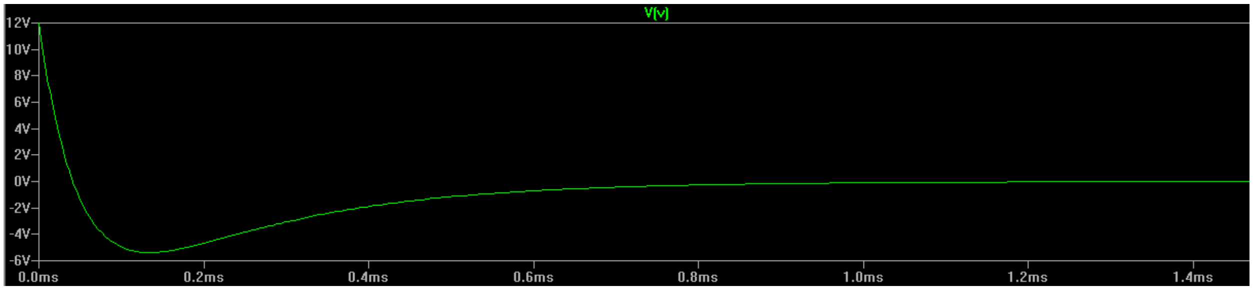

Figure. Transient Response

Solution:

$$\alpha = \frac{1}{2RC} = 12500$$

$$\omega_0 = \frac{1}{\sqrt{LC}} = 10000$$

$$s_1 = 3507.81\frac{rad}{s}$$

$$s_2 = -28507.81\frac{rad}{s}$$

$$v(0^+) = A_1 + A_2 = 12$$

$$from\ KCL: i_C(0^+) + i_L(0^+) + i_R(0^+) = 0 = i_C(0^+) + 30mA + \frac{12}{200}$$

$$i_C(0^+) = -90mA$$

$$C\frac{dv(0^+)}{dt} = i_C(0^+) = -90mA$$

$$\frac{dv(0^+)}{dt} = -450\frac{kV}{s}$$

$$Hence\ we\ end\ up\ with\ a\ system\ of\ equation:$$

$$\boxed{\begin{aligned}&(1)\ A_1 + A_2 = 12\\&(2)\ A_1 s_1 + A_2 s_2 = -450000\end{aligned}}$$

$$\begin{bmatrix} s_1 & s_2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} -450000 \\ 12 \end{bmatrix}$$

$$\begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} -14 \\ 26 \end{bmatrix}$$

$$\therefore v(t) = -14\exp(-5000t) + 26\exp(-20000t)$$

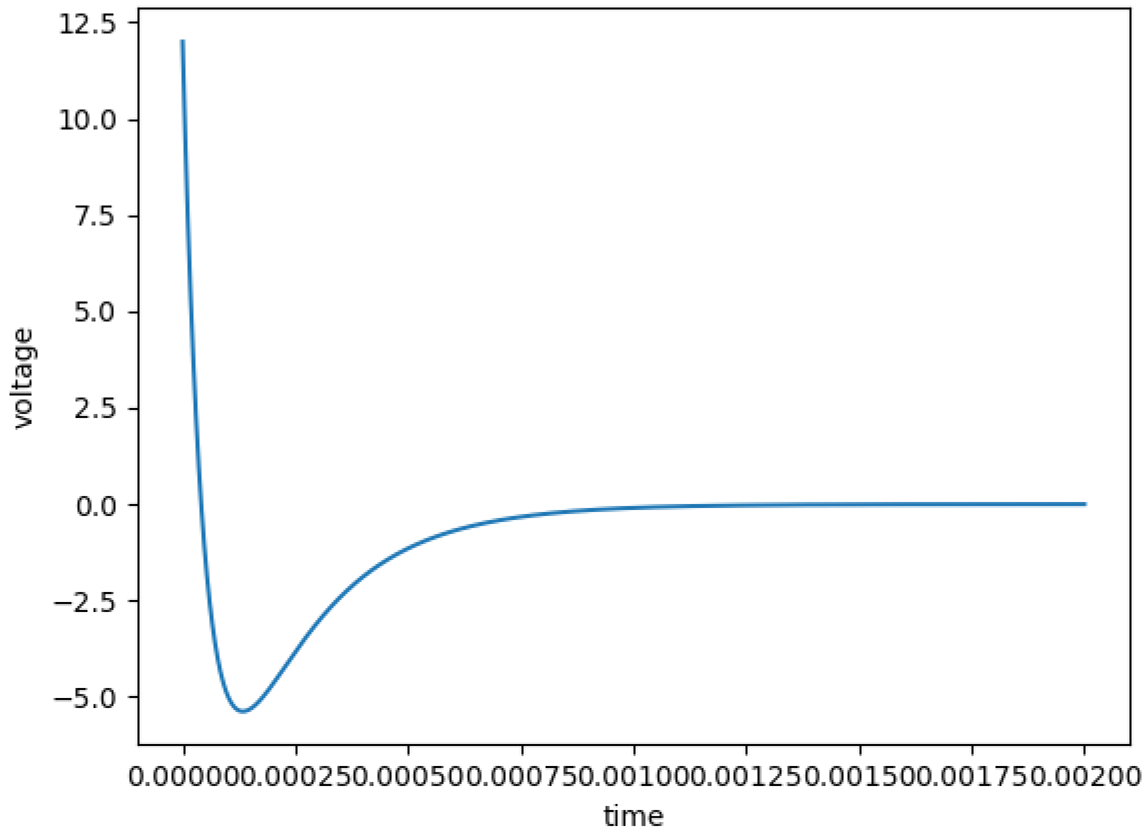

Figure. Plot of $v(t) = -14\exp(-5000t) + 26\exp(-20000t)$

The steady-state point in this circuit yields $V = 0$ since there's no external power supply.

ii. Example 8.4 (Underdamped Response)

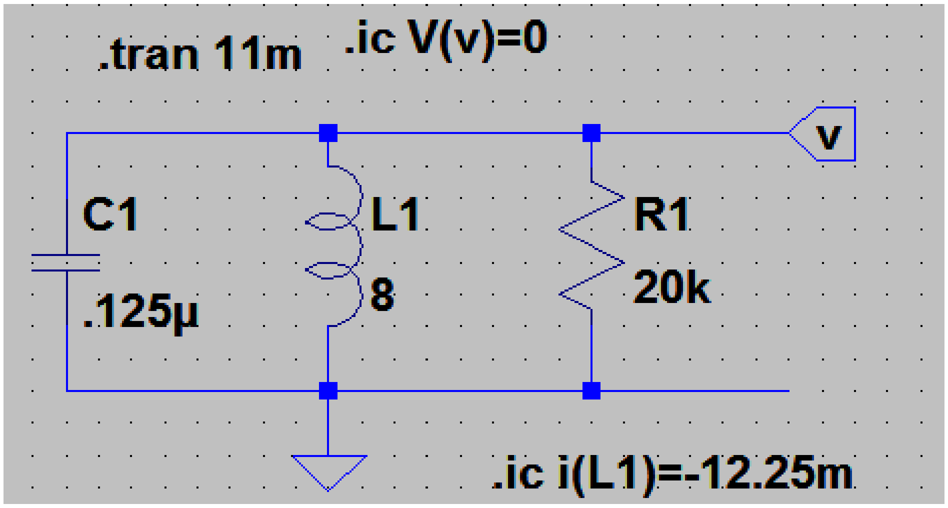

Figure. Example 8.4 Circuit with initial conditions

After going through the same procedure as Example 8.2,

$$v(t) = 100\exp(-200t)\sin(980t)$$

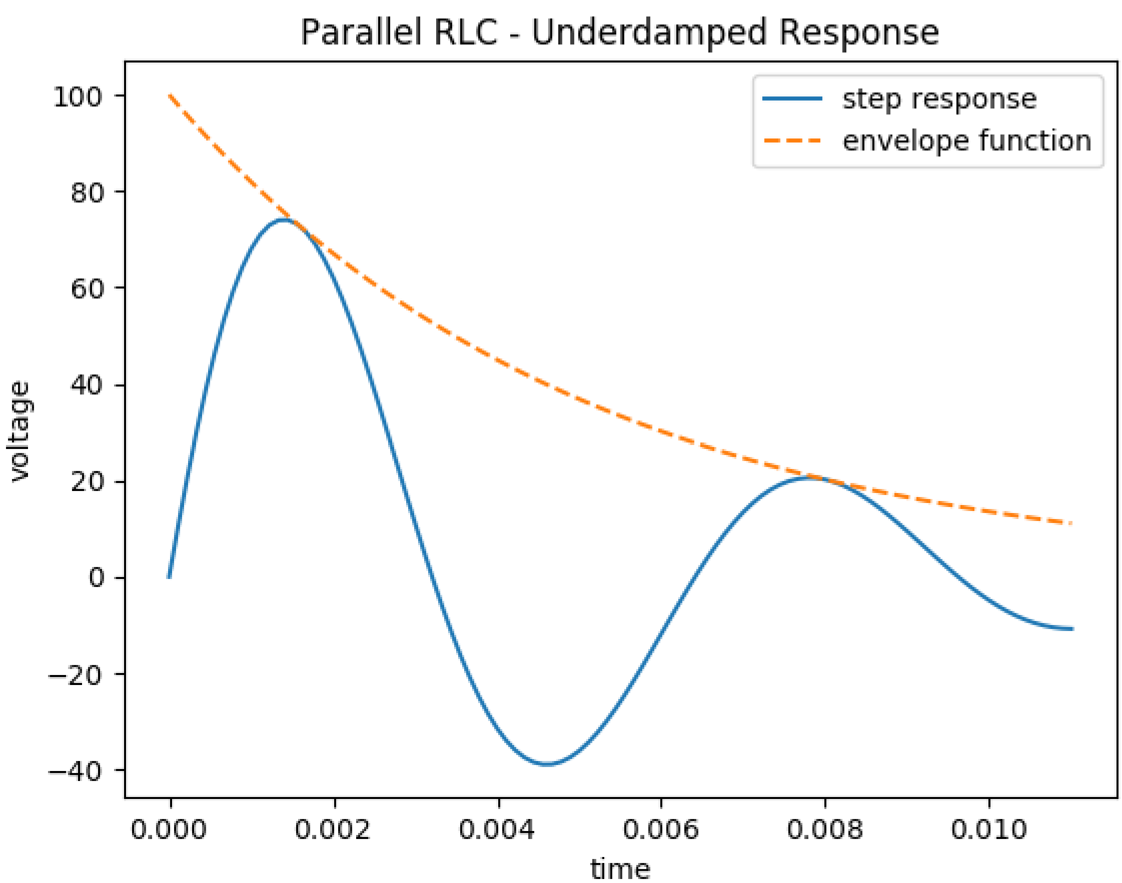

Figure. Python Output (Please note that it’s natural response)

Note that the Natural response:

$$v(t) = B_2 \exp(-\alpha t)\sin(\omega_d t)$$

The envelope function:

$$f(t) = B_2 \exp(-\alpha t)$$

Now simulating the natural response of the circuit:

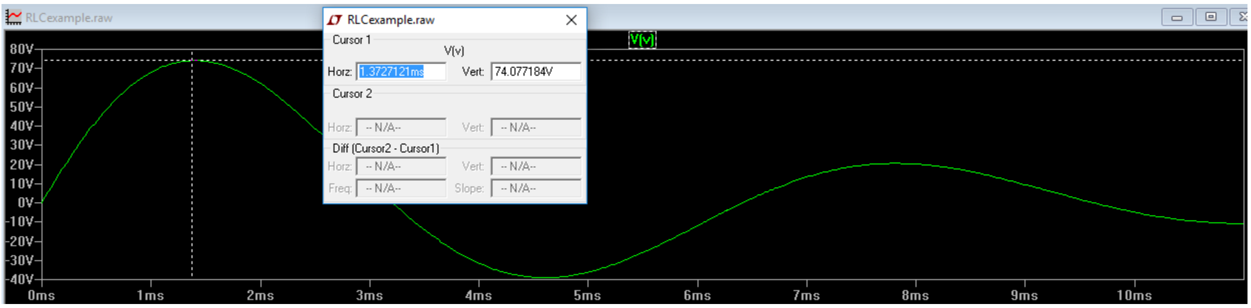

Figure. LTspice Transient Response

Important Characteristic of Underdamped Response?

As dissipative element (Resistor) goes to infinity,

$$\lim_{R\to\infty} \frac{v^2}{R} = 0 \ \left(p = \frac{v^2}{R}\right)$$

Hence, there's no power dissipation. In addition,

$$\lim_{R\to\infty} \omega_d = \omega_0 \ (\because \alpha = 0)$$

Which means as the resistance goes larger and larger, the oscillation tends to sustain its resonant frequency.

From the Example 8.4, we can observe the above story

$$\lim_{R\to\infty} v(t) = 100\sin(980t)$$

The natural response has become just a simple sinusoidal wave.

Difference between Underdamped and Overdamped?

As shown above, underdamped response oscillation bounces about its final values – called "ringing". In overdamped response, there's no ringing. The response simply approaches to its final value without ringing.

When to design underdamped or overdamped?

(1) Underdamped:

When you want to hit the final value in the shortest possible time period and don't have to concern about the small oscillation about its final value.

(2) Overdamped:

When concerned the component not exceeding the final value, perhaps to avoid component damage. However, you have to accept the fact that response would approach to its final value slowly.

Meaning of "final value"?

The terminology of “final value” kept arose in the context. This simply refers to the “steady-state” point. Meaning both overdamped and underdamped responses would settle down at some point and stop oscillating because of the dissipation (or Neper frequency).

iii. Example 8.5 (Critically damped Response)

Unlike the overdamped or underdamped, the solution to the differential equation takes the following form:

$$v(t) = D_1 t e^{-\alpha t} + D_2 e^{-\alpha t}$$

However, the procedure of finding the coefficients is the same as described before for underdamped and overdamped cases.

This time the problem asked to find the resistance that yields the critically damped response of the same circuit in Example 8.4. Hence, capacitance and inductance remain the same (as well as the initial conditions). Using the fact that $\omega_0 = \alpha$:

$$R = 4kohm$$

$$\alpha = \frac{1}{2RC} = 1000\frac{rad}{s}$$

$$D1 = 98000$$

Hence, the natural response came out to be:

$$v(t) = 98000te^{-1000t}$$

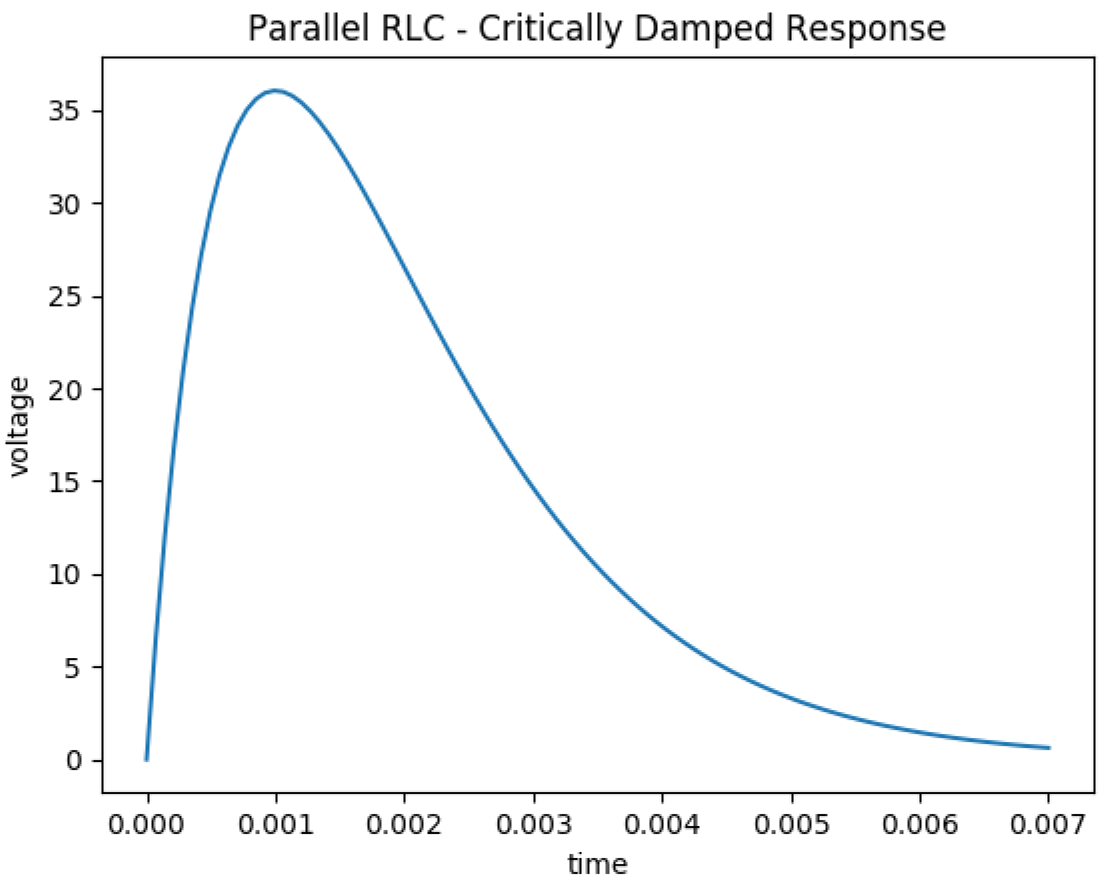

Figure. Python output

Now, simulating the LTspice circuit:

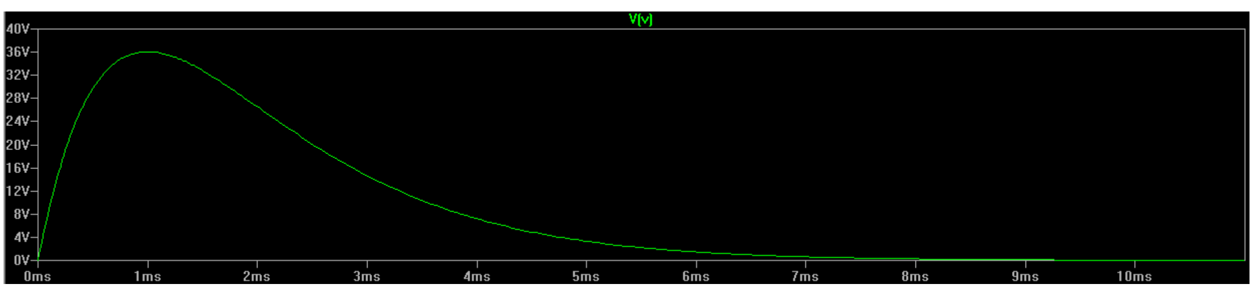

Figure. Transient Response – Critically Damped Response

From the python script, the maximum voltage found to be:

$$V_{max} = 36.05\ V$$

From the LTspice graph, the maximum voltage found to be:

$$V_{max} = 36.04\ V$$

The steady-state point for voltage:

$$V_{steady-state} = 0$$

V. Natural Response of Series RLC Circuit

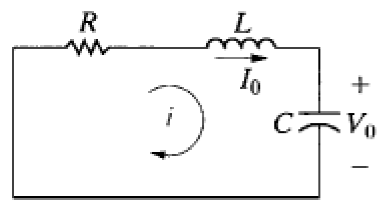

Figure 8.3 ▲ A circuit used to illustrate the natural response of a series *RLC* circuit.

The procedures for finding the natural or step responses of series RLC are **the same** for parallel RLC. By writing the KVL around the loop,

$$iR + L\frac{di}{dt} + \frac{1}{C}\int_0^t i d\tau + V_0 = 0$$

At the end of day, we end up with:

$$\frac{d^2 i}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{i}{LC} = 0$$

Using the characteristic equation method, we can find the two solutions:

$$s_1 = -\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

$$s_2 = -\frac{R}{2L} - \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

Define the Neper and Resonant radian frequencies for Series RLC circuit:

$$\alpha = \frac{R}{2L}$$

$$\omega_0 = \frac{1}{\sqrt{LC}}$$

Recall the solutions for parallel RLC were in voltage form because voltages for every component were the same. In series RLC, currents are the same for every component and the solutions to the differential equation are also in current form.

Underdamped	$\omega_0^2 > \alpha^2$	$i(t) = B_1 e^{-\alpha t}\cos(\omega_d t) + B_2 e^{-\alpha t}\sin(\omega_d t)$
Overdamped	$\omega_0^2 < \alpha^2$	$i(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$ $(s_1\ and\ s_2\ are\ real\ and\ distinct)$
Critically Damped	$\omega_0^2 = \alpha^2$	$i(t) = D_1 t e^{-\alpha t} + D_2 e^{-\alpha t}$
$\omega_d = \sqrt{\omega_0^2 - \alpha^2}$		

i. Example 8.11 – Underdamped Natural Response in Series RLC

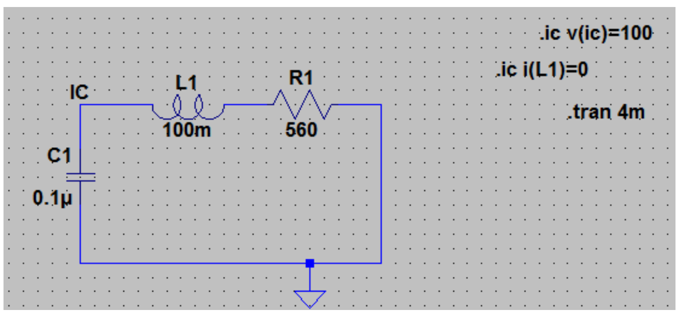

Figure.

It's series RLC hence the solution is in current dimension (ampere). From the initial condition,

$$B_1 = 0$$

$$i(t) = B_2 e^{-\alpha t}\sin(\omega_d t)$$

$$\alpha = \frac{R}{2L} = 2800$$

$$\omega_0 = \frac{1}{\sqrt{LC}} = 10000$$

$$\omega_d = 9600$$

Writing the KVL around the loop:

$$-v_C + v_L + v_R = 0 = -\frac{1}{C}\int idt - V_0 + L\frac{di}{dt} + iR$$

$$-V_0 + L\frac{di(0^+)}{dt} + i(0^+)R = 0$$

$$\frac{di(0^+)}{dt} = \frac{V_0}{L}$$

We need to know the information about $\frac{di}{dt}$:

$$\frac{di(t)}{dt} = -\alpha B_2 e^{-\alpha t}\sin(\omega_d t) + B_2\omega_d e^{-\alpha t}\cos(\omega_d t)$$

$$B_2\omega_d = \frac{V_0}{L}$$

$$i(t) = \frac{V_0}{\omega_d L} e^{-\alpha t}\sin(\omega_d t)$$

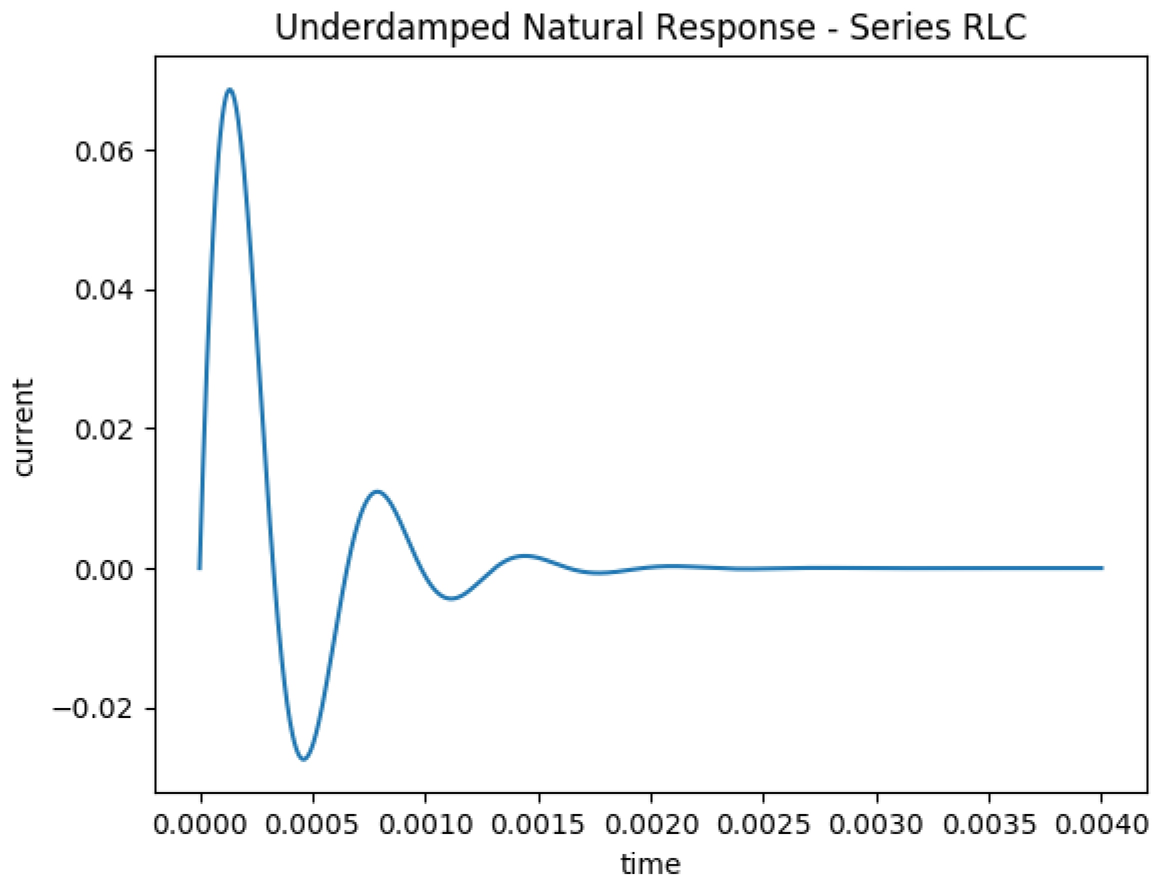

Figure. Python output

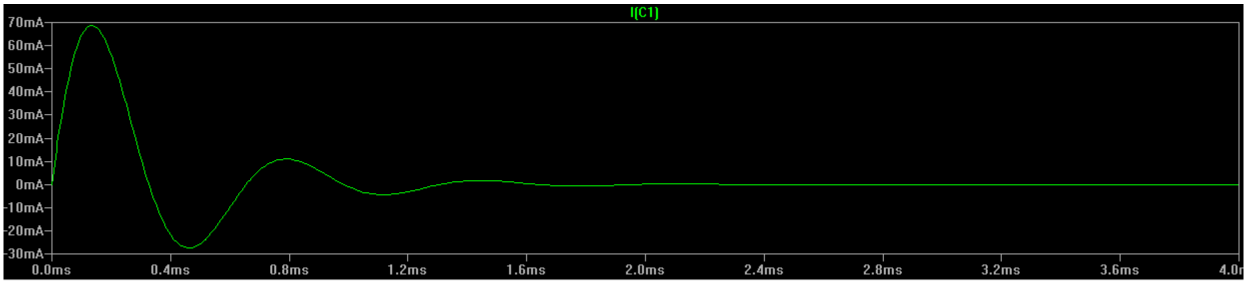

Figure. Transient Response for current

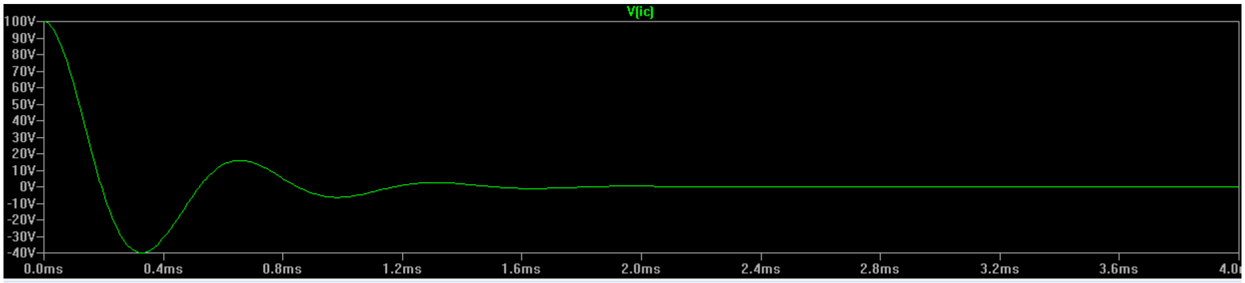

Figure. Transient Response for voltage

Note that the two graphs may look the same but their starting points are different because of the initial conditions.

ii. Example 8.12 – Underdamped Step Response of Series RLC

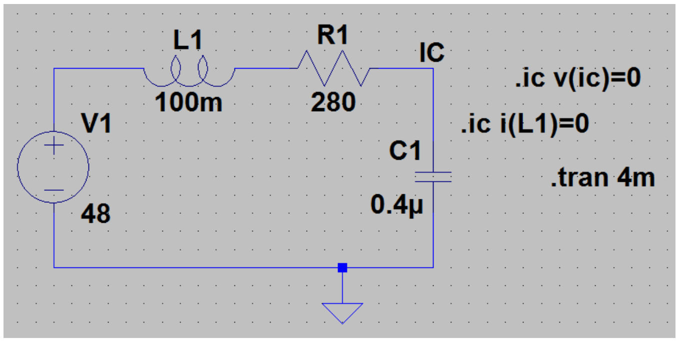

Figure.

$$\alpha = 1400$$

$$\omega_d = 4000$$

$$B_1 = 0$$

$$i(t) = B_2 e^{-\alpha t} \sin(\omega_d t)$$

Writing the KVL:

$$-48 + v_C + v_L + v_R = 0$$

$$-48 + L\frac{di(0^+)}{dt} = 0$$

$$\frac{di(0^+)}{dt} = \frac{48}{L}$$

Since

$$\frac{di(t)}{dt} = -\alpha B_2 e^{-\alpha t}\sin(\omega_d t) + B_2\omega_d e^{-\alpha t}\cos(\omega_d t)$$

$$\frac{di(0^+)}{dt} = \frac{48}{L} = B_2\omega_d$$

$$B_2 = \frac{48}{L\omega_d}$$

$$i(t) = \frac{48}{\omega_d L}e^{-\alpha t}\sin(\omega_d t)$$

To find the voltage across the capacitor:

$$v_C(t) = \frac{1}{C}\int_0^t i(\tau)d\tau = \frac{48}{\omega_d LC}\int_0^t e^{-\alpha\tau}\sin(\omega_d\tau)\,d\tau$$

$$= -\frac{48}{\omega_d LC}\left[\frac{e^{-\alpha\tau}\{\alpha\sin(\omega_d\tau) + \omega_d\cos(\omega_d\tau)\}}{\alpha^2 + \omega_d^2}\right]\begin{matrix} t \\ 0 \end{matrix}$$

$$= 48 - \frac{48}{\omega_d LC}\left[\frac{e^{-\alpha t}\{\alpha\sin(\omega_d t) + \omega_d\cos(\omega_d t)\}}{\alpha^2 + \omega_d^2}\right]$$

$$= 48 - 14e^{-\alpha t}\sin(\omega_d t) - 48e^{-\alpha t}\cos(\omega_d t)$$

This is a complicated way to solve the problem. The book already introduced the solution forms for the voltage step response of series RLC circuits:

$$v_C = V_f + A_1' e^{s_1 t} + A_2' e^{s_2 t} \quad \text{(overdamped)}, \qquad (8.67)$$

$$v_C = V_f + B_1' e^{-\alpha t}\cos\omega_d t + B_2' e^{-\alpha t}\sin\omega_d t \quad \text{(underdamped)}, \qquad (8.68)$$

$$v_C = V_f + D_1' t e^{-\alpha t} + D_2' e^{-\alpha t} \quad \text{(critically damped)}, \qquad (8.69)$$

◀ **Capacitor voltage step response forms in series *RLC* circuits**

To check the result, let's simulate the circuit:

Figure. Transient response for capacitor voltage

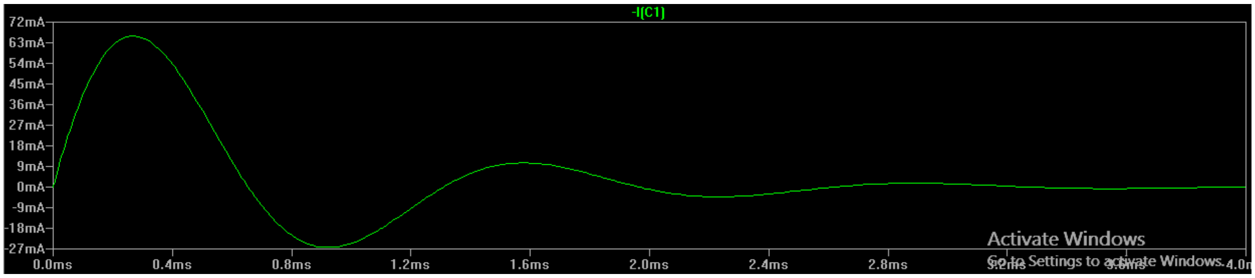

Figure. Transient response for current

Since our current solution was:

$$i(t) = \frac{48}{\omega_d L} e^{-\alpha t} \sin(\omega_d t)$$

The steady-state value should be zero and the graph shows the same result.

VI. Conclusion

We have looked at the RC and RL circuit and their natural response. Their responses look exactly the same except that the response for RC is voltage type whereas current for the RL. The graphs look reasonable, let's say, we have an initially fully charged capacitor, which can act like a voltage source. Now, let’s say we’d want to transfer this stored energy to some load (resistor). This is a perfect example of RC circuit and its response can be introduced by the above work.

The steady-state for natural responses are always approaching to zero if time approaches to infinity since there’s no external power supply. For step responses, if there’s DC power supply in the circuit, step responses will go through some perturbations at first then come to the steady-state value, which is the DC value supplied.

The forms of solutions for both parallel and series RLC are the same except voltage solution for parallel RLC whereas current solution for series RLC. This fact is reasonable since voltages are the same in parallel circuit whereas currents are the same in series circuit.

VII. References

[1] Nilsson & Riedel Electric Circuits, 9th Edition