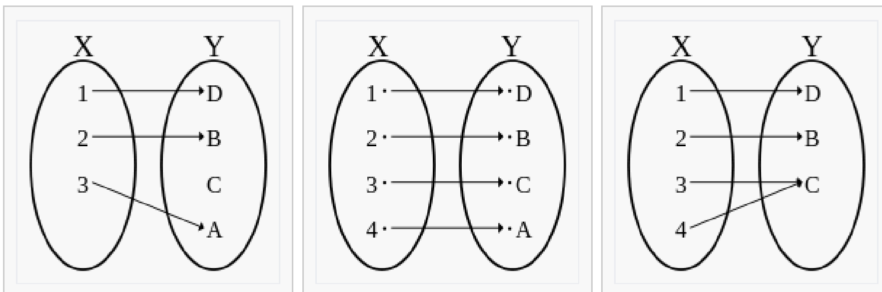
Type of Functions

I. Introduction

Set theory and other mathematics area are interested in bijective functions. A bijective function from a set to itself is known as “permutation”, which is a widely used concept in probability theory.

II. Functions



An injective non-surjective function (injection, not a bijection)

An injective surjective function (bijection)

A non-injective surjective function (surjection, not a bijection)

Figure. [2]

a. Bijection

A function can be identified as ***bijective*** if the function between a set to another is one-to-one correspondence. This means every element in the first set is mapped to every element in the other set.

Since every element is related between a set and another, this concept often arises when the function between two sets: isomorphism (vector spaces), homeomorphism (topological spaces), homomorphism (rings), etc.

Another interesting and powerful property that a bijective function has, is that there always exists an inverse function.

b. Injective [2]

Injective function is also known as “one-to-one”. Unlike bijection, not every element may have an image in the codomain. In this case, the number of elements in codomain may be larger than that of domain.

Definition:

Let f be a function whose domain is a set X. The function f is said to be injective provided that for all a and b in X, whenever $f(a) = f(b)$, then $a = b$; that is, $f(a) = f(b)$ implies $a = b$.

c. Surjective

Surjective function is also known as “onto”. Every element in domain are mapped to those in codomain; however, there’s should not be one-to-one relation between them, meaning that one of the element in domain may have multiple images in the codomain.

Definition:

A function whose image is equal to its codomain. Equivalently, a function f with domain X and codomain Y is surjective if for every y in Y there exists at least one x in X with $f(x) = y$.

$$f: X \twoheadrightarrow Y$$

This two-headed rightwards arrow represents surjection.

i. Example – Richard’s transformation

Richard’s transformation is used when one wants to convert passive elements into stubs (Open- or short-circuit). The function that enables such conversion can be expressed in tangent function, which is periodic. [4]

$$\omega \to \tan(k\omega)$$

where

$$\omega = angular\ frequency$$

$$k = wavenumber$$

In transmission line theory, short-circuit and open-circuit stubs can be easily shown:

$$Z_{SC} = jZ_0 \tan(k\omega)$$

$$Z_{OC} = -jZ_0 \cot(k\omega)$$

where the argument inside the function should be less than $\pi/2$.

One may feel comfortable with the following normalized (or, a.k.a. omega) domain.

$$\Omega = \tan(k\omega)$$

where $\Omega = 1$ if $\omega = \omega_c$.

Because of this periodic nature of new domain, the filter response derived from this transformation would yield repetitive behaviors on frequency domain. (e.g. I checked that Butterworth response with $N = 3$ at 4 GHz would also be seen around 16 GHz.)

III. Proof

There are many ways to proof whether a function is injective or not.

One method:

Take the derivative of the function on the interval under interest and show it's always positive or always negative on the same interval.

For this reason, exponential and logarithm functions are injective but not surjective. (i.e. Exponential function is not defined on the negative interval)

IV. References

[1] https://en.wikipedia.org/wiki/Bijection

[2] https://en.wikipedia.org/wiki/Injective_function

[3] https://en.wikipedia.org/wiki/Surjective_function

[4] https://en.wikipedia.org/wiki/Commensurate_line_circuit