Integration Techniques

I. Introduction

There are many ways to perform integrals over single-variable functions and multi-variable functions. In this document, rather dealing with some examples than conveying the fundamental ideas of integral.

As a matter of fact, mathematicians already formulate a list of integrals that appear frequently in science and engineering fields. Reference [1] contains the list of integrals; however, one can find more on internet at their convenience.

A friend of mine motivated me to write this document. Special thanks to him.

II. Types of Integration Techniques

From my experience, there are about 4 different integral methods (there could be more):

(1) v-substitution
(2) Integration by parts
(3) Trigonometry substitution
(4) Partial Fraction

a. Basic Integral Rules

REVIEW OF BASIC INTEGRATION RULES ($a > 0$)

1. $\int kf(u)\,du = k\int f(u)\,du$
2. $\int [f(u) \pm g(u)]\,du = \int f(u)\,du \pm \int g(u)\,du$
3. $\int du = u + C$
4. $\int u^n\,du = \frac{u^{n+1}}{n+1} + C,\ n \neq -1$
5. $\int \frac{du}{u} = \ln|u| + C$
6. $\int e^u\,du = e^u + C$
7. $\int a^u\,du = \left(\frac{1}{\ln a}\right)a^u + C$
8. $\int \sin u\,du = -\cos u + C$
9. $\int \cos u\,du = \sin u + C$
10. $\int \tan u\,du = -\ln|\cos u| + C$
11. $\int \cot u\,du = \ln|\sin u| + C$
12. $\int \sec u\,du = \ln|\sec u + \tan u| + C$
13. $\int \csc u\,du = -\ln|\csc u + \cot u| + C$
14. $\int \sec^2 u\,du = \tan u + C$
15. $\int \csc^2 u\,du = -\cot u + C$
16. $\int \sec u \tan u\,du = \sec u + C$
17. $\int \csc u \cot u\,du = -\csc u + C$
18. $\int \frac{du}{\sqrt{a^2 - u^2}} = \arcsin\frac{u}{a} + C$
19. $\int \frac{du}{a^2 + u^2} = \frac{1}{a}\arctan\frac{u}{a} + C$
20. $\int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a}\operatorname{arcsec}\frac{|u|}{a} + C$

Figure. [2]

Although it's recommended deriving all these integrals to fully understand the steps taken, one can just refer to these basic rules to apply integrals.

b.Special Integrals

Not all single-variable function can be performed integral. Here's a famous example.

$$f(x) = \exp(x^2)$$

Consider indefinite integral of the function:

$$F(x) = \int f(x)dx = \int \exp(x^2)\,dx = ?$$

The following is Wolframalpha's answer of the integral:

Indefinite integral:

$$\int \exp(x^2)\, dx = \frac{1}{2} \sqrt{\pi}\, \mathrm{erfi}(x) + \text{constant}$$

Figure. [3]

where $erfi(x)$ function is known as "*imaginary error function*" in Wolfram language that has a derivative which simply implies that such integral does not exist in real world.

c. v-substitution

Let's look at the basic idea first. Consider the chain rule:

$$f(x) = u\big(v(x)\big)$$

$$\frac{d\Big(u\big(v(x)\big)\Big)}{dx} = \frac{du}{dv}\frac{dv}{dx} = u'\big(v(x)\big) \cdot v'(x)$$

then, the anti-derivative of the result is.

$$f(x) = \int d\Big(u\big(v(x)\big)\Big) = \int u'\big(v(x)\big) \cdot v'(x) dx = u\big(v(x)\big)$$

In differential form,

$$\int u'\big(v(x)\big) \cdot v'(x) dx = \int \frac{du}{dv}\frac{dv}{dx} dx = \int du = u$$

Hence, u-substitution is the reverse process of the chain rule of calculus. [4]

i. Example

Consider a function

$$f(x) = \cos(7x - 4) = w\big(s(x)\big)$$

where

$$w(s) = \cos(s)$$

$$s(x) = 7x - 4$$

Now, calculate the differential quantities:

$$ds = 7dx$$

Perform integral:

$$\int f(x)dx = \int w\big(s(x)\big)dx = \int [w(s)ds]dx \cdot \frac{1}{ds} = [w(s)ds]\frac{dx}{7dx}$$

Here we insert the differential quantity ds to perform the integral over s-domain rather than over x-domain.

Therefore, the result:

$$\int \frac{1}{7}\cos(s)\, ds = \frac{1}{7}\sin(s) + C$$

Retrieving the variable s to the x-domain

$$\therefore \int \cos(7x-4) = \frac{1}{7}\sin(7x-4) + C$$

d.Integration by Parts

From [2], the integration by parts can be expressed as:

$$\int udv = uv - \int vdu$$

Derivation process already formulated on the reference [2] as:

$$\frac{d(uv)}{dx} = u'v + uv'$$

$$\int d(uv) = \int [u'v + uv']dx$$

$$= \int v\frac{du}{dx}dx + \int u\frac{dv}{dx}dx$$

$$uv = \int vdu + \int udv$$

Hence,

$$\boxed{\int udv = uv - \int vdu}$$

i. Example – Integration by Parts

Rather going over a trivial example, let’s do some interesting integral.

$$\int sec^3(x)\, dx$$

If one let

$$u = sec(x)$$

$$dv = sec^2(x)dx$$

Then,

$$du = sec(x)\, tan(x)\, dx$$

$$v = tan(x)$$

Hence,

$$= \sec(x)\tan(x) - \int \tan(x)\sec(x)\tan(x)\, dx$$

$$\int sec^3(x)\, dx = sec(x)\, tan(x) - \int sec^3(x) - sec(x)\, dx$$

$$\because \tan^2(x) = \sec^2(x) - 1$$

$$2\int sec^3(x)\, dx = sec(x)\, tan(x) + \int sec(x)\, dx$$

Thus,

$$\int sec^3(x)\, dx = \frac{1}{2} sec(x)\, tan(x) + \frac{1}{2}\ln|sec(x) + tan(x)| + C$$

ii. Example - Tabular Method

Tabular method is part of the integration by parts but allows simpler approach to the integral.

$$\int x^2 \sin(3x)\, dx$$

First, one must identify which factor can be differentiated to be zero. Since sine functions are transcendental function, one can't obtain zero by differentiating with respect to the independent variable x. However, x^2, in this case, is a polynomial function, which yields zero by differentiating many times eventually. Now, let's set up a table.

Signs	D	I
+	x^2	$\sin(3x)$
-	$2x$	$-\frac{1}{3}\cos(3x)$
+	2	$-\frac{1}{3^2}\sin(3x)$
-	0	$\frac{1}{3^3}\cos(3x)$

Multiplying the same colored terms and add will yield the result.

$$\int x^2 \sin(3x)\, dx = -\frac{1}{3}x^2 \cos(3x) + \frac{2}{9}x sin(3x) + \frac{2}{27}\cos(3x) + C$$

e. Trigonometry Substitution [5]

The main idea for Trig substitution is the Pythagorean theorem. There are three possible forms for Trig substitution:

(1) The integrand contains $A^2 - x^2$
(2) The integrand contains $A^2 + x^2$
(3) The integrand contains $x^2 - A^2$

where a is a constant.

If use see	use the sub	so that	and
$\sqrt{a^2 - x^2}$	$x = a\sin\theta$	$dx = a\cos\theta\, d\theta$	$\sqrt{a^2 - x^2} = a\cos\theta$
$\sqrt{a^2 + x^2}$	$x = a\tan\theta$	$dx = a\sec^2\theta\, d\theta$	$\sqrt{a^2 + x^2} = a\sec\theta$
$\sqrt{x^2 - a^2}$	$x = a\sec\theta$	$dx = a\sec\theta\tan\theta\, d\theta$	$\sqrt{x^2 - a^2} = a\tan\theta$

Figure. [6]

The above relation can be easily done by using the Pythagorean theorem. Let's do the first case for illustration.

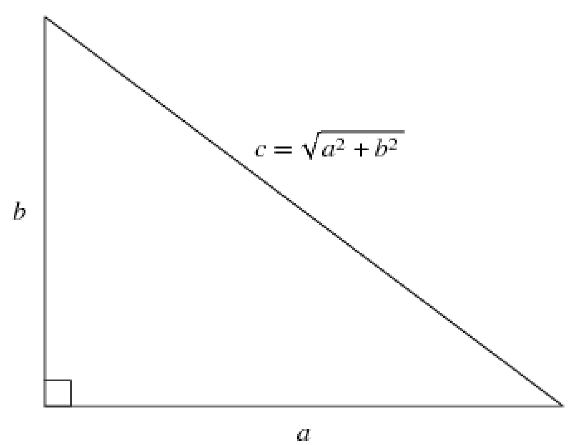

Figure. Pythagorean Theorem

A sine function can be defined from the triangle as

$$sin\theta = \frac{x}{A}$$

Let's assign:

$$c = A$$

$$a = x$$

Once two of three variables have been determined, the rest one should be determined from the Pythagorean theorem.

$$b = \sqrt{A^2 - x^2}$$

Assigning a, b, c (of a triangle) to appropriate variable is an important process to determine other trig functions such as secant and cosecant functions if needed.

f. Example – Trig Substitution

Let's derive arctangent function using the Trig substitution.

$$\int \frac{1}{A^2 + u^2} du = ?$$

By looking at the form, it should be easier if one assign:

$$a = A$$

$$b = u$$

Then,

$$c = \sqrt{A^2 + u^2}$$

$$tan\theta = \frac{u}{A}$$

which leads to the following relations:

$$u = Atan\theta$$

$$du = Asec^2\theta d\theta$$

$$\theta = \arctan\left(\frac{u}{A}\right)$$

Hence,

$$\int \frac{1}{A^2 + u^2} du = \int \frac{1}{A^2(1 + \tan^2\theta)} Asec^2\theta d\theta$$

$$= \frac{1}{A} \int \frac{\sec^2 \theta}{\sec^2 \theta} d\theta = \frac{\theta}{A}$$

Thus,

$$\int \frac{1}{A^2 + u^2} du = \frac{1}{A} \arctan\left(\frac{u}{A}\right) + C$$

g. Partial Fraction

DECOMPOSITION OF $N(x)/D(x)$ INTO PARTIAL FRACTIONS

1. **Divide if improper:** If $N(x)/D(x)$ is an improper fraction (that is, if the degree of the numerator is greater than or equal to the degree of the denominator), divide the denominator into the numerator to obtain

$$\frac{N(x)}{D(x)} = (\text{a polynomial}) + \frac{N_1(x)}{D(x)}$$

where the degree of $N_1(x)$ is less than the degree of $D(x)$. Then apply Steps 2, 3, and 4 to the proper rational expression $N_1(x)/D(x)$.

2. **Factor denominator:** Completely factor the denominator into factors of the form

$$(px + q)^m \quad \text{and} \quad (ax^2 + bx + c)^n$$

where $ax^2 + bx + c$ is irreducible.

3. **Linear factors:** For each factor of the form $(px + q)^m$, the partial fraction decomposition must include the following sum of m fractions.

$$\frac{A_1}{(px + q)} + \frac{A_2}{(px + q)^2} + \cdots + \frac{A_m}{(px + q)^m}$$

4. **Quadratic factors:** For each factor of the form $(ax^2 + bx + c)^n$, the partial fraction decomposition must include the following sum of n fractions.

$$\frac{B_1x + C_1}{ax^2 + bx + c} + \frac{B_2x + C_2}{(ax^2 + bx + c)^2} + \cdots + \frac{B_nx + C_n}{(ax^2 + bx + c)^n}$$

Figure. [2]

i. Example – Distinct Linear Factors

$$\int \frac{1}{x^2 - 5x + 6} dx$$

For distinct linear factor example, integral becomes easier if one can decompose the quadratic equation into two linear factors to use the following relation:

$$\int \frac{1}{x}dx = \ln|x| + C$$

Here, we can decompose the quadratic equation:

$$x^2 - 5x + 6 = (x-2)(x-3)$$

and equate

$$\frac{1}{x^2 - 5x + 6} = \frac{A}{x-2} + \frac{B}{x-3}$$

which leads to

$$1 = A(x-3) + B(x-2)$$

To find A, let $x = 2$

$$A = -1$$

Likewise, for B, let $x = 3$

$$B = 1$$

Therefore, we obtain an equivalent equation

$$\frac{1}{x^2 - 5x + 6} = -\frac{1}{x-2} + \frac{1}{x-3}$$

Thus,

$$\int \frac{1}{x^2 - 5x + 6}dx = \int -\frac{1}{x-2}dx + \int \frac{1}{x-3}dx$$

$$= -\ln|x-2| + \ln|x-3| + C$$

III. References

[1] https://en.wikipedia.org/wiki/Lists_of_integrals

[2] Calculus Early Transcendental Functions, Larson and Edwards, 5th Edition

[3] https://www.wolframalpha.com/input/?i=integral+exp(x%5E2)dx

[4] https://www.khanacademy.org/math/ap-calculus-ab/ab-antiderivatives-ftc/ab-u-substitution/a/review-applying-u-substitution

[5] https://en.wikipedia.org/wiki/Trigonometric_substitution

[6] https://www.quora.com/What-is-the-easiest-way-to-do-remember-how-to-do-trig-substitution-integrals