Filter Circuits

I. Introduction

First, we will be looking at Low Pass filters, RC and RL circuits.

II. Low Pass Filters

Both RL and RC circuits can be a candidate for low pass filter circuit.

a. Series RL circuit

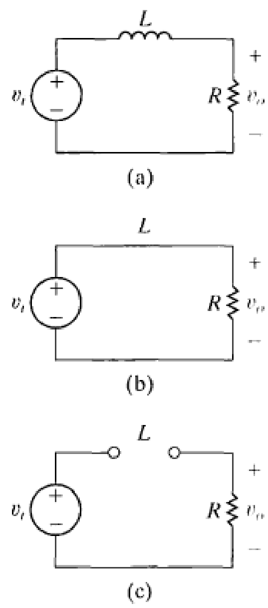


Figure 14.4 ▲ (a) A series *RL* low-pass filter. (b) The equivalent circuit at $\omega = 0$, and (c) The equivalent circuit at $\omega = \infty$.

Note that the **impedance of inductor** is very small at low frequency; therefore, it acts like a short circuit at low frequency. On the other hand, it acts like open circuit for high frequency.

Let's define what it means high and low frequency:

High Frequencies?	$\omega L \gg R$
Low Frequencies?	$\omega L \ll R$

Phase shift caused by inductor?

The phase of output voltage will be 90° more negative than that of input voltage.

Let's look at the transfer function plot for both magnitude and phase.

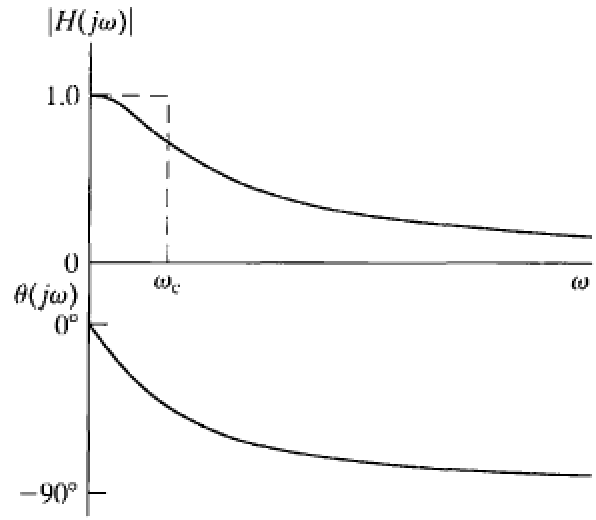


Figure 14.5 ▲ The frequency response plot for the series *RL* circuit in Fig. 14.4(a).

Note that how the magnitude plot shows a gradual transition from passband to stopband. Abrupt transition does not occur in real circuit. The cutoff frequency (ω_c) indicates the boundary between passband and stopband. So how do we define the cutoff frequency?

$$|H(j\omega_c)| = \frac{1}{\sqrt{2}} H_{max}$$

Now, we've defined the cutoff frequency. The factor (~70.7%) seems little arbitrary choice. As a matter of fact, it is not. Consider the power when the input is sinusoidal it may become evident.

Let's consider the voltage transfer function of RL circuit (consider voltage divider method):

$$Z_L = sL$$

$$H(s) = \frac{\frac{R}{L}}{s + \frac{R}{L}}$$

If you want to know about the frequency response, then make a simple substitution, $s = j\omega$

Since the transfer function contains real and imaginary parts, we can analyze them in polar form. Before jumping onto the exact form, let's consider the general magnitude and phase plot for the following complex number:

$$z = \frac{a}{a + jb} = \frac{a^2 - jab}{a^2 + b^2}$$

Magnitude Plot	$\lvert z \rvert = \frac{a}{\sqrt{a^2 + b^2}}$
Phase Plot	$\measuredangle z = \tan^{-1}\left(-\frac{b}{a}\right) = -\tan^{-1}\left(\frac{b}{a}\right)$

Note that arctangent function is an odd function since

$$\tan^{-1}(x) + \tan^{-1}(y) = \tan^{-1}\left(\frac{x + y}{1 - xy}\right)$$

Which implies that if $f(x) = \tan^{-1}(x)$, then $f(x) + f(-x) = 0$

Now back to our RL circuit, the magnitude and phase plot for RL circuit can be simplified:

Magnitude Plot	$\lvert H(j\omega) \rvert = \frac{R/L}{\sqrt{(R/L)^2 + \omega^2}}$
Phase Plot	$\measuredangle z = \tan^{-1}\left(-\frac{\omega}{R/L}\right) = -\tan^{-1}\left(\frac{\omega}{R/L}\right)$

Let's define the cutoff frequency of RL circuit. Recall the cutoff frequency occurs

$$\lvert H(j\omega_c) \rvert = \frac{1}{\sqrt{2}} H_{max}$$

And the maximum magnitude is 1 and it can be shown in the figure14.5. Solving the equation will yield:

$$\omega_c = \frac{R}{L}$$

i. Example 14.1 – Design Low Pass Filter

We look for a filter that passes human heart beats signal (~1Hz) but there's noise in the signal whose fundamental frequency is 60Hz. Design a filter that filters out any signal above 10Hz. Compute the output voltage level at 1Hz, 10Hz, and 60Hz.

From the problem statement, the following items can be identified:

$f_c = 10Hz$

To determine the cutoff frequency for an RL low pass filter, the following equation should be examined:

$$\omega_c = \frac{R}{L}$$

However, there are two independent variables for a single equation; therefore, we need to choose the size of an inductor with a popular value that we can use. Let's choose 100mH.

Hence, the resistance came out to be:

$$R = 6.283\ ohm$$

f	$\lvert V_i(\omega)\rvert$	$\lvert V_o(\omega)\rvert$
$1Hz$	1	0.995
$10Hz$	1	0.707
$60Hz$	1	0.164

Conversely, we can induce a fact that 10Hz is the cutoff frequency in the above table by looking at the value of the output since it is $.707|V_i(\omega)|$.

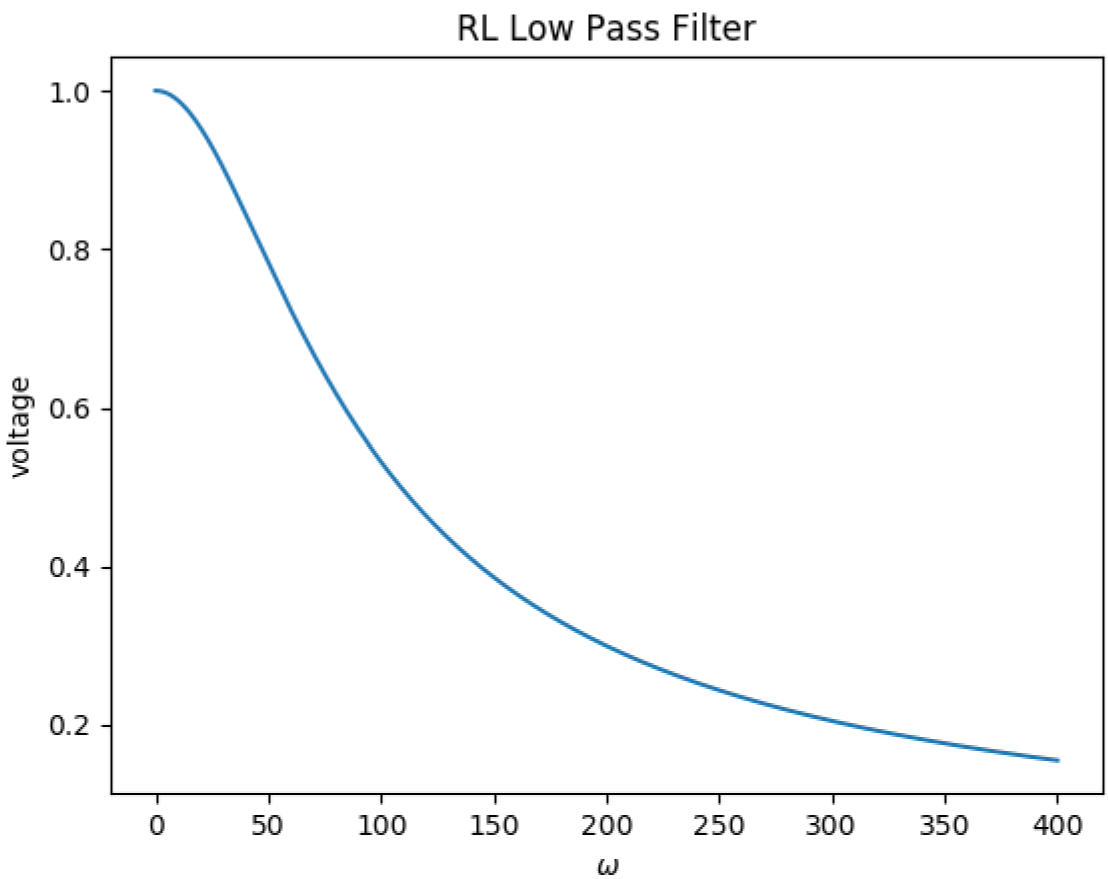


Figure. Graph of $|V_o(\omega)|$ when input is 1V

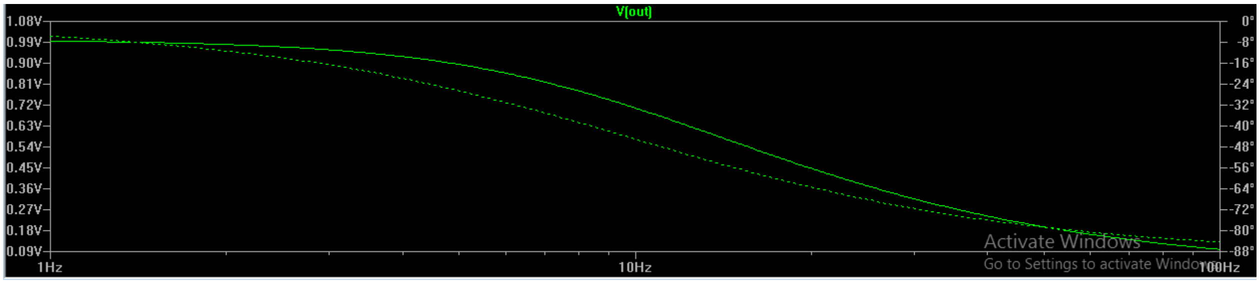


Figure. Frequency response of output voltage at resistor

$V_o = 0.70774\, V\ at\ 9.980 Hz$

b.Series RC circuit

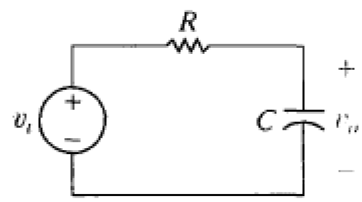


Figure 14.7 ▲ A series *RC* low-pass filter.

Note that output voltage is measured at capacitor which is unlike RL circuit case where the output voltage was measure at resistor. The output voltage is just the voltage divider between resistor and impedance of capacitor. Impedance of capacitor is inversely proportional to frequency.

The transfer function can be easily found from the voltage divider:

$$Z_C = \frac{1}{sC}$$

$$H(s) = \frac{1/RC}{s + 1/RC}$$

We already develop the general forms of magnitude and phase function for this type:

$$a = \frac{1}{RC}\ and\ b = \omega$$

Magnitude Plot	$\lvert H(j\omega)\rvert = \dfrac{1/RC}{\sqrt{(1/RC)^2 + \omega^2}}$
Phase Plot	$\measuredangle z = \tan^{-1}(-\omega RC) = -\tan^{-1}(\omega RC)$

Cutoff Frequency	$\omega_c = \dfrac{1}{RC}$

i. Example 14.2 – Series RC filter

Just like for the low pass RL filter, we first set the capacitance and then look for resistance that yields the cutoff frequency. Let $C = 1\mu F$ then $R = 53.05\ ohm$ for $\omega_c = 2\pi(3kHz)$

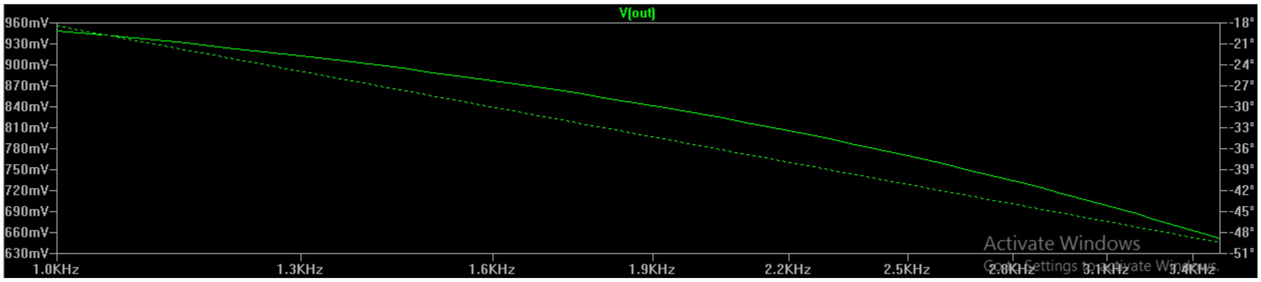


Figure. Frequency response of output voltage

$V_o = 0.70763\ V\ at\ 3.023kHz$

c. Conclusion

The conclusion of low pass filter:

$$H(s) = \frac{\omega_c}{s + \omega_c}$$

As you might notice, the cutoff frequency is highly related to the **time constant** if you recall RL and RC circuit for natural responses.

$$\tau = \frac{1}{\omega_c}$$

For further study, read section 13.6 in the textbook.

III. High Pass Filters

Let's look at RC and RL circuits first. They are the same circuits in the Low pass filter section. See how they can differ. RC high pass filter shall be examined first. Transfer functions for both series RL and RC take the same format except for their time constants.

a. RC Circuit

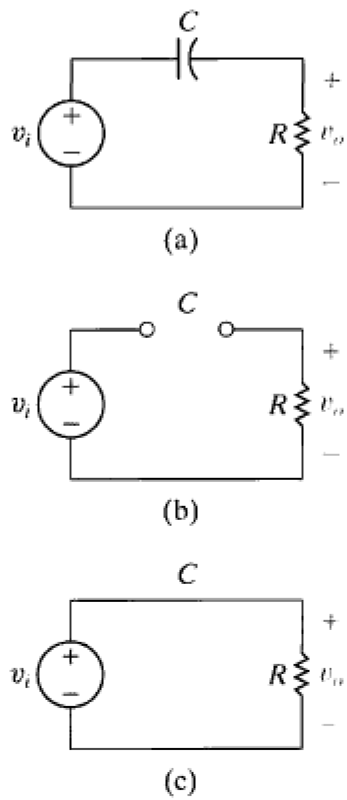


Figure 14.10 ▲ (a) A series *RC* high-pass filter; (b) the equivalent circuit at $\omega = 0$; and (c) the equivalent circuit at $\omega = \infty$.

Recall from the RC low pass filter configuration, the output voltage was measured for the capacitor voltage. For RC high pass filter, the output should be measured across the resistor.

Let $j\omega = jb$ and $\frac{1}{RC} = a$

$$H(j\omega) = \frac{jab + b^2}{a^2 + b^2}$$

Rewrite the equation in polar form:

$$H(j\omega) = \frac{1}{a^2 + b^2}\sqrt{b^4 + a^2b^2}\,\exp\left(jtan^{-1}\left(\frac{ab}{b^2}\right)\right)$$

$$= \frac{b}{\sqrt{a^2 + b^2}}\exp\left(jtan^{-1}\left(\frac{a}{b}\right)\right)$$

$$= \frac{b}{\sqrt{a^2 + b^2}}\exp[j\{\frac{\pi}{2} - \tan^{-1}\left(\frac{b}{a}\right)\}]$$

$$(\because \tan^{-1}(x) = \frac{\pi}{2} - \tan^{-1}\left(\frac{1}{x}\right))$$

$$H(j\omega) = \frac{\omega}{\sqrt{\left(\frac{1}{RC}\right)^2 + \omega^2}} \exp[j\{\frac{\pi}{2} - \tan^{-1}(\omega RC)\}]$$

Magnitude	$\frac{\omega}{\sqrt{\left(\frac{1}{RC}\right)^2 + \omega^2}}$
Phase	$\frac{\pi}{2} - \tan^{-1}(\omega RC)$

i. RC High pass filter example

Let's examine the same values used for RC low pass filter. They were $C = 1\mu F$ *and* $R = 53.05$ *ohm*. The cutoff frequency should occur around $3kHz$ since the **time constant** for both high- and low-pass filters should be the same(?) <<Is this the correct reason?>>

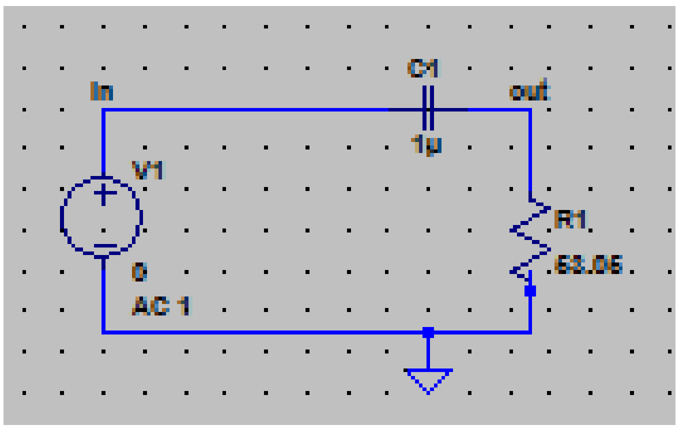


Figure. RC High pass filter circuit

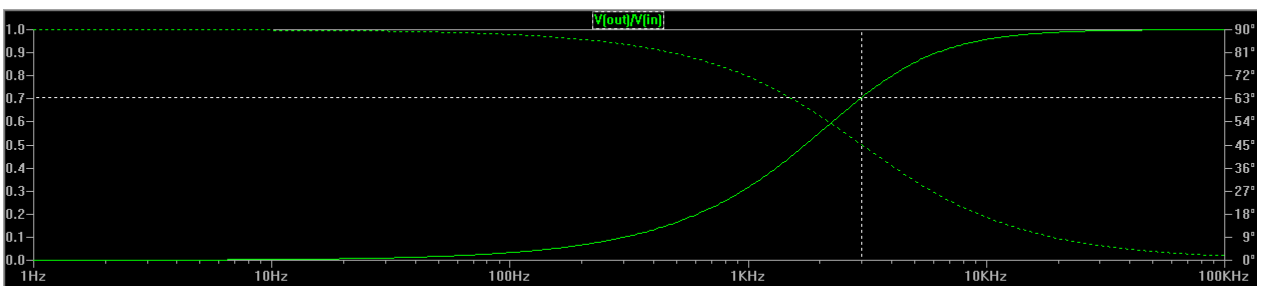


Figure. Frequency Response of the Output

$V_o = 0.7062\ V\ at\ f = 2.993kHz\ and\ the\ phase = +45.07°$

Or

$$at\ -2.999dB \rightarrow f = 3.008kHz$$

In general, the phase difference is 45 degrees at the cutoff frequency. Also note that there is phase difference between input and output. Phase difference occurs with frequency variation.

$$\omega = 0 \rightarrow \theta_{in} - \theta_{out} = +90°$$
$$\omega = \infty \rightarrow \theta_{in} - \theta_{out} = 0°$$

In other words, as frequency approaches infinity, the input waveform is the same as that of output.

b.RL Filters

Since the configuration has changed, the transfer function would look little different now. Applying the voltage division,

$$H(s) = \frac{s}{s + 1/RL}$$

$$H(j\omega) = \frac{j\omega}{j\omega + \frac{1}{RL}}$$

Let $j\omega = jb$ and $\frac{R}{L} = a$

$$H(j\omega) = \frac{jab + b^2}{a^2 + b^2}$$

Rewrite the equation in polar form:

$$H(j\omega) = \frac{1}{a^2 + b^2}\sqrt{b^4 + a^2b^2}\,\exp\left(jtan^{-1}\left(\frac{ab}{b^2}\right)\right)$$

$$= \frac{b}{\sqrt{a^2 + b^2}}\exp\left(jtan^{-1}\left(\frac{a}{b}\right)\right)$$

$$= \frac{b}{\sqrt{a^2 + b^2}}\exp[j\{\frac{\pi}{2} - \tan^{-1}\left(\frac{b}{a}\right)\}]$$

$$(\because \tan^{-1}(x) = \frac{\pi}{2} - \tan^{-1}\left(\frac{1}{x}\right))$$

$$H(j\omega) = \frac{\omega}{\sqrt{\left(\frac{R}{L}\right)^2 + \omega^2}}\exp[j\{\frac{\pi}{2} - \tan^{-1}\left(\frac{\omega L}{R}\right)\}]$$

i. Example 14.4 – Series RL with loaded and unloaded cases

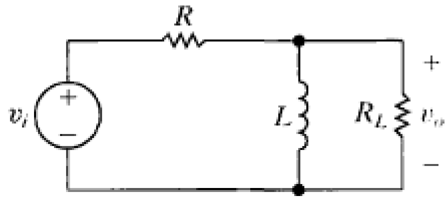


Figure 14.15 ▲ The circuit for Example 14.4.

First, let's derive the transfer function of this circuit. The derivation is rather simple.

$$H(s) = \frac{\frac{R_L}{R_L + R} s}{s + \frac{R_L}{R_L + R}\left(\frac{R}{L}\right)} = \frac{Ks}{s + \omega_c}$$

Where $K = \frac{R_L}{R_L + R}$ and $\omega_c = K\frac{R}{L}$

Note that K factor is always less than zero; hence, the loaded case's cutoff frequency and magnitude will always be less than the unloaded case. In this case, let's say we were given the $R = 500\ ohm$ and $\omega_c = 15kHz\ (unloaded)$

Importantly, we first compute the inductance from unloaded circuit.

$$\omega_{c,unloaded} = \frac{R}{L} \to L = 5.31mH$$

K factor can be any value between 0 and 1. If $K = \frac{1}{2}$, then $R_L = R = 500\ ohm$

Let's look at unloaded frequency response first.

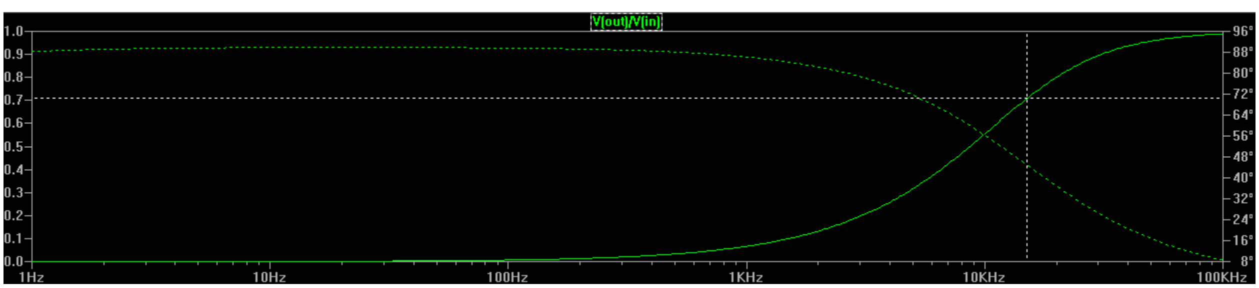


Figure. Unloaded Series RL filter Frequency Response

As intended, $V_o = 0.7082\ V\ at\ f = 15.04kHz$

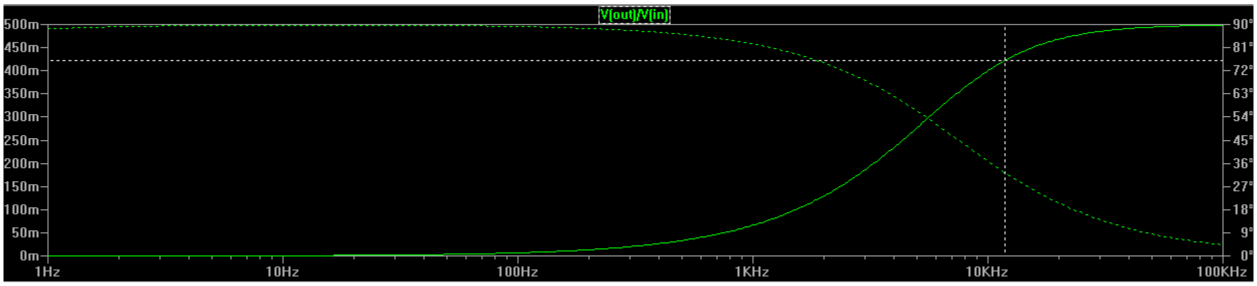


Figure. Loaded Series RL filter

Note that the maximum output voltage it can get is KV_{in} for the loaded case. The $-3dB$ point in this plot is $\frac{K}{\sqrt{2}} = \frac{1}{2\sqrt{2}} = 0.3535V_{in}\ at\ f = 7.53kHz$. Recall that loaded cutoff frequency is also less than the unloaded cutoff frequency and they are related by the following relation:

$$f_{c,loaded} = Kf_{c,unloaded}$$

As a matter fact, if you look at the transfer function, you may see all the relations at once.

Both plots are together:

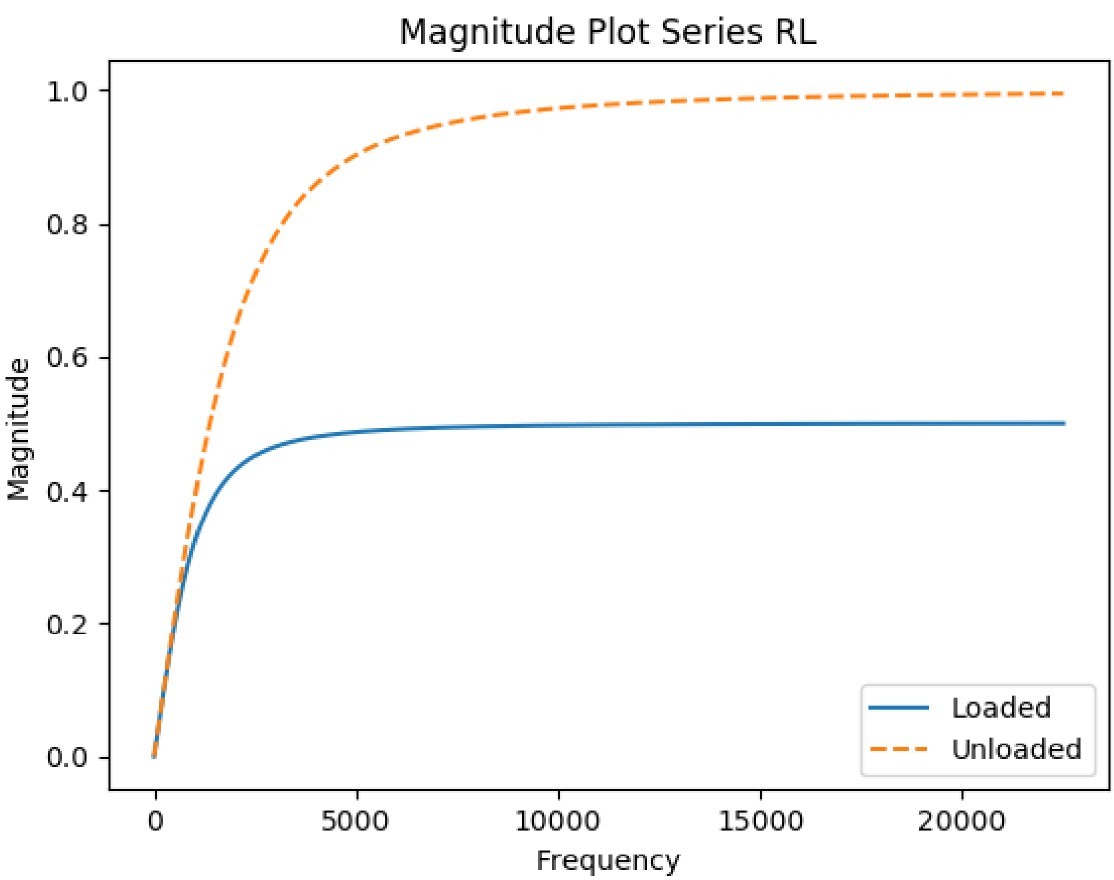


Figure. Loaded and Unloaded – Python plot

	Unloaded	Load

Transfer Function $H(s)$	$\frac{s}{s+\omega_c}$	$\frac{Ks}{s+K\omega_c}$
Magnitude	$\frac{\omega}{\sqrt{\omega^2+\omega_c^2}}$	$\frac{K\omega}{\sqrt{\omega^2+\omega_c^2}}$
Cutoff Frequency	$\omega_c = \frac{R}{L}$	$K\frac{R}{L}$

As it can be shown in the above table, important quantities of loaded case are in terms of those of unloaded one. Hence, it makes sense when it comes to analyze a series loaded RL filter, one should start with unloaded case first and find out the K factor which will complete the analysis. However, this fact poses a problem of passive filter design when the load is not specified.

IV. Bandpass Filters

a. Introduction to BPF

Which element should we use prior to the other when designing passive RLC (bandpass filter)? The answer is ***Capacitor***! Since the availability of capacitor is stricter than that of inductor in market.

The forcing function (input with some frequency) to the BPF can be divided with two cases: the first one is when the input source is ideal voltage source, that is when there is no series resistance connected with the source. The second one is when the input voltage source is non-ideal, that is there is a series resistance. This series resistance is non-ideal case that usually occurs when designing BPF. The effects of series resistance lead to (1) decrease in magnitude of transfer function and (2) increase in the bandwidth. Details are covered as it goes. The center frequ

Ideal bandpass filters have two cutoff frequencies. The definition of cutoff frequency is the same as those of low- and high-pass filters, the frequency yields $\frac{1}{\sqrt{2}}H_{max}$. Unlike low- and high-pass filters, Bandpass filters contain other important parameters:

(1) Center Frequency (a.k.a Resonant Frequency)
(2) Bandwidth
(3) Quality Factor

Definition of ***Center Frequency*** (ω_0)? It is ***the geometric mean*** of the two cutoff frequencies. $$\omega_o = \sqrt{\omega_{c1}\omega_{c2}}$$ This frequency is the same frequency that characterizes the natural response of second-order circuit! At the center frequency, the magnitude of transfer function is the maximum. Both series and parallel RLC share the same equation for the resonant frequency, $\omega_0 = \frac{1}{\sqrt{LC}}$ Meaning of circuits ***in resonance***?

When the driven frequency (the forcing frequency) is the same as the frequency of natural response of the circuit.

Bandwidth (β)?
The width of the passband.

Quality Factor (Q)?
The ratio of the center frequency to the bandwidth.

Strategy to design?

There are total 5 parameters, $\omega_{c1}, \omega_{c2}, \omega_0, \beta, and\ Q$. However, only two of them can be calculated independently. The other three can be calculated from the dependent relations which will be stated later.

b.Series RLC circuit

For bandpass filter, analysis begins with series RLC circuit:

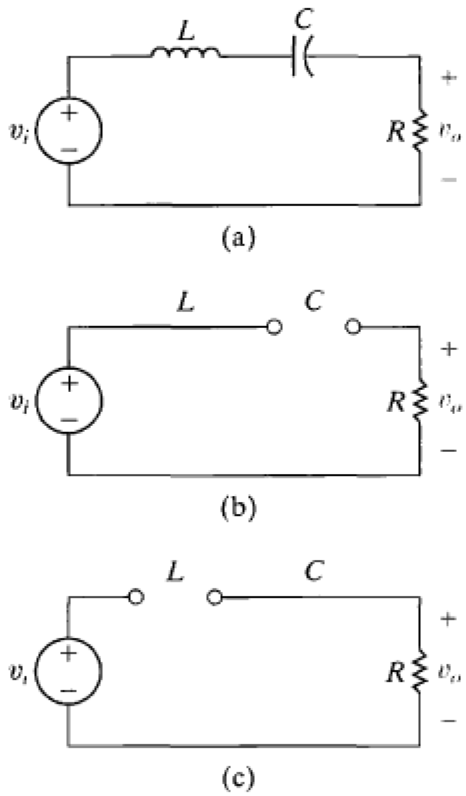


Figure 14.19 ▲ (a) A series *RLC* bandpass filter; (b) the equivalent circuit for $\omega = 0$; and (c) the equivalent circuit for $\omega = \infty$.

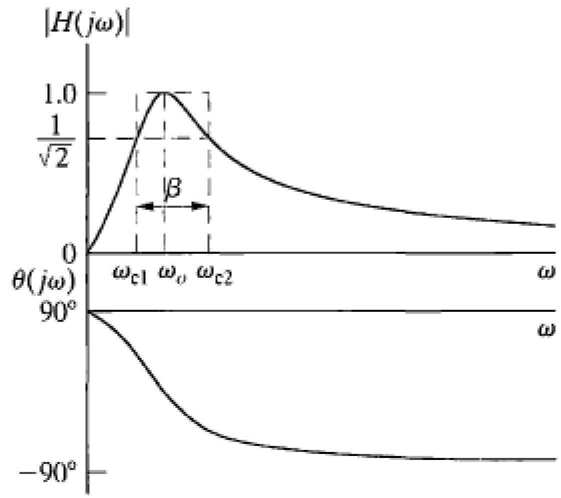


Figure 14.20 ▲ The frequency response plot for the series *RLC* bandpass filter circuit in Fig. 14.19.

The idea of resonance occurs with the following story. The impedance of capacitor is negative whereas that of inductor is positive. At some frequency, the two impedances cancel each other causing zero impedance; hence, output voltage across the resistor is equal to the input voltage. That frequency is called the resonant frequency. For phase concern, the output phase is the same as the input at the center frequency. However, as frequency decreases, capacitor impedance will dominate the inductor impedance. Recall that for capacitor, current leads voltage, which is positive phase shift; therefore, the net output phase is also positive. Figure 14.20 describes this with graphical representation. The opposite story goes on if frequency increases.

Let’s look at the s-domain transfer function:

$$H(s)=\frac{\frac{R}{L}}{s^2+\left(\frac{R}{L}\right)s+\left(\frac{1}{LC}\right)}$$

Magnitude	$\dfrac{\omega\left(\frac{R}{L}\right)}{\sqrt{\left(\frac{1}{LC}-\omega^2\right)+\left(\omega\left(\frac{R}{L}\right)\right)^2}}$
Phase	$\dfrac{\pi}{2}-\tan^{-1}\left[\dfrac{\frac{\omega R}{L}}{\left(\frac{1}{LC}\right)-\omega^2}\right]$

Finding the five parameters?

Center frequency? Recall the idea to find the center frequency. The two impedances cancel each other.

$$j\omega_0 L + \frac{1}{j\omega_0 C} = 0$$

Hence,

$$\omega_0 = \frac{1}{\sqrt{LC}}$$

Two cutoff frequencies (Series RLC)?

These frequencies can be found by equating the magnitude of transfer function to $\frac{1}{\sqrt{2}}$. The whole derivation is lengthy.

$$\omega_{c1} = -\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \left(\frac{1}{LC}\right)}$$

$$\omega_{c2} = \frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + (\frac{1}{LC})}$$

Bandwidth?

$$\beta = \omega_{c2} - \omega_{c1} = \frac{R}{L}$$

Quality Factor?

$$Q = \frac{\omega_0}{\beta} = \sqrt{\frac{L}{CR^2}}$$

i. Example 14.5 – Design BPF with RLC circuit

There are many ways for design for RLC BPF circuit. For this example, we'd like to pass $1 \sim 10kHz$ frequency range of input signals. For the design rule, we should select the capacitor value prior to inductor value as mentioned in the beginning. Here, let's choose $C = 1\mu F$. Then, let's start with the bandwidth.

$$\beta = 2\pi(10kHz - 1kHz) = 18\pi \times 10^3\ \frac{rad}{s}$$

$$f_0 = \sqrt{f_{c1} f_{c2}} = 3162.28\ Hz$$

$$Q = \frac{\omega_0}{\beta} = 0.3514$$

From the resonant frequency equation,

$$L = \frac{1}{\omega_0^2 C} = 2.533\ mH$$

Now, we can find the resistor value,

$$R = \sqrt{\frac{Q^2 C}{L}} = 143.24\ ohm$$

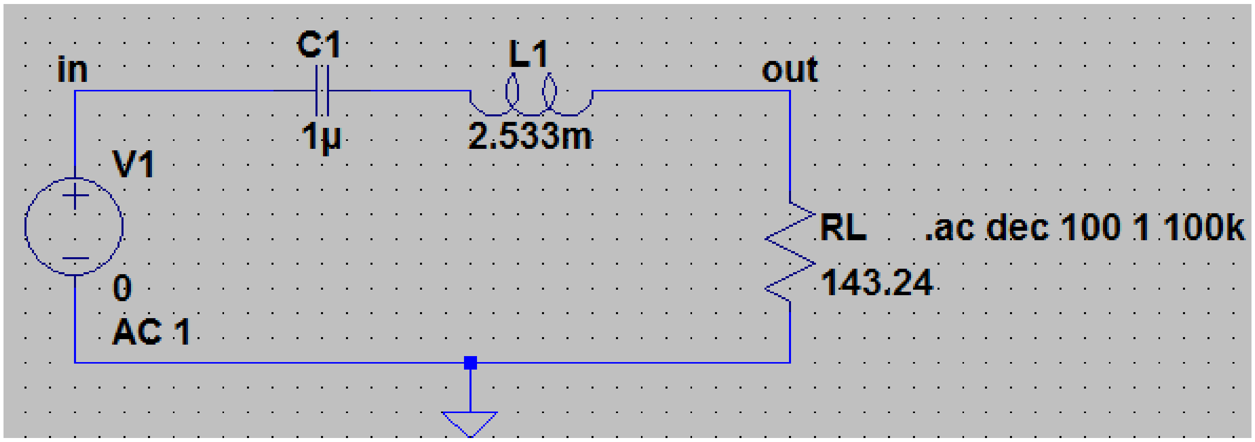


Figure. BPF RLC Circuit

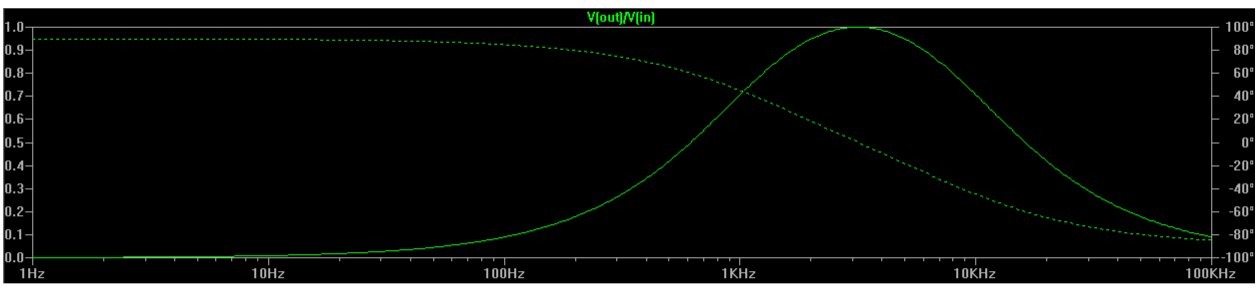


Figure. Transfer Function Plots (Both Mag. and Phase)

Simulation Results	$\|H(j\omega)\|$	f
f_{c1}	$705.397\ mV$	$996.102\ Hz$
f_{c2}	$707.96\ mV$	$9.980\ kHz$
f_0	$999.98\ mV$	$3.153\ kHz$

From the simulation, we confirm that the circuit provides the exact BPF properties that we want.

c. Parallel RLC BPF Circuit

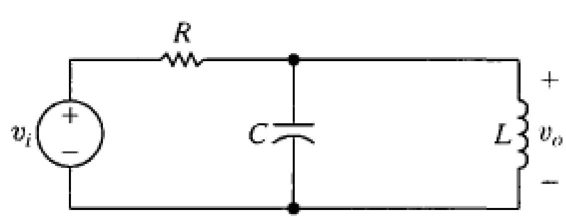


Figure 14.22 ▲ The circuit for Example 14.6.

Let's first find the transfer function in s-domain,

$$H(s) = \frac{\frac{s}{RC}}{s^2 + \frac{s}{RC} + \frac{1}{LC}}$$

Two cutoff frequencies (Parallel RLC)?

These frequencies can be found by equating the magnitude of transfer function to $\frac{1}{\sqrt{2}}$. The whole derivation is lengthy.

$$\omega_{c1} = -\frac{1}{2RC} + \sqrt{\left(\frac{1}{2RC}\right)^2 + \left(\frac{1}{LC}\right)}$$

$$\omega_{c2} = \frac{1}{2RC} + \sqrt{\left(\frac{1}{2RC}\right)^2 + (\frac{1}{LC})}$$

Bandwidth?

$$\beta = \omega_{c2} - \omega_{c1} = \frac{1}{RC}$$

Quality Factor?

$$Q = \frac{\omega_0}{\beta} = \sqrt{\frac{R^2C}{L}}$$

i. Example 14.6 – Parallel RLC BPF

Find $R\ and\ L$ given that $C = 5\mu F, f_0 = 5kHz, and\ \beta = 200Hz$

Simulation Results	$\lvert H(j\omega)\rvert$	f
f_{c1}	$708.36\ mV$	$4.8972\ kHz$
f_{c2}	$711.534\ mV$	$5.096\ kHz$
f_0	$979.33\ mV$	$5.010\ kHz$

From the simulation result, the bandwidth can be calculated (approximately):

$$\beta = 2\pi(5.096 - 4.8972)\times 1000 \frac{rad}{s} = 2\pi(198.7Hz) \approx 2\pi(200Hz)$$

Hence, the circuit behaves as we want.

Non-Ideal Voltage source?

Ideal voltage source model does not have a series resistance. The problem arose when the input voltage source is non-ideal voltage source, that is it contains a series resistance. This is a common situation whenever we hook up our power supply from the function generator sitting in lab. The function generator is modeled with a series resistance, which is $50\ ohm$.

ii. Example 14.7 – Effect of Series Resistance in Series RLC BPF

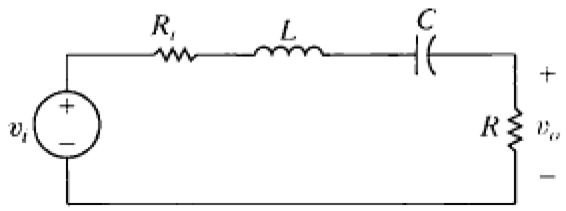


Figure 14.24 ▲ The circuit for Example 14.7.

Transfer function when the series resistance included:

$$H(s) = \frac{\frac{R}{L}s}{s^2 + \left(\frac{R + R_i}{L}\right)s + \frac{1}{LC}}$$

The magnitude of the transfer function:

$$|H(j\omega)| = \frac{\frac{R}{L}\omega}{\sqrt{\left(\frac{1}{LC} - \omega^2\right)^2 + \left(\omega\frac{R + R_i}{L}\right)^2}}$$

The maximum value of the magnitude of the transfer function:

$$H_{max} = |H(j\omega_0)| = \frac{R}{R + R_i}$$

The cutoff frequencies can be found with the following equations:

$$\omega_{c1} = -\frac{R + R_i}{2L} + \sqrt{\left(\frac{R + R_i}{2L}\right)^2 + \frac{1}{LC}}$$

$$\omega_{c2} = \frac{R + R_i}{2L} + \sqrt{\left(\frac{R + R_i}{2L}\right)^2 + \frac{1}{LC}}$$

The bandwidth:

$$\beta = \frac{R + R_i}{L}$$

The Quality Factor:

$$Q = \frac{\sqrt{\frac{L}{C}}}{R + R_i}$$

The center frequency stays the same and it is given:

$$\omega_0 = \frac{1}{\sqrt{LC}}$$

Now, we can express the transfer in more compact form:

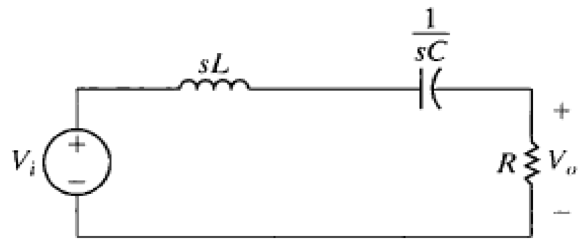
$$H(s) = \frac{K\beta s}{s^2 + \beta s + \omega_0^2}$$

Where $K = \frac{R}{R+R_i}$

If $R_i = 0$ and $K = 1$, then we obtain:

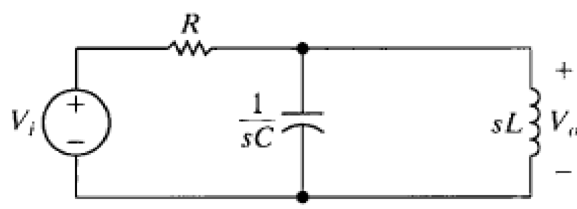
$$H(s) = \frac{\beta s}{s^2 + \beta s + \omega_0^2}$$

d.Quick Summary



$$H(s) = \frac{(R/L)s}{s^2 + (R/L)s + 1/LC}$$

$$\omega_o = \sqrt{1/LC} \qquad \beta = R/L$$



$$H(s) = \frac{s/RC}{s^2 + s/RC + 1/LC}$$

$$\omega_o = \sqrt{1/LC} \qquad \beta = 1/RC$$

Figure 14.27 ▲ Two *RLC* bandpass filters, together with equations for the transfer function, center frequency, and bandwidth of each.

V. References

[1] Electric Circuits, 9^{th} Edition