# Mathematical Spaces

# I. Introduction

Motivation: I need to get this thing straight. This topic frequently appears as I research more. A few import topics must be taken down before preceding reading the document. In addition, this document may lack mathematical rigors for proofs, etc. It may provide general ideas of items.

## a. Cauchy Sequences

It's a sequence whose elements become arbitrarily close to each other as the sequence progresses. [3] Look at the below figures and it would make more sense:

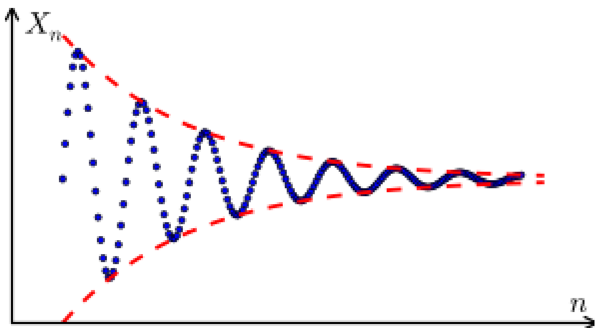


Figure. Example of Cauchy sequence [3]

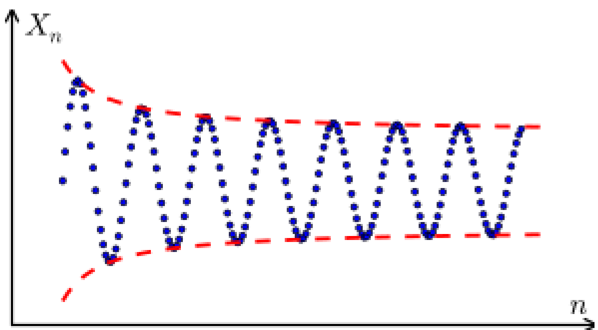


Figure. Example of non-Cauchy sequence [3]

***Completeness?***

A metric space $X$ in which every Cauchy sequences converges to an element of $X$ is called complete. [3]

Cauchy sequence requires not only each term getting close to the preceding term but also ***all terms getting close to each other***. This is the most important part to understand. (e.g. harmonic series does not converge!) In mathematics [3],

$$for\ given\ \varepsilon > 0\ (an\ arbitrarily\ small\ value), \exists\ an\ N\ such\ that\ for\ any\ pair\ m, n > N, we\ have\ |a_m - a_n| < \varepsilon\ (whereas\ |a_{n+1} - a_n| < \varepsilon\ is\ not\ sufficient!)$$

# b. Complex Coordinate Space [4]

$$\mathbb{C}^n = \{(z_1, \ldots, z_n) | z_i \in \mathbb{C}\}$$

or

$$\mathbb{C}^n = \underbrace{\mathbb{C} \times \mathbb{C} \times \cdots \times \mathbb{C}}_{n}.$$

Figure. Symbolical notation complex coordinate space

The n-dimensional complex coordinate space is the set of all ordered n-tuples of complex numbers. Or, it is the n-fold Cartesian product of complex plane $\mathbb{C}$ with itself. <<See above figure.>>

Tuple? [5]
It’s a finite ordered list (sequence) of elements. Empty sequence means “zero” tuple.
For example,

$$(3,5,4,1)$$

It contains 4 tuples.

Note that complex coordinate space is a vector space over complex numbers. The real and imaginary parts of the coordinates set up a bijection of $\mathbb{C}^n$ with the real coordinate space $\mathbb{R}^{2n}$.

# c. Lebesgue Measure [6]

The standard way of assigning a measure to subsets of n-dimensional Euclidean space. It is important because it is used to define Lebesgue integration.

Lebesgue Measurable?

Sets that can be assigned to a Lebesgue measure are called “*Lebesgue measurable*”.

The measure of Lebesgue measurable set A is denoted by:

$$\lambda(A)$$

# d. Lebesgue Integration [7]

We know area and integration is highly related to each other. Lebesgue integration is an extension of integral. It considers not only integration on the real line but also on ***spaces***. This concept is especially important in probability theory.

# II. Spaces

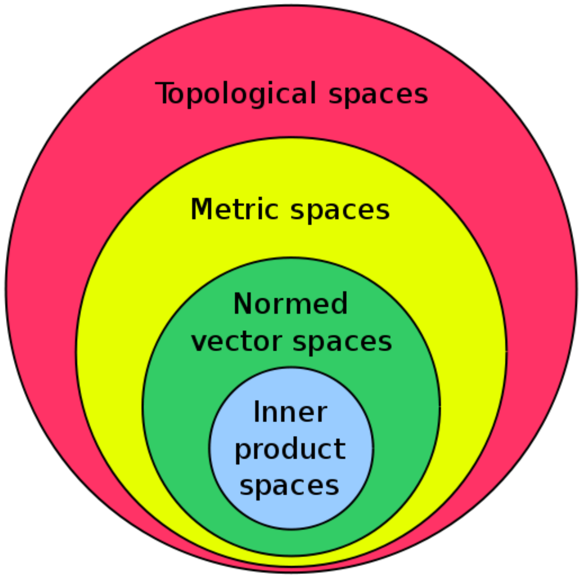


Figure. A hierarchy of mathematical spaces [2]

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| Linear spaces vs. Topological spaces?<br><br>Background knowledge: open interval does not contain the end points whereas closed interval does.<br><br>(1) Linear spaces: algebraic nature. Two linear spaces over the same filed are isomorphic iff they are of the same dimension. Dimension of a linear space can be defined by the maximal number of linearly independent vectors (, or equivalently, minimal number of vectors that span the space). Real number is also a complex number (hence every complex linear space is also real linear space).<br>(2) Topological spaces: analytic nature. Isomorphism between topological spaces is called "homeomorphism" (one-to-one correspondence continuous in both directions). E.g. open interval (0,1) is homeomorphic to the whole real line $(-\infty, \infty)$ but not homeomorphic to the closed interval [0,1], nor to a circle.<br>Dimension of a topological space is hard to define: "inductive dimension" and "Lebesque covering dimension" are used to define.<br>Closed intervals are compact; open intervals are not.<br>(3) Manifolds? These are topological spaces locally homeomorphic to Euclidean spaces. |

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| Affine vs. Projective spaces? |

Background knowledge: $n$ dimensional linear subspace of a $(n+1)$ dimensional linear subspace is not homogenous for it contains the origin. In this sense, affine spaces are homogenous. <<but shouldn’t affine spaces be called “not homogenous”? since it does not contain the origin which means they system does not have the trivial solution for the homogenous matrix equation. I’m confused here. And affine spaces are not vector space since it does not contain the origin.>>

(1) Affine: John Baez said “an affine space is a vector space that’s forgotten its origin”.
(2) Projective spaces: It sounds little confusing but here it goes. A set of all one dimensional subspace (line through the origin), that are parallel to the affine space, of $(n+1)$ dimensional linear space is called projective space.

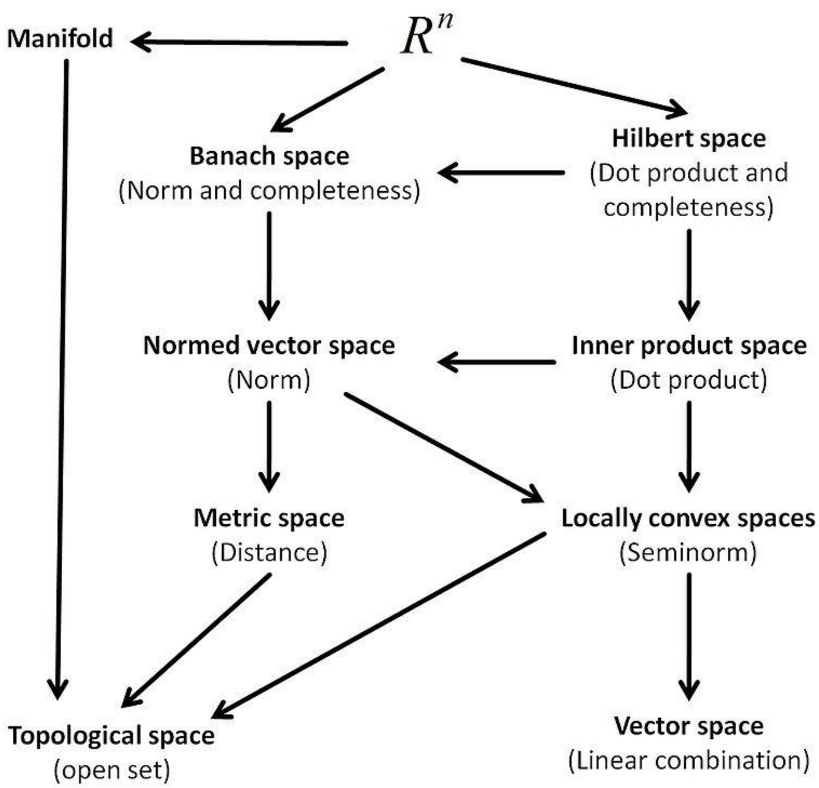


Figure. Types of abstract spaces [2]

“The arrow from A to B refers to space A is kind of space B, for example, normed vector space is also a metric space.”

Metric vs. Uniform spaces

(1) Metric space: distances between points are defined in a metric space. Metric space is also a topological space. Isomorphisms between metric spaces are called isometries.

Bounded sets and Cauchy sequences are defined in metric space. <<I don’t know what they are.>>

Every compact metric space is ***complete***; the real line is non-compact but complete, the open interval $(0,1)$ incomplete. (why is it incomplete?)

Every Euclidean space is also a ***complete*** metric space.

(2) Uniform spaces: Uniform spaces do not introduce distances, but still allow to use uniform continuity, Cauchy sequences, completeness and completion.

Normed, Banach, inner product, and Hilbert spaces

(1) Normed space: A linear (real or complex) space endowed with a norm is a normed space. The parallelogram law fails in normed space but holds for vectors in Euclidean space. The parallelogram law: $\|x - y\|^2 + \|x + y\|^2 = 2\|x\|^2 + 2\|y\|^2$

(2) Banach space: A banach space is a ***complete*** normed space.

(3) Inner product space: It is a linear (real or complex) space endowed with a bilinear form satisfying some conditions and called inner product. Every inner product space is also a normed space. A normed space underlies an inner product space iff the parallelogram law does not fail.

(4) Hilbert space: Hilbert space is defined as a ***complete*** inner product space.

# III. References


[1] https://en.wikipedia.org/wiki/Inner_product_space

[2] https://en.wikipedia.org/wiki/Space_(mathematics)

[3] https://en.wikipedia.org/wiki/Cauchy_sequence

[4] https://en.wikipedia.org/wiki/Complex_coordinate_space

[5] https://en.wikipedia.org/wiki/Tuple

[6] https://en.wikipedia.org/wiki/Lebesgue_measure

[7] https://en.wikipedia.org/wiki/Lebesgue_integration