Multi-section Matching Transformers

I. Introduction

Recall from a single unit of quarter-wave transformer, one can easily design a matching network when the load is purely real. However, single unit may not yield a desired bandwidth, or etc. Considering multiple units (commensurate) of the same kind of transformer, one may obtain a better response (i.e. attenuation) and a better bandwidth (i.e. wider).

II. Quarter-wavelength Transformer

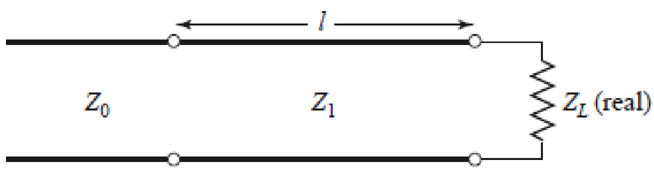

FIGURE 5.10 A single-section quarter-wave matching transformer. $\ell = \lambda_0/4$ at the design frequency f_0.

$$Z_1 = \sqrt{Z_0 Z_L}$$

We may find the input impedance as

$$Z_{in} = Z_1 \frac{Z_L + jZ_1 \tan(\beta l)}{Z_1 + jZ_L \tan(\beta l)}$$

where

$$\beta l = {}^{\pi}\!/_{2} \ at\ f_0$$

The reflection coefficient can be found from

$$\Gamma_{in} = \frac{Z_{in} - Z_0}{Z_{in} + Z_0} = \frac{Z_L - Z_0}{Z_L + Z_0 + j2\tan(\beta l)\sqrt{Z_0 Z_L}}$$

The magnitude of this reflection is expressed as

$$|\Gamma_{in}| = \frac{1}{\left[1 + \left\{{}^{4Z_L Z_0}\!/_{(Z_L - Z_0)^2}\right\} \sec^2\theta\right]^{0.5}}$$

The passband response plot looks like the following:

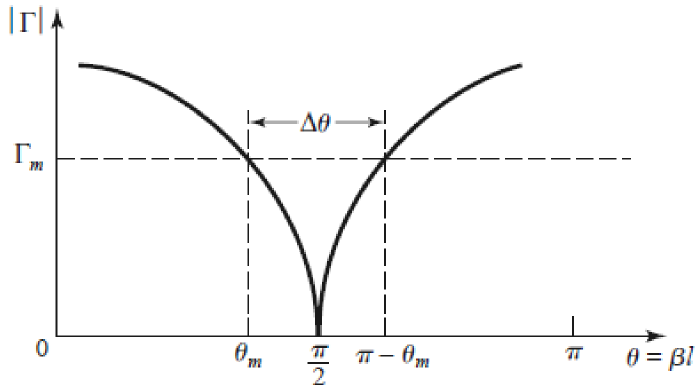

FIGURE 5.11 Approximate behavior of the reflection coefficient magnitude for a single-section quarter-wave transformer operating near its design frequency.

If we consider the maximum reflection Γ_m at θ_m:

$$\frac{1}{|\Gamma_m|^2} = 1 + \left\{ {}^{4Z_L Z_0}\big/_{(Z_L - Z_0)^2} \right\} \sec^2 \theta_m$$

$$= 1 + \left(\frac{2\sqrt{Z_L Z_0}}{(Z_L - Z_0)\cos\theta_m} \right)^2$$

$$\rightarrow \sqrt{\frac{1 - |\Gamma_m|^2}{|\Gamma_m|^2}} = \frac{2\sqrt{Z_L Z_0}}{(Z_L - Z_0)\cos\theta_m}$$

$$\cos\theta_m = \frac{|\Gamma_m|}{\sqrt{1 - |\Gamma_m|^2}} \frac{2\sqrt{Z_L Z_0}}{(Z_L - Z_0)}$$

$$\boxed{\theta_m = \cos^{-1}\left(\frac{|\Gamma_m|}{\sqrt{1 - |\Gamma_m|^2}} \frac{2\sqrt{Z_L Z_0}}{(Z_L - Z_0)} \right)}$$

Now, we assume that the transmission line propagates in TEM mode which leads us to find the electrical length expression (the length of line is quarter-wavelength!):

$$\boxed{\theta = \beta l = \frac{2\pi f}{v_p} \frac{v_p}{4 f_0} = \frac{\pi f}{2 f_0}}$$

From the above plot, if we call f_m at θ_m:

$$f_m = \frac{2\theta_m f_0}{\pi}$$

The fractional bandwidth is defined as [4]

$$\frac{\Delta f}{f_0} = \frac{2(f_0 - f_m)}{f_0} = 2 - \frac{4\theta_m}{\pi}$$

$$= 2 - \frac{4}{\pi}\cos^{-1}\left(\frac{|\Gamma_m|}{\sqrt{1-|\Gamma_m|^2}}\frac{2\sqrt{Z_L Z_0}}{(Z_L - Z_0)}\right)$$

The fractional bandwidth is usually expressed a percentage:

$$100\frac{\Delta f}{f_0} \rightarrow [\%]$$

In conclusion, the above expressions are all valid if transmission line is TEM mode line. This means that if lines are non-TEM mode line (i.e. waveguides), then propagation constant $\gamma = \alpha + j\beta$ is no longer linear function of frequency (i.e. wave impedance will be dependent on frequency) and expressions are not valid anymore.

In addition, discontinuities between step changes (i.e. step changes between each section of transformer when multi-section transformer is considered which will be discussed in the next section of this document) would yield reactances that could affect the result; however, this point can also be adjusted by making a small change in the length (e.g. playing around with length a little).

a. Example

Let’s try an exercise. Here’s just random component values for quarter-wavelength transformer.

$$Z_s = 50\ ohm$$

$$Z_L = 100\ ohm$$

$$f_0 = 300\ MHz$$

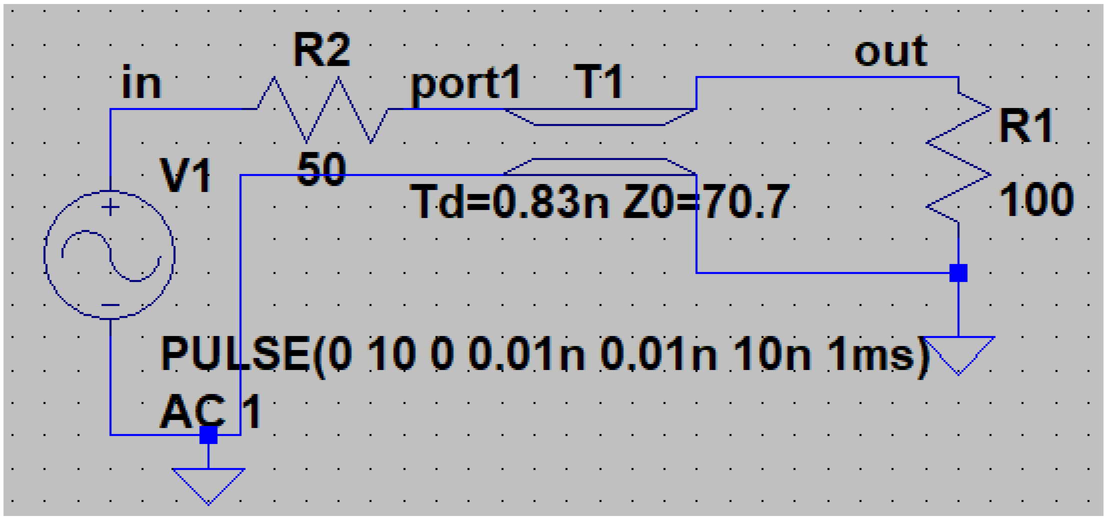

Figure. Quarter-wave Transformer

I assumed that the propagation time is just the speed of light since I didn't specify the information of the medium on LTspice. The variable of "tline", T_d, is the time delay due to that transmission line. This time variable can be easily found by the simple relation between the distance and speed which is shown in the circuit already.

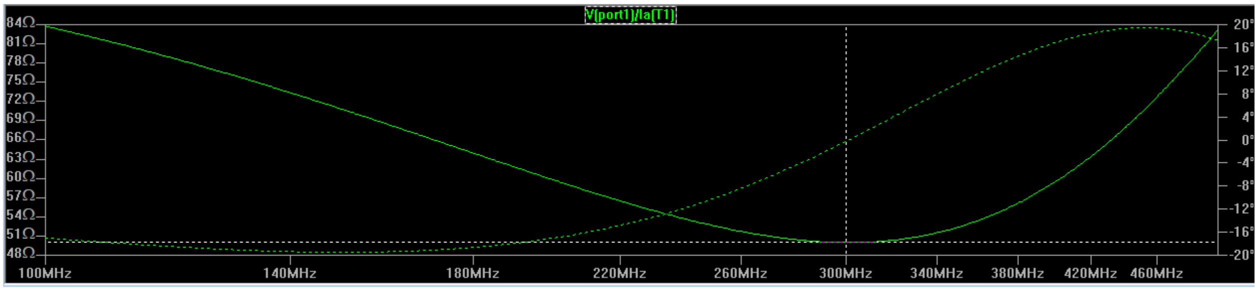

Figure. Input Impedance plot

From the plot,

$$Z_{in}|_{300.1MHz} = 49.99\ ohm$$

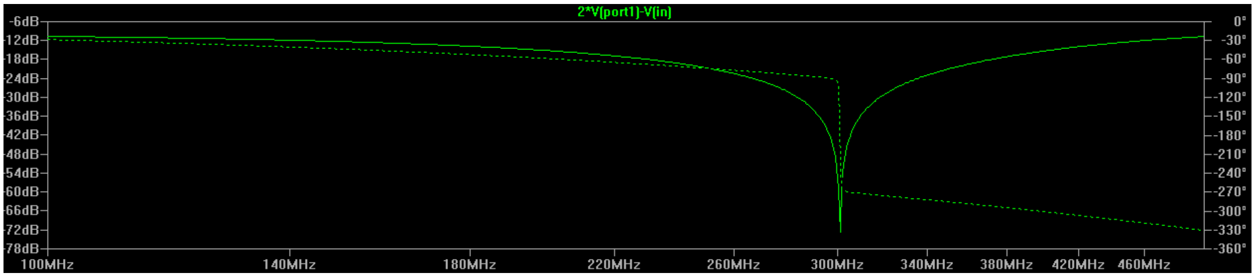

Figure. Reflection Coefficient

If zooming in closely, then one may find out that

$$|\Gamma|_{301.32MHz} = -71.56\ dB$$

$$|\Gamma|_{300.01MHz} = -53.14\ dB$$

Frankly, the circuit is rather matched at $301.32MHz$ than at $300.01MHz$.

For phase,

$$\angle\Gamma|_{299.79MHz} = -92.96°$$

$$\angle\Gamma|_{302.06MHz} = -264.6°$$

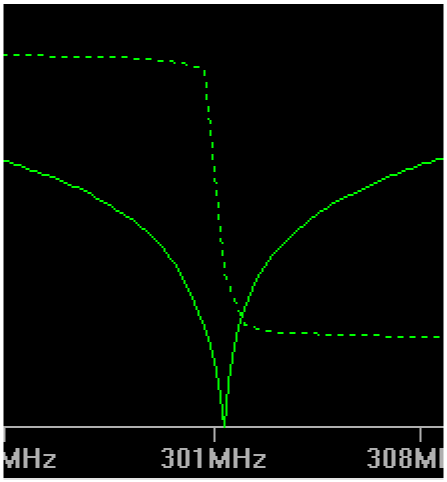

Figure. Reflection coefficient (both magnitude and phase) around 300 MHz

However, if I measured the transient response, I get unexpected result.

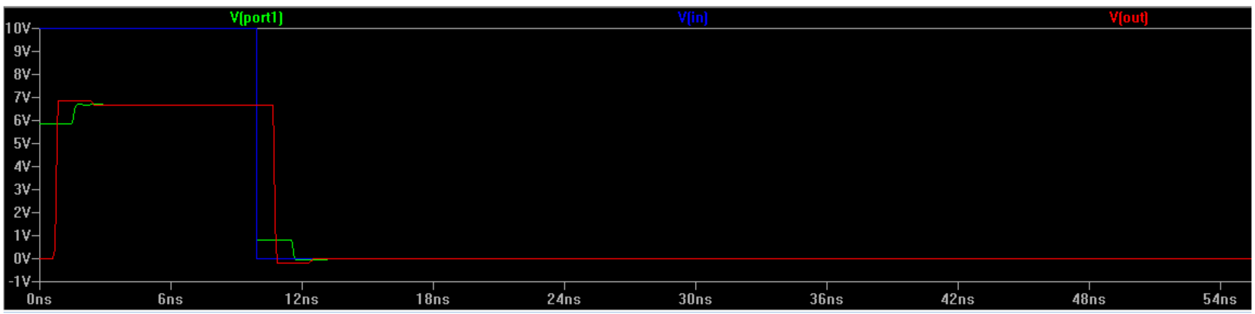

Figure. Transient Response

The input pulse's length is 10 ns and its magnitude is 10 V. Both voltages at "port 1" and at the load is around 6.67 V. There's been around 33% attenuation of the original signal which baffles me. Let me find out where this attenuation has come from.

III. Theory of Small Reflections

I had no idea why the professor emphasized that the bounce diagram is important when I first took the electromagnetic course a few years ago. I still remember that I said to myself the bounce diagram could be one of the useless theory I had even seen. Now, please forgive the past of me and appreciate the beauty of this theory.

The small reflection theory starts with considering multi-sections of transformer for larger bandwidth purpose. It would have been nice if wave just propagates through multi-section transformer regardless of the number of sections; however, there would be partial reflections due to discontinuities generated each section of transformer. Luckily, there's a model to account for all the small partial reflections in each unit and the overall response while accounting for all the partial reflections.

For developing some terminologies, we should go over a single-section transformer first.

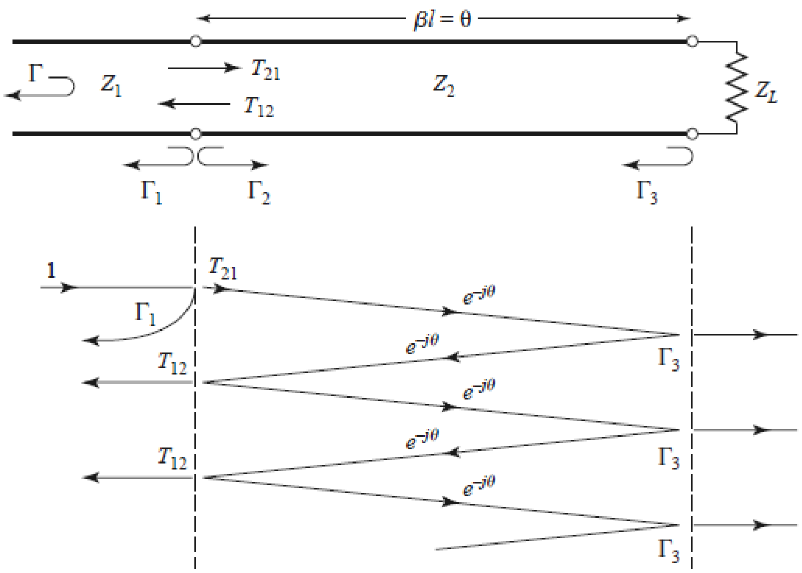

FIGURE 5.13 Partial reflections and transmissions on a single-section matching transformer.

The general equation for a reflection:

$$\Gamma = \frac{Z_L - Z_0}{Z_L + Z_0}$$

Let us find all the reflections shown in the above figure.

$$\Gamma_1 = \frac{Z_2 - Z_1}{Z_2 + Z_1}$$

Note that Γ_2 is facing the opposite direction than Γ_1. One can easily view this as a wave coming from Z_2 side and seeing Z_1 as the load. Because of differences in impedances, reflections occur.

$$\Gamma_2 = \frac{Z_1 - Z_2}{Z_1 + Z_2}$$

We may recognize a simple relation developed between the two reflections:

$$\Gamma_2 = -\Gamma_1$$

Finally,

$$\Gamma_3 = \frac{Z_L - Z_2}{Z_L + Z_2}$$

In telecommunication theory, the reflection and transmission coefficients are related in the following way:

$$T_{21} = 1 + \Gamma_1 = \frac{2Z_2}{Z_2 + Z_1}$$

$$T_{12} = 1 + \Gamma_2 = \frac{2Z_1}{Z_1 + Z_2}$$

Now, let's account for all the small reflections in a single transformer. So, how should we look at this? The overall picture may become easy if you keep in mind that the wave is bouncing back and forth indefinitely and we measure only the reflected ones. The bouncing diagram helps developing the reflection equation. If you are still not sure how to see the diagram, just simply follow each term I wrote in the following equation:

$$\begin{aligned}\Gamma(\theta) = \Gamma_1 &+ T_{21}e^{-j\theta}\Gamma_3 e^{-j\theta}T_{12} + T_{21}e^{-j\theta}\Gamma_3 e^{-j\theta}\Gamma_2 e^{-j\theta}\Gamma_3 e^{-j\theta} \\ &+ T_{21}e^{-j\theta}\Gamma_3 e^{-j\theta}\Gamma_2 e^{-j\theta}\Gamma_3 e^{-j\theta}\Gamma_2 e^{-j\theta}\Gamma_3 e^{-j\theta}T_{12} + \cdots + T_{21}T_{12}\Gamma_3^N\Gamma_2^N e^{-j2N\theta}\end{aligned}$$

$$= \Gamma_1 + T_{21}T_{12}\Gamma_3 e^{-j2\theta}\sum_{n=0}^{\infty}\Gamma_3^N\Gamma_2^N e^{-j2N\theta}$$

$$= \Gamma_1 + T_{21}T_{12}\Gamma_3 e^{-j2\theta}\sum_{n=0}^{\infty}\left(\Gamma_3\Gamma_2 e^{-j2\theta}\right)^N$$

Hence, if we consider the geometric series,

$$\Gamma(\theta) = \Gamma_1 + T_{21}T_{12}\Gamma_3 e^{-j2\theta}\frac{1}{1 - \Gamma_3\Gamma_2 e^{-j2\theta}}$$

$$= \Gamma_1 + (1 - \Gamma_1^2)\Gamma_3 e^{-j2\theta}\frac{1}{1 + \Gamma_3\Gamma_1 e^{-j2\theta}}$$

$$\Gamma(\theta) = \frac{\Gamma_1 + \Gamma_3 e^{-j2\theta}}{1 + \Gamma_3\Gamma_1 e^{-j2\theta}}$$

The above result is important because we found a way to express the total reflection in terms of the length of line, and the reflections at each junction.

As you might sense, $e^{-j2\theta}$ term account for the phase delay. We've looked at the total reflection expression in a single unit of transformer. This idea will become the building block of multi-section transformer.

In addition, we could approximate the total reflection if the discontinuities between consecutive units are small. (i.e. $|\Gamma_3 \Gamma_1| = 1$)

$$\boxed{\Gamma(\theta) \approx \Gamma_1 + \Gamma_3 e^{-j2\theta}}$$

a. Multi-section Transformer

Now that we are good to jump on the topic of the multi-section transformer. One should note that the lengths of each unit are the same, or ***commensurate***.

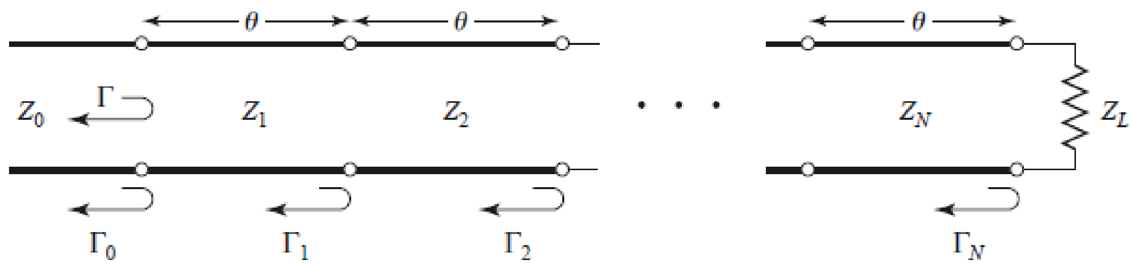

FIGURE 5.14 Partial reflection coefficients for a multisection matching transformer.

Before trying to find the total reflection of this commensurate line, we need the following assumption:

(1) As N increases, Z_N either increases or decreases monotonically and they are all real.
(2) Symmetric, meaning that $\Gamma_0 = \Gamma_N$, $\Gamma_1 = \Gamma_{N-1}$, and so on (i.e. $\Gamma_n = \Gamma_{N-n}$). However, Z_n's are not necessarily symmetrical.

<<In the previous section, we didn't account for the transmitted wave after the load (, or absorbed one (if you want to be precise) at the load because the load is terminated for the single unit!) One may use the equation

$$\Gamma(\theta) \approx \Gamma_1 + \Gamma_3 e^{-j2\theta}$$

and expand idea from it to get the following approximation.>>

$$\Gamma(\theta) \approx \Gamma_0 + \Gamma_1 e^{-j2\theta} + \Gamma_2 e^{-j4\theta} + \cdots + \Gamma_N e^{-j2N\theta}$$

Using the assumption that the commensurate line reflections are symmetric:

$$= \Gamma_0\left(1 + e^{-j2N\theta}\right) + \Gamma_1\left(e^{-j2\theta} + e^{-j2(N-1)\theta}\right) + \Gamma_2\left(e^{-j4\theta} + e^{-j2(N-2)\theta}\right) + \cdots$$

$$= e^{-jN\theta}\Big[\Gamma_0\left(e^{+jN\theta} + e^{-jN\theta}\right) + \Gamma_1\left(e^{+j(N-2)\theta} + e^{-j(N-2)\theta}\right) + \Gamma_2\left(e^{+j(N-4)\theta} + e^{-j(N-4)\theta}\right) + \cdots + \Gamma_n\left(e^{+j(N-2n)\theta} + e^{-j(N-2n)\theta}\right) + \cdots\Big]$$

We need to consider two cases: (1) $N = odd$ (2) $N = even$

Because the term located at the very center is $\frac{N+1}{2}$

(1) $N = odd$

$$= e^{-jN\theta}\left[\Gamma_0\cos(N\theta) + \Gamma_1\cos\big((N-2)\theta\big) + \cdots + \Gamma_n\cos\big((N-2n)\theta\big) + \cdots + \Gamma_{\frac{N}{2}}\right]$$

(2) $N = even$

$$= e^{-jN\theta}\left[\Gamma_0\cos(N\theta) + \Gamma_1\cos\big((N-2)\theta\big) + \cdots + \Gamma_n\cos\big((N-2n)\theta\big) + \cdots + \Gamma_{\frac{(N-1)}{2}}\cos(\theta)\right]$$

Now, we are going to investigate two famous passband responses: (1) binomial (maximally-flat) (2) Chebyshev (equal-ripple) response

IV. Binomial Multi-Sections Matching Transformers

Binomial responses can be used to analyze the passband response which reveals an optimum response in which the response is as flat as possible in each number of sections of transformer. This type of response is also known as maximally-flat response. From the small reflection model,

$$\Gamma(\theta) = A\left(1 + e^{-j2\theta}\right)^N$$

<<This has been found by equating the first $N-1$ derivatives of the total reflection to zeros; just like you do in Taylor series.>>

Since what we need is the magnitude response,

$$|\Gamma(\theta)| = |A|\left|e^{-j\theta}\right|^N\left|e^{+j\theta} + e^{-j\theta}\right|^N$$

$$= |A|\left|e^{-j\theta}\right|^N|2\cos\theta|^N$$

$$= 2^N|A||\cos\theta|^N$$

There are two distinguishable characteristics of this function:

(1) There's a point (or multiple points in the entire domain but we only consider finite size domain for practical reason (i.e. matching network for finite bandwidth)) where the function becomes zero.

$$\Gamma(\theta)|_{\theta={\pi}/{2}} = 0$$

This binomial function is maximally flat at $\theta = {}^{\pi}/_{2}$ where $\theta = {}^{\pi}/_{2}$ represents the center frequency f_0. Recall that $\theta = {}^{\pi}/_{2} = \beta l$
Hence, the length should be

$$l = {}^{\lambda}/_{4}$$

How beautiful is this!

(2) For the order of function N, $N - 1$ derivatives are all zero.

$$\frac{d^n \Gamma(\theta)}{d\theta^n} = 0$$

where $n = 1,2, \cdots, N - 1$

a. Finding the Constant

It's easy to find the constant in the binomial function by letting the argument of the function to zero:

$$\Gamma(0) = 2^N A$$

Since the electrical lengths are zeros,

$$2^N A = \frac{Z_L - Z_0}{Z_L + Z_0}$$

$$A = 2^{-N} \frac{Z_L - Z_0}{Z_L + Z_0}$$

b. Binomial Expansion

$$\Gamma(\theta) = A\left(1 + e^{-j2\theta}\right)^N$$

We see this in the form of a binomial series and the binomial expansion is given as [2]:

$$(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^n$$

where

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

Therefore, the binomial function can be re-written in the following form:

$$\Gamma(\theta) = A \sum_{n=0}^{N} \binom{N}{n} e^{-j2n\theta}$$

Now, note the following important characteristic of the binomial coefficient:

$$\binom{N}{n} = \binom{N}{N-n}$$

In addition to that, we've seen the above series before when we approximated the total reflection from the bouncing diagram for multi-section transformer. This is why one of the reasons I love engineering.

Recall the approximated total reflection:

$$\Gamma(\theta) \approx \Gamma_0 + \Gamma_1 e^{-j2\theta} + \Gamma_2 e^{-j4\theta} + \cdots + \Gamma_N e^{-j2N\theta}$$

With this being held, if we could find the relation between the two different coefficient expressions, we may design the transformer just as we expect:

$$\Gamma_n = A \binom{N}{n}$$

c. Finding the General Characteristic Impedance Z_n

For any n, the nth reflection coefficient can be expressed as the following:

$$\Gamma_n = \frac{Z_{n+1} - Z_n}{Z_{n+1} + Z_n}$$

$$\approx \frac{1}{2} \ln\left(\frac{Z_{n+1}}{Z_n}\right)$$

where we used the following approximating relation (you might have seen this from rudimentary calculus course):

$$\ln(x) \approx 2\frac{x-1}{x+1}$$

This relation holds whenever the variable x is small and indeed $\frac{Z_{n+1}}{Z_n}$ is small because Γ_n is small.

Recall that

$$\Gamma_n = A \binom{N}{n}$$

$$\ln\left(\frac{Z_{n+1}}{Z_n}\right) \approx 2\Gamma_n = 2A\binom{N}{n} = 2\left(2^{-N}\frac{Z_L - Z_0}{Z_L + Z_0}\right)\binom{N}{n}$$

$$\approx 2\left(2^{-N}\frac{1}{2}\ln\left(\frac{Z_L}{Z_0}\right)\right)\binom{N}{n}$$

$$= 2^{-N}\ln\left(\frac{Z_L}{Z_0}\right)\binom{N}{n}$$

$$\boxed{\ln\left(\frac{Z_{n+1}}{Z_n}\right) \approx 2^{-N}\ln\left(\frac{Z_L}{Z_0}\right)\binom{N}{n}}$$

Recall that we assumed that the characteristic impedance of each section either monotonically increases or decreases. This means that we need to consider two cases:

(1) $\frac{Z_L}{Z_0} > 1$
(2) $\frac{Z_L}{Z_0} < 1$

d.Bandwidth of the Binomial Transformer

Recall the magnitude of total reflection was found as

$$\Gamma_N = 2^N|A||\cos\theta|^N$$

Now, we let the maximum value of this magnitude to

$$\Gamma_m$$

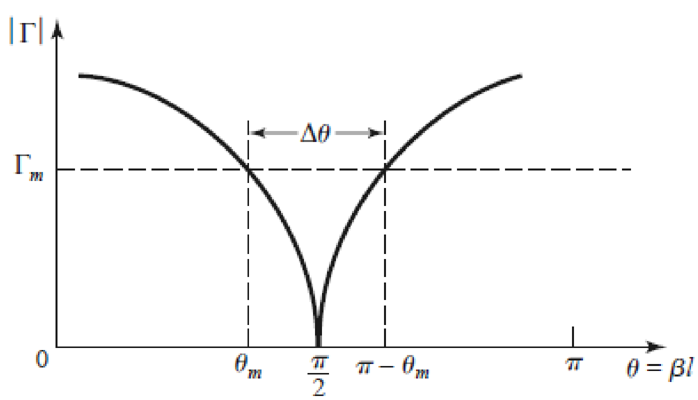

FIGURE 5.11 Approximate behavior of the reflection coefficient magnitude for a single-section quarter-wave transformer operating near its design frequency.

The angle θ_m can be easily found by (for finding θ_m purpose, we may consider the interval $\left[0,\frac{\pi}{2}\right]$ and let remove that the absolute value.)

$$\Gamma_m = 2^N|A|\cos^N\theta_m$$

$$\theta_m = \cos^{-1}\left(\frac{\Gamma_m}{2^N|A|}\right)^{1/N} = \cos^{-1}\left[\frac{1}{2}\left(\frac{\Gamma_m}{|A|}\right)^{1/N}\right]$$

By setting the maximum value of Γ_m, we may define the bandwidth in the following way:

$$\Delta\theta = 2\left(\frac{\pi}{2} - \theta_m\right)$$

The fractional bandwidth is defined as

$$\frac{\Delta f}{f_0} = \frac{2(f_0 - f_m)}{f_0} = 2 - \frac{4\theta_m}{\pi}$$

$$\boxed{\frac{\Delta f}{f_0} = 2 - \frac{4}{\pi}\cos^{-1}\left[\frac{1}{2}\left(\frac{\Gamma_m}{|A|}\right)^{1/N}\right]}$$

where

$$\frac{f_m}{f_0} = \frac{2\theta_m}{\pi}$$

was derived from the single unit of quarter-wavelength transformer.

V. References

[1] Microwave Engineering, David Pozar, 4th Edition

[2] https://en.wikipedia.org/wiki/Binomial_coefficient

[3] http://www.ittc.ku.edu/~jstiles/723/handouts/The%20Binomial%20Multisection%20Matching%20Transformer.pdf

[4] https://en.wikipedia.org/wiki/Bandwidth_(signal_processing)