Tangent Vector

I. Introduction

This concept is useful in vector calculus and tensor calculus. In this document, I would like to show why the derivative of a vector valued function is perpendicular to the original function. This property is used in Tensor calculus.

II. Theory

Let vector valued functions be $\vec{r}(t) = f(t)\hat{e}_1 + g(t)\hat{e}_2 + h(t)\hat{e}_3$ and $\vec{u}(t) = l(t)\hat{e}_1 + k(t)\hat{e}_2 + m(t)\hat{e}_3$.

a. Property 1

We first need to show the follow:

$$\boxed{D_t[r(t) \cdot u(t)] = \vec{r}(t) \cdot \vec{u}'(t) + \vec{r}'(t) \cdot \vec{u}(t)}$$

where D_t is the derivative operator with respect to t.

$$r(t) \cdot u(t) = f(t)l(t) + g(t)k(t) + h(t)m(t)$$

Now applying the derivative operator on $r(t) \cdot u(t)$, we obtain:

$$D_t[r(t) \cdot u(t)] = f(t)l'(t) + f'(t)l(t) + g(t)k'(t) + g'(t)k(t) + h(t)m'(t) + h'(t)m(t)$$

where the chain rule has been applied. Now, re-arrange the terms:

$$\{f(t)l'(t) + g(t)k'(t) + h(t)m'(t)\} + \{f'(t)l(t) + g'(t)k(t) + h'(t)m(t)\}$$

$$= \vec{r}(t) \cdot \vec{u}'(t) + \vec{r}'(t) \cdot \vec{u}(t)$$

b. Property 2

Then, we are going to show that

$$\boxed{if\ r(t) \cdot r(t) = c, then\ r(t) \cdot r'(t) = 0}$$

Let $r(t) \cdot r(t) = c$, and then apply the derivative operator, we get:

$$D_t[r(t) \cdot r(t)] = \vec{r}(t) \cdot \vec{r}'(t) + \vec{r}'(t) \cdot \vec{r}(t) = D_t[c]$$

$$2\vec{r}(t) \cdot \vec{r}'(t) = 0$$

Hence,

$$\vec{r}(t) \cdot \vec{r}'(t) = 0$$

This leads to the following conclusion:

$\vec{r}(t)$ and $\vec{r}'(t)$ are perpendicular to each other if $\vec{r}(t) \cdot \vec{r}(t) = c$ where c is some constant.

III. References

[1] https://en.wikipedia.org/wiki/Tangent_vector

[2] Calculus Early Transcendental Functions, 5th Edition, Larson and Edwards