FREQUENCY RESPONSE OF A SECOND ORDER SYSTEM

- $$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$$

- The sinusoidal transfer function of the system is

$$\frac{C(j\omega)}{R(j\omega)} = \frac{\omega_n^2}{(j\omega)^2+2\zeta\omega_n(j\omega)+\omega_n^2}$$

$$= \frac{1}{-\frac{\omega^2}{\omega_n^2}+j2\zeta\frac{\omega}{\omega_n}+1}$$

- $$\frac{C(j\omega)}{R(j\omega)} = \frac{1}{(1-u^2)+j2\zeta u}$$

where $u = \frac{\omega}{\omega_n}$

$$\left|\frac{C(j\omega)}{R(j\omega)}\right| = M = \frac{1}{\sqrt{(1-u^2)^2+(2\zeta u)^2}}$$

$$\angle\frac{C(j\omega)}{R(j\omega)} = \phi = -\tan^{-1}\left(\frac{2\zeta u}{1-u^2}\right)$$

→ When $u=0$, $M=1$, $\phi=0$

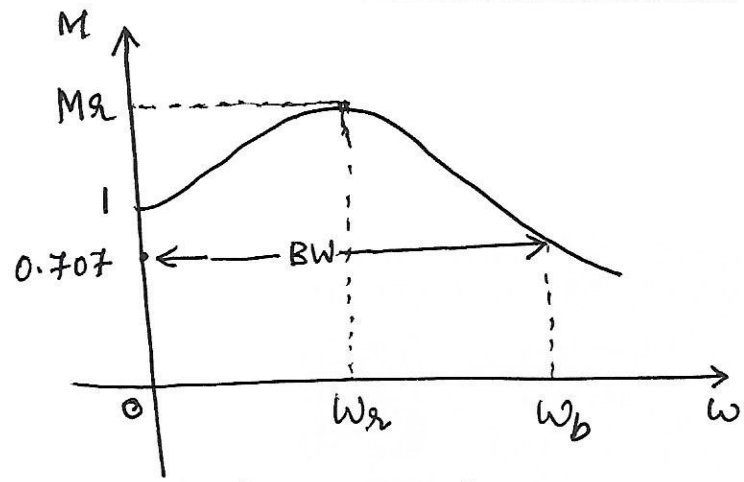
$u=1$, $M=\frac{1}{2\zeta}$, $\phi=-\frac{\pi}{2}$

$u\to\infty$, $M\to 0$, $\phi\to -\pi$

→ The steady-state output of the system for a sinusoidal input of unit magnitude and variable frequency ω is given by

$$c(t) = \frac{1}{\sqrt{(1-u^2)^2+(2\zeta u)^2}} \cdot 1 \cdot \sin\left(\omega t - \tan^{-1}\left(\frac{2\zeta u}{1-u^2}\right)\right)$$

FREQUENCY RESPONSE SPECIFICATIONS



① Resonant Peak (M_r)

The resonant peak is defined as the maximum value of $|G(j\omega)|$.

② Resonant Frequency (ω_r)

The resonant frequency is defined as the frequency at which the resonant peak, M_r occurs.

③ Bandwidth ω_b

The bandwidth is defined as the frequency at which the magnitude of $|G(j\omega)|$ drops to 70.7 percent of its zero frequency value or 3dB down from the zero frequency gain.

④ Cutoff rate

The bandwidth alone is inadequate in the indication of the system characteristics, in distinguishing signals from noise. Sometimes it may be necessary to specify cutoff rate of the frequency response at higher frequencies.

DERIVATION OF FREQUENCY RESPONSE SPECIFICATIONS

① RESONANT FREQUENCY (ω_r)

* At the resonant frequency, the slope of the magnitude curve is zero.

* Let ω_r be the resonant frequency and $u_r = \frac{\omega_r}{\omega_n}$.

* $$\frac{dM}{du} = \frac{d}{du}\left[\left\{(1-u^2)^2 + (2\zeta u)^2\right\}^{-\frac{1}{2}}\right]$$

$$= -\frac{1}{2}\left[(1-u^2)^2 + (2\zeta u)^2\right]^{-\frac{3}{2}}\left[2(1-u^2)(-2u) + 4\zeta^2 2u\right]$$

$$= -\frac{1}{2}\left[\frac{-4u(1-u^2) + 8\zeta^2 u}{\left[(1-u^2)^2 + (2\zeta u)^2\right]^{3/2}}\right]$$

$$\left.\frac{dM}{du}\right|_{u=u_r} = -\frac{1}{2}\left[\frac{-4u_r(1-u_r^2) + 8\zeta^2 u_r}{\left[(1-u_r^2)^2 + (2\zeta u_r)^2\right]^{3/2}}\right] = 0$$

$$-4u_r + 4u_r^3 + 8\zeta^2 u_r = 0$$

$$u_r = \sqrt{1-2\zeta^2}$$

$$\frac{\omega_r}{\omega_n} = \sqrt{1-2\zeta^2}$$

$$\boxed{\omega_r = \omega_n\sqrt{1-2\zeta^2}}$$

② RESONANT PEAK (M_r)

* The resonant peak may be found by substituting $u = u_r$ in the expression for M

$$M_r = \frac{1}{\sqrt{(1-u_r^2)^2 + (2\zeta u_r)^2}}$$

$$\boxed{M_r = \frac{1}{2\zeta\sqrt{1-\zeta^2}}}$$

* M_r (like M_p) depends only on ζ.

③ BANDWIDTH (u_b)

* Let $u_b = \frac{\omega_b}{\omega_n}$

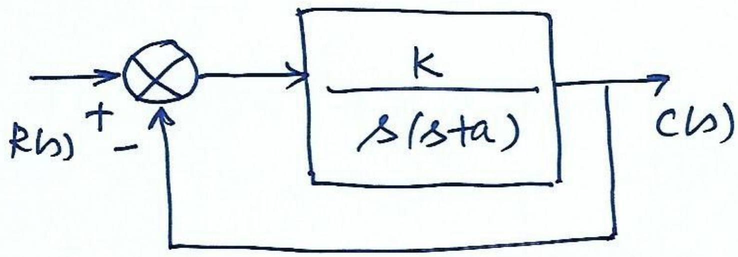
* $$M = \frac{1}{\sqrt{(1-u_b^2)^2 + (2\zeta u_b)^2}} = \frac{1}{\sqrt{2}}$$

$$u_b^4 - 2(1-2\zeta^2)u_b^2 - 1 = 0$$

$$u_b = \left[1 - 2\zeta^2 + \sqrt{2 - 4\zeta^2 + 4\zeta^4}\right]^{1/2}$$

$$\boxed{\omega_b = \omega_n\left[1 - 2\zeta^2 + \sqrt{2 - 4\zeta^2 + 4\zeta^4}\right]^{1/2}}$$

1b For the system shown in fig below, find K and a to satisfy the following frequency response specifications. $M_r = 1.04$, $W_r = 11.55$ rad/sec



Solution

(1) $$\frac{C(s)}{R(s)} = \frac{K}{s^2 + as + K}$$

(2) $$M_r = \frac{1}{2\zeta\sqrt{1-\zeta^2}} = 1.04$$

$$\zeta = 0.7983 \quad \text{or} \quad \zeta = 0.6022$$

(3) When $\zeta = 0.7983$,

$$W_r = W_n\sqrt{1-2\zeta^2}$$

$$W_n = \frac{W_r}{\sqrt{1-2\zeta^2}} = \frac{11.5}{\sqrt{1-2(0.7983)^2}}$$

$$W_n = \frac{11.5}{\sqrt{1-1.27}}$$, $W_n \rightarrow$ Imaginary

$\therefore$ We take $\zeta = 0.6022$

$$\omega_n = \frac{\omega_r}{\sqrt{1-2\zeta^2}} = 21.95 \text{ rad/sec}$$

$$\frac{C(s)}{R(s)} = \frac{k}{s^2+as+k} = \frac{\omega_n^2}{s^2+2\zeta\omega_n^2+\omega_n^2}$$

$$k = \omega_n^2 = (21.95)^2$$

$$\boxed{k = 481.8025}$$

$$a = 2\zeta\omega_n$$

$$= 26.44$$

$$\boxed{a = 26.44}$$