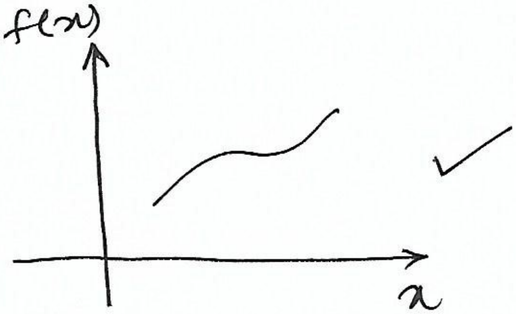
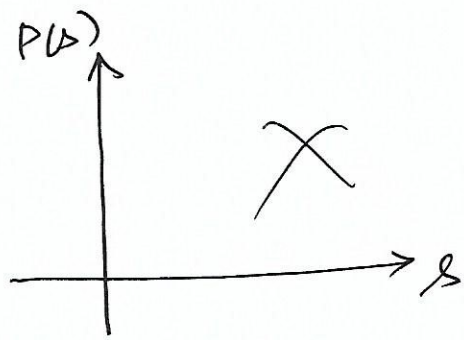
Complex Functions Of A Complex Variable

* A real function of a real variable is easily plotted on a single set of co-ordinate axes.

* The real function $f(x)$ with x real is easily plotted in rectangular co-ordinates with x as the abscissa and $f(x)$ as the ordinate.



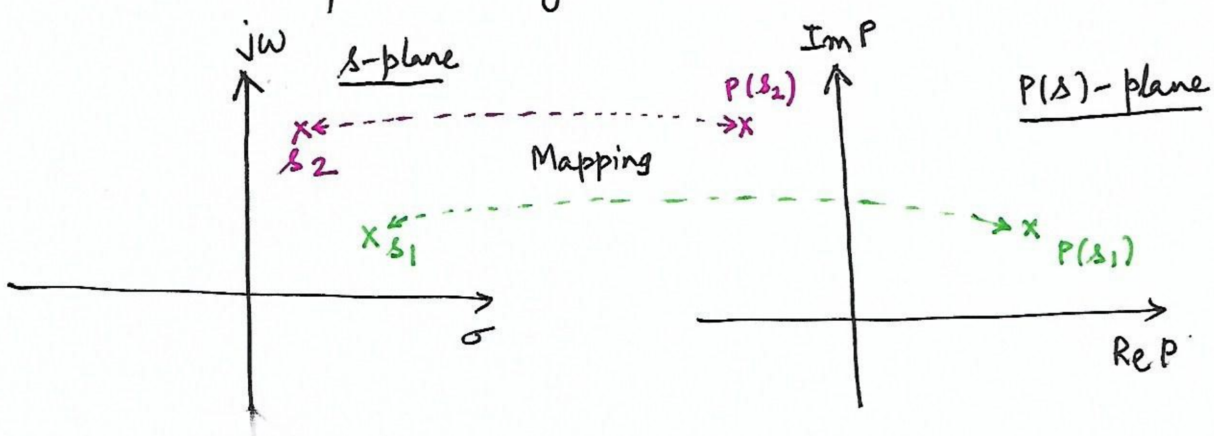
* A complex function of a complex variable such as the transfer function $P(s)$ with $s=\sigma+j\omega$ cannot be plotted on a single set of co-ordinates.



* The complex variable $s=\sigma+j\omega$ itself is dependent upon two independant quantities, the real and imaginary parts. (of s)

- The complex function $P(s)$ also has real and imaginary parts.
- In order to plot $P(s)$ as a function of $s = \sigma + j\omega$, two sets of co-ordinate axes are required.
- In the first set a graph of $j\omega$ versus σ is plotted (called the s-plane)
- In the second set a graph of Imaginary part of $P(s)$ ($\mathrm{Im}\,P$) versus the real part of $P(s)$ ($\mathrm{Re}\,P$) is plotted (called $P(s)$-plane)

- There is a correspondance between points in the two planes that is called Mapping or transformation.
- Points in the s-plane are mapped onto points in the $P(s)$ plane by the function P



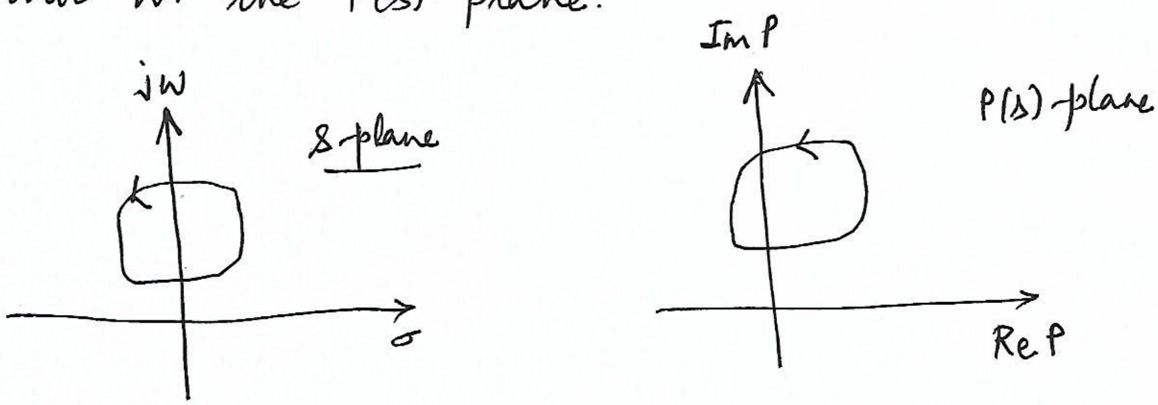
* A complex function P(s) is said to be ANALYTIC in a region if P(s) and all its derivatives exist in that region.

* Points in the s-plane at which the function P(s) is analytic are called ORDINARY points.

* Points in the s-plane at which the function P(s) is not analytic are called SINGULAR points.

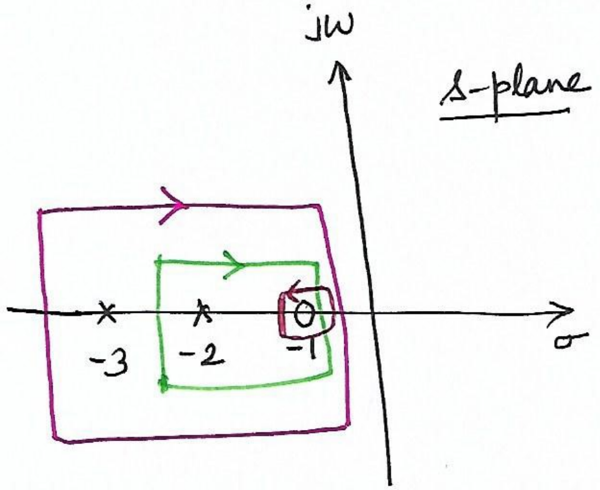
(Eg: Poles and zeros may be considered as SINGULAR points for a given transfer function)

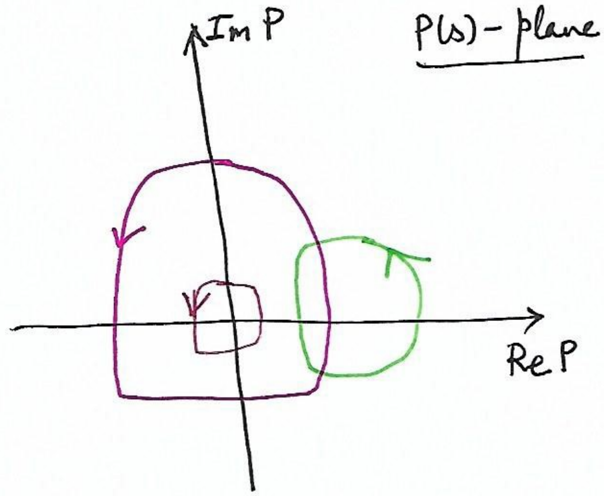
* For a given closed path in the s-plane. Which does not go through any singular points, there corresponds a ~~closed~~ closed curve in the P(s)-plane.



* Eg: Consider a complex function P(s) given by $P(s) = \dfrac{(s+1)}{(s+2)(s+3)}$

- A closed curve in the s-plane is mapped on to a closed curve in the P(s)-plane.





- $N = Z - P$

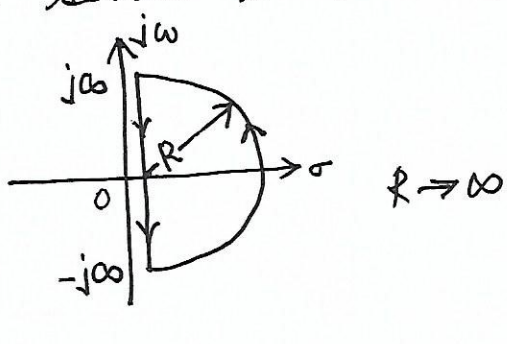
MAPPING THEOREM (PRINCIPLE OF ARGUMENT)

- Let P(s) be a ratio of two polynomials in s. Let P be the number of poles and Z be the number of zeros of P(s) which lie inside some closed contour in the s-plane.
- Let this contour be such that it does not pass through any poles or zeros of P(s).
- This closed contour in the s-plane is then mapped into the P(s) plane as a closed curve.

- The total number of encirclements of the origin of the P(s)-plane (N), as a representative point s traces out the entire contour in the s-plane is equal to Z-P.
- In general N can be positive (Z>P), zero (Z=P) or negative (Z<P).
- If N is positive, the direction of encirclement of the origin of the P(s)-plane is the same as that of the s-plane path.
- If N is negative, the direction of encirclement is opposite to that of the s-plane path.
- If N=0, there is no encirclement.

NYQUIST PATH

- The Mapping theorem can be used to solve stability problems if the s-plane path is taken to be one that encircles the entire right half of s-plane.
- P(s) is equal to 1+G(s) H(s).
- The fig below shows a s-plane path with an anticlockwise sense which encircles the entire RHSP. It is called the Nyquist path.



- The Nyquist path consists of the entire $j\omega$ axis $j\omega$ axis from $\omega = +\infty$ to $-\infty$ and a semicircular path of infinite radius in the RHSP.

- The Nyquist path encloses the entire RHSP and thereby encloses all the zeros and poles of $1+G(s)H(s)$ that have positive real parts.

- For convenience, the Nyquist path is divided into a minimum of three sections.
  (a) Positive Imaginary axis
  (b) Negative Imaginary axis
  (c) Semicircle of infinite radius.

Nyquist Criterion and the $G(s)H(s)$-plot

- The Nyquist Criterion is a direct application of the Mapping theorem, when the s-plane path is the Nyquist path.

- The stability of a closed loop system can be determined by plotting the $P(s) = 1+G(s)H(s)$ locus, when s taken on values along the Nyquist path and observing the behaviour of the $P(s)$ plot wrt the origin of the $P(s)$-plane.

  This is called Nyquist Plot of $P(s)$

* Since usually the function $G(s)H(s)$ is given, a simpler approach is to construct the Nyquist plot of $G(s)H(s)$ and the same result of $P(s)$ can be determined from the behaviour of the $G(s)H(s)$ plot wrt the $(-1, j0)$ point in the $G(s)H(s)$-plane.

* This is because the origin of the $P(s)$ plane corresponds to the $(-1, j0)$ point of the $G(s)H(s)$-plane

$$1 + G(s)H(s) = 0 + j0$$
$$G(s)H(s) = -1 + j0$$

* stability criterion is $N = -P$

Nyquist Stability Criterion

For a closed loop system to be stable, the Nyquist plot of $G(s)H(s)$ must encircle the $(-1, j0)$ point as many times as the number of poles of $G(s)H(s)$ that are in the RHSP and the encirclement must be made in a direction opposite to the Nyquist Path.

If the function $G(s)H(s)$ has no poles in the RHSP, for the closed loop system to be stable the Nyquist plot of $G(s)H(s)$ must not encircle the critical point $(-1, j0)$.