CONTROL SYTEMS

TOPIC TWO

TRANSFER FUNCTION

Transfer Function plays an important role in the characterization of Linear Time-invariant Systems.

The starting point of defining the Transfer Function is the Differential equation

A transfer function between an input variable & an output variable of a system is defined as the ratio of the Laplace Transform of the Output to the Laplace Transform of the Input

Transfer Function is defined only for a Linear System & strictly only for Time-invariant Systems

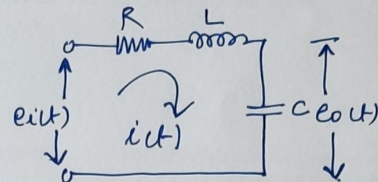
All initial conditions of the system are assumed to be zero.

A Transfer Function is independent of the Input excitation

Example

- The following example illustrates how transfer function for a linear system can be derived.

- A series RLC circuit is shown in Fig below



- The input variable is designated by $e_i(t)$
- The output variable is the current $i(t)$
- The loop equation of the network is

$$e_i(t) = R\,i(t) + L\frac{di(t)}{dt} + \frac{1}{C}\int i(t)\,dt$$

- Taking Laplace transform on both sides and assuming zero initial conditions,

$$E_i(s) = RI(s) + sLI(s) + \frac{1}{Cs}I(s)$$

$$= \left[R + sL + \frac{1}{Cs}\right]I(s)$$

$$\frac{I(s)}{E_i(s)} = \frac{1}{R + sL + \frac{1}{Cs}} = \frac{Cs}{LCs^2 + RCs + 1}$$

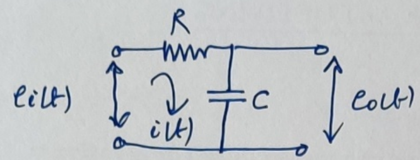
- If the voltage across the capacitor $e_o(t)$ is considered as an output, the transfer function between $e_i(t)$ and $e_o(t)$ is obtained as

$$E_o(s) = \frac{1}{Cs} I(s)$$

$$= \frac{1}{\cancel{C}s} \cdot \frac{\cancel{C}s}{LCs^2 + RCs + 1} E_i(s)$$

$$\boxed{\frac{E_o(s)}{E_i(s)} = \frac{1}{LCs^2 + RCs + 1}}$$

Transfer function of RC network



$$e_i(t) = R\,i(t) + \frac{1}{C}\int i(t)\,dt$$

$$E_i(s) = R\,I(s) + \frac{1}{Cs} I(s)$$

$$E_i(s) = \left[R + \frac{1}{Cs}\right] I(s) = \left[\frac{RCs+1}{Cs}\right] I(s)$$

$$e_o(t) = \frac{1}{C}\int i(t)\,dt$$

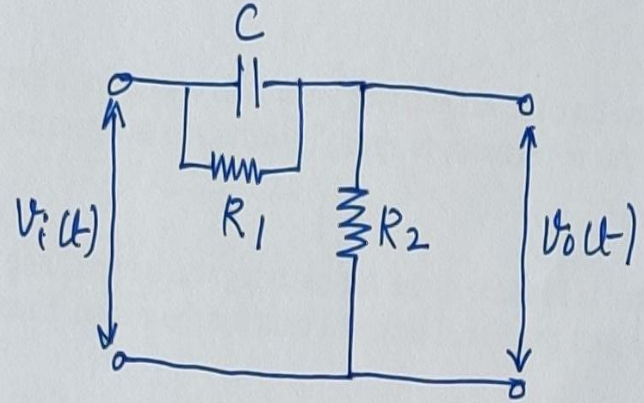
$$= \frac{1}{Cs} I(s)$$

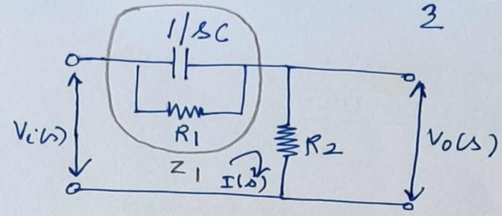
$$= \frac{1}{C\cancel{s}} \cdot C\cancel{s} \cdot \frac{E_i(s)}{RCs+1}$$

$$\boxed{\frac{E_o(s)}{E_i(s)} = \frac{1}{RCs+1}}$$

Pb For the circuit given in Fig below, if $C = 1\,\mu F$, what values of R_1 and R_2 will give $T = 0.6$ sec and $a = 0.1$. The expression for transfer function is

$$\frac{V_o(s)}{V_i(s)} = \frac{a(1+sT)}{1+asT}$$





$$Z_1(s) = \frac{R_1 \cdot \frac{1}{sC}}{R_1 + \frac{1}{sC}} = \frac{R_1}{R_1Cs+1}$$

$$V_i(s) = \left[\left(\frac{R_1}{R_1Cs+1}\right) + R_2\right] I(s) \longrightarrow (1)$$

$$V_o(s) = R_2 I(s)$$

$$\therefore V_o(s) = R_2 \left[\frac{V_i(s)}{\frac{R_1}{R_1Cs+1} + R_2}\right] \longrightarrow \text{From } (1)$$

$$\frac{V_o(s)}{V_i(s)} = \frac{R_2(R_1Cs+1)}{R_1 + R_2(R_1Cs+1)}$$

$$= \frac{R_2(R_1Cs+1)}{R_1 + R_2R_1Cs + R_2}$$

$$= \frac{R_2(R_1Cs+1)}{(R_1+R_2)\left(1 + \frac{R_1R_2}{R_1+R_2} \cdot Cs\right)}$$

$$= \frac{\left(\frac{R_2}{R_1+R_2}\right)(1+R_1Cs)}{\left[1 + \left(\frac{R_1R_2}{R_1+R_2}\right)Cs\right]}$$

$$= \frac{a(1+sT)}{1+aTs}$$

Comparing with given equation

$$a = \frac{R_2}{R_1+R_2} \quad , \quad T = R_1 C$$

$$T = 0.6 \text{ sec} \quad , \quad C = 1\mu f$$

$$R_1 = \frac{0.6}{10^{-6}} = 600k$$

$$\boxed{R_1 = 600k}$$

$$a = 0.1 = \frac{R_2}{R_1 + R_2}$$

$$= \frac{R_2}{600k + R_2}$$

$$\boxed{R_2 = 66.67 K}$$

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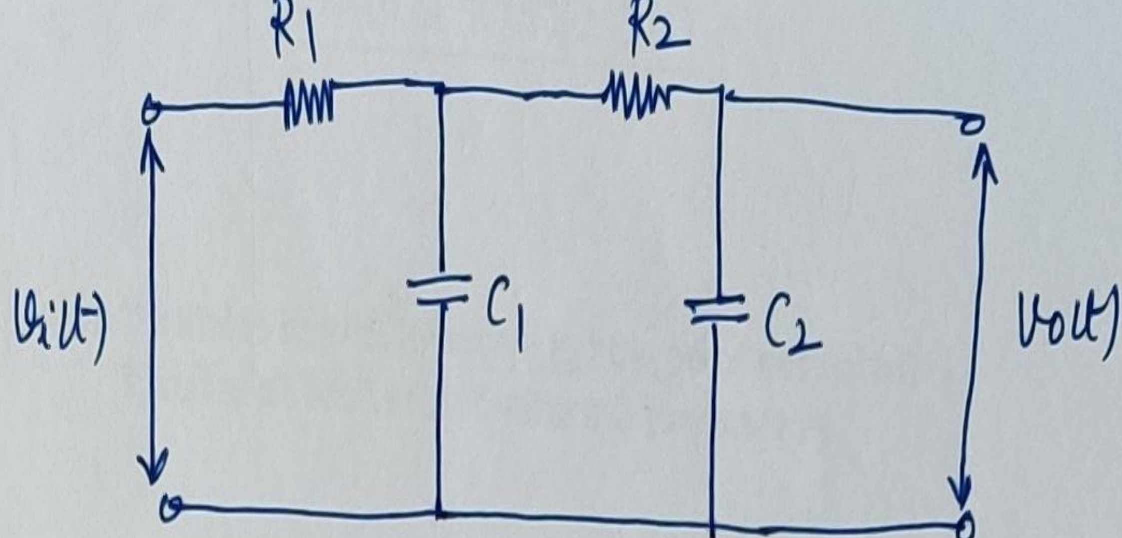

Find $\frac{V_o(s)}{V_i(s)}$

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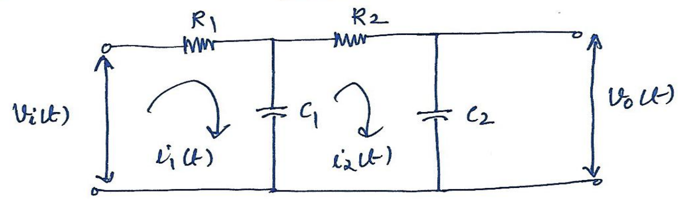
SOLUTION



→ Writing Mesh equations

$$V_i = R_1 i_1 + \frac{1}{C_1}\int i_1\,dt - \frac{1}{C_1}\int i_2\,dt \longrightarrow (1)$$

$$0 = -R_2 i_2 - \frac{1}{C_2}\int i_2\,dt - \frac{1}{C_1}\int i_2\,dt + \frac{1}{C_1}\int i_1\,dt \longrightarrow (2)$$

→ Taking Laplace Transforms

$$V_i(s) = R_1 I_1(s) + \frac{1}{C_1 s} I_1(s) - \frac{1}{C_1 s} I_2(s) \longrightarrow (3)$$

$$0 = -R_2 I_2(s) - \frac{1}{C_2 s} I_2(s) - \frac{1}{C_1 s} I_2(s) + \frac{1}{C_1 s} I_1(s) \longrightarrow (4)$$

→ Combining terms in $I_1(s)$ and $I_2(s)$

$$V_i(s) = I_1(s)\left[R_1 + \frac{1}{C_1 s}\right] - I_2(s)\frac{1}{C_1 s} \longrightarrow (5)$$

$$0 = -I_1(s)\frac{1}{C_1 s} + I_2(s)\left[R_2 + \frac{1}{C_2 s} + \frac{1}{C_1 s}\right] \longrightarrow (6)$$

→ Using Cramer's Rule,

$$I_2(s) = \frac{\begin{vmatrix} R_1 + \frac{1}{C_1 s} & +V_i(s) \\ -\frac{1}{C_1 s} & 0 \end{vmatrix}}{\begin{vmatrix} R_1 + \frac{1}{C_1 s} & -\frac{1}{C_1 s} \\ -\frac{1}{C_1 s} & R_2 + \frac{1}{C_2 s} + \frac{1}{C_1 s} \end{vmatrix}}$$

$$I_2(s) = \frac{V_i(s)\dfrac{1}{C_1 s}}{\left(R_1 + \dfrac{1}{C_1 s}\right)\left(R_2 + \dfrac{1}{C_1 s} + \dfrac{1}{C_2 s}\right) - \dfrac{1}{C_1^2 s^2}}$$

$$V_o(s) = \frac{1}{C_2 s} I_2(s) \longrightarrow (7)$$

$$V_o(s) = \frac{\dfrac{1}{C_2 s} \cdot \dfrac{1}{C_1 s} V_i(s)}{\left(R_1 + \dfrac{1}{C_1 s}\right)\left(R_2 + \dfrac{1}{C_1 s} + \dfrac{1}{C_2 s}\right) - \dfrac{1}{C_1^2 s^2}}$$

$$\frac{V_o(s)}{V_i(s)} = \frac{\dfrac{1}{C_1 C_2 s^2}}{\dfrac{(R_1 C_1 s + 1)}{C_1 s} + \dfrac{(R_2 C_1 C_2 s^2 + C_2 s + C_1 s)}{C_1 C_2 s^2} - \dfrac{1}{C_1^2 s^2}}$$

$$\frac{V_o(s)}{V_i(s)} = \frac{\dfrac{1}{C_1 C_2 s^2}}{\dfrac{1}{C_1^2 s^2}\left[\dfrac{(R_1 C_1 s + 1)(R_2 C_1 C_2 s^2 + C_2 s + C_1 s)}{C_2 s} - 1\right]}$$

$$= \frac{\dfrac{1}{\cancel{C_1} \cancel{C_2} \cancel{s^2}}}{\dfrac{1}{C_1^{\cancel{2}} \cancel{s^2}}\left[\dfrac{(R_1 C_1 s + 1)(R_2 C_1 C_2 s^2 + C_2 s + C_1 s) - C_2 s}{\cancel{C_2} s}\right]}$$

$$\frac{V_o(s)}{V_i(s)} = \frac{C_1 s}{(R_1C_1s+1)(R_2C_1C_2s^2+C_1s+C_2s) - C_2 s}$$

$$\frac{V_o(s)}{V_i(s)} = \frac{C_1 s}{\left[R_1C_1s\cdot R_2C_1C_2s^2 + R_1C_1s\cdot C_1s + R_1C_1s\cdot C_2s + R_2C_1C_2s^2 + C_1s + \cancel{C_2s} - \cancel{C_2s}\right]}$$

$$= \frac{\cancel{C_1s}}{\cancel{C_1s}\left[R_1R_2C_1C_2s^2 + R_1C_1s + R_1C_2s + R_2C_2s + 1\right]}$$

$$\frac{V_o(s)}{V_i(s)} = \frac{1}{R_1R_2C_1C_2s^2 + (R_1C_1 + R_1C_2 + R_2C_2)s + 1}$$

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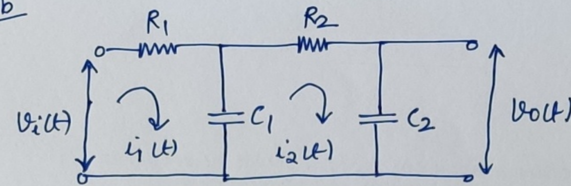
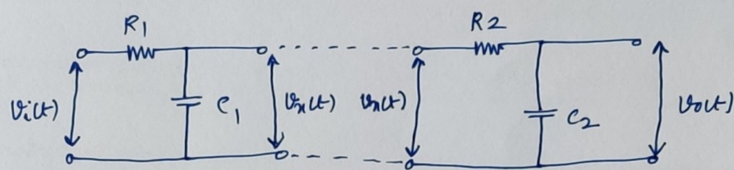


Ans $$\frac{V_o(s)}{V_i(s)} = \frac{1}{R_1C_1R_2C_2s^2 + (R_1C_1 + R_2C_2 + R_1C_2)s + 1}$$



$$\frac{V_x(s)}{V_i(s)} = \frac{1}{R_1C_1s+1} \quad , \quad \frac{V_o(s)}{V_x(s)} = \frac{1}{R_2C_2s+1}$$

$$\frac{V_x(s)}{V_i(s)} \times \frac{V_o(s)}{V_x(s)} = \frac{1}{(R_1C_1s+1)} \times \frac{1}{(R_2C_2s+1)}$$

$$\frac{V_o(s)}{V_i(s)} = \frac{1}{(R_1C_1s+1)(R_2C_2s+1)}$$

$$= \frac{1}{R_1C_1R_2C_2s^2 + (R_1C_1 + R_2C_2)s + 1}$$

$$\frac{V_o(s)}{V_i(s)} = \frac{1}{R_1C_1R_2C_2s^2 + (R_1C_1 + R_2C_2)s + 1}$$

IMPULSE RESPONSE

- For a Linear Time-Invariant System , the transfer function T(s) is the ratio of C(s) / R(s) , where R(s) & C(s) are the Laplace transforms of the Input & Output respectively.

  Therefore, C(s) = T(s) R(s)

- Consider the Output response of a system to a **Unit Impulse Input**, where initial conditions are zero.

- Since the Laplace transform of the Unit impulse function is Unity, the Laplace transform of the Output of the system is just C(S) = T(S).

- The Impulse Response of a system is thus the response of a Linear system to a Unit Impulse Input ,when the initial conditions are zero.

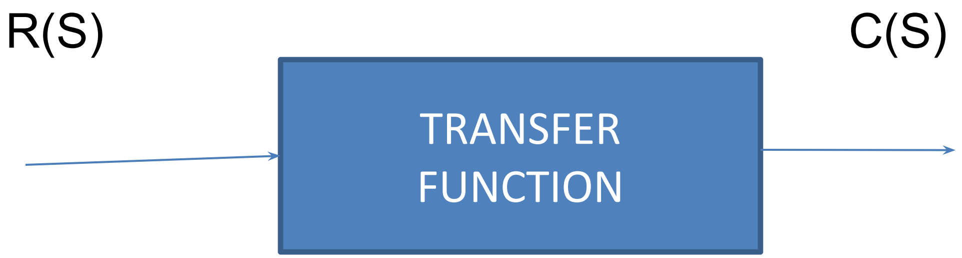
- Thus the transfer function & the Impulse response of a LTI system , contain the same information about the system performance.

- Hence it is possible to obtain complete information about the system by exciting it with an Impulse input & measuring the response.

- In practice a pulse input with a very short duration compared with the significant time constants of the system can be used.

BLOCK DIAGRAMS

- A block diagram of a system is a pictorial representation of the functions performed by each component & of the flow of signals.
- The transfer function of the components are usually entered in the corresponding blocks , which are connected by arrows to indicate the direction of the flow of signals.
- Fig below shows an element of the block diagram



- The output signal is the input signal multiplied by the transfer function in the block.
- The **Error Detector or Summing Point** produces a signal which is the difference between the Reference Input & the Feedback Signal of the control system.
- A circle with a cross is the symbol which indicates a Summing operation
- The plus or minus sign at the arrow head indicates whether that signal is to be added or subtracted.

BLOCK DIAGRAMS CONT------

- The quantities being added or subtracted should have the same dimension & same units.