CONTROL SYTEMS

TOPIC TWO

TRANSFER FUNCTION

Transfer Function plays an important role in the characterization of Linear Time-invariant Systems.

The starting point of defining the Transfer Function is the Differential equation

A transfer function between an input variable & an output variable of a system is defined as the ratio of the Laplace Transform of the Output to the Laplace Transform of the Input

Transfer Function is defined only for a Linear System & strictly only for Time-invariant Systems

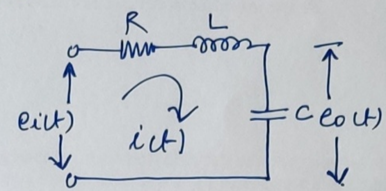
All initial conditions of the system are assumed to be zero.

A Transfer Function is independent of the Input excitation

Example

* The following example illustrates how transfer function for a linear system can be derived.

* A series RLC circuit is shown in Fig below



* The input variable is designated by $e_i(t)$
* The output variable is the current $i(t)$
* The loop equation of the network is

$$e_i(t) = R\,i(t) + L\frac{di(t)}{dt} + \frac{1}{C}\int i(t)\,dt$$

* Taking Laplace transform on both sides and assuming zero initial conditions,

$$E_i(s) = R I(s) + sL I(s) + \frac{1}{Cs} I(s)$$

$$= \left[R + sL + \frac{1}{Cs}\right] I(s)$$

$$\frac{I(s)}{E_i(s)} = \frac{1}{R + sL + \frac{1}{Cs}} = \frac{Cs}{LCs^2 + RCs + 1}$$

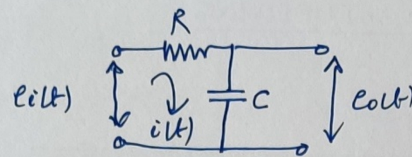
* If the voltage across the capacitor $e_o(t)$ is considered as an output, the transfer function between $e_i(t)$ and $e_o(t)$ is obtained as

$$E_o(s) = \frac{1}{Cs} I(s)$$

$$= \frac{1}{\cancel{Cs}} \cdot \frac{\cancel{Cs}}{LCs^2 + RCs + 1} E_i(s)$$

$$\boxed{\frac{E_o(s)}{E_i(s)} = \frac{1}{LCs^2 + RCs + 1}}$$

Transfer function of RC network



$$e_i(t) = R\,i(t) + \frac{1}{C}\int i(t)\,dt$$

$$E_i(s) = R\,I(s) + \frac{1}{Cs}I(s)$$

$$E_i(s) = \left[R + \frac{1}{Cs}\right]I(s) \qquad = \left[\frac{RCs+1}{Cs}\right]I(s)$$

$$e_o(t) = \frac{1}{C}\int i(t)\,dt$$

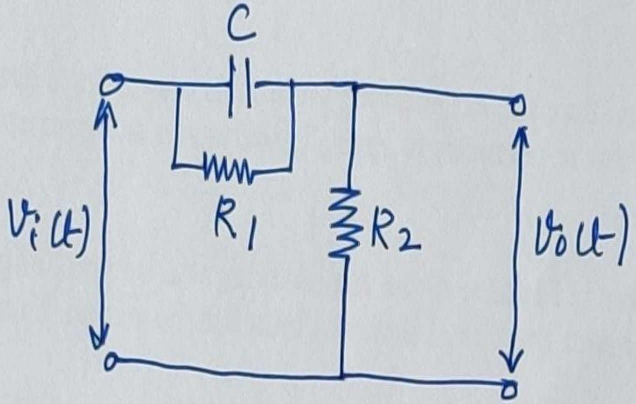
$$= \frac{1}{Cs}I(s)$$

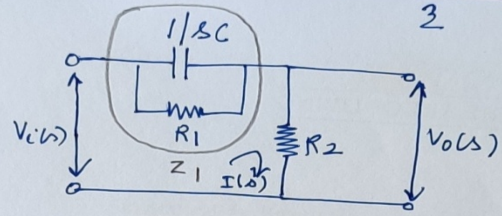
$$= \frac{1}{C\cancel{s}} \cdot C\cancel{s} \cdot \frac{E_i(s)}{RCs+1}$$

$$\boxed{\frac{E_o(s)}{E_i(s)} = \frac{1}{RCs+1}}$$

Pb For the circuit given in Fig below, if $C = 1\,\mu F$, what values of R_1 and R_2 will give $T = 0.6$ sec and $a = 0.1$. The expression for transfer function is

$$\frac{V_o(s)}{V_i(s)} = \frac{a(1+sT)}{1+asT}$$





$$Z_1(s) = \frac{R_1 \cdot \frac{1}{sC}}{R_1 + \frac{1}{sC}} = \frac{R_1}{R_1Cs+1}$$

$$V_i(s) = \left[\left(\frac{R_1}{R_1Cs+1}\right) + R_2\right] I(s) \longrightarrow ①$$

$$V_o(s) = R_2 I(s)$$

$$\therefore V_o(s) = R_2 \left[\frac{V_i(s)}{\frac{R_1}{R_1Cs+1} + R_2}\right] \longrightarrow \text{From } ①$$

$$\frac{V_o(s)}{V_i(s)} = \frac{R_2(R_1Cs+1)}{R_1 + R_2(R_1Cs+1)}$$

$$= \frac{R_2(R_1Cs+1)}{R_1 + R_2R_1Cs + R_2}$$

$$= \frac{R_2(R_1Cs+1)}{(R_1+R_2)\left(1 + \frac{R_1R_2}{R_1+R_2} \cdot Cs\right)}$$

$$= \frac{\left(\frac{R_2}{R_1+R_2}\right)(1+R_1Cs)}{\left[1 + \left(\frac{R_1R_2}{R_1+R_2}\right)Cs\right]}$$

$$= \frac{a(1+sT)}{1+aTs}$$

Comparing with given equation

$$a = \frac{R_2}{R_1+R_2} , \quad T = R_1 C$$

$$T = 0.6 \text{ sec} , \quad C = 1\mu f$$

$$R_1 = \frac{0.6}{10^{-6}} = 600k$$

$$\boxed{R_1 = 600k}$$

$$a = 0.1 = \frac{R_2}{R_1+R_2}$$

$$= \frac{R_2}{600k + R_2}$$

$$\boxed{R_2 = 66.67k}$$

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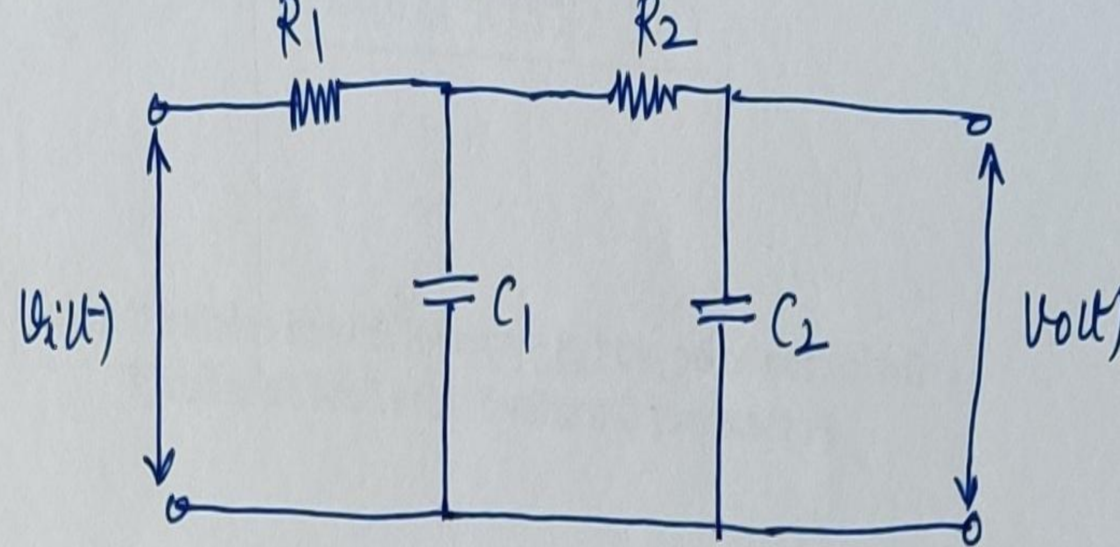
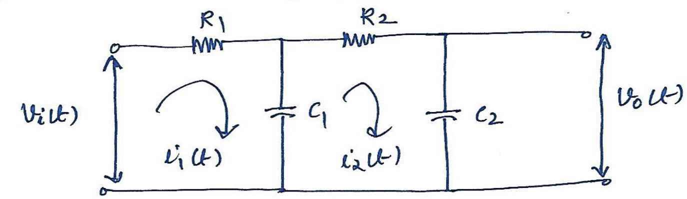


Find $\dfrac{V_o(s)}{V_i(s)}$

Pb2

SOLUTION



→ Writing Mesh equations

$$v_i = R_1 i_1 + \frac{1}{C_1}\int i_1\,dt - \frac{1}{C_1}\int i_2\,dt \longrightarrow (1)$$

$$0 = -R_2 i_2 - \frac{1}{C_2}\int i_2\,dt - \frac{1}{C_1}\int i_2\,dt + \frac{1}{C_1}\int i_1\,dt \longrightarrow (2)$$

→ Taking Laplace Transforms

$$V_i(s) = R_1 I_1(s) + \frac{1}{C_1 s} I_1(s) - \frac{1}{C_1 s} I_2(s) \longrightarrow (3)$$

$$0 = -R_2 I_2(s) - \frac{1}{C_2 s} I_2(s) - \frac{1}{C_1 s} I_2(s) + \frac{1}{C_1 s} I_1(s) \longrightarrow (4)$$

→ combining terms in $I_1(s)$ and $I_2(s)$

$$V_i(s) = I_1(s)\left[R_1 + \frac{1}{C_1 s}\right] - I_2(s)\frac{1}{C_1 s} \longrightarrow (5)$$

$$0 = -I_1(s)\frac{1}{C_1 s} + I_2(s)\left[R_2 + \frac{1}{C_2 s} + \frac{1}{C_1 s}\right] \rightarrow (6)$$

→ Using Cramer's Rule.

$$I_2(s) = \frac{\begin{vmatrix} R_1 + \frac{1}{C_1 s} & +V_i(s) \\ -\frac{1}{C_1 s} & 0 \end{vmatrix}}{\begin{vmatrix} R_1 + \frac{1}{C_1 s} & -\frac{1}{C_1 s} \\ -\frac{1}{C_1 s} & R_2 + \frac{1}{C_2 s} + \frac{1}{C_1 s} \end{vmatrix}}$$

$$I_2(s) = \frac{V_i(s)\,\frac{1}{C_1 s}}{\left(R_1 + \frac{1}{C_1 s}\right)\left(R_2 + \frac{1}{C_1 s} + \frac{1}{C_2 s}\right) - \frac{1}{C_1^2 s^2}}$$

$$V_o(s) = \frac{1}{C_2 s} I_2(s) \longrightarrow (7)$$

$$V_o(s) = \frac{\frac{1}{C_2 s} \cdot \frac{1}{C_1 s} V_i(s)}{\left(R_1 + \frac{1}{C_1 s}\right)\left(R_2 + \frac{1}{C_1 s} + \frac{1}{C_2 s}\right) - \frac{1}{C_1^2 s^2}}$$

$$\frac{V_o(s)}{V_i(s)} = \frac{\frac{1}{C_1 C_2 s^2}}{\frac{(R_1 C_1 s + 1)}{C_1 s} + \frac{(R_2 C_1 C_2 s^2 + C_2 s + C_1 s)}{C_1 C_2 s^2} - \frac{1}{C_1^2 s^2}}$$

$$\frac{V_o(s)}{V_i(s)} = \frac{\frac{1}{C_1 C_2 s^2}}{\frac{1}{C_1^2 s^2}\left[\frac{(R_1 C_1 s + 1)(R_2 C_1 C_2 s^2 + C_2 s + C_1 s)}{C_2 s} - 1\right]}$$

$$= \frac{\frac{1}{\cancel{C_1} C_2 \cancel{s^2}}}{\frac{1}{C_1^{\cancel{2}} \cancel{s^2}}\left[\frac{(R_1 C_1 s + 1)(R_2 C_1 C_2 s^2 + C_2 s + C_1 s) - C_2 s}{\cancel{C_2} s}\right]}$$

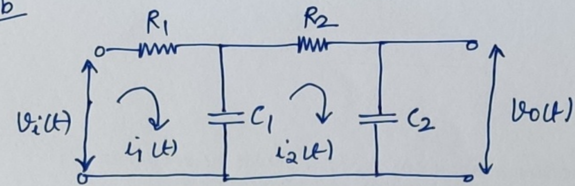
$$\frac{V_o(s)}{V_i(s)} = \frac{C_1 s}{(R_1C_1s+1)(R_2C_1C_2s^2+C_1s+C_2s)-C_2s}$$

$$\frac{V_o(s)}{V_i(s)} = \frac{C_1 s}{\left[R_1C_1s \cdot R_2C_1C_2s^2 + R_1C_1s \cdot C_1s + R_1C_1s \cdot C_2s + R_2C_1C_2s^2 + C_1s + \cancel{C_2s} - \cancel{C_2s}\right]}$$

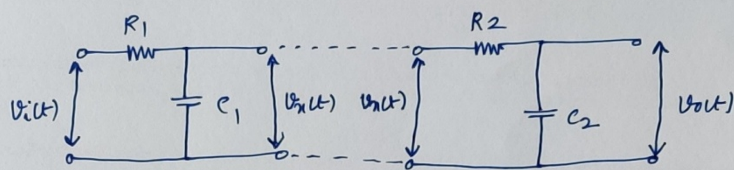
$$= \frac{\cancel{C_1s}}{\cancel{C_1s}\left[R_1R_2C_1C_2s^2 + R_1C_1s + R_1C_2s + R_2C_2s + 1\right]}$$

$$\frac{V_o(s)}{V_i(s)} = \frac{1}{R_1R_2C_1C_2s^2 + (R_1C_1 + R_1C_2 + R_2C_2)s + 1}$$

Pb



Ans $$\frac{V_o(s)}{V_i(s)} = \frac{1}{R_1C_1R_2C_2s^2 + (R_1C_1 + R_2C_2 + R_1C_2)s + 1}$$



$$\frac{V_x(s)}{V_i(s)} = \frac{1}{R_1C_1s+1}, \quad \frac{V_o(s)}{V_x(s)} = \frac{1}{R_2C_2s+1}$$

$$\frac{V_x(s)}{V_i(s)} \times \frac{V_o(s)}{V_x(s)} = \frac{1}{(R_1C_1s+1)} \times \frac{1}{(R_2C_2s+1)}$$

$$\frac{V_o(s)}{V_i(s)} = \frac{1}{(R_1C_1s+1)(R_2C_2s+1)}$$

$$= \frac{1}{R_1C_1R_2C_2s^2 + (R_1C_1 + R_2C_2)s + 1}$$

$$\frac{V_o(s)}{V_i(s)} = \frac{1}{R_1C_1R_2C_2s^2 + (R_1C_1 + R_2C_2)s + 1}$$

IMPULSE RESPONSE

- For a Linear Time-Invariant System , the transfer function T(s) is the ratio of C(s) / R(s) , where R(s) & C(s) are the Laplace transforms of the Input & Output respectively.

  Therefore, C(s) = T(s) R(s)

- Consider the Output response of a system to a **Unit Impulse Input**, where initial conditions are zero.

- Since the Laplace transform of the Unit impulse function is Unity, the Laplace transform of the Output of the system is just C(S) = T(S).

- The Impulse Response of a system is thus the response of a Linear system to a Unit Impulse Input ,when the initial conditions are zero.

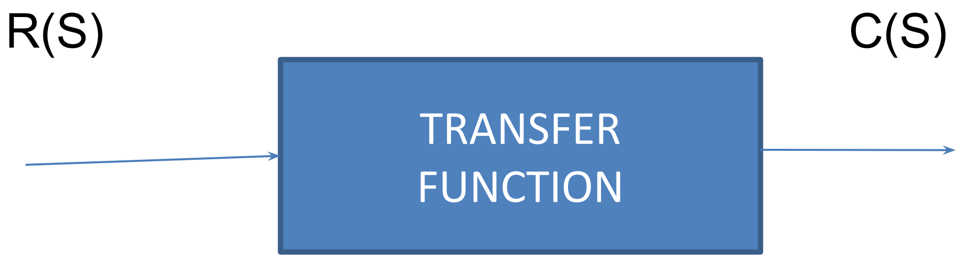
- Thus the transfer function & the Impulse response of a LTI system , contain the same information about the system performance.

- Hence it is possible to obtain complete information about the system by exciting it with an Impulse input & measuring the response.

- In practice a pulse input with a very short duration compared with the significant time constants of the system can be used.

BLOCK DIAGRAMS

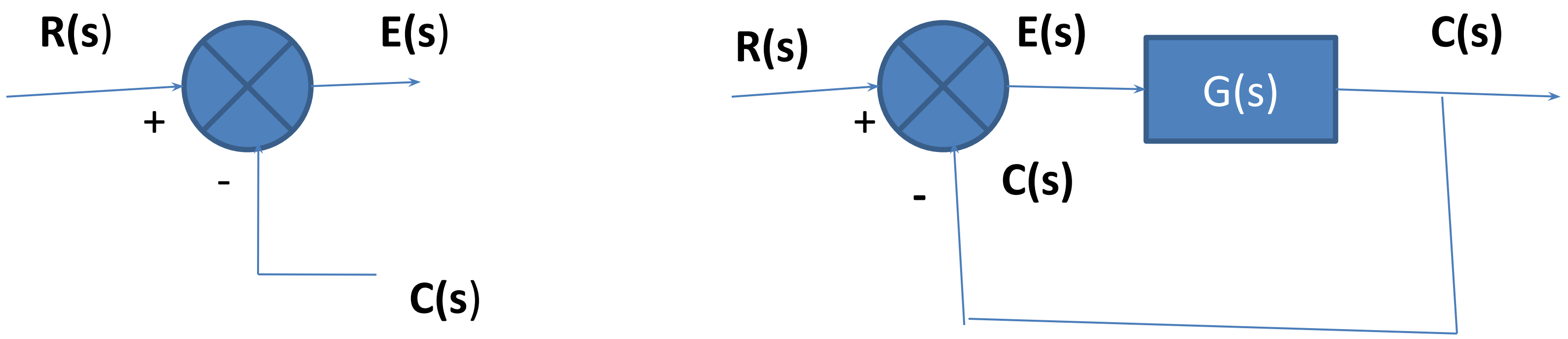
- A block diagram of a system is a pictorial representation of the functions performed by each component & of the flow of signals.
- The transfer function of the components are usually entered in the corresponding blocks , which are connected by arrows to indicate the direction of the flow of signals.
- Fig below shows an element of the block diagram



- The output signal is the input signal multiplied by the transfer function in the block.
- The **Error Detector or Summing Point** produces a signal which is the difference between the Reference Input & the Feedback Signal of the control system.
- A circle with a cross is the symbol which indicates a Summing operation
- The plus or minus sign at the arrow head indicates whether that signal is to be added or subtracted.

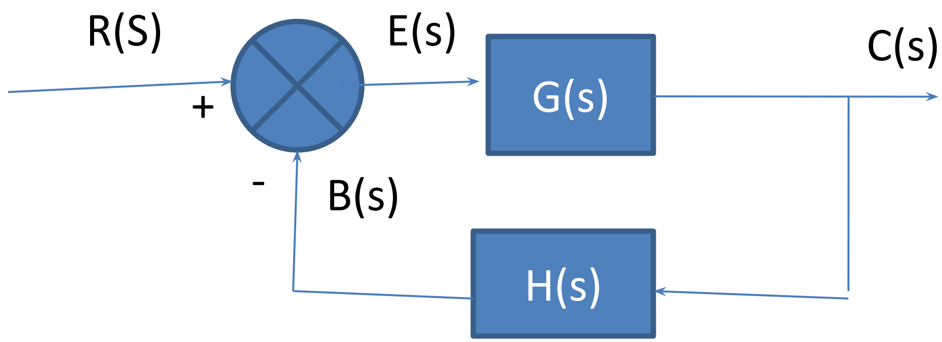
BLOCK DIAGRAMS CONT------

- The quantities being added or subtracted should have the same dimension & same units.



- When the output is fed back to the summing point for comparison with the input ,it is necessary to convert the form of the output signal to that of the input signal.
- This conversion is accomplished by the Feedback element whose Transfer function is **H(s)**.

BLOCK DIAGRAMS CONT------



- Thus the feedback Signal **B(s) = H(s) C(s)**
- The ratio of the Feedback Signal **B(s)** to the Actuating Error Signal **E(s)** is called the **Open Loop Transfer Function (OLTF)**

  OLTF = B(s) / E (s) = H(s) C(s) / E(s) = H(s) G(s)
- Thus **OLTF = G(s) H(s)**
- The ratio of output **C(s)** to the error signal **E(s)** is called the Feed Forward Transfer function

  Feed forward Transfer Function = **C(s) / E(s)**
- The ratio of **C(s) / R(s)** is called the **Closed Loop Transfer Function.**

BLOCK DIAGRAMS CONT------

- The Closed Loop Transfer Function **(CLTF)** may be derived as follows:

$$C(s) / E(s) = G(s)$$

$$C(s) = G(s) E(s)$$

But $$E(s) = R(s) – B(s)$$

$$= R(s) – H(s) C(s)$$

Therefore $$C(s) = G(s) [R(s) – H(s) C(s)]$$

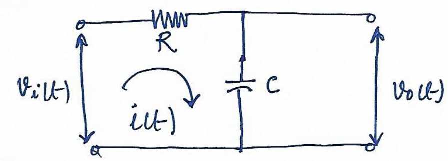
$$= G(s) R(s) – G(s) H(s) C(s)$$

$$C(s) [1+ G(s) H(s)] = G(s) R(s)$$

$$\mathbf{C(S) / R(s) = G(s) / [1+ G(s) H(s)]}$$

For positive Feedback , $\mathbf{C(S) / R(s) = G(s) / [1 - G(s) H(s)]}$

REPRESENTING RC NETWORK IN BLOCK DIAGRAM FO



The equations are

$$i(t) = \frac{v_i(t) - v_o(t)}{R} \longrightarrow ①$$

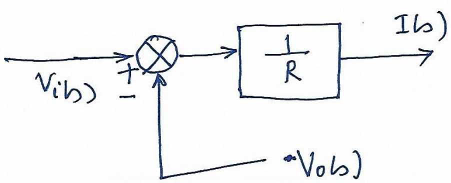
$$v_o(t) = \frac{1}{C}\int i(t)\,dt \longrightarrow ②$$

Taking Laplace Transforms

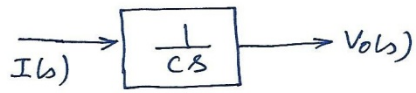
$$I(s) = \frac{V_i(s) - V_o(s)}{R} \longrightarrow ③$$

$$V_o(s) = \frac{1}{Cs} I(s) \longrightarrow ④$$

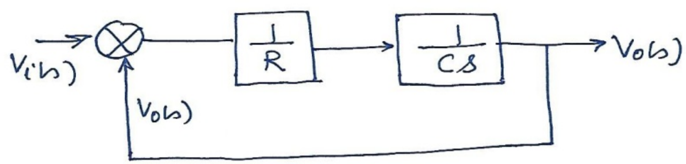
Eqn ③ can be represented as a block shown below

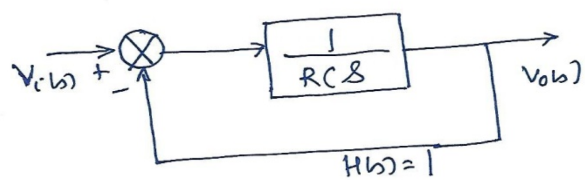


→ Eqn (4) can be represented as



→ Combining the two block, we get



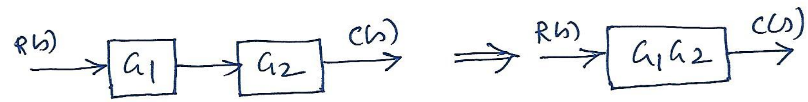


$$\frac{V_o(s)}{V_i(s)} = \frac{\frac{1}{RCs}}{1 + \frac{1}{RCs} \cdot 1}$$

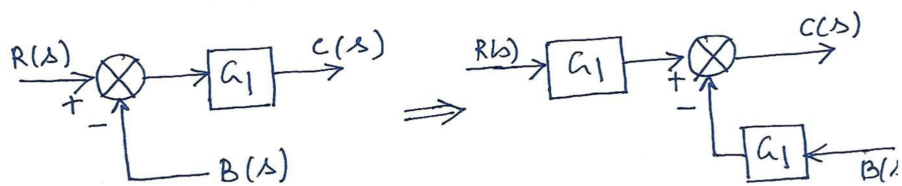
$$\boxed{\frac{V_o(s)}{V_i(s)} = \frac{1}{RCs+1}}$$

RULES FOR BLOCK DIAGRAM REDUCTION

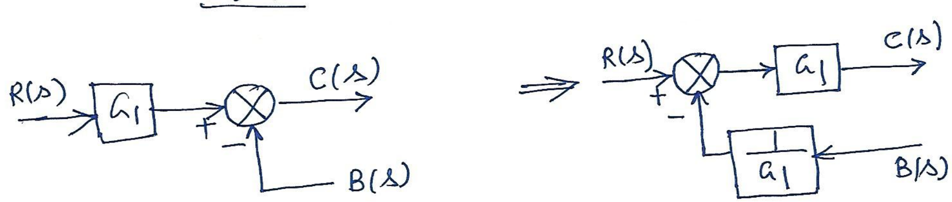
(1) Two blocks in cascade



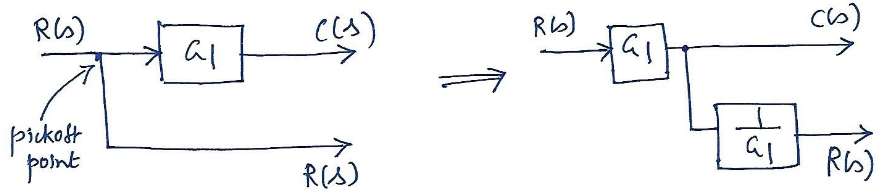
(2) Moving a summing point ahead of a block



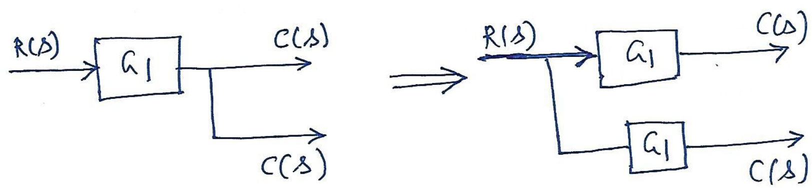
(3) Moving a summing point behind a block



4 Moving a pickoff point ahead of a block



5 Moving a pickoff point behind a block



6 Elimination of a feedback loop