FREQUENCY RESPONSE

- The term frequency response means the steady state response of a system to a sinusoidal input

- If $r(t) = A \sin \omega t$, the steady-state output may be written as $c(t) = B \sin(\omega t + \phi)$

- The magnitude and phase relationship between the sinusoidal input and the steady-state output of a system is termed the frequency response.

- The frequency response test on a system is normally performed by keeping the amplitude A fixed and determining B and ϕ for a suitable range of frequencies.

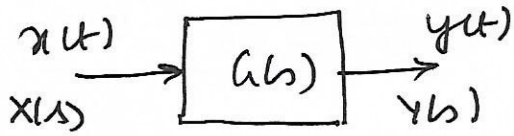
- Signal generation and precise measuring instruments are readily available for various range of frequencies and amplitudes.

- Whenever it is not possible to obtain the form of the transfer function of a system through mathematical techniques, the necessary information to find its transfer function can be obtained by performing the frequency response test on the system.

OBTAINING STEADY-STATE SOLUTIONS TO SINUSOIDAL INPUT

→ The frequency-response characteristic of a system can be obtained directly from the SINUSOIDAL TRANSFER function ie the transfer function in which s is replaced by $j\omega$, where ω is the frequency.

→ Consider the stable linear time-invariant system shown below:



→ The input and output of the system whose transfer function is $G(s)$, are denoted by $x(t)$ and $y(t)$ respectively.

→ If the input $x(t)$ is a sinusoidal signal, the steady-state output will also be a sinusoidal signal of the same frequency but with possibly different magnitude and phase angle

→ Let the input signal be given by

$$x(t) = X \sin \omega t$$

→ Let the transfer function $G(s)$ be written as a ratio of two polynomials in s, ie

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$$G(s) = \frac{p(s)}{q(s)} = \frac{p(s)}{(s+s_1)(s+s_2)\cdots\cdots(s+s_n)}$$

→ The Laplace-transformed output $Y(s)$ is then

$$Y(s) = G(s)\,X(s) = \frac{p(s)}{q(s)}\,X(s)$$

Where $X(s)$ is the Laplace transform of the input $x(t)$.

→ $$Y(s) = G(s)\,\frac{\omega X}{s^2+\omega^2}$$

$$= \frac{p(s)}{(s+s_1)(s+s_2)\cdots\cdots(s+s_n)} \cdot \frac{X\omega}{s^2+\omega^2}$$

→ Expanding into partial fractions,

$$Y(s) = \frac{a}{s+j\omega} + \frac{a^*}{s-j\omega} + \frac{b_1}{s+s_1} + \cdots\cdots + \frac{b_n}{s+s_n}$$

where a and the b_i $(i = 1, 2, \cdots\cdots n)$ are Constants and a^* is the complex conjugate of a.

→ Taking inverse Laplace transforms

$$y(t) = a e^{-j\omega t} + a^* e^{j\omega t} + b_1 e^{-s_1 t} + \cdots\cdots + b_n e^{-s_n t}$$

→ For a stable system, at steady-state, the terms $e^{-s_1 t}, e^{-s_2 t} \cdots\cdots e^{-s_n t}$ approach zero.

→ Thus all the terms on the right hand side, except the first two, drop out at steady state.

→ Thus $y_{ss}(t) = a e^{-j\omega t} + a^* e^{j\omega t}$,

Where a can be evaluated as

$$a = G(s) \frac{\omega X}{s^2+\omega^2} (s+j\omega) \Big|_{s=-j\omega}$$

$$= G(s) \frac{\omega X}{\cancel{(s+j\omega)} (s-j\omega)} \cancel{(s+j\omega)} \Big|_{s=-j\omega}$$

$$a = \frac{G(-j\omega)\, \omega X}{-j\omega - j\omega} = \frac{X\, G(-j\omega)\, \cancel{\omega}}{-2j\cancel{\omega}}$$

$$= -\frac{X\, G(-j\omega)}{2j}$$

→ $$a^* = G(s) \frac{\omega X}{(s^2+\omega^2)} (s-j\omega) \Big|_{s=j\omega}$$

$$= G(s) \frac{\omega X \cancel{(s-j\omega)}}{(s+j\omega) \cancel{(s-j\omega)}} \Big|_{s=j\omega}$$

$$= \frac{G(j\omega)\, \cancel{\omega} X}{2j\cancel{\omega}} = \frac{X\, G(j\omega)}{2j}$$

→ Since $G(j\omega)$ is a complex quantity, it can be written as

$$G(j\omega) = |G(j\omega)| e^{j\phi}$$

where $|G(j\omega)|$ represents the magnitude and ϕ represents the angle of $G(j\omega)$

ie $\phi = \angle G(j\omega)$

→ Similarly $G(-j\omega) = |G(-j\omega)| e^{-j\phi} = |G(j\omega)| e^{-j\phi}$

→ Therefore $y_{ss}(t) = \frac{-X G(-j\omega)}{2j} e^{-j\omega t} + \frac{X G(j\omega)}{2j} e^{j\omega t}$

$$→ y_{ss}(t) = \frac{-X|G(-j\omega)| e^{-j\phi} e^{-j\omega t}}{2j} + \frac{X|G(j\omega)| e^{j\phi} e^{j\omega t}}{2j}$$

$$= X|G(j\omega)| \left[\frac{e^{j(\omega t+\phi)} - e^{-j(\omega t+\phi)}}{2j} \right]$$

$$= X|G(j\omega)| \sin(\omega t+\phi)$$

$$= Y \sin(\omega t+\phi)$$

→ Thus a stable linear time-invariant system subjected to a sinusoidal input will, at steady steady state have a sinusoidal output of the same frequency as the input.

→ But the amplitude and phase of the output will, in general, be different from those of the input.

→ In fact, the amplitude of the output is given by the product of that of the input and $|G(j\omega)|$, while the phase angle differs from that of the input by the amount $\phi = \angle G(j\omega)$

→ A negative phase angle is called phase lag and a positive phase angle is called phase lead.

→ The function $G(j\omega)$ is called the sinusoidal transfer function.

→ It is obtained by substituting $j\omega$ for s in the transfer function.