Quantathon Problem and Solution

February 18, 2017

Consider a game in which you make repeated bets. When you have \$$x$, you may bet any amount of money \$$y$, where $0 \leq y \leq x$. If you win the bet, you will then have \$$(x+y)$, but if you lose you will have \$$(x-y)$. If you ever reach \$0, will not be permitted to bet again.

On each bet, the probability you win is p, where p is a number in the interval $(1/2, 1)$. The probability you lose on a given bet is $q = 1 - p$. The outcomes of the successive bets are independent.

You begin with an initial amount of money \$$X_0$, where $0 < X_0 < 1$. Your goal is to eventually have \$1.

(i) Suppose you are permitted to bet at most n times. Describe a betting strategy that maximizes the probability you reach \$1, and determine the probability this strategy succeeds.

(ii) Show that if $X_0 = 1/2$, then as $n \to \infty$, the probability you can reach \$1 by betting at most n times converges to 1.

Solution. Let's first simplify the language by assuming that if you are permitted to bet at most n times, then you bet exactly n times, although some of the bets you make might be zero. We next observe that if you have x and bet y, then the average of your wealth after the bet is,

$$\frac{1}{2}(x+y) + \frac{1}{2}(x-y) = x.$$

Suppose if you win the first bet, you then bet z_W, but if you lose the first bet, you then bet z_L. After two bets there are four possible amounts of money you have,

$$x + y + z_W, \quad x + y - z_W, \quad x - y + z_L, \quad x - y - z_L.$$

The average of these four numbers is

$$\frac{1}{4}(x+y+z_W) + \frac{1}{4}(x+y-z_W) + \frac{1}{4}(x-y+z_L) + \frac{1}{4}(x-y-z_L) = x.$$

Continuing in this way, we see that if we consider all the possible amounts of wealth you have after any number of bets, the average of these numbers is x.

Once you choose a betting strategy, then you can specify the amount of money X_n you will have after n bets as a function of the outcome of the bets. In other words, X_n can be written as a function of ω, where ω is a sequence of length n of the letters W, for win, and L, for lose. There are 2^n possible such sequences, and we have just argued that

$$\frac{1}{2^n} \sum_{\omega} X_n(\omega) = X_0.$$

You want to choose a betting strategy that makes $\mathbb{P}\{X_n = 1\}$ as large as possible.

For example, if you are permitted at most $n = 2$ bets, your initial capital is $X_0 = 1/2$, and your strategy is to always bet all my money or the amount needed to reach 1, whichever is smaller (this is called the *bold* strategy), then after one bet you have

$$X_1(W) = 1 \text{ or } X_1(L) = 0,$$

depending on whether you win or lose the bet, respectively. After two bets you have

{1.1}
$$X_2(WW) = 1, \quad X_2(WL) = 1, \quad X_2(LW) = 0, \quad X_2(LL) = 0. \tag{1.1}$$

This is because you initially bet $1/2$, and if you win you have 1. Then on the second bet you bet 0, so regardless of whether the second bet results in W or L, you still have 1. On the other hand, if you lose on the first bet, you have 0 after the bet and are forced to bet 0 on the second bet. Therefore, regardless of whether the second bet results in W or L, you have 0 after two bets.

Continuing this example, we see that

{1.2}
$$\frac{1}{4}X_2(WW) + \frac{1}{4}X_2(WL) + \frac{1}{4}X_2(LW) + \frac{1}{4}X_2(LL) = \frac{1}{2} = X_0, \tag{1.2}$$

In fact, as we argued previously, this equation must hold no matter what betting strategy you use. For the bold strategy, starting with initial capital $1/2$, the probability of getting to \$1 with at most two bets is

{1.3}
$$\mathbb{P}\{X_2 = 1\} = \mathbb{P}\{WW, WL\} = p^2 + pq = p(p+q) = p. \tag{1.3}$$

If you use some betting strategy other than the bold strategy, the amount of money X_2 you have after two bets depends on whether you win and lose each of those bets and must still satisfy (1.2). Furthermore, there is no reason to ever bet in such a way that X_2 is strictly larger than 1, since you get no extra credit for exceeding 1. Therefore, the only betting strategies that make sense are those in with $X_2(\omega) = 1$ for two values of ω in $\{WW, WL, LW, LL\}$ and $X_2 = 0$ for the other two values of ω. Since $p > 1/2$, we have

$$p^2 = \mathbb{P}\{WW\} > pq = \mathbb{P}\{WL\} = \mathbb{P}\{LW\} > q^2 = \mathbb{P}\{LL\}.$$

Thus, in order to maximize the probability that $X_2 = 1$, you should choose X_2 as in (1.1).

> *In conclusion, if you begin with $X_0 = 1/2$ and you can make only two bets, then the bold strategy is optimal and the probability this strategy succeeds is given by p.*

We generalize this argument to the case of n bets. For $k = 1, 2, \ldots, n$, let Ω_k be the set of all sequences $\omega_1 \cdots \omega_k$ of length k, where each ω_i is either W or L. The set Ω_k represents all possible outcomes of k successive bets, and there are 2^k sequences in Ω_k. For $\omega_1 \cdots \omega_k$ in Ω_k, we define

$$\begin{aligned} W_k(\omega_1 \cdots \omega_k) &= \text{The number of } W\text{s in the sequence } \omega_1 \cdots \omega_k, \\ L_k(\omega_1 \cdots \omega_k) &= \text{The number of } L\text{s in the sequence } \omega_1 \cdots \omega_k. \end{aligned}$$

Then the probability the sequence $\omega_1 \cdots \omega_k$ occurs is

$$\mathbb{P}\{\omega_1 \cdots \omega_k\} = p^{W_k(\omega_1 \cdots \omega_k)} q^{L_k(\omega_1 \cdots \omega_k)}.$$

A betting strategy begins with initial capital X_0 and produces a random variable $X_n(\omega_1 \cdots \omega_n)$ of money possessed after n bets. We wish to choose a betting strategy that maximizes

$$\sum_{\{\omega_1 \cdots \omega_n : X_n(\omega_1 \cdots \omega_n) \geq 1\}} P\{\omega_1 \cdots \omega_n\}.$$

We are not permitted to ever have less than 0 and there is no reason to ever have more than 1, and hence we restrict ourselves to betting strategies for which

$$0 \leq X_n(\omega_1 \cdots \omega_n) \leq 1 \text{ for all } \omega_1 \cdots \omega_n \in \Omega_n. \tag{1.4}$$
{1.4}

We argued earlier that regardless of the betting strategy we use,

$$\frac{1}{2^n} \sum_{\omega_1 \cdots \omega_n \in \Omega_n} X_n(\omega_1 \cdots \omega_n) = X_0. \tag{1.5}$$
{1.6}

With these constraints, we seek to maximize

$$\sum_{\{\omega_1 \cdots \omega_n : X_n(\omega_1 \cdots \omega_n) = 1\}} \mathbb{P}\{\omega_1 \cdots \omega_n\}. \tag{1.6}$$
{1.5}

Let us consider the problem of maximizing the sum in (1.6) subject to the constraints (1.4) and (1.5). Clearly we should take $X_n(\omega) = 1$ for as many values of ω as we can, and (1.5) says that the number of ωs for which we can take $X_n(\omega_1 \cdots \omega_n) = 1$ is $\lfloor 2^n X_0 \rfloor$, where $\lfloor 2^n X_0 \rfloor$ denotes the greatest integer less than or equal to $2^n X_0$. The $\omega_1 \cdots \omega_n$s we choose for which we want $X_n(\omega_1 \cdots \omega_n)$ to be 1 are those with the largest values of $\mathbb{P}\{\omega_1 \cdots \omega_n\}$ so that (1.6) is maximized. Because $p > 1/2$, the $\omega_1 \cdots \omega_n$ with the largest probability is the sequence $\omega_1 \cdots \omega_n$ for which $\omega_i = W$ for all $i = 1, 2, \ldots, n$. The sequences with the next highest probability are the n sequences with $n-1$ Ws and 1 L. After that are $\binom{n}{n-2}$ sequences that have $(n-2)$ Ws and 2 Ls.

We construct a candidate random variable X_n^* that maximizes (1.6) among all the random variables that satisfy (1.4) and (1.5). Define

$$k_n(X_0) = \max \left\{ k \in \{0, 1, \ldots, n\} : \sum_{j=0}^{k} \binom{n}{n-j} \leq \lfloor 2^n X_0 \rfloor \right\}. \tag{1.7}$$
{1.7}

The Binomial Theorem says that

$$2^n = (1+1)^n = \sum_{j=0}^{n} \binom{n}{n-j},$$

and because $0 < X_0 < 1$, we have $k_n(X_0) \leq n-1$. The number of values of $\omega_1 \cdots \omega_n$ for which we can have $X_n(\omega_1 \cdots \omega_n) = 1$ is $\lfloor 2^n X_0 \rfloor$, and the $\omega_1 \cdots \omega_n$s we should designate for this honor are all those for which the number of losses is $k_n(X_0)$ or fewer, and in addition, $\lfloor 2^n X_0 \rfloor - k_n(X_0)$ of those for which the number of losses is $k_n(X_0)+1$. Then the probability that $X_n^* = 1$ is

$$\mathbb{P}\{X_n^* = 1\} = \sum_{j=0}^{k_n(X_0)} \binom{n}{n-j} p^{n-j} q^j + \big(\lfloor 2^n X_0 \rfloor] - k_n(X_0)\big) p^{n-k_n(X_0)-1} q^{k_n(X_0)+1}. \quad (1.8)$$ {1.8}

There is more than one way to choose the $(\lfloor 2^n X_0 \rfloor - k_n(X_0))$ sequences $\omega_1 \cdots \omega_n$ with $(k_n(X_0)+1)$ Ls to which we assign the value $X_n^*(\omega_1 \cdots \omega_n) = 1$. It does not matter which of these sequences we choose, so we make a choice and use it for the rest of this solution.

Once we have made these choices and assigned the value 1 to $X_n^*(\omega_1 \cdots \omega_n)$ for the chosen sequences, we have

{1.9}
$$\frac{1}{2^n} \sum_{\{\omega_1 \cdots \omega_n : X_n^*(\omega_1 \cdots \omega_n) = 1\}} 1 = \frac{1}{2^n} \lfloor 2^n X_0 \rfloor. \quad (1.9)$$

If $2^n X_0$ is an integer, then $\frac{1}{2^n} \lfloor 2^n X_0 \rfloor = X_0$, we assign the value 0 to $X_n^*(\omega_1 \cdots \omega_n)$ for the remaining sequences, and we have (1.5). However, if $2^n X_0$ is not an integer, then

$$0 < \frac{1}{2^n} \lfloor 2^n X_0 \rfloor < X_0.$$

We choose one of the remaining sequences $\omega_1 \cdots \omega_n$ with $(k_n(X_0)+1)$ Ls and set X_n^* for that sequence to be $2^n X_0 - \lfloor 2^n X_0 \rfloor$, which is strictly less than 1. For all other sequences, we set $X_n^* = 0$. Then we have as a modification of (1.9) that

$$\frac{1}{2^n} \sum_{\omega_1 \cdots \omega_n \in \Omega_n} X_n^*(\omega_1 \cdots \omega_n) = \frac{1}{2^n} \left(\sum_{\{\omega_1 \cdots \omega_n : X_n^*(\omega_1 \cdots \omega_n) = 1\}} 1 + \big(2^n x - \lfloor 2^n x \rfloor\big) \right) = X_0.$$

We have constructed a random variable X_n^* satisfying both the constraints (1.4) and (1.5). We cannot have a greater probability of success with n bets than the probability given in (1.8). We have maximized (1.6), subject to the constraints (1.4) and (1.5).

However, we are not done yet. We still need to show that there is a betting strategy that generates the random variable X_n^*. To aid in this, we define recursively (backward in the index k)

{1.11}
$$X_k^*(\omega_1 \cdots \omega_k) = \frac{1}{2} X_{k+1}^*(\omega_1 \cdots \omega_k W) + \frac{1}{2} X_{k+1}^*(\omega_1 \cdots \omega_k L) \quad (1.10)$$

for all sequences $\omega_1 \cdots \omega_k \in \Omega_k$ and for $k = n-1, n-2, \ldots, 1$. We then define

{1.11b}
$$X_0^* = \frac{1}{2} X_1^*(W) + \frac{1}{2} X_2^*(L). \quad (1.11)$$

These random variables have the following properties.

Theorem. *For $k = n, n-1, \ldots, 1$,*

{1.11a}
$$\frac{1}{2^k} \sum_{\omega_1 \cdots \omega_k \in \Omega_k} X_k^*(\omega_1 \cdots \omega_k) = X_0, \tag{1.12}$$

and for every pair of sequences $\omega_1 \cdots \omega_k$ and $\omega_1' \cdots \omega_k'$ in Ω_k with

$$W_k(\omega_1 \cdots \omega_k) \geq W_k(\omega_1' \cdots \omega_k'), \tag{1.13}$$ {1.12}

we have

$$X_k^*(\omega_1 \cdots \omega_k) \geq X_k^*(\omega_1' \cdots \omega_k'). \tag{1.14}$$ {1.13}

In addition,

$$X_0^* = X_0. \tag{1.15}$$ {1.14a}

Proof: We have constructed X_n^* so that

$$\frac{1}{2^n} \sum_{\omega_1 \cdots \omega_n \in \Omega_n} X_n^*(\omega_1 \cdots \omega_n) = X_0,$$

and for every pair of sequences $\omega_1 \cdots \omega_n$ and $\omega_1' \cdots \omega_n'$ in Ω_n with

$$W_n(\omega_1 \cdots \omega_n) \geq W_n(\omega_1' \cdots \omega_n'),$$

we have

$$X_n^*(\omega_1 \cdots \omega_n) \geq X_n^*(\omega_1' \cdots \omega_n')$$

We prove (1.12) and the implication (1.13)$\Longrightarrow$(1.14) by induction backward in k, from $k = n$ to $k = 1$, with the base case $k = n$ just established.

We make the induction hypothesis that (1.12) and the implication (1.13)$\Longrightarrow$(1.14) hold for some $k+1 \in \{2, 3, \ldots, n\}$, i.e.,

$$\frac{1}{2^{k+1}} \sum_{\omega_1 \cdots \omega_k \omega_{k+1} \in \Omega_{k+1}} X_{k+1}^*(\omega_1 \cdots \omega_k \omega_{k+1}) = X_0, \tag{1.16}$$ {1.14}

and for every pair of sequences $\omega_1 \cdots \omega_k \omega_{k+1}$ and $\omega_1' \cdots \omega_k' \omega_{k+1}'$ in Ω_{k+1} with

$$W_{k+1}(\omega_1 \cdots \omega_k \omega_{k+1}) \geq W_{k+1}(\omega_1' \cdots \omega_k' \omega_{k+1}') \tag{1.17}$$ {1.15}

we have

$$X_{k+1}^*(\omega_1 \cdots \omega_k \omega_{k+1}) \geq X_{k+1}^*(\omega_1' \cdots \omega_k' \omega_{k+1}'). \tag{1.18}$$ {1.16}

We then prove (1.12) and the implication (1.13)$\Longrightarrow$(1.14) for k. From (1.16) and using (1.10), we have

$$
\begin{aligned}
\frac{1}{2^k}\sum_{\omega_1\cdots\omega_k\in\Omega_k} X_k^*(\omega_1\cdots\omega_k) &= \frac{1}{2^k}\sum_{\omega_1\cdots\omega_k\in\Omega_k}\left(\frac{1}{2}X_{k+1}^*(\omega_1\cdots\omega_k W)+\frac{1}{2}X_{k+1}^*(\omega_1\cdots\omega_k L)\right)\\
&= \frac{1}{2^{k+1}}\sum_{\omega_1\cdots\omega_k\omega_{k+1}\in\Omega_{k+1}} X_{k+1}^*(\omega_1\cdots\omega_k\omega_{k+1})\\
&= X_0,
\end{aligned}
$$

which is (1.12). We next let $\omega_1\cdots\omega_k$ and $\omega_1'\cdots\omega_k'$ be a pair of sequences in Ω_k for which (1.13) holds. Therefore,

$$
\begin{aligned}
W_{k+1}(\omega_1\cdots\omega_k W) &\geq W_{k+1}(\omega_1'\cdots\omega_k' W),\\
W_{k+1}(\omega_1\cdots\omega_k L) &\geq W_{k+1}(\omega_1'\cdots\omega_k' L),
\end{aligned}
$$

and by the induction hypothesis,

$$
\begin{aligned}
X_{k+1}^*(\omega_1\cdots\omega_k W) &\geq X_{k+1}^*(\omega_1'\cdots\omega_k' W),\\
X_{k+1}^*(\omega_1\cdots\omega_k L) &\geq X_{k+1}^*(\omega_1'\cdots\omega_k' L).
\end{aligned}
$$

From these inequalities and (1.10) we have

$$
\begin{aligned}
X_k^*(\omega_1\cdots\omega_k) &= \frac{1}{2}X_{k+1}^*(\omega_1\cdots\omega_k W)+\frac{1}{2}X_{k+1}^*(\omega_1\cdots\omega_k L)\\
&\geq \frac{1}{2}X_{k+1}^*(\omega_1'\cdots\omega_k' W)+\frac{1}{2}X_{k+1}^*(\omega_1'\cdots\omega_k' L)\\
&= X_k^*(\omega_1'\cdots\omega_k').
\end{aligned}
$$

We have established (1.14).

Finally, (1.12) with $k=1$ implies

$$
\frac{1}{2}X_1^*(W)+\frac{1}{2}X_1^*(L)=X_0.
$$

Comparing this we (1.11), we obtain (1.15). □

Armed with the theorem, we can create a betting strategy that generates the random variable X_n^*, and indeed, generates the random variables X_k^*, $k=1,2,\ldots,n$, at each step along the way to the final nth bet. We want to take the initial bet to be $X_1^*(W)-X_0^*=X_1^*(W)-X_0$. Because of (1.11) and the fact, just proved in the theorem, that $X_1^*(W)\geq X_1^*(L)$, we see that this quantity is nonnegative, so we can bet this amount. Because of (1.11), we also see that this bet is

$$
\begin{aligned}
X_1^*(W)-X_0^* &= X_1^*(W)-\left(\frac{1}{2}X_1^*(W)+\frac{1}{2}X_1^*(L)\right)\\
&= \left(\frac{1}{2}X_1^*(W)+\frac{1}{2}X_1^*(L)\right)-X_1^*(L)\\
&= X_0^*-X_1^*(L).
\end{aligned}
$$

If we win this bet, then we have

$$X_0^* + \big(X_1^*(W) - X_0^*\big) = X_1^*(W).$$

If we lose, we have

$$X_0^* - \big(X_0^* - X_1^*(L)\big) = X_1^*(L).$$

This is the base case for an induction argument, this time forward in in k from $k = 1$ to $k = n$.

Suppose that for all $\omega_1 \cdots \omega_k \in \Omega_k$, the amount of money we have if the first k bets result in $\omega_1 \cdots \omega_k$ is $X_k^*(\omega_1 \cdots \omega_k)$. We want to make the $(k+1)$st bet of size

$$X_{k+1}^*(\omega_1 \cdots \omega_k W) - X_k^*(\omega_1 \cdots \omega_k).$$

Because $X_{k+1}^*(\omega_1 \cdots \omega_k W) \geq X_{k+1}^*(\omega_1 \cdots \omega_k L)$ and (1.10) holds, this quantity is nonnegative, so we can bet this amount. Because of (1.10), this bet is

$$\begin{aligned}
&X_{k+1}^*(\omega_1 \cdots \omega_k W) - X_k^*(\omega_1 \cdots \omega_k)\\
&= X_{k+1}^*(\omega_1 \cdots \omega_k W) - \left(\frac{1}{2}X_{k+1}^*(\omega_1 \cdots \omega_k W) + \frac{1}{2}X_{k+1}^*(\omega_1 \cdots \omega_k L)\right)\\
&= \left(\frac{1}{2}X_{k+1}^*(\omega_1 \cdots \omega_k W) + \frac{1}{2}X_{k+1}^*(\omega_1 \cdots \omega_k L)\right) - X_{k+1}^*(\omega_1 \cdots \omega_k L)\\
&= X_k^*(\omega_1 \cdots \omega_k) - X_{k+1}^*(\omega_1 \cdots \omega_k L).
\end{aligned}$$

If we win, we have

$$X_k^*(\omega_1 \cdots \omega_k) + \big(X_{k+1}^*(\omega_1 \cdots \omega_k W) - X_k^*(\omega_1 \cdots \omega_k)\big) = X_{k+1}^*(\omega_1 \cdots \omega_1 W).$$

If we lose, we have

$$X_k^*(\omega_1 \cdots \omega_k) - \big(X_k^*(\omega_1 \cdots \omega_k) - X_{k+1}^*(\omega_1 \cdots \omega_k L)\big) = X_{k+1}^*(\omega_1 \cdots \omega_k L).$$

Because $\omega_1 \cdots \omega_k$ is an arbitrary sequence in Ω_k, we have shown that for every sequence $\omega_1 \cdots \omega_k \omega_{k+1} \in \Omega_{k+1}$, the amount of money we have after $k+1$ bets is $X_{k+1}^*(\omega_1 \cdots \omega_k \omega_{k+1})$. In particular, with the betting strategy just described, after n bets we have $X_n^*(\omega_1 \cdots \omega_n)$, regardless of the outcome $\omega_1 \cdots \omega_n$ of the n bets. This ends part (i) of the problem.

We turn to part (ii) of the problem, where we assume that $X_0 = 1/2$. For the moment, we consider only the case that n is odd. Then

$$\sum_{j=0}^{(n-1)/2} \binom{n}{n-j} = \sum_{j=(n+1)/2}^{n} \binom{n}{n-j} = 2^{n-1}.$$

This means that

$$k_n(X_0) = \max\left\{k \in \{0, 1, \dots, n\} : \sum_{j=0}^{k} \binom{n}{n-j} \leq 2^{n-1}\right\} = \frac{n-1}{2}. \qquad (1.19) \quad \texttt{\{1.19d\}}$$

For $i = 1, \ldots, n$, let $Y_i = 1$ if we win on the i-th bet and $Y_i = 0$ if we lose on the i-th bet. Then $Y_1, \ldots, Y_n$ are independent, identically distributed random variables with

$$\mathbb{P}\{Y_i = 1\} = p, \quad \mathbb{P}\{Y_i = 0\} = q, \quad i = 1, \ldots, n.$$

We define $S_n = \sum_{i=1}^n Y_i$ to be the number of bets won among the n bets. We saw just prior to (1.8) that if the number of losses is $k_n(X_0)$ or fewer, i.e., the number of wins is $n - k_n(X_0) = (n+1)/2$ or greater, then $X_n^* = 1$. This implies

{1.19} $$\mathbb{P}\{S_n \geq (n+1)/2\} \leq \mathbb{P}\{X_n^* = 1\}. \tag{1.20}$$

The random variable S_n has the binomial distribution with parameter p:

$$\mathbb{P}\{S_n = j\} = \binom{n}{j} p^j q^{n-j}, \quad j = 0, 1, \ldots, n.$$

We can use the Central Limit Theorem to estimate the probability on the left-hand side of (1.20). Because

$$\mathbb{E}Y_i = p, \quad \mathrm{Var}(Y_i) = pq, \quad i = 1, \ldots, n,$$

$(Y_i - p)/\sqrt{pq}$ has expected value 0 and standard deviation 1. The Central Limit Theorem states that as n approaches infinity, the distribution of

$$\frac{1}{\sqrt{n}} \sum_{i=1}^n \frac{Y_i - p}{\sqrt{pq}} = \frac{S_n - np}{\sqrt{npq}}$$

approaches the standard normal distribution. In particular, for every $z \in \mathbb{R}$,

{1.21} $$\lim_{n \to \infty} \mathbb{P}\left\{\frac{S_n - np}{\sqrt{npq}} \geq z\right\} = 1 - N(z) := \frac{1}{\sqrt{2\pi}} \int_z^\infty e^{-\zeta^2/2} d\zeta. \tag{1.21}$$

We have

{1.22} $$\begin{aligned} \mathbb{P}\left\{S_n \geq (n+1)/2\right\} &= \mathbb{P}\left\{\frac{S_n - np}{\sqrt{npq}} \geq \frac{(n+1)/2 - np}{\sqrt{npq}}\right\} \\ &= \mathbb{P}\left\{\frac{S_n - np}{\sqrt{npq}} \geq -\sqrt{n}\left(\frac{p - 1/2}{\sqrt{pq}}\right) + \frac{1}{2\sqrt{npq}}\right\}. \end{aligned} \tag{1.22}$$

Given $\varepsilon > 0$, there exists z_ε (for small ε, z_ε is negative with large absolute value) such that $1 - N(z_\varepsilon) \geq 1 - \varepsilon$. Because $p > 1/2$, we can then choose n_ε such that

$$-\sqrt{n}\left(\frac{p - 1/2}{\sqrt{pq}}\right) + \frac{1}{2\sqrt{npq}} \leq z_\varepsilon \text{ for all } n \geq n_\varepsilon.$$

For $n \geq n_\varepsilon$, n odd, we have from (1.20) and (1.22) that

$$\mathbb{P}\{X_n^* = 1\} \geq \mathbb{P}\{S_n \geq (n+1)/2\} \geq \mathbb{P}\left\{\frac{S_n - np}{\sqrt{npq}} \geq z_\varepsilon\right\}.$$

Letting $n \to \infty$ through the odd integers, we obtain

$$\lim_{n\to\infty} \mathbb{P}\{X_n^* = 1\} \geq \lim_{n\to\infty} \mathbb{P}\left\{ \frac{S_n - np}{\sqrt{npq}} \geq z_\varepsilon \right\} = 1 - N(z_\varepsilon) \geq 1 - \varepsilon.$$

We have shown that

$$\lim_{n\to\infty} \mathbb{P}\{X_n^* = 1\} \geq 1 - \varepsilon$$

for every $\varepsilon > 0$, which means we must in fact have

$$\lim_{n\to\infty} \mathbb{P}\{X_n^* = 1\} = 1. \tag{1.23}$$

{1.23}

The limit in (1.23) is through the odd integers, but it is always better to have more betting opportunities, so

$$\mathbb{P}\{X_n^* = 1\} \leq \mathbb{P}\{X_{n+1}^* = 1\},$$

and when n is odd, $n + 1$ is even. Therefore, (1.23) must also hold if we take the limit through the even integers.