Quantathon Problem

February 3, 2018

Consider a symmetric random walk X_n on the integers. We repeatedly toss a fair coin. If after n tosses the state of the random walk is X_n, then after the $(n+1)$-st toss, the state is either X_n+1 or X_n-1, depending on whether the outcome of the toss is a Head or a Tail, respectively.

(i) You are invited to play a game for as many tosses of the fair coin as you wish. At any time, you may choose to stop playing, and if you do that, the game is over. As long as you play the game, you pay 1/4 before each toss of the coin. If you stop after the n-th toss (n can be 0, in which case you stop immediately and pay nothing), you receive the payoff $|X_n|$. Your goal is to find a stopping strategy that maximizes your expected payoff.

For example, suppose $X_0 = 0$ and you decide to pay 1/4 and play for the first toss. If that toss results in Tail, which happens with probability 1/2, then $X_1 = -1$. Suppose you decide to stop and receive $|X_1| = 1$. On the other hand, if the toss results in Head so that $X_1 = 1$, then you decide to pay another 1/4 and play for the second toss, after which you stop. After the second toss, you have $X_2 = 0$ or $X_2 = 2$. The probability of each of these outcomes is 1/4 (1/2 probability for Head on the first toss and Tail on the second toss and 1/2 probability for Head on the first toss and Head on the second toss). Therefore, the expected payoff, not taking fees into account, is

$$\frac{1}{2} \cdot 1 + \frac{1}{4} \cdot 0 + \frac{1}{4} \cdot 2 = 1.$$

However, you paid 1/4 for the first toss, and with probabilty 1/2 there is a second toss for which you pay another 1/4, so the expected value of the fees paid is

$$\frac{1}{4} + \frac{1}{2} \cdot \frac{1}{4} = \frac{3}{8}.$$

Therefore, the expected payoff of this strategy, taking the fees into account is

$$1 - \frac{3}{8} = \frac{5}{8}.$$

Starting with initial condition $X_0 = 0$, what is the largest expected payoff you can receive, taking the fees for playing into account, and what is a stopping strategy that achieves it?[1]

(ii) Playing the same game, suppose you pay a rather than 1/4 before each toss, where $a = 2^{-(m+1)}$ for some positive integer m. In terms of a, what is the largest expected payoff you can receive, and what is a stopping strategy that achieves it?

(iii) Suppose in part (ii) that $a = 0$. What can you say about the largest expected payoff and a stopping strategy that achieves it?

[1] You may use the following fact. Denote by τ the number of tosses you purchase. Then τ is a random variable whose expected value is defined to be $\mathbb{E}\tau = \sum_{j=0}^{\infty} j \times (\text{Probability that } \tau \text{ equals } j)$. This might be infinite. However, if it is, then $\mathbb{E}\big[|X_\tau| - a\tau\big] = -\infty$ for every $a > 0$, and this is true no matter what is the initial condition for X_0.