5 A. a. $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$

b. $|0\rangle$

c. $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$.

B. a. $\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$

b. $|1\rangle$

c. $\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$.

Both the parts have a similar solution for (d.). The first beam splitter creates superposition. For $|0\rangle$ it creates $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$, for $|1\rangle$ it creates $\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$.

The second beam splitter however can produce the initial input as the output. The math is as follows:

Initial input for B.S. 2 : $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$.

$$B.S\left(\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)\right) = \frac{1}{\sqrt{2}}\left(B.S(|0\rangle) + B.S(|1\rangle)\right).$$

$$= \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) + \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)\right)$$

$$= \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}\right)\Big(|0\rangle + |1\rangle + |0\rangle - |1\rangle\Big)$$

$$= \frac{1}{2}\left(2|0\rangle\right) = |0\rangle$$

Similarly for $\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$ we get $|1\rangle$.

$$B.S\left(\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)\right) = \frac{1}{\sqrt{2}}\Big(B.S(|0\rangle) - B.S(|1\rangle)\Big)$$

$$= \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) - \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)\right)$$

$$= \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}\right)\Big(|0\rangle + |1\rangle - (|0\rangle + |1\rangle)\Big)$$

$$= \frac{1}{2}\left(2|1\rangle\right) = |1\rangle.$$

Again if we pass this through another B.S, we get a state of superposition. Here, in this context, $|0\rangle$ and $|1\rangle$ individually are called as "basis states".

If you wonder how a beam splitter works, since we learned that $|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $|1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ in the vector representation.

The beam splitter can be represented as $\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

which is a similar matrix representation of H gate.

8. To find: $\langle 0|\psi\rangle = ?$

Given: $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$

$$\langle 0|\psi\rangle = \langle 0 (\alpha|0\rangle + \beta|1\rangle)$$
$$= \alpha\langle 0|0\rangle + \beta\langle 0|1\rangle$$

We know $|0\rangle = \begin{bmatrix}1\\0\end{bmatrix}$ and $|1\rangle = \begin{bmatrix}0\\1\end{bmatrix}$. Similarly.

$\langle 0| = \begin{bmatrix}1 & 0\end{bmatrix}$ and $\langle 1| = \begin{bmatrix}0 & 1\end{bmatrix}$.

$$\langle 0|\psi\rangle = \alpha \begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} + \beta \begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix}$$
$$= \alpha\big(1(1) + (0)(0)\big) + \beta\big(1(0) + 0(1)\big)$$
$$= \alpha(1) + \beta(0) = \alpha.$$

Probability of $|\langle 0|\psi\rangle|^2 = \underline{\underline{\alpha^2}}$