Basic and Free Variables

$x_1\ x_2\ x_3\ x_4$

$$\left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & 3 & 4 & 5 \\ 0 & 0 & 1 & 4 & 5 \\ 0 & 0 & 0 & 1 & 5 \end{array}\right]$$

Basic vars
Free vars
○ Pivots
≥1 free var → ∞ soln
0 free var → 1 soln

$x_1\ x_2\ x_3\ x_4\ x_5\ x_6\ x_7$

$$\left[\begin{array}{ccccccc|c} 1 & 2 & 3 & 4 & 5 & 6 & 7 & b_1 \\ 0 & 0 & 1 & 4 & 5 & 6 & 7 & b_2 \\ 0 & 0 & 0 & 1 & 5 & 6 & 7 & b_3 \\ 0 & 0 & 0 & 0 & 0 & 1 & 7 & b_4 \end{array}\right]$$

Matrix Multiplication

For each row of A, multiply and sum for each col of B

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{bmatrix}$$

Matrix Vector Mult

A m×n, $\vec{x}$ n×1, $\vec{b}$ m×1

$$A\vec{x} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{bmatrix} = x_1\begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2\begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + x_n\begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

2 Definitions of Linear Dependence

① $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ LD if $\alpha_1\vec{v}_1 + \cdots + \alpha_n\vec{v}_n = \vec{0}$

② $\vec{v}_i = \sum_{j \neq i} \alpha_j \vec{v}_j$

Span

- $\text{span}\{v_1, \ldots, v_n\} = \left\{\sum_{i=1} \alpha_i \vec{v}_i \mid \alpha_i \in \mathbb{R}\right\}$
- set of all linear combos of $\{v_1, \ldots, v_n\}$
- span of a set of vectors is a subspace
- span(A) = range(A)
- span(A) = columnspace(A)

Is $\vec{v}$ in the span $\{[\vec{a}], [\vec{b}], [\vec{c}]\}$?

To solve this, aug matrix & GE

$\left[\begin{array}{ccc|c} \vec{a} & \vec{b} & \vec{c} & \vec{v} \end{array}\right]$ → if no soln, $\vec{v}$ not in the span

Are $\vec{v}_1, \vec{v}_2, \vec{v}_3$ Lin ind?

need to find nullspace. If trivial, LI. If nontrivial, LD

$\left[\begin{array}{ccc|c} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 & \vec{0} \end{array}\right]$

State-transition matrix

$$A = \begin{bmatrix} P_{1\to1} & P_{2\to1} & \cdots & P_{n\to1} \\ P_{1\to2} & P_{2\to2} & \cdots & P_{n\to2} \\ \vdots & & \ddots & \vdots \\ P_{1\to n} & P_{2\to n} & & P_{n\to n} \end{bmatrix}$$

=1 → conservative system

$A\vec{x}[n] = \vec{x}[n+1]$ (time n, time n+1)

$A^{-1}\vec{x}[n] = \vec{x}[n-1]$

if the inverse exists
↳ note inverse is UNIQUE

Calculating Matrix Inv.

$\left[\begin{array}{c|c} A & I_n \end{array}\right] \to ge \to \left[\begin{array}{c|c} I_n & A^{-1} \end{array}\right]$

note: A^{-1} doesn't have to be in RREF

note: for $A = BC$ (m×n, n×m), A^{-1} DNE because C has more columns than rows → LD

Basis

- for $\{v_1, \ldots, v_n\} = S$, the vectors in S are a basis for $\mathbb{V}$ if ① they're LI ② their span is $\mathbb{V}$
- minimal set of spanning vectors
- for $\mathbb{R}^N$, N LI vectors form a basis

Dimension

- dimension($\mathbb{V}$) equals # vectors in its basis
- $\dim(\mathbb{R}^N) = N$

Subspace

$\mathbb{U}$ is a subspace of $\mathbb{V}$ if: ① contains $\vec{0}$ ② closed under vector + ③ closed under scalar ×

subspace basis = LI vectors spanning $\mathbb{U}$
subspace dimension = # vectors in basis

Columnspace

Col(A) where $A_{m\times n}$
= span n columns of A
= range(A)
form a subspace for $\mathbb{R}^m$

Rowspace

= span n rows of A

Rank

= dim(col(A)) = # pivots in RREF
= dim(range(A)) can be at most min(m,n) if A an n×m matrix
= dim(span(A)) ← columns

Rank-Nullity Thm

dim(range(A)) + dim(N(A)) = n (A matrix m×n) (row, col)

Nullspace

set of all $\vec{x}$ s.t. $A\vec{x} = \vec{0}$
- if $\vec{x} = \vec{0}$ is only soln, trivial nullspace
- solve for free vars, write as a vector, tada!

ex:

$$\left[\begin{array}{cccccc|c} 1 & 1 & 0 & -2 & 3 & & 0 \\ 0 & 0 & 1 & -1 & 1 & & 0 \\ 0 & 0 & 0 & 0 & 0 & & 0 \end{array}\right]$$

x_2, x_4, x_5 free vars

$x_2 = \alpha$, $x_4 = \beta$, $x_5 = \gamma$

$x_1 = -\alpha + 2\beta - 3\gamma$
$x_3 = \beta - \gamma$
~~$x_4 = \beta$~~
~~$x_5 = \gamma$~~

write as vector sum

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}\alpha + \begin{bmatrix} 2 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}\beta + \begin{bmatrix} -3 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}\gamma$$

$N(A) = \left\{\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}\right\}$

dim(N(A)) = 3

Eigenstuff

- to calculate λs, solve $\det(A - \lambda I) = 0$ for λ
- to calculate $\vec{v}_i$ solve $(A - \lambda I)\vec{v} = \vec{0}$ for $\vec{v}$

$A\vec{x} = \lambda\vec{x}$
$(A - \lambda I)\vec{x} = 0$

① Find λs — for an n×n matrix, we should have n λ's
② Find eigenvectors corresponding by plugging in $\lambda_1, \ldots, \lambda_n$ into $(A - \lambda I)$
 — if a matrix has m eigenvalues, all eigenvectors are LI
- if λ = 0 → not invertible, nontrivial nullspace
 λ = 1 → steady state
- for a 2×2 matrix: $\lambda^2 - (a+d)\lambda + (ad - bc) = 0$ — basis for N(A) = eigenvectors
- distinct eigenvectors form a subspace — repeated e-values can have 1 or 2 e-vecs

Steady-state $\vec{x}^* = P\vec{x}^*$

- to find steady state, substitute λ = 1, solve for nullspace and that's the st-state

ex: $(A - \lambda I)\vec{x}^* = 0$

$A - \lambda I$ (λ=1)

$$\begin{bmatrix} -1 & 0 & 0 & 1/2 \\ 1/3 & -1 & 0 & 0 \\ 1/3 & 1/2 & -1 & 0 \\ 1/3 & 1/2 & 0 & -1/2 \end{bmatrix} \rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & -2 & 0 \\ 0 & 1 & 0 & -2/3 & 0 \\ 0 & 0 & 1 & -3/2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \rightarrow \begin{array}{l} x_1 = 2x_4 \\ x_2 = 2/3\, x_4 \\ x_3 = 3/2\, x_4 \end{array} \rightarrow \vec{x}^* = \begin{bmatrix} 2 \\ 2/3 \\ 3/2 \\ 1 \end{bmatrix} x_4$$

If you start pumps with A_0, B_0 and C_0, what's the associated steady-state?

① $A_0 + B_0 + C_0 = D$ ② given $x^* = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\alpha$, $x_1 + x_2 + x_3 = E$ ③ $D = E\alpha$. Solve for α. ④ Multiply x^* by α to get ss for A_0, B_0 & C_0

Predicting system behavior for initial states

$A^n\vec{x} = \alpha(\lambda^n\vec{x})$

- |λ| > 1 : $\vec{x}[n] \to \infty$ exponential growth
- λ = 1 : $\vec{x}[n] \to k\vec{v}$ (steady-state)
- |λ| < 1 : $\vec{x}[n] \to \vec{0}$ exponential decay
- λ = 1 : $\vec{x}[n] = 0\vec{v} \to \vec{0}$ instant disappearance

$\vec{x}[0] = \alpha_1\vec{v}_1 + \alpha_2\vec{v}_2 + \cdots + \alpha_n\vec{v}_n$ $\vec{x}[n] = \alpha_1\lambda_1^n\vec{v}_1 + \alpha_2\lambda_2^n\vec{v}_2 + \cdots + \alpha_n\lambda_n^n\vec{v}_n$

① given an initial state, solve for $\alpha_1, \ldots, \alpha_n$ $\left[\begin{array}{cc|c} \vec{v}_1 & \vec{v}_2 & \vec{s}[0] \end{array}\right]$ ② Plug $\alpha_1, \ldots, \alpha_n$ into $\vec{x}[n]$ eqn and lambdas and $\vec{v}_i$

Rotation Matrix

$$A_R = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

Change of Basis

$\vec{F}^{(v)} = V^{-1}U\vec{F}^{(u)}$ $\vec{r}^{(v)} = V^{-1}\vec{r}$ $T = V^{-1}U$ (transformation matrix)

Equivalent Statements — LI for $A_{n\times n}$ matrix

- $A\vec{x} = \vec{0}$ only when $\vec{x} = 0$
- $A^2 \neq 0$

- A has n pivot positions
- columns/rows form a basis for $\mathbb{R}^n$
- spans $\mathbb{R}^n$
- doesn't have an e-val of 0
- rank(A) = n
- A invertible
- A has trivial nullspace
- A has LI columns
- A is full rank
- det(A) ≠ 0
- $A\vec{x} = \vec{b}$ has a unique soln
- col(A) = $\mathbb{R}^n$

Diagonalization

- matrix $T_{n\times n}$ is diagonalizable iff it has n LI e-vectors with corresponding e-vals $\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n$ ↳ $\lambda_1, \lambda_2, \lambda_3$

$A = [\vec{a}_1\ \vec{a}_2\ \cdots\ \vec{a}_n]$ D = diagonal matrix of e-vals

$D = \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \lambda_3 \end{bmatrix}$

$T = ADA^{-1}$
$T^k = AD^kA^{-1}$

Procedure
① compute $(\vec{v}, \lambda)$ pairs of T
② make sure e-vecs are LI (can use GE and nullspace to make sure it's trivial)
③ make A and A^{-1} using e-vecs
④ make D out of $\lambda_1, \ldots, \lambda_n$
⑤ multiply and get T^k

Thm: Span $\{[\begin{smallmatrix}1\\1\end{smallmatrix}], [\begin{smallmatrix}1\\-1\end{smallmatrix}]\} = \mathbb{R}^2$

Known: $\text{span}\{[\begin{smallmatrix}1\\1\end{smallmatrix}], [\begin{smallmatrix}1\\-1\end{smallmatrix}]\}$ set of all $\vec{b}$ that can be written as $S = \alpha[\begin{smallmatrix}1\\1\end{smallmatrix}] + \beta[\begin{smallmatrix}1\\-1\end{smallmatrix}]$, $\alpha, \beta \in \mathbb{R}$

To show: $\text{span}\{[\begin{smallmatrix}1\\1\end{smallmatrix}], [\begin{smallmatrix}1\\-1\end{smallmatrix}]\} = \mathbb{R}^2$; any $\vec{b}$ in $\mathbb{R}^2$ can be represented using the span $\vec{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$

Want: All $\vec{b}$ to belong to the set S

$\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \alpha\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \beta\begin{bmatrix} 1 \\ -1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \rightarrow \left[\begin{array}{cc|c} 1 & 1 & b_1 \\ 1 & -1 & b_2 \end{array}\right]$

$\rightarrow ge \rightarrow \left[\begin{array}{cc|c} 1 & 0 & \frac{b_1 + b_2}{2} \\ 0 & 1 & \frac{b_1 - b_2}{2} \end{array}\right] \rightarrow \alpha = \frac{b_1 + b_2}{2}$ $\beta = \frac{b_1 - b_2}{2}$ ⇒ every $\vec{b} \in \mathbb{R}$ can be represented as a linear combo of α, β ⇒ $\vec{b} \in S$ ■

Thm: If col(A) are LD, then $A\vec{x} = \vec{b}$ doesn't have a unique soln

Known: $A = [\vec{a}_1\ \vec{a}_2\ \cdots\ \vec{a}_n]$

Want: $A\vec{x} = \vec{b}$ doesn't have unique soln → either 0 or 2+. If possible, let a soln be $\vec{s}$.

$\vec{a}_k = c_1\vec{a}_1 + c_2\vec{a}_2 + \cdots + c_{k-1}\vec{a}_{k-1} + c_{k+1}\vec{a}_{k+1} + \cdots + c_n\vec{a}_n$ (rewrite as matrix)

$\begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_n \end{bmatrix}\begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_n \end{bmatrix}\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} \rightarrow A\left(\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} - \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix}\right) = \vec{0} \rightarrow A\begin{bmatrix} c_1 \\ \vdots \\ -1 \\ \vdots \\ c_n \end{bmatrix} = \vec{0}$ (k-th position) call the vector $\vec{\omega}$: $A\vec{\omega} = 0$

we know $A\vec{s} = \vec{b} \rightarrow A\vec{s} + \vec{0} = \vec{b} \rightarrow A\vec{s} + A\vec{\omega} = \vec{b} \rightarrow A(\vec{s} + \vec{\omega}) = \vec{b}$ $\vec{s} \neq \vec{\omega}$ → $\vec{s} + \vec{\omega}$ also a soln!
→ $\vec{s}$ is not a unique soln ■

Thm: $A\vec{x} = \vec{b}$ has 2+ solns → columns of A are LD

Known: $A\vec{x} = \vec{b}$ has 2 distinct solutions ↳ $\vec{s} \neq \vec{\omega}$ but both solns

Want: columns of A are LD: $\vec{a}_k = c_1\vec{a}_1 + \cdots + c_n\vec{a}_n$

$A\vec{s} = \vec{b}$, $A\vec{\omega} = \vec{b} \rightarrow A\vec{\omega} = A\vec{s}$
$\rightarrow A(\vec{\omega} - \vec{s}) = \vec{0}$
Let $\vec{\omega} - \vec{s} = \vec{y}$ → $\vec{y} \neq \vec{0}$ → $\vec{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$ → $A\vec{y} = \vec{0}$

$\begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_n \end{bmatrix}\begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = \vec{0} \rightarrow y_1\vec{a}_1 + y_2\vec{a}_2 + \cdots + y_n\vec{a}_n = \vec{0}$

At least one $y \neq 0$. call this y_k.
$y_k\vec{a}_k = y_1\vec{a}_1 + y_2\vec{a}_2 + \cdots + y_n\vec{a}_n$
$\vec{a}_k = \frac{y_1\vec{a}_1 + y_2\vec{a}_2 + \cdots + y_n\vec{a}_n}{y_k}$ ⇒ $\vec{a}_k$ is lin. combo of other cols ■

Thm: If col(A) are LD, A is not invertible

Known: col(A) are LD
$c_1\vec{a}_1 + c_2\vec{a}_2 + \cdots + c_n\vec{a}_n = 0$

Want: A^{-1} DNE. Assume A^{-1} does exist

Let A^{-1} exist. → $A^{-1}A = AA^{-1} = I$ → $[\vec{a}_1\ \vec{a}_2\ \cdots\ \vec{a}_n]\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = \vec{0}$
$\rightarrow A\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = \vec{0} \Rightarrow A^{-1}A\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = A^{-1}\vec{0} \rightarrow I\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = \vec{0} \rightarrow \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = \vec{0}$

contradiction! → A^{-1} can't exist ■

Thm: If A is invertible, unique soln exists

Known: $A^{-1}A = AA^{-1} = I$

To show: ① at least 1 soln ② any other soln equivalent

Guess: $\vec{s} = A^{-1}\vec{b}$ is a soln
Check: $A\vec{s} = A(A^{-1}\vec{b}) = (AA^{-1})\vec{b} = I\vec{b} = \vec{b}$ ⇒ $\vec{s}$ is a soln

Say $\vec{\omega} \neq \vec{s}$ is also a soln

Tips for proofs

- unique soln? proof by contradiction
- want columns of A? matrix mult
- if $A\vec{x} = \vec{0}$ then you can add this to pretty much anything

Given 2 eigenvectors $\vec{v}_1$ and v_2 corresponding to two uniq...

form a basis

Known:
$A\vec{v}_1 = \lambda_1\vec{v}_1$, $A\vec{v}_2 = \lambda_2\vec{v}_2$, $\lambda_1 \neq \lambda_2$, $\vec{v}_1 \neq \vec{v}_2 \neq \vec{0}$
$A \in \mathbb{R}^{2\times 2}$

Want:
$\vec{v}_1, \vec{v}_2$ form a basis for $\mathbb{R}^2$
① $\vec{v}_1, \vec{v}_2$ are LI ② $\vec{v}_1, \vec{v}_2$ span all of $\mathbb{R}^2$

If possible, let $\vec{v}_1$ & $\vec{v}_2$ be LD
$\alpha_1\vec{v}_1 + \alpha_2\vec{v}_2 = \vec{0}$ → say $\alpha_1 \neq 0$ → $\vec{v}_1 = -\frac{\alpha_2}{\alpha_1}\vec{v}_2$
Multiply by A.
$A\vec{v}_1 = -\frac{\alpha_2}{\alpha_1}A\vec{v}_2$
$A_1\vec{v}_1 = -\frac{\alpha_2}{\alpha_1}\lambda_2\vec{v}_2$
$\lambda_1\vec{v}_1 = -\frac{\alpha_2}{\alpha_1}\lambda_2\vec{v}_2$
→ plug in $\vec{v}_1$ into the eqn.
$-\frac{\alpha_2}{\alpha_1}\lambda_1\vec{v}_2 = -\frac{\alpha_2}{\alpha_1}\lambda_2\vec{v}_2$
$\lambda_1 = \lambda_2$
↳ contradiction!
→ therefore $\vec{v}_1, \vec{v}_2$ are LI
To show they span all of $\mathbb{R}^2$:
$[\vec{v}_1\ \vec{v}_2 \mid \vec{x}]$ → $V = [\vec{v}_1\ \vec{v}_2]$
→ V is an invertible matrix
⇒ $[V \mid \vec{x}]$ has a unique soln
→ therefore $\vec{x} \in \text{span}\{\vec{v}_1, \vec{v}_2\}$
⇒ $\{\vec{v}_1, \vec{v}_2\}$ form a basis for $\mathbb{R}^2$

If $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ are LD vectors in $\mathbb{R}^n$, then $\{A\vec{v}_1, A\vec{v}_2, \dots, A\vec{v}_n\}$ are LD.
Known:
$\vec{v}_i = \sum_{j\neq i} \alpha_j\vec{v}_j$
Show:
$A\vec{v}_k = \sum \beta_\ell\vec{v}_\ell$
$A\vec{v}_i = A\left(\sum_{j\neq i}\alpha_j\vec{v}_j\right)$
$= \sum_{j\neq i} A(\alpha_j\vec{v}_j)$
$A\vec{v}_i = \sum_{j\neq i}\alpha_j(A\vec{v}_j)$ ⇒ linear combo exists
QED

If $\vec{v}_1, \vec{v}_2$ solns to $A\vec{x} = \vec{b}$, $\vec{b}$ must be $\vec{0}$.
$\vec{v}_1 + \vec{v}_2$
Known:
$A\vec{v}_1 = \vec{b}$, $A\vec{v}_2 = \vec{b}$
$A(\vec{v}_1 + \vec{v}_2) = \vec{b}$ → $A\vec{v}_1 + A\vec{v}_2 = \vec{b}$
⇒ $\vec{b} + \vec{b} = \vec{b}$ ⇒ $\vec{b} = \vec{0}$
Want: $\vec{b} = 0$
QED

$\text{span}\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\} = \text{span}\{\vec{v}_1 + \vec{v}_2, \vec{v}_2, \dots, \vec{v}_n\}$
say $\vec{q} \in \text{span}\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$
$\vec{q} = \alpha_1\vec{v}_1 + \alpha_2\vec{v}_2 + \dots + \alpha_n\vec{v}_n$
$= \alpha_1(\vec{v}_1 + \vec{v}_2) + (-\alpha_1 + \alpha_2)\vec{v}_2 + \dots + \alpha_n\vec{v}_n$
⇒ $\vec{q} \in \text{span}\{\vec{v}_1 + \vec{v}_2, \vec{v}_2, \dots, \vec{v}_n\}$
say $\vec{r} \in \text{span}\{\vec{v}_1 + \vec{v}_2, \vec{v}_2, \dots, \vec{v}_n\}$
$\vec{r} = \beta_1(\vec{v}_1 + \vec{v}_2) + \beta_2(\vec{v}_2) + \dots + \beta_n\vec{v}_n$
$= \beta_1\vec{v}_1 + (\beta_1 + \beta_2)\vec{v}_2 + \dots + \beta_n\vec{v}_n$
$\vec{r} \in \text{span}\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$

If A invertible, unique A^{-1}
Known: $AA^{-1} = A^{-1}A = I$
Want: A^{-1} unique
say B_1, B_2 inverses of A, $B_1 \neq B_2$
$AB_1 = B_1A = I$ $AB_2 = B_2A = I$
$B_2AB_1 = B_2B_1A$
$(B_2A)B_1 = B_2(B_1A)$
$B_1 = B_2$
→ contradiction, A^{-1} must be unique

If $QP = I$ and $RQ = I$, then $P = R$.
$QP = RQ$
$RQP = RRQ$
$(RQ)P = R(RQ)$
$I\ P = R\ \ I$

If a system of K reservoirs has columns that sum to one, then it's is the total amount of water at timestep n, then the total amount of water is S at timestep n+1
Known
Want
$x_1[n+1] + x_2[n+2] = S$
$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ $\xrightarrow{\text{mat mul}}$ $\begin{bmatrix} a_{11}x_1[n] + a_{12}x[n] \\ a_{21}x_1[n] + a_{22}x_2[n] \end{bmatrix}$
$a_{11} + a_{21} = 1$
$a_{12} + a_{22} = 1$
$x_1[n] + x_2[n] = S$
$\vec{x}[n+1] = A\vec{x}[n]$
Consider product
$A\vec{x}[n] = \vec{x}[n+1]$

Transpose
- eigenvalues remain the same across transposes

Matrix Inverse Properties
- $AA^{-1} = A^{-1}A = I$
- $(A^{-1})^{-1} = A$
- $(KA)^{-1} = K^{-1}A^{-1}$ $K \in \mathbb{R}$
- $(AB)^{-1} = B^{-1}A^{-1}$
- $(A^T)^{-1} = (A^{-1})^T$
- $(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = I$

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$
$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

Applying Matrices
- go from right to left. ex.
$ABCD\vec{x}$
$(A(B(C(D\vec{x}))))$
④ ③ ② ①

Given unknown Matrix A, given $A\vec{v}_1 = A\vec{v}_2 = \vec{p}$, find $\vec{u}$ s.t. $A\vec{u} = \vec{0}$ where $\vec{u} \neq \vec{0}$.
$A\vec{v}_1 - A\vec{v}_2 = \vec{0}$
$A(\underbrace{\vec{v}_1 - \vec{v}_2}) = \vec{0}$
$\vec{u} = \vec{v}_1 - v_2$

$\lambda_i\vec{v}_i = A\vec{v}_i$
$\Rightarrow A^{-1}\lambda_i\vec{v}_i = \vec{v}_i$
$A^{-1}\vec{v}_i = \frac{1}{\lambda_i}\vec{v}_i$

More steady-state:
$\vec{s}[0] = \alpha\vec{v}_1 + \beta\vec{v}_2 + \gamma\vec{v}_3$
to decompose $\vec{s}[0]$ into the eqn (given $\vec{s}[0]$):
$\begin{bmatrix} | & | & | \\ \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \\ | & | & | \end{bmatrix}\begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \vec{s}[0]$
↳ do GE

- $A\vec{s}[0]$ has lambdas
($A\vec{s}[0] = \lambda\vec{s}[0]$)

~~Is ... in the column space of A when ...~~
Is $v = \begin{bmatrix} 3 \\ -8 \end{bmatrix}$ in C(A) when a = 3?
$A = \begin{bmatrix} 2 & 1 \\ -1 & a \end{bmatrix}$ a = 3 $v = \begin{bmatrix} 3 \\ -8 \end{bmatrix}$
$\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ & $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ are LI? span $\mathbb{R}^2$ → yes

4b

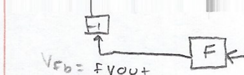
Solve for smallest possible identical e-vals for A.
$A = \begin{bmatrix} 2 & 1 \\ -1 & a \end{bmatrix}$ → $\det(A - \lambda I) = 0$ → $\lambda^2 - (2+a)\lambda + (2a+1) = 0$
→ $\lambda = \frac{2 + a \pm \sqrt{(2a+a)^2 - 4(2a+1)}}{2}$ → since we want identical evals, everything under sqrt = 0.
→ $(2a+a)^2 - 4(a2+1) = 0$ → solve for a → a = 0, 4
→ want a minimizing e-vals → plug in a to the quadratic eqn formula w/ the λs → (a = 4 → λ = 3);
(a = 0 → λ = 1) → $\boxed{a=0}$

Find all vals for x s.t. A has a trivial nullspace
$\begin{bmatrix} 1 & 0 & 4 \\ 1 & 2 & 6 \\ 0 & 1 & x \end{bmatrix}$ → GE → $\begin{bmatrix} 1 & 0 & 4 \\ 0 & 2 & 2 \\ 0 & 2 & 2x \end{bmatrix}$ → want LI so $x \neq 0$

Pluto — 10c

Given a transformation, what is the transformation matrix that created the transform?
ex: $\begin{bmatrix} 0 \\ 2 \end{bmatrix} \to \begin{bmatrix} 0 \\ 3 \end{bmatrix}$
and
$\begin{bmatrix} -2 \\ -1 \end{bmatrix} \to \begin{bmatrix} -3 \\ -1.5 \end{bmatrix}$
rewrite:
$\begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 \\ 3 \end{bmatrix}$
$\begin{bmatrix} -2 \\ -1 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} -3 \\ -1.5 \end{bmatrix}$
→ do mat mul, solve for a, b, c & d and plug into A
$a = \frac{2}{3}$ b = 0 c = 0 $d = \frac{2}{3}$
$\Rightarrow A = \begin{bmatrix} 2/3 & 0 \\ 0 & 2/3 \end{bmatrix}$

$= \frac{C_1 C_2}{C_1 + C_2}$



$V_{fb} = fV_{out}$
$V_{out} = A V_{err}$
$V_{err} = V_{in} - fV_{out}$
$V_{out} = A(V_{in} - fV_{out})$
$\boxed{V_{out} = \frac{A}{1+Af} V_{in}}$

large
$\frac{A/A}{(1+Af)/A} = \frac{1}{1/A + f}$

$\lim_{A\to\infty} \frac{1}{\frac{1}{A} + f} = \frac{1}{f}$

$V_{out} = \frac{1}{f} V_{in}$

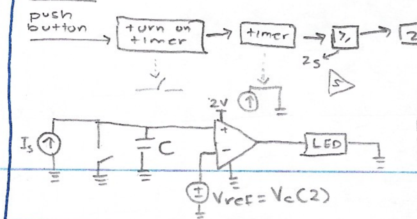
$V_{in} = fV_{out}$
$V_{fb} = fV_{out}$

- If $V_{out} \uparrow$ and $V_{in} \downarrow$, we are in neg fb.

$V_- = V_{out}$ $V_+ = V_{in}$
$V_{out} = A(V_+ - V_-)$
$V_{out} = AV_{in} - AV_{out}$
$V_{out} = [A/(1+A)]\, V_{in}$
$V_{out} = \frac{1}{\frac{1}{A}+1} V_{in}$

as $A \to \infty$ $V_{out} = V_{in}$
↳ this is neg fb with $f = 1$

block

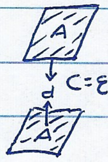


After button pushed:
$V_C(t) = \frac{I_s t}{C}$, $V_{ref} = V_C(2) = \frac{I_s 2}{C}$

Note: Before button is pushed, S is closed to short circuit and make sure there is no current across C.
- gives us $V_C(0) = 0$
- $I_C(C) = 0$
(path of least resistance → wire of S)

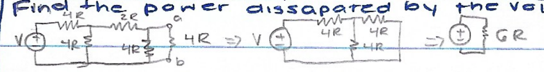
"wiggle"
↳ if $V_{out} \uparrow$, then $V_- \uparrow$
↳ since V_- is subtracted from V_+, then $(V_+ - V_-) \downarrow$
↳ this pulls $V_{out} \downarrow$, cancelling the increase

Restated:
if $V_{out} \uparrow$ then $(V_+ - V_-) \downarrow$ in neg fb
so $V_{out} = A(V_+ - V_-) \downarrow$



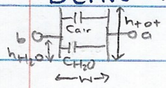
$C = \varepsilon \frac{A}{d}$

Find the power dissapated by the voltage source.



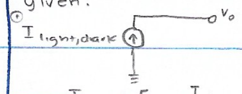
$P = VI = -\frac{V^2}{R} = -\frac{V^2}{6R}$

Derive an exp for C_{tank}.



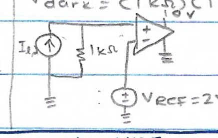
$C_{tank} = C_{air} + C_{H_2O}$
$C_{air} = \frac{\varepsilon(h_{tot} - h_{H_2O})w}{w}$
$= \varepsilon h_{tot} - h_{H_2O}$
$C_{H_2O} = 81\varepsilon h_{H_2O}$
$C_{tank} = \varepsilon h_{tot} + 80\, h_{H_2O}$

Circuit that outputs 10V w/light 0V w/o.
① given:



$I_{light} = 5mA$ $I_{dark} = 1mA$
② one op amp
③ one resistor
④ two voltage sources

$V_{light} = (1k\Omega)(5mA) = 5V$
$V_{dark} = (1k\Omega)(1mA) = 1V$



- ~~Circuit~~ light sensor can only provide max of 40mW. Is this a problem?
- $P = IV = (5mA)(-5V) = -25mW$
- No, not a problem bc our circuit can generate 25mW which is less than P_{max}.
- Resistances that meet the
- P_{max} req: $|P| \le 40mW \Rightarrow$
- $|I^2 V| \le 40mW \Rightarrow |(5mA)^2 R| \le 40mW$
  $R \le 1.6k\Omega$

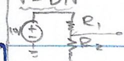
What to use as a refr. voltage?
Halfway between the 2 options.
$V_{ref} = \frac{V^+ + V^-}{2}$

Design a motor driving circuit that outputs a decreasing positive motor voltage as the robot moves toward the light.

Specs
- Distance ↓, speed ↓
- $V_m \ge 5V$ when far (D↑)
- "far away": $R_{PH} = 10k\Omega$
  "close by": $R_{PH} = 100\,\Omega$

Given
op amps, 10V, −10V, resistors

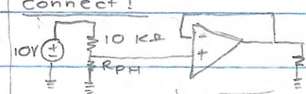
Strategy
- decr. as a func of distance
D↑ L↓ R_{PH}↑
Question
- Can we build something to measure resistance?
↳ output voltage
A/ let's try a voltage divider
V-DIV



$V_x = \frac{R_2}{R_1 + R_2}(10V) = \frac{1}{1 + \frac{R_1}{R_2}}$

$R_1 \uparrow$ $(\frac{R_1}{R_2})\uparrow$ $V_x \downarrow$
$R_2 \uparrow$ $(\frac{R_1}{R_2})\downarrow$ $V_x \uparrow$ } make $R_2 = R_{PH}$

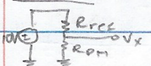
Now we have
D↑ L↓ R_{PH}↑ $(\frac{R_1}{R_{PH}})\downarrow$ $V_x \uparrow$
From spec
$V_x = 5V$ when $R_{PH} = 10k\Omega$
↳ plug R_{PH} into V_x eqn and solve for R_1.
⇒ $R_1 = 10k\Omega$
Connect!



Need buffer to avoid loading.

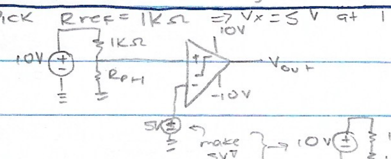
Design a comparator circuit that outputs a positive motor voltage when Petbot exceeds 1m in distance, making the robot move toward it, and a negative voltage when Petbot is w/in 1m (making Petbot move away)
Specs
① "stop" at 1m/1kΩ
- output > 0 when > 1m (↑D)
- output < 0 when < 1m (↓D)
② Recall $R_{PH}\uparrow$, D↑ and $R_{PH}\downarrow$ when D↑

Given
resistors, comparator, 10V, −10V
Design

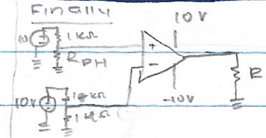


$V_x = \frac{R_{PH} V_{in}}{R_{ref} + R_{PH}}$
D↑, R_{PH}↑ V_x↑

Pick $R_{ref} = 1k\Omega$ ⇒ $V_x = 5V$ at 1m



Finally



"gain"
usually voltage gain
$G = \frac{V_{out}}{V_{in}}$

charge on a cap after time t
$Q = It$

$I = \frac{dQ}{dt}$

~~4c~~ ~~U did~~ I set comparator up right?
(+ or − terminals)

6a → No C source $V_{out} = 0$?

6b → ~~d~~ rederive

6e

7a, $\boxed{7b}$

DNA

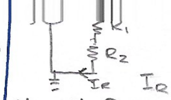
hecto (h) : E2
deca (da) : E1
base (b) : E0
deci (d) : E-1
centi (c) : E-2
milli (m) : E-3
micro (μ) : E-6
nano (n) : E-9
pico (p) : E-12
Femto (f) : E-15

0.001

$x^a x^n = x^{m+n}$
$(xy)^n = x^n y^n$
$(x^m)^n = x^{mn}$

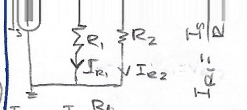
series eq.	$R_1 + R_2 + \dots$	$C_1 \| C_2$ $\left(\frac{1}{C_1}+\frac{1}{C_2}\right)^{-1}$
parallel eq.	$R_1 \| R_2$	$C_1 + C_2$
energy store	none	$E = \frac{1}{2} C V_c^2$

$Q = CV_c$ $\quad i = C\frac{dv}{dt}$

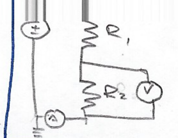


$R_1 + R_2$ $\quad I_R = V_s(R_1+R_2)$

$V_{R_1} = V_s \frac{R_1}{R_1+R_2}$



$I_{R_1} = I_s \frac{R_2}{R_1+R_2}$



$V_{R_1} = V_{R_2}$

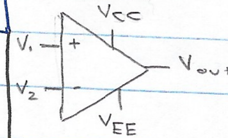
KCL
$I_{in} = I_{out}$

KVL
$\sum_{loop} V_K = 0$; + → − add. − → + subtract

ohm
$V = IR$

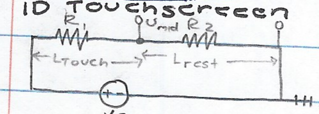
Power
$P = VI = \frac{V^2}{R} = I^2 R$

Comparator



$V_{out} = \begin{cases} V_{CC} & \text{if } V_1 > V_2 \\ V_{EE} & \text{if } V_1 < V_2 \end{cases}$

1D Touchscreen



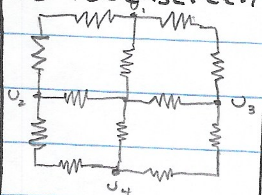
$R_1 = \frac{\rho L_{Touch}}{A}$ $\quad R_2 = \frac{\rho L_{rest}}{A}$

$U_{mid} = V_s \dfrac{\frac{\rho L_{rest}}{A}}{\frac{\rho L_{rest}}{A} + \frac{\rho L_{Touch}}{A}}$

$U_{mid} = V_s \frac{L_{rest}}{L_{rest}+L_{Touch}}$

$U_{mid} = V_s \frac{L_{rest}}{L_{TOTAL}}$

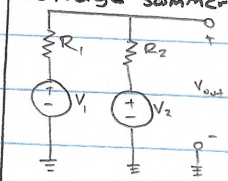
2D Touchscreen



Powering $U_1 \to U_4$: measure x position
$V_{out} = V_s \frac{L_{Touch,vertical}}{L}$

Powering $U_2 \to U_3$: measure y position
$V_{out} = V_s \frac{L_{Touch,horizontal}}{L}$

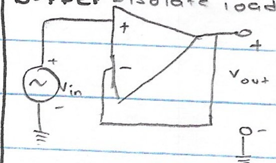
Voltage Summer



$V_{out} = \frac{V_1 R_2}{R_1+R_2} + \frac{V_2 R_1}{R_1+R_2}$

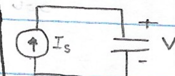
- negative gain
- does NOT have ∞ input resistance
→ will load input signal
- recommended to use after a buffer or if you know your input/source has no source resistance

Buffer – isolate load



$V_{out} = V_{in}$

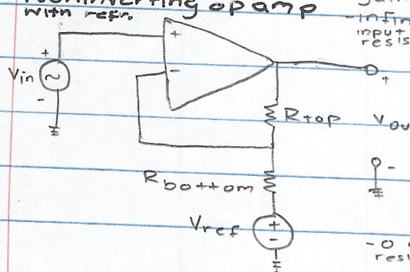
Capacitor
"create a voltage which increases/decreases linearly with respect to time" or a voltage ramp with a given slope.



$I_s = C\frac{dV_c}{dt} + V_c \frac{dC}{dt}$

$V_c(t) = \frac{I_s}{C}(t - t_0) + V_c(t_0)$

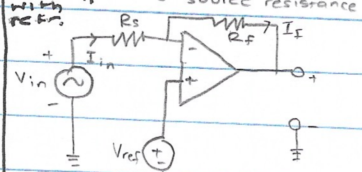
Noninverting op amp with refr.



positive gain
- infinite input resistance
- 0 output resistance

$V_{out} = V_{in}\left(1 + \frac{R_{top}}{R_{bottom}}\right) - V_{ref}\left(\frac{R_{top}}{R_{bottom}}\right)$

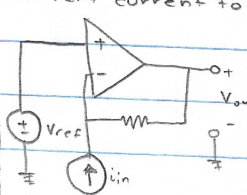
Inverting opamp with refr.



$V_{out} = V_{in}\left(-\frac{R_f}{R_s}\right) + V_{ref}\left(\frac{R_f}{R_s} + 1\right)$

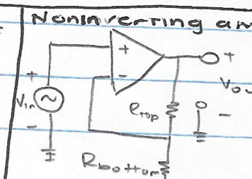
$I_{in} = I_{out}$

Transresistance amp
convert current to voltage



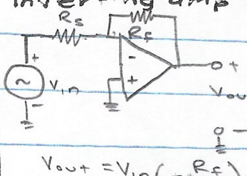
$V_{out} = i_{in}(-R) + V_{ref}$

Noninverting amp



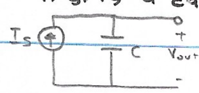
$V_{out} = V_{in}\left(1 + \frac{R_{top}}{R_{bottom}}\right)$

Inverting amp



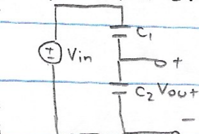
$V_{out} = V_{in}\left(-\frac{R_f}{R_s}\right)$

Charging a capacitor:



$V_{out} = \frac{I_s t}{C}$

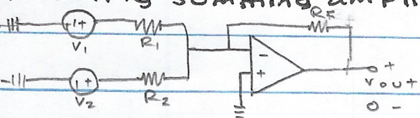
Capacitive Divider:



$V_{out} = \frac{C_1 V_{in}}{C_1 + C_2}$

ideal op amp
$A \to \infty$ $\quad U^+ - U^- = 0$ $\quad U^+ = U^-$

Inverting Summing amplifier

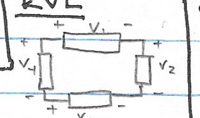


$V_{out} = -R_f\left(\frac{V_{in1}}{R_1} + \frac{V_{in2}}{R_2}\right)$

inner product

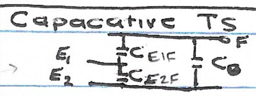
Op amps and golden rules
① $I^+ = I^- = 0$
② (NF only) $V^+ = V^-$ (note $V^+ \neq 0 \neq V^-$)

KVL



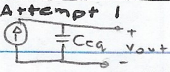
$-V_1 - V_2 + V_3 + V_4 = 0$

Capacative TS



Cap Derivation
$I_s = C\frac{dV(t)}{dt}$
$\int_a^b I_s\,dt = \int_a^b C\frac{dV(t)}{dt}dt$
$I_s t = C(V(t) - V(0))$
$V(t) = \frac{I_s t}{C} + V(0)$
Note: to use this, I_s must be const → choose intervals where it's

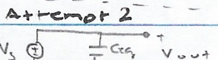
Attempt 1



$C_{EQ} = C_0 + C_\Delta$
$C_\Delta = \left(\frac{1}{C_{E1F}} + \frac{1}{C_{E2F}}\right)^{-1}$

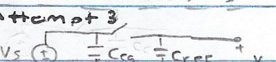
$I_s = C\frac{dV_c}{dt}$ $\quad V_c = \frac{I_s t}{C_{eq}}$

Attempt 2



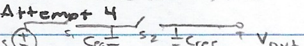
Vout (touch) = Vs
Vout (no touch) = Vs

Attempt 3



Vout (open switch) = 0
Vout (closed switch) = Vs

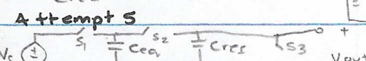
Attempt 4



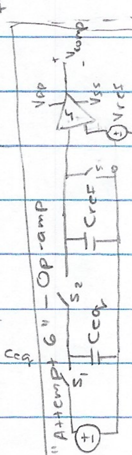
Φ1: close S1 to charge Ceq up. S2 open
Φ2: open S1, close S2

Prob: potential charge on Cref

Attempt 5



Φ1: S3 & S1 on, S2 off
discharge anything off Cref)
Vcref = 0 $\quad$ VCEQ = Vs
Qcref = CV = 0 $\quad$ QCEQ = Ceq Vs
Φ2: S3 & S1 off, S2 on
VCEQ = Vout $\quad$ VCref = Vout
QCEQ = Ceq Vout $\quad$ QCref = Cref Vout
Total Q @ phase 2: Ceq Vout + Cref Vout



"Attempt 6" – Op-amp

If touch:
Vout = (Co + CΔ) Vs / (Cref + Co + CΔ)

If no touch:
Vout = Co Vs / (Cref + Co)

More power
- negative power dissapated P < 0
- positive power dissipated P > 0
- negative power generated P > 0
- positive power generated P < 0

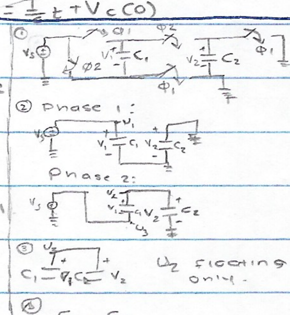
Note:
6W generated by Vs ⟺ −6W dissapated by Vs

Have 2 diff sized cap plates?
measure the smaller plate for the capacitance!

Capacitor Principles
- charge at floating nodes is conserved
- if sum of charges on a floating node = 0, the sum of their charges at steady state will equal zero.
- the charge of a capacitor ⇒ +Q = +CV
- if 2 caps are initially uncharged, then connected in series, the charges on both caps are equal at steady state
- voltage across capacitors in parallel is equal at steady state.
- in steady state, DC capacitors act as open circuits (no current flows through them)

Derivation
$I = C\frac{dV_c}{dt}$, $I\,dt = C\,dV_c$, $\int_0^t I\,dt = \int_0^{V_c} C\,dV_c$,
$It = C V_c(t) - V_c(0)$ $\quad V_c(t) = \frac{I}{C}t + V_c(0)$

Charge Sharing
$Q^{\Phi 1} = Q^{\Phi 2}$
① Label all voltages across capacitors
② Draw the equivalent circuit in each phase
③ Identify all floating nodes in your circuit during phase 2. For each node u_i:
Ⓐ identify all cap plates attached to the node during phase 2.
Ⓑ calculate the charge on each of the plates in the steady state of phase 1
Ⓒ calculate the charge on each plate during phase 2's steady state.
Ⓓ set $Q^{\Phi 1} = Q^{\Phi 2}$. Solve for the unknown node voltage.
Ⓔ Repeat.



① ② phase 1: phase 2: ③ u_2 floating only. ④ C_1, C_2

Ⓑ $Q^{\Phi 1}_{u_2} = C_1 V_1^{\Phi 1} + C_2 V_2^{\Phi 1} = (V_s - 0)C_1 + 0 = V_s C_1$
Ⓒ $Q^{\Phi 2}_{u_2} = C_1 V_1^{\Phi 2} + C_2 V_2^{\Phi 2} = (u_2 - u_3)C_1 + u_2 C_1 = (u_2 - V_s)C_1 + u_2 C_1 = u_3(C_1 + C_2) - V_s C_1$
Ⓓ $Q^{\Phi 1}_{u_2} = Q^{\Phi 2}_{u_2} \Rightarrow 2V_s C_1 = u_3(C_1 + C_2)$ $\quad u_3 = \frac{2 V_s C_1}{C_1 + C_2}$

Checking for negative feedback
① zero out all independent sources, other than the power supply

$1\,k\Omega = 1000\,\Omega$
$= 1{,}000{,}000\,m\Omega$
$1\,mA = 0.001\,A$
$= 0.000001\,kA$

Power → W
Energy → J

- voltage drop
↳ voltage across the base minus the tip. $(V_b - V_a)$
$a \to b$ $\quad P = IV$
$a \to b$ $\quad (V_b - V_a)$ $\quad P = -IV$

consuming power – entering higher node

Generating power → exiting higher

- power sources supply power
- resistors consume
- capacitors consume when charging, generate when discharging