Today

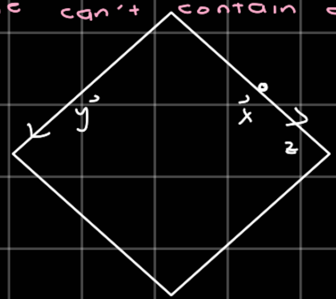
- LPs continued; Simplex algo
- QPs

Extreme Point / Vertex

- $\vec{x}$ is a vertex of P (polyhedron $\{\vec{x} \in \mathbb{R}^n \mid A\vec{x} \geq \vec{b}\}$)
  if it can't be expressed as a convex combination of
  $\vec{y}, \vec{z} \in P$, $\vec{y}, \vec{z} \neq \vec{x}$, ie

  $$\vec{x} = \lambda\vec{y} + (1-\lambda)\vec{z} \quad \lambda \in [0, 1] \text{ is NOT possible}$$

  ↳ ie can't contain a line



Thm: Consider $\left.\begin{matrix} \min \ \vec{c}^T\vec{x} \\ \text{s.t } A\vec{x} \leq b \end{matrix}\right\}$ linear program

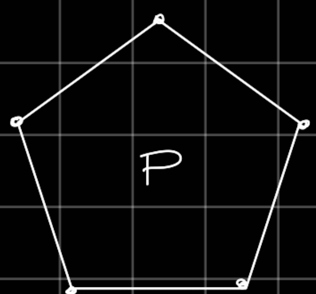
① Assume P has an extreme point.

② Assume optimal soln exists & is finite

Then: $\exists$ an optimal soln that's an extreme point of P

(note: we aren't saying $\forall$ optimal solns are extreme points; just that there's one)

Proof:



$$P = \{x \mid A\vec{x} \leq \vec{b}\}$$

$$p^* = \min_{x \in P} \vec{c}^T\vec{x}$$

Consider Q: set of all optimal solutions

$$Q = \{\vec{x} \mid \underbrace{A\vec{x} \leq b}_{\text{bc } \vec{x} \text{ is in } P}, \underbrace{\vec{c}^T\vec{x} = p^*}_{\text{bc } \vec{x} \text{ is optimal soln}}\}$$

Q : nonempty (bc we assumed an optimal soln exists)

- also a polyhedron → set defined by linear constraints

↳ want to argue that Q has an extreme point

(bc some polyhedrons, like half-spaces, don't has any extreme points)

$Q \subset P$, P can't contain a line $\Rightarrow$ Q can't contain a line (bc it's a subset) $\Rightarrow$ Q has a vertex

↳ wts vertex of Q is also vertex of P.

↳ do a convex combination thing

$\vec{v}$ is a vertex of Q, wts : $\vec{v}$ is a vertex of P.

↳ Proof by contradiction

IF possible $\vec{v}$ is not a vertex of P, eg:

$\exists \vec{y}, \vec{z} \in P$, $\vec{y}, \vec{z} \neq \vec{v}$ s.t. $\lambda\vec{y} + (1-\lambda)\vec{z} = \vec{v}$ $\lambda \in [0,1]$

↳ want to use fact that $\vec{v}$ is part of Q

$\vec{v} \in Q \Rightarrow \vec{c}^T\vec{v} = p^*$

$p^* = c^T\vec{v} = c^T(\lambda\vec{y} + (1-\lambda)\vec{z})$

$= \lambda c^T\vec{y} + (1-\lambda)\vec{c}^T\vec{z}$

Ayah: contradiction: Q contains a line

$c^T\vec{y} \geq p^*$ $\vec{c}^T\vec{z} \geq p^*$

↳ $\lambda c^T\vec{y} + (1-\lambda)c^Tz \leq \lambda p^* + (1-\lambda)p^*$

but we know $\lambda c^T\vec{y} + (1-\lambda)c^Tz = p^*$

so $\vec{c}^T\vec{y} = p^* = \vec{c}^T\vec{z}$

→ since $\vec{y}$ & $\vec{z}$ are optimal $\Rightarrow$ $\vec{y}, \vec{z} \in Q$

$\Rightarrow$ $\vec{v}$ cannot be a vertex of Q (contradiction)

(contains a line consistent of linear combo of points in Q) CONTRADICTION

$\therefore$ $\vec{v}$ must be a vertex of P ▨ QED.

"strong duality always holds for LPs that are feasible)

Quadratic Programs

- general form: $\min_{\substack{A\vec{x} \leq \vec{b} \\ C\vec{x} = \vec{d}}} \frac{1}{2}\vec{x}^T H \vec{x} + \vec{c}^T\vec{x} = p^*$

→ if you have that $H = H^T$ (symmetric) & H PSD, then this is also convex.

Unconstrained QP

Cases for eigenvalues of H (symmetric), unconstrained minimization

① H has at least one negative eigenvalue

$p^* = -\infty$ choose e-vector corresponding to negative eigenvalue as your $\vec{x}^*$.

② $H \succeq 0$ (PSD) $\vec{c} \in R(H)$

↳ $\vec{c}$ belongs to the range of H

$f(\vec{x}) = \frac{1}{2}\vec{x}^T H \vec{x} + \vec{c}^T\vec{x}$

lets us write in this form

Rewrite: $f(\vec{x}) = \frac{1}{2}(\vec{x} - \vec{x}_0)^T H (\vec{x} - \vec{x}_0) + \alpha$

$= \frac{1}{2}\vec{x}^T H \vec{x} + \frac{1}{2}\vec{x}_0^T H \vec{x}_0 - \vec{x}_0^T H \vec{x} + \alpha$

choose:

$\vec{c}^T = -\vec{x}_0^T H \Rightarrow \vec{c} = -H\vec{x}_0$

$\alpha = -\frac{1}{2}\vec{x}_0 H \vec{x}_0$ ← so it cancels out

Ideally choose $\vec{x} = \vec{x}_0$

then $\vec{c} = -H\vec{x}_0$ but couple cases

① case 1: H invertible

$\vec{x}^*_? = -H^{-1}\vec{c}$

② Case 2: H has nullspace (not invertible)

↳ then any $\vec{x}_0$ s.t. $-H\vec{x}_0 = \vec{c}$

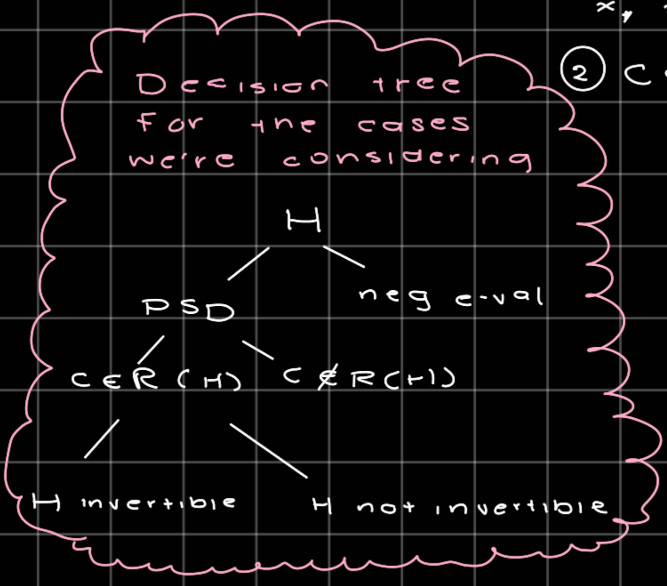
Key idea: pseudoinverse

$\underset{n\times n}{H} = U\Sigma V^T \quad rk(H) = r$

$$H = U \begin{bmatrix} \Sigma_{r\times r} & 0 \\ 0 & 0 \end{bmatrix} V^T$$

Moore-penrose pseudoinverse H^+

$$H^+ = V \begin{bmatrix} \Sigma_{r\times r}^{-1} & 0 \\ 0 & 0 \end{bmatrix} U^T$$



Properties of Moore-penrose pseudoinverse:

- $HH^{\dagger} = U_r U_r^T$
- $H^{\dagger} H = V_r V_r^T$
- $HH^{\dagger} H = H$

... continuing case 2:

$$\vec{x}_* = -H^{\dagger}\vec{c} + \vec{\xi} \qquad \vec{\xi} \in N(H)$$

3 Case 3: $\vec{c} \notin R(H)$, $H \succeq 0$, symmetric

$$\vec{c} = -H\vec{x}_0 + \vec{r} \qquad \vec{r} \in N(H^T)$$

$$f(k\vec{r}) = k\left(\tfrac{1}{2}\vec{r}^T H \vec{r} + \vec{c}^T\vec{r}\right) \qquad k \in \mathbb{R}$$

vector from nullspace

$$= 0 + k\vec{c}^T\vec{r}$$

$$= k(-H\vec{x}_0 + \vec{r})^T\vec{r}$$

$$= -kH\vec{x}_0^T\vec{r} + k\vec{r}^T\vec{r}$$

$$= \cancel{-\vec{x}_0 H \vec{r}}^{\,0} + k\vec{r}^T\vec{r}$$

$$= k\|\vec{r}\|_2^2$$

↳ if we choose k negative → −∞ unbounded below

$$p^* = -\infty$$

$$x^* = \text{any vector} \in N(H)$$

Equality Constrained QP

↳ can make eq-constrained QP into unconstrained QP using change of var.

Applications

1 Linear control LQR

$$x[t+1] = Ax[t] + \underbrace{Bu[t]}$$

Goal: reach $\vec{s}$ goal by time t

want to choose controls u to reach your goal

$$x[t] = A^t x[0] + \sum_{i=1}^{t-1} A^{t-i-1} Bu[i]$$

→ to reach goal:

optimization problm: quadratic

$$\min \; \underbrace{\|\vec{x}(T) - \vec{g}\|_2^2 + \sum_{\tau=0}^{T} \|u(\tau)\|_2^2}$$

↑ optimization vars are u's

constraints: linear

$$\begin{cases} x(t) : 0, \cdots, T \\ u(t) : 0, \cdots, T-1 \\ \text{s.t.} \\ x(t) = A^t x[0] + \sum_{i=1}^{t-1} A^{t-i-1} B u[i] \end{cases}$$

② Functional Fitting

observation

piecewise constant fcn
want to estimate what $\vec{x}$ is

$\vec{y} = \vec{x} + \text{noise}$

↳ want to find $\hat{x}$ s.t. x doesn't change much (want smooth transitions)

→ often expressed as a cardinality constraint

↳ can be reduced to L1 → can be reduced to QP

$$D = \begin{bmatrix} -1 & 1 & 0 & \cdots & & & 0 \\ 0 & -1 & 1 & 0 & \cdots & & 0 \\ 0 & 0 & -1 & 1 & 0 & \cdots & 0 \\ \cdots & & 0 & -1 & 1 & 0 & \cdots 0 \end{bmatrix} \qquad D\vec{x} = \begin{bmatrix} x_2 - x_1 \\ x_3 - x_2 \\ \vdots \\ x_n - x_{n-1} \end{bmatrix}$$

$$p^* = \min_{\vec{x}} \|\vec{y} - \vec{x}\|_2^2$$
$$\text{s.t. } \operatorname{card}(Dx) \le k$$

↳ can relax this to L1 constraints

$$p^* = \min_{\vec{x}} \|\vec{y} - \vec{x}\|_2^2$$
$$\text{s.t. } \|D\vec{x}\| \le \alpha$$