# Convex Optimization

$$p^* = \min \ f_0(\vec{x})$$
$$\text{s.t.} \ f_i(\vec{x}) \leq 0 \qquad i = 1, 2, \ldots, m$$

↳ convex problem if $f_0(\vec{x})$ convex & $f_i(\vec{x})$ convex

## Ex

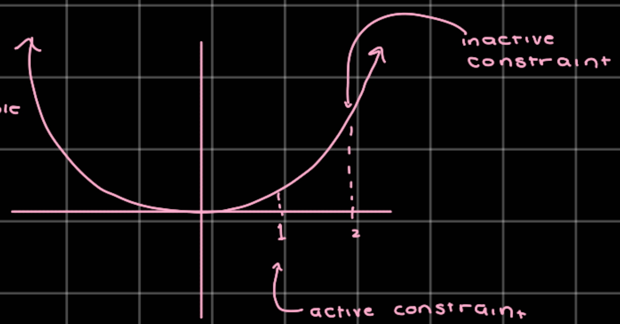
$\min \ x^2$

$x \in \mathbb{R}$

$x \geq 1$

$x \leq 2$

feasible set




↳ $p^* = 1$

$x^* = 1$

↳ $\nabla f_0(x) = 0$ → with our domain this isn't helpful

$\min \ x^2 = p^*$

$x \geq 1$

$x \leq 1$

infeasible

feasible set $= \emptyset$

$p^* = \infty$

# Linear Programming

$$\min \ \vec{c}^T \vec{x}$$
$$\text{s.t.} \ A\vec{x} = \vec{b}$$
$$P\vec{x} \leq \vec{q}$$

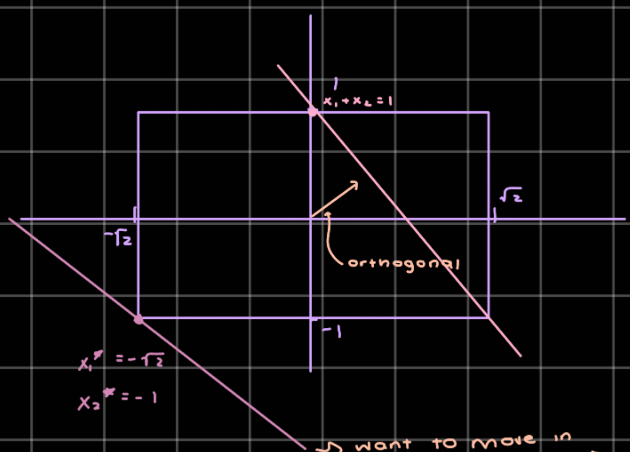
## Consider

$\min \ x_1 + x_2$

$x \in \mathbb{R}$

$x_1^2 \leq 2$

$x_2^2 \leq 1$




↳ want to move in direction of negative gradient (ie $-\nabla f = -c$)

↳ both constraints are active

eg:

$$\min x_1 \qquad = p^*$$

$$x_1 + x_2 \geq 0$$

↳ unbounded

$p^* = -\infty$


**Theorem:** If $\vec{x}_*$ is an optimum solution, then $\vec{x}_* \in \text{Boundary}(\mathcal{X})$ for

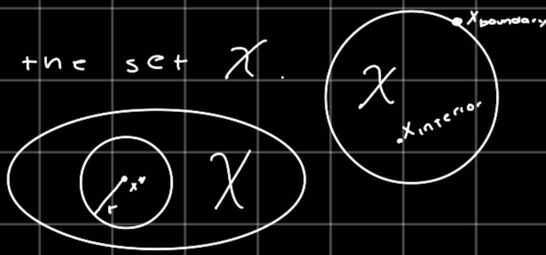
$$\min \vec{c}^T \vec{x}$$

$\vec{x} \in \mathcal{X}$ , $\mathcal{X}$ convex set, closed ($\mathcal{X}$ contains its boundary)

**Proof:** Assume $\vec{x}^*$ in the interior of the set $\mathcal{X}$.

$\exists$ some ball of radius $r>0$, s.t. Ball $\in \mathcal{X}$

$$\forall \vec{z} \quad \|\vec{x} - \vec{y}\|_2 \leq r \implies \vec{y} \in \mathcal{X}$$




choose $\vec{y} = \vec{z} = -\alpha \vec{c}$

$$\alpha = \frac{r}{\|\vec{c}\|_2}$$

consider : $\vec{x}_* + \vec{z}$

$$f_0(\vec{x}_* + \vec{z}) = \vec{c}^T(\vec{x}^* + \vec{z})$$

$$= \underbrace{\underbrace{\vec{c}^T \vec{x}^*}_{f_0(\vec{x}^*)} - \alpha \underbrace{\vec{c}^T \vec{c}}_{>0}}_{< f_0(\vec{x}^*)}$$

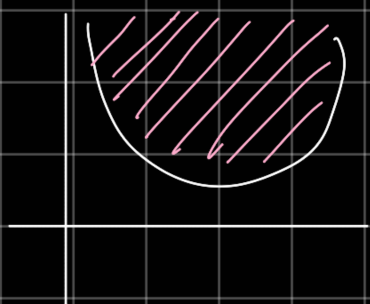
↳ smaller than what we claimed our minimum was

$\implies$ contradiction $\therefore$ QED

## Problem transformations

① Addition of "slack" variables

(epigraph reformulations)



$$\min_{\vec{x} \in \mathcal{X}} f_0(\vec{x}) \iff \min_t t$$
$$f(\vec{x}) < t$$
$$x \in \mathcal{X}$$

slack variable

this is equivalent bc we're minimizing an upper bound of $f_0(x)$

↳ reformulates convex fcn into linear problem

Consider:

$$\min_x \|A\vec{x} - \vec{b}\|_2^2 + \|\vec{x}\|_1$$ LASSO problem

$$= \min_x \|A\vec{x} - \vec{b}\|_2^2 + \sum_{i=1}^n |x_i|$$

↳ consider equiv problem:

$$= \min_{\vec{x}, t} \|A\vec{x} - \vec{b}\|_2^2 + \sum_{i=1}^n t_i$$
$$|x_i| \le t_i$$
$$-t_i \le x_i \le t_i$$

minimizing upper bound of abs. value of $x_i$

this is now a quadratic problem

## ② Monotone transformation

$\phi(x)$ : continuous & strictly increasing

$$g^* = \min \phi(f_0(\vec{x}))$$
$$\text{s.t. } f_i(\vec{x}) \le 0$$

$$p^* = \min f_0(\vec{x})$$
$$\text{s.t. } f_i(\vec{x}) = 0$$

Consider:

$$p^* = \min \alpha x_1^{\alpha_1} x_2^{\alpha_2}$$
$$\text{s.t. } x_1^{\beta_1} x_2^{\beta_2} \le b$$
$$x_i \ge 0 \quad \forall i$$

← not convex ⇒ not a convex prb

↳ take the log() to make it convex!

$$\min \log\alpha + \alpha_1 \log x_1 + \alpha_2 \log x_2$$
$$\text{s.t. } \beta_1 \log x_1 + \beta_2 \log x_2 \le \log b$$

$\log x_i = y_i$

$$= \min \log\alpha + \alpha_1 y_1 + \alpha_2 y_2$$
$$\text{s.t. } \beta_1 y_1 + \beta_2 y_2 \le \log b$$

this is now linear!

③ Logistic Regression

↳ way of doing classification

$\vec{x}_1, \vec{x}_2, \dots, \vec{x}_n$ ← points

$y_1, y_2, \dots, y_n$ ← labels (+1, -1)

↳ Predict $\mathbb{P}[Y = 1 \mid \vec{X} = \vec{x}_i]$