Lecture 11

- seperating hyperplane thm
- convex fcns

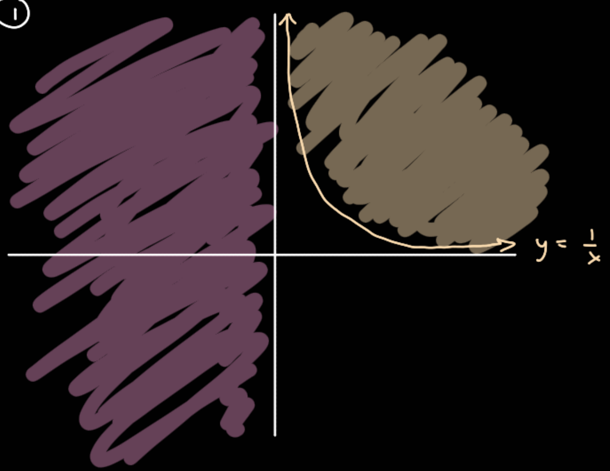
Seperating hyperplane thm

- let C & D be convex sets. Let them be disjoint $C \cap D = \emptyset$. Then $\exists$ a hyperplane $\vec{a}^T \vec{x} = b$ s.t.

$$\forall \vec{x} \in C \quad \vec{a}^T \vec{x} \geq b \qquad \vec{a}^T(\vec{x} - \vec{x}_0) \geq 0$$
$$\forall \vec{x} \in D \quad \vec{a}^T \vec{x} \leq b \qquad \vec{a}^T(\vec{x} - \vec{x}_0) \leq 0$$

Some tricky casts

①



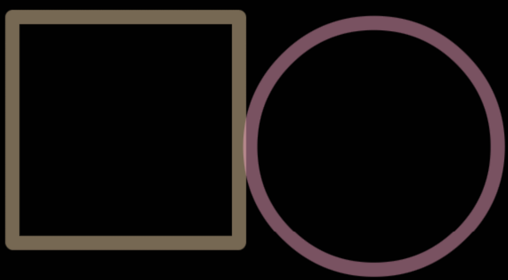
$C = \{(x,y) \; x \leq 0\}$

$D = \{(x,y) \; xy \geq 1\}$

$y \geq \frac{1}{x}$

Unbounded sets

② open sets



Seperating hyperplane thm (cont from last time)

distance btwn C & D

- $\text{dist}(C, D) = \inf \{ \| \vec{c} - \vec{d} \|_2 \mid \vec{c} \in C, \vec{d} \in D \}$

infimum

(largest lower bound)

$y \leq f(x) \quad \forall x$

y is the largest such lower bound

- let $\vec{c}, \vec{d}$ be the points that are closest to each other

Consider

- $\vec{d} - \vec{c}$ as normal that passes through $\frac{\vec{d} + \vec{c}}{2}$
  - the equation for this is:

$$f(\vec{x}) = (\vec{d} - \vec{c})^T \left(\vec{x} - \frac{\vec{d} + \vec{c}}{2}\right) = 0$$ } eqn for a hyperplane

$f(\vec{x}) = 0$ is a hyperplace

$$f(\vec{d}) = (\vec{d} - \vec{c})^T \left(\vec{d} - \frac{\vec{d} + \vec{c}}{2}\right) = \frac{1}{2} \| \vec{d} - \vec{c} \|^2$$

$$f(\vec{c}) = (\vec{d} - \vec{c})^T \left(\vec{c} - \frac{\vec{d} + \vec{c}}{2}\right) = -\frac{1}{2} \| \vec{d} - \vec{c} \|^2$$

Want to show: $\forall \vec{x} \in D$, $f(\vec{x}) \geq 0$

$\forall \vec{x} \in C$, $f(\vec{y}) \leq 0$

- Proof by contradiction

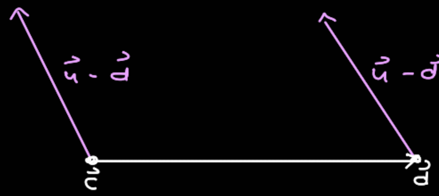
Assume $\exists\, \vec{u} \in D$ s.t $f(\vec{u}) < 0$

$$f(\vec{u}) = (\vec{d} - \vec{c})(\vec{u} + \left(\frac{\vec{d} + \vec{c}}{2}\right))$$

$$= (\vec{d} - \vec{c})^T \left(\vec{u} - \vec{d} + \vec{d} - \frac{\vec{d} + \vec{c}}{2}\right)$$

$$= \langle \vec{d} - \vec{c}, \vec{u} - \vec{d} \rangle + \underbrace{\frac{1}{2} \| \vec{d} - \vec{c} \|_2^2}_{\geq 0}$$

$f(\vec{u}) < 0 \implies \langle \vec{d} - \vec{c}, \vec{u} - \vec{d} \rangle < 0$



If I move along the vector $(\vec{u}-\vec{d})$ for some distance t, then I may get closer to $\vec{c}$.

consider: $\vec{p} = \vec{d} + t(\vec{u}-\vec{d})$

$$= t\vec{u} + (1-t)\vec{d}$$

D is a convex set: $\vec{u} \in D$, $\vec{d} \in D$

if $t \in [0,1] \Rightarrow \vec{p} \in D$

consider

$$\|\vec{c}-\vec{p}\|_2^2 = \|\vec{c}-\vec{d} - t(\vec{u}-\vec{d})\|^2$$

$$= ((\vec{c}-\vec{d}) - t(\vec{u}-\vec{d}))^T((\vec{c}-\vec{d}) - t(\vec{u}-\vec{d}))$$

$$= \|\vec{c}-\vec{d}\|^2 + \boxed{t^2\|\vec{u}-\vec{d}\|^2 - 2t\langle \vec{c}-\vec{d}, \vec{u}-\vec{d}\rangle}$$

want to show that this is negative to show that $\vec{p}$ is closer to $\vec{c}$ than $\vec{d}$ is

$$t^2\|\vec{u}-\vec{d}\|^2 - 2t\langle \vec{c}-\vec{d}, \vec{u}-\vec{d}\rangle$$

$$= t\|\vec{u}-\vec{d}\|^2 + 2t\langle \vec{d}-\vec{c}, \vec{u}-\vec{d}\rangle \quad \text{want } < 0$$

$$\underbrace{t\|\vec{u}-\vec{d}\|^2}_{} + \underbrace{2\langle \vec{d}-\vec{c}, \vec{u}-\vec{d}\rangle}_{\text{strictly } < 0} < 0$$

First, recall $t[0,1]$. So t can be arbitrarily close to 0.

If

$$t < \frac{-2\langle \vec{d}-\vec{c}, \vec{u}-\vec{d}\rangle}{\|\vec{u}-\vec{d}\|^2}$$

then we can infer that

$$t\|\vec{u}-\vec{d}\|^2 + 2\langle \vec{d}-\vec{c}, \vec{u}-\vec{d}\rangle < 0$$

which implies that

$$\|\vec{c}-\vec{p}\|^2 < \|\vec{c}-\vec{d}\|^2$$

This contradicts the minimality of the distance btwn $\vec{c}$ & $\vec{d}$

$\Rightarrow f(\vec{u}) \geq 0$ & $f(\vec{x})$ must be a separating hyperplane.

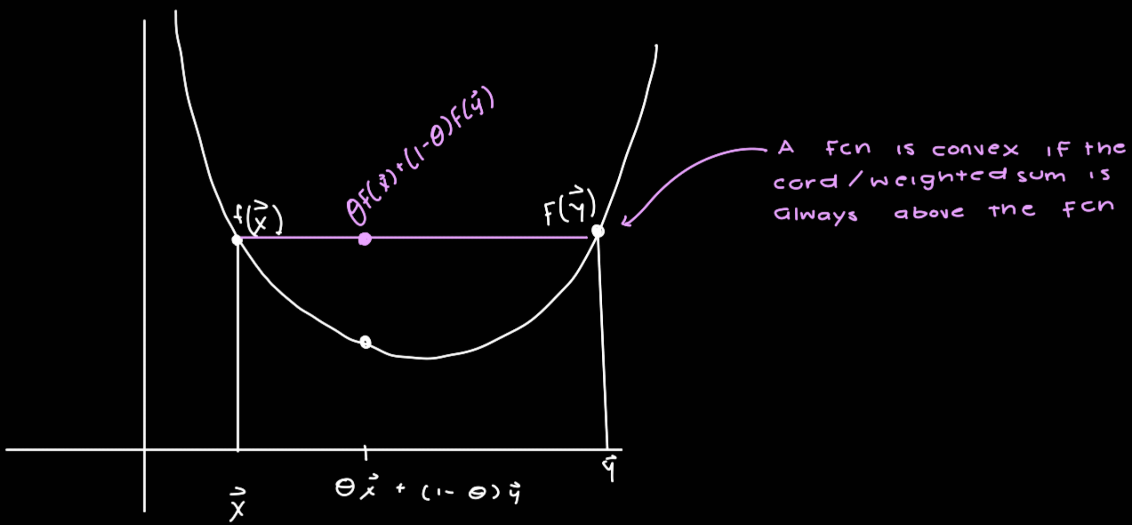
■

Convex Functions

$f: \mathbb{R}^n \to \mathbb{R}$, convex if domain of f is a convex set &

$$0 \leq \theta \leq 1$$

$$f(\theta\vec{x} + (1-\theta)\vec{y}) \leq \theta f(\vec{x}) + (1-\theta) f(\vec{y}) \quad \Big\} \text{ Jensen's Inequality}$$



Epigraph: Epi $f \{(x,t) \mid x \in \text{dom } f,\ f(x) \leq t\}$

Property: f is a convex fcn iff epi(f) is a convex set

First-order conditions: $f: \mathbb{R}^n \to \mathbb{R}$

- f: diff'able fcn
- Then f is convex iff

$$f(\vec{y}) \geq f(\vec{x}) + \nabla f(\vec{x})^T(\vec{y}-\vec{x})$$

$$\forall \vec{y}, \vec{x} \in \mathrm{dom}(f)$$

Implications: If $\nabla f(\vec{x}_*) = 0$ & f is convex

then

$$f(\vec{y}) \geq f(\vec{x}_*) + 0(\vec{y} - \vec{x})$$

$$f(\vec{y}) \geq f(\vec{x}_*) \quad \forall y \text{ in the domain}$$

$\therefore \vec{x}_*$ is a global minimum

Proof of First-Order Condition

1. Proving if: If f is convex, then 1st order cond holds

$$f((1-t)x + ty) \leq (1-t)f(x) + tf(y)$$ Know: f convex, so Jensen's inequality holds

↳ want to isolate $f(y)$

$$\frac{f((1-t)x+ty) - (1-t)f(x)}{t} \leq f(y)$$

$$f(y) \geq \frac{f((1-t)x+ty) - f(x) + tf(x)}{t}$$

$$f(y) \geq \frac{1}{t}f(x - xt + ty) - \frac{1}{t}f(x) + f(x)$$

$$f(y) \geq f(x) + \frac{1}{t}\left(f(x - xt + ty) - f(x)\right)$$

$$f(y) \geq f(x) + \frac{1}{t}\left(f(x + t(y-x)) - f(x)\right)$$

Recall

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

↳ Take lim as $t \to 0$

$$f(y) \geq f(x) + \frac{1}{t}\Big(f(x + \underbrace{t(y-x)}_{\Delta x}) - f(x)\Big)$$

Δx

$$f'(x) = \lim_{t \to 0} \frac{f(x + t(y-x)) - f(x)}{t(y-x)}$$

substitute

$$f'(x)(y-x) = \lim_{t \to 0} \frac{f(x + t(y-x)) - f(x)}{t}$$

$$\boxed{f(y) \geq f(x) + f'(x)(y-x)}$$

1st order condition

② Proving only if : If 1st order cond holds $\Rightarrow$ f convex

$x, y \in \mathbb{R}$

$z = \theta x + (1-\theta)y$

want to show Jensen's holds ie that:

$$f(z) \leq \theta f(x) + (1-\theta) f(y)$$

Know : $\forall p, q \in$ domain

$$f(p) \geq f'(q) + \nabla f(q)^T (p-q)$$

multiply:

θ $\quad f(x) \geq f(z) + f'(z)(x-z)$

$(1-\theta)$ $\quad f(y) \geq f(z) + f'(z)(y-z)$

$$\theta f(x) + (1-\theta) f(y) \geq \theta f(z) + (1-\theta) f(z) + \theta f'(z)(x-z) + (1-\theta) f'(z)(y-z)$$

$$= f(z) + f'(z)(\theta x - \theta z + (1-\theta) y - (1-\theta) z)$$

$$= f(z) + f'(z)(\theta x + (1-\theta) y - z)$$

$$= f(z)$$