EECS 127

- shadow prices recap
- duality certificate
- complementary slackness
- KKT conditions

Recall

200 kilos merlot

300 kilos shiraz

Blend 1: 4 kilos merlot + 1 kilo shiraz } \$20/bottle

Blend 2: 2 kilos merlot + 3 kilos shiraz } \$15/bottle

→ make q_1 of Blend 1

q_2 of Blend 2

$$p^* = \max_{q_1, q_2 \geq 0} \quad 20q_1 + 15q_2$$

$$4q_1 + 2q_2 \leq 200$$

$$q_2 + 3q_2 \leq 300$$

$$\max_{q_1, q_2 \geq 0} \quad 20q_1 + 15q_2 + \lambda_1(200 - 4q_1 - 2q_2) + \lambda_2(300 - q_2 - 3q_2)$$

$$= \max_{q_1, q_2} \quad \underbrace{(20 - 4\lambda_1 - \lambda_2)}_{\text{if} < 0 \Rightarrow \text{don't make blend 1}} q_1 + \underbrace{(15 - 2\lambda_1 - 3\lambda_2)}_{\text{if} < 0 \Rightarrow \text{don't make blend}} q_2 + 200\lambda_1 + 300\lambda_2$$

what if these both equal 0?

↳ break-even point ⇒ don't care whether you sell off or make the juice

→ these are the shadow prices of the grapes

For LPs, as long as the problem is feasible, strong duality holds

→ minimum profit:

$$d^* = \min_{\lambda_1, \lambda_2 \geq 0} \quad 200\lambda_1 + 300\lambda_2$$

→ correspond to shadow prices (ie, prices to make it worth your while)

$$20 - 4\lambda_1 - \lambda_2 = 0$$

$$15 - 2\lambda_1 - 3\lambda_2 = 0$$

Certificate

(λ_1, ν_1) : dual feasible point

x_1 : primal feasible

$$p^* \geq g(\lambda, \nu) \implies f_0(x_1) - p^* \leq \underbrace{f_0(x_1) - g(\lambda_1, \nu_1)}_{\leq \varepsilon}$$

$p^* \in [g(\lambda_1,\nu_1), f_0(x_1)]$ if strong duality $d^* \in [g(\lambda_1,\nu_1), f_0(x_1)]$

Complementary Slackness

consider primal optimal $\vec{x}^*$ & dual optimal $(\vec{\lambda}_*, \vec{\nu}_*)$

↳ Assume strong duality holds $(p^* = d^*)$

$$p^* = f_0(\vec{x}^*) = d^* = g(\lambda^*, \nu^*)$$

$$g(\lambda^*, \nu^*) = \min_{\vec{x}} \left(f_0(\vec{x}) + \sum_{i=1}^{m} \lambda_i^* f_i(\vec{x}) + \sum_{i=1}^{p} \nu_i^* h_i(\vec{x}) \right)$$

↑ Lagrangian

① $$\leq f_0(\vec{x}^*) + \underbrace{\sum_{i=1}^{m} \lambda_i^* f_i(\vec{x}^*)}_{\leq 0} + \underbrace{\sum_{i=1}^{p} \nu_i^* h_i(\vec{x}_*)}_{=0}$$

$p^* = \min f_0(\vec{x})$ $\qquad$ $d^* = \max_{\vec{\lambda} \geq 0} g(\vec{\lambda}, \vec{\nu})$

$f_i(\vec{x}) \leq 0 \quad \forall i \quad 1 \leq i \leq m$

$h_i(\vec{x}) = 0 \quad \forall i \quad 1 \leq i \leq p$

$$L(x, \lambda, \nu) = f_0(x) + \sum \lambda_i f_i(x) + \sum \nu_i h_i(\vec{x})$$

$\min_{\vec{x}} L(x, \lambda, \nu)$ ← this is an unconstrained minimization. $\vec{x}$ doesn't have to be in feasible set.

② $$\leq f_0(\vec{x}_*) + 0 + 0 = f_0(\vec{x}_*) = g(\lambda^*, \nu^*)$$

Both inequalities ① & ② must be equalities! (in special case of strong duality.

① $\implies \min_{\vec{x}} L(\vec{x}^*, \vec{\lambda}^*, \vec{\nu}^*) = L(\vec{x}^*, \vec{\lambda}^*, \vec{\nu}^*)$

non-positive term

② $\implies \sum_{i=1}^{m} \underbrace{\lambda_i^* f(\vec{x}_*)}_{\leq 0} = 0 \implies \lambda_i^* f_i(\vec{x}_*) = 0 \quad \forall i \; 1 \leq i \leq m$

not-neg term

if : $\lambda_i^* > 0 \implies f_i(\vec{x}_*) = 0$

$f_i(\vec{x}_*) < 0 \implies \lambda_i^* = 0$

} Complementary slackness (applies to non-convex fcns → note; we only assumed strong duality, not convexity?)

Karush Kuhn Tukker
KKT Conditions

↳ these are necessary conditions (ie. if $\vec{x}^*, \vec{\lambda}^*, \vec{\nu}^*$ are optimal)

- primal optimal $\vec{x}^*$ & dual optimal $(\vec{\lambda}_*, \vec{\nu}_*)$
- strong duality holds
- diff'able f_i's, h_i's
- non-convex problems ok

$\vec{x}_*, \vec{\lambda}_*, \vec{\nu}_*$ optimality ⇒

1. $f_i(\vec{x}_*) \leq 0 \quad i=1,\dots,m$
2. $h_i(\vec{x}_*) = 0 \quad i=1,\dots,p$
3. $\lambda_i^* \geq 0 \quad \forall i=1,\dots,m$
4. $\lambda_i^* f_i(\vec{x}_*) = 0 \quad \forall i=1,\dots,m$ (complementary slackness
5. $\nabla_x f_0(x^*) + \sum \lambda_i^* \nabla f_i(\vec{x}_*) + \sum \nu_i^* \nabla h_i(\vec{x}^*) = 0$

   ↳ x^* minimizes $L(\vec{x}, \vec{\lambda}^*, \vec{\nu}^*)$

   ↳ proved earlier if strong duality holds using complementary slackness. implied by the main thm

KKT Conditions II - Sufficient Conds

- convex problems
- diff'able
- strong duality doesn't necessarily hold

Sufficient cond: $\tilde{x}, \tilde{\lambda}, \tilde{\nu}$ are points

$f_i \quad i=0,\dots,m$ convex $\qquad \{x \mid f_i(x) \leq 0\}$

h_i : affine

1. $f_i(\tilde{x}) \leq 0 \quad i=1,\dots,m$
2. $h_i(\tilde{x}) = 0 \quad i=1,\dots,p$
3. $\tilde{\lambda} \geq 0 \quad i=1,\dots,m$
4. $\tilde{\lambda} \cdot f_i(\tilde{x}) = 0$
5. $\nabla f_0(\tilde{x}) + \sum \tilde{\lambda} \nabla f_i(\tilde{x}) + \sum \tilde{\nu} \nabla h_i(\tilde{x}) = 0$

↳ then $\tilde{x}, \tilde{\lambda}, \tilde{\nu}$ are primal + dual optimum

↳ if $\tilde{x}, \tilde{\lambda}, \tilde{\nu}$ satisfy KKT

+ convex

+ h_i affine $\iff$ $\tilde{x}, \tilde{\lambda}, \tilde{\nu}$ are primal & dual optimal

+ Slater condition

+ strong duality?