Lecture 3

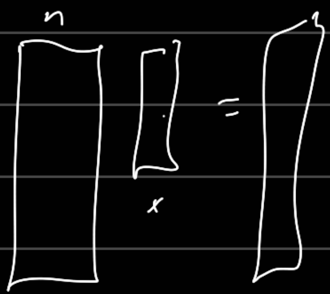
- Gram-Schmidt
- Fund. Thm of Lin Alg
  - ↳ min norm problem
- Symmetric Matrices

Fund Thm of Lin Alg. ECG 3.1

$\mathbb{R}^n \qquad N(A) \oplus R(A^T) = \mathbb{R}^n \qquad A \in \mathbb{R}^{m \times n}$

↑ direct sum ⇒ implies uniqueness of decomposition

$R(A) \oplus N(A^T) = \mathbb{R}^m$



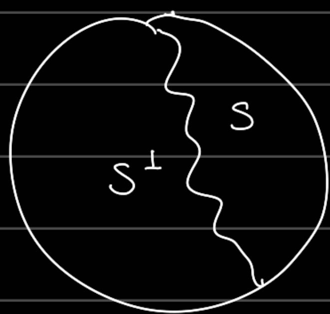
Orthogonal Decomposition Theorem (Thm 2.1 ECG)

$\mathcal{X}$: vector space

S : subspace. Then $\forall \vec{x} \in \mathcal{X}$

$\vec{x} = \vec{s} + \vec{r}$

$\vec{s} \in S, \quad \vec{r} \in S^\perp$



$S^\perp = \{ \vec{r} \mid \langle \vec{r}, \vec{s} \rangle = 0, \ \forall \vec{s} \in S \}$

$\mathcal{X} = S \oplus S^\perp$

- using the orthogonal decomposition thm to prove the FTLA:

Want: $N(A)^\perp = R(A^T)$

WTS: ① $N(A) \subseteq R(A^T)^\perp$; ② $R(A^T)^\perp \subseteq N(A)$

↳ subsets of each other

To show ①: Let $\vec{x} \in N(A)$, show $\vec{x} \in R(A^T)^\perp$

$$\underbrace{R(A^T)}$$
can be re-written as
$\vec{\omega} = A^T\vec{y}$

$$A\vec{x} = \vec{0}$$

$$\Rightarrow \vec{x}^T A^T = 0$$

$\forall \vec{\omega}$ s.t. $\vec{\omega} = A^T\vec{y}$

$\langle \vec{x}, \vec{w} \rangle = 0$

$$\langle \vec{x}, \vec{\omega} \rangle = \langle \vec{x}, A^T\vec{y} \rangle$$
$$= \vec{x}^T A^T \vec{y}$$
$$= \vec{y}^T A \vec{x}$$
$$= 0$$

$\vec{x} \in \mathbb{R}^n$

$\vec{x}^T$:

Aside

A scalar transposed is just that same scalar

To show ②: Let $\vec{x} \in R(A^T)^\perp$

$\forall \vec{\omega}$ s.t. $\vec{\omega} = A^T\vec{y}$

$$\langle \vec{x}, \vec{w} \rangle = 0$$
$$= \langle \vec{x}, A^T\vec{y} \rangle$$
$$= \vec{x}^T A^T \vec{y}$$
$$= \vec{y}^T A \vec{x}$$
$$= 0$$

Want: $\vec{x} \in N(A)$

$A\vec{x} = 0$

$\rightsquigarrow$ this is true for $\forall \vec{y}$ because $\vec{\omega} = A^T\vec{y}$ is valid for $\forall \vec{\omega}$

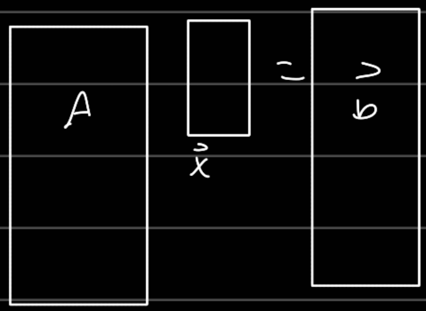
$\Rightarrow$ since true $\forall \vec{y}$, $A\vec{x} = 0$.

$a^T \vec{b} = 0 \quad \forall \vec{a}^T$

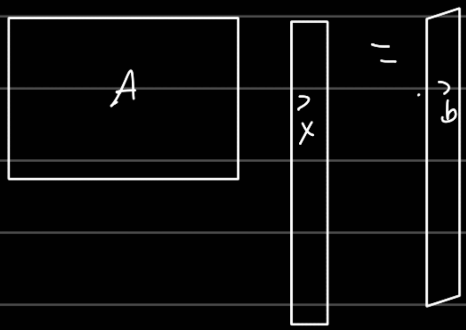
$$\underbrace{\begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \end{bmatrix}}_{\forall a} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = \begin{bmatrix} b_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \Rightarrow \vec{b} \text{ must be } 0$$

Minimum Norm Problem

$A\vec{x} = \vec{b}$



overdetermined
↳ generally 0 solutions



underdetermined
↳ generally ∞ solutions

- minimize the 2 norm of $\vec{x}$ subject to $A\vec{x} = \vec{b}$

$$\min \|\vec{x}\|_2^2$$
$$\text{s.t. } A\vec{x} = \vec{b}$$

$A\vec{x} = \vec{b}$

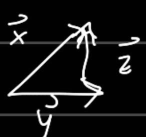
FTLA $\Rightarrow$ $\vec{x} = \vec{y} + \vec{z}$ $\quad \vec{y} \in N(A)$, $z \in R(A^T)$

$A\vec{x} = \vec{b}$

$A\vec{y} + A\vec{z} = \vec{b}$

$A\vec{z} = \vec{b}$

$\vec{y} \perp \vec{z}$ $\quad \|\vec{x}\|_2^2 = \|\vec{y}\|_2^2 + \|z\|_2^2$



$\vec{z} = A^T\vec{\omega}$

$A\vec{z} = \vec{b}$ $\quad \Rightarrow \underbrace{A \cdot A^T}_{\text{square matrix}}\vec{\omega} = \vec{b}$

$\in \mathbb{R}^{m \times n}$

→ if full row rank, invertible

$\vec{\omega} = (AA^T)^{-1}\vec{b}$

$$\boxed{\vec{z} = A^T(AA^T)^{-1}\vec{b}}$$

↳ care about full row rank

In the case of least squares, care about full column rank

Symmetric Matrices

$A = A^T$ ⇒ all scalars are symmetric matrices

$A \in S^n$

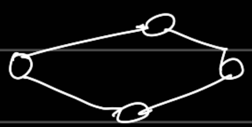
$A_{ij} = A_{ji}$

$\begin{bmatrix} a & b \\ b & c \end{bmatrix}$ is symmetric

eg. covariance matrices are symmetric $A = BB^T$

$A^T = (BB^T)^T = BB^T$

eg. graph laplacians



- have nice eigenvalues
- are always diagonalizable

eg $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ → not $\underbrace{\text{diagnolizable}}_{A = U\Lambda U^{-1}}$

- "algebraic multiplicity" = "geometric multiplicity"

$\det(\lambda I - A)$

↳ # of times λ appears as a root

↳ dimension of $N(\lambda I - A)$

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \Rightarrow \det(\lambda I - A) = (\lambda - 1)^2$$

$\hookrightarrow \lambda = 1$

$\hookrightarrow$ multiplicity of 2

$\mu = 2$

$$\text{Null}(\lambda I - A) = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -y \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

is the eigenvector corresponding to $\lambda = 1$

Spectral Theorem : $A \in S^n$

① $\lambda_i \in \mathbb{R}$

② eigenspaces corresponding to distinct eigenvalues $\lambda_i \neq \lambda_j$ are orthogonal

$$\Phi_i = N(\lambda_i I - A)$$

③ $\dim(\Phi_i) = \mu_i$ (μ_i is algebraic multiplicity)

$A \in S^n$

$A = U \Lambda U^T$

U: orthonormal

Λ: diagonal

Proof of ③

Lemma: $(\lambda, \vec{u})$ be an eigenpair for A.

Then $\exists$ orthonormal U s.t. $U^T A U = \begin{bmatrix} \lambda & 0 \cdots 0 \\ 0 & \\ \vdots & \boxed{B} \\ 0 & \end{bmatrix}$

and $B \in S^{n-1}$

Proof by Construction:

Consider: $U = \begin{bmatrix} \vec{v} & \boxed{U_1} \end{bmatrix}$

$\underbrace{}$ gram-schmidt to fill out the rest of the matrix

$$U^TAU = \begin{bmatrix} - \vec{v} - \\ U_1^T \end{bmatrix} \begin{bmatrix} A \end{bmatrix} \begin{bmatrix} | & \\ \vec{v} & U_1 \\ | & \end{bmatrix}$$

$$= \begin{bmatrix} - \vec{v} - \\ U_1^T \end{bmatrix} \begin{bmatrix} A\vec{v} & AU_1 \end{bmatrix}$$

$$= \begin{bmatrix} - \vec{v} - \\ U_1^T \end{bmatrix} \begin{bmatrix} \lambda\vec{v} & AU_1 \end{bmatrix}$$

$$= \begin{bmatrix} \lambda & 0 \cdots 0 \\ 0 & \\ \vdots & B \\ 0 & \end{bmatrix}$$

$B = U_1^T A U_1$

$B^T = B$ (bc $A = A^T$)

$B \in S^{n-1}$

Say $B \in S^{n-1}$

$B = V^T \underbrace{\Lambda}_{\text{diagonal}} \overbrace{V}^{\text{orthonormal}}$ $\quad \Lambda \in \mathbb{R}^{(n-1)(n-1)}$

$$U^TAU = \begin{bmatrix} \lambda & \cdots 0 \\ \vdots & B \\ 0 & \end{bmatrix}$$

$$VU^TAU = \begin{bmatrix} \lambda & \\ & VBV^T \end{bmatrix} = \begin{bmatrix} \lambda & 0 \\ 0 & \Lambda \end{bmatrix}$$

$VBV^T = \Lambda$