Today

- SOCPs
- Newton's Method

SOCPs

"easier to solve"

Def: cone

- set of points $C \subseteq \mathbb{R}^n$ is a cone iff

$\alpha \vec{x} \in$ if $\vec{x} \in C \quad \forall \alpha \geq 0$

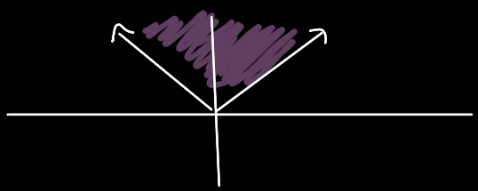
Def: Convex Cone $\quad C$ convex cone iF

$\alpha \vec{x} \in C$ for $\alpha \geq 0$; $\theta_1, \theta_2 \geq 0$

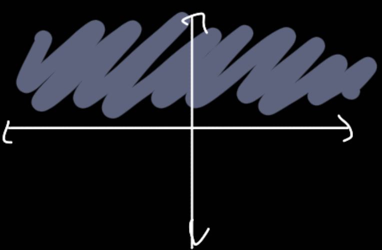
then

$$\theta_1 \vec{x_1} + \theta_2 \vec{x_2} \in C$$

eg $\quad C = \{(x, y) \mid |x| \leq y\}$

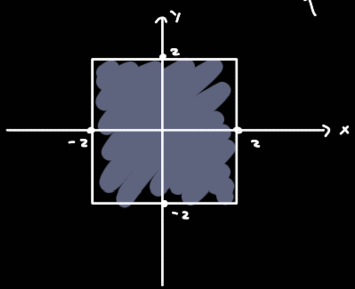


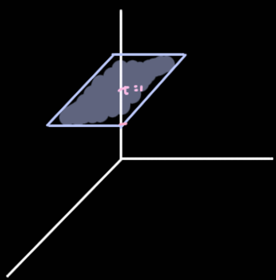
$C = \{(x, y) \mid y \geq 0\}$



Polyhedral Cone

- Polyhedron : $\{\vec{x} \mid A\vec{x} \leq \vec{b}\}$
- Polyhedral cone : $\{(\vec{x}, t) \mid A\vec{x} \leq \vec{b}t \quad t \in \mathbb{R}, t \geq 0\}$
- Polyhedron: $\left\{ \vec{x} \mid \begin{array}{l} x \leq 2, x \geq -2 \\ y \leq 2, y \geq -2 \end{array} \right\}$





at t=1, you get the og polyhedron

Ellipsoidal Cone $x^T P x + q^T x + r \leq 0 \quad P \succeq 0$

Consider $\|Ax + b\|_2^2 \leq c^2$

$x^T A^T A x + 2b^T A x + b^T b \leq 0$ ellipse

$C = \{(x, t) \mid \|Ax + bt\|_2 \leq ct\}$ ← Ellipsoidal cone

↳ $(x, t) \in C$ ⇒ Q / Does $(\alpha x, \alpha t) \in C$?

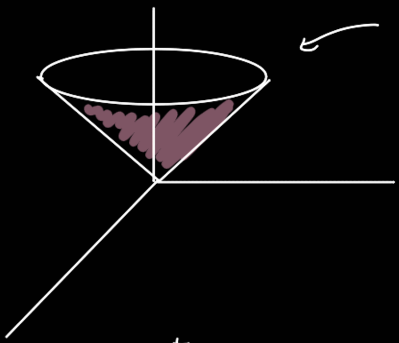
Check $\|A(\alpha x) + b(\alpha t)\|_2 \leq c\alpha t$

$\alpha \|Ax + bt\|_2 \leq \alpha c t$

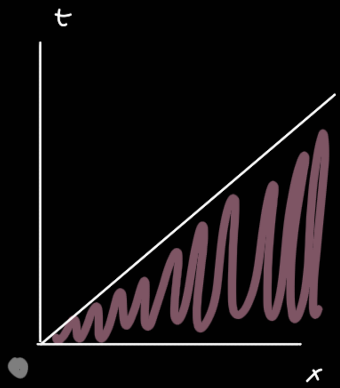
yes ⇒ $(\alpha x, \alpha t) \in C$

Special case of Ellipsoidal cone : 2nd order cone ∈ $\mathbb{R}^3$

$\{(x_1, x_2, t) \mid \sqrt{x_1^2 + x_2^2} \leq t\}$



SOC in 3D "ice cream cone"



2D SOC

Define: SOCP

$\min \ \vec{q}^T \vec{x}$

s.t. $\|A_i \vec{x} + \vec{b}_i\|_2 \leq \vec{c}_i^T \vec{x} + d_i \quad i = 1, \dots, m$

$(\underbrace{A_i \vec{x}_i + \vec{b}_i}_{(\vec{x}}, \underbrace{\vec{c}_i^T x + d_i}_{t\ )})$ must belong to SOC $\|x\|_2 \leq t$

Reformulating optimization problems to look like SOCPs

eg ① $\min_x \sum_{i=1}^{m} \|A_i x - b_i\|_2$

$\min_x \|Ax - b\|_2$ (Least squares)

$\min_x \sum_{i=1}^{m} \|A_i x - b_i\|_2$ ← no longer least squares

AA reformulating:

$\min_x \sum_{i=1}^{m} \|A_i \vec{x} - \vec{b}_i\|_2$

↳ $\sum_{i=1}^{m} \|A_i \vec{x} - \vec{b}_i\|_2 \leq \sum_{i=1}^{m} \vec{c}_i^T \vec{x} + d_i$

$\min_x \vec{q}^T x$

s.t. $\sum_{i=1}^{m} \|A_i \vec{x} - \vec{b}_i\|_2 \leq \sum_{i=1}^{m} \vec{c}_i^T \vec{x} + d_i$

+

$\min \vec{q}^T \vec{x}$

s.t. $\|A_i \vec{x} - \vec{b}_i\|_2 \qquad \vec{c}_i^T \vec{x} + d_i$

Ranade:

$\|A_i \vec{x} - \vec{b}_i\| = y_i$

↳ $\min_x \sum_{i=1}^{m} y_i$

$\|A_i \vec{x} - \vec{b}_i\|_2 = y_i$

↓ relax the problem

$\min_{x, y_i} \sum_{i=1}^{m} y_i$

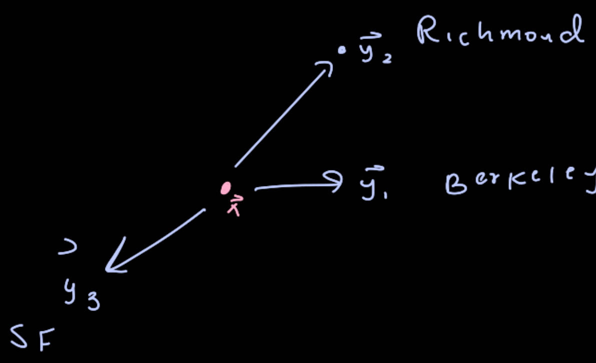
$\|A_i x - b_i\|_2 \leq y_i$

② $\min_x \max_{i=1,\dots,m} \|A_i \vec{x} - \vec{b}_i\|_2$

$\min_{\vec{x}, \vec{y}} y$

$\|A_i \vec{x} - \vec{b}_i\|_2 \leq y \quad \forall i = 1, \dots, m$

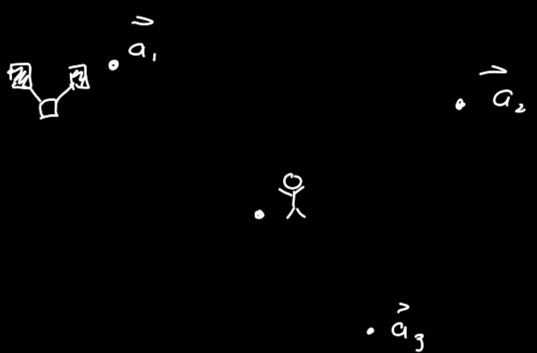
Facility Location Problem



$$\min \frac{1}{m} \sum_{i=1}^{m} \|\vec{x} - \vec{y}_i\|_2$$

$$\min_x \max_i \|x - y_i\|_2$$

Trilateration / GPS



Packet transmission times : t_i^T

Packet received times : t_i^R

offset : δ

$\hookrightarrow t_i^{true} = t_i^R + \delta$

Time of Flight $f_i = t_i^{True} - t_i^T = t_i^R + \delta - t_i^T = \Delta_i + \delta$

Distance: $cf_i = c\underbrace{\Delta_i}_{\Delta_i = t_i^R - t_i^T} + c\delta$

where satellite are (know)

offset (unknown)

$\|\vec{x} - \vec{a}_i\|_2 = c\Delta_i + c\delta$

where I am (unknown)

speed of light (known)

unknowns : $n+1$ — δ, $x_i \ldots x_n$

$x \in \mathbb{R}^n$

squaring gives: $(\vec{x} - \vec{a}_i)^T(\vec{x} - \vec{a}_i) = (c\Delta_i + c\delta)^2$

4 satellites $\longrightarrow$ square all the eqns $\rightarrow$ subtract them from eqn from satellite 4

$$\|\vec{x} - \vec{a}_4\|_2^2 - \|\vec{x} - \vec{a}_1\|_2^2 = \underbrace{x^T x - x^T x}_{\text{gets rid of quadratic term}} + \overset{\text{const}}{K} x + \text{constant}$$

$\hookrightarrow$ gives linear eqn

3 linear eqns in the form:

$$2(\vec{a}_4 - \vec{a}_i)^T \vec{x} + 2c^2(\Delta_4 + \Delta_i)\delta = c^2(\Delta_i^2 + \Delta_4^2) + \|\vec{a}_4\|_2^2 - \|\vec{a}_i\|_2^2$$

$\hookrightarrow \in \mathbb{R}^2$

$2+1 = 3$ unknowns, 3 eqns ✓

Q/ What happens if we lose another satellite?

↳ 3 unknowns, 2 eqns ∴ → will solve with SOCPs! :)

eqn ① $2(\vec{a}_3 - \vec{a}_1)^T x + 2c^2(\Delta_3 - \Delta_1)\delta = c^2(\Delta_1^2 - \Delta_3^2) + \|\vec{a}_3\|_2^2 - \|\vec{a}_1\|_2^2$

② $2(\vec{a}_3 - \vec{a}_2)^T x + \dots$

③ $\|x - a_3\|_2 = c\Delta_3 + c\delta$ } nonlinear

$\min \ \delta$

s.t. ① } linear equality constraints don't have to be relaxed

②

③ $\|x - a_3\|_2 \leq c\Delta_3 + c\delta$ ← need to relax this to make it into an SOCP & to make it solvable

↳ since we have a linear objective, we know that this can be relaxed & will achieve the same solution. we know the objective will always be attained on the boundary.

Newton's Method "big brother of gradient descent"

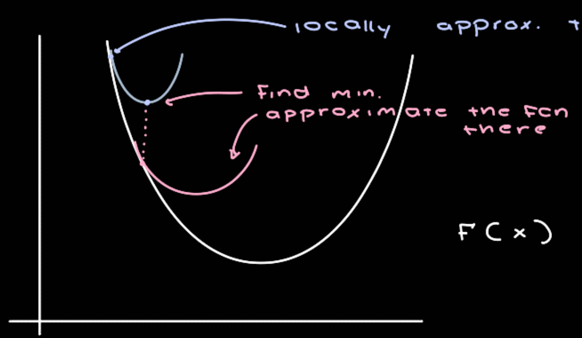
- second order

$F(\vec{x})$ want $\vec{x}_0, \vec{x}_1$ converging to $\vec{x}_*$ (optimal)

$$\min_{\vec{x}} \ F(\vec{x})$$

↳ Taylor's

$$F(\vec{x} + \vec{v}) = \underbrace{F(\vec{x}) + \nabla F(\vec{x})^T \vec{v} + \frac{1}{2}\vec{v}^T \nabla^2 F(\vec{x})\vec{v}}_{\text{quadratic approx}} + \dots \text{ HOT}$$



x_0 : initial start

- $H = \nabla^2 F$ is PD → invertible

↳ general quadratic

$$\frac{1}{2}\vec{x}^T H \vec{x} + \vec{c}^T \vec{x} + d$$

$$\vec{x}_{min} = -H^{-1}c$$

Best $\vec{v}$ direction to take step to minimize quadratic

↳ $\vec{v}^* = -H^{-1}\vec{c}$

$= -(\nabla^2 F(\vec{x}))^{-1} \nabla F(\vec{x})$ by pattern matching

Newton step:

$$\vec{x}_{k+1} = x_k - (\nabla^2 F(x_k))^{-1} \nabla F(\vec{x_k})$$

↳ Note: if H not PD, then you have to use Quasi-Newton methods (recall from last class; how to deal w/ not PD cases)

Q/ Why Newton's method instead of gradient?

Pros
- converges much faster

Cons
- Hessian inversion

Damped Newton's method (add a step η)

og: $\vec{x}_{k+1} = x_k - (\nabla^2 F(x_k))^{-1} \nabla F(\vec{x_k})$

damped: $\vec{x}_{k+1} = x_k - \eta (\nabla^2 F(x_k))^{-1} \nabla F(\vec{x_k})$