Today

- More duality
- minmax interpretation
- strong duality / Slater's condition
- price interpretation

Recall (last time):

$p^* = \min f_0(\vec{x})$ $\longleftrightarrow$ Lagrangian

$f_i(\vec{x}) \leq 0 \quad 1 \leq i \leq m$

$h_i(\vec{x}) = 0 \quad 1 \leq i \leq p$

$g(\vec{\lambda}, \vec{\nu}) = \min_{\vec{x}} L(\vec{x}, \vec{\lambda}, \vec{\nu})$

↳ unconstrained optimization problem ⇒ "might" be easier to solve

maximize the dual Fcn

$d^* = \max_{\vec{\lambda} \geq 0} g(\vec{\lambda}, \vec{\nu})$

↳ maximization of a concave fcn ⇒ convex problem

↳ dimension = # constraints of the primal

$d^* \leq p^*$ ← weak duality

duality gap $:= p^* - d^*$

$d^* = p^*$ ← strong duality

↳ slater's condition

Note

all of these hold even when $f_0(\vec{x})$, $f_i(\vec{x})$ aren't convex

Minimax Inequality Proof

↳ describes the first mover's advantage

sets $\mathcal{X}, \mathcal{Y}$ F : any fcn

$$\min_{x \in \mathcal{X}} \max_{y \in \mathcal{Y}} F(x,y) \geq \max_{y \in \mathcal{Y}} \min_{x \in \mathcal{X}} F(x,y)$$

↳ doing minimization first gets you a smaller value

<u>proof</u>: Fix $x_0 \in \mathcal{X}$ $y_0 \in \mathcal{Y}$

Define : $h(y_0) := \min_{x \in \mathcal{X}} F(x, y_0)$

$$g(x_0) := \max_{y \in \mathcal{Y}} F(x_0, y)$$

Ayan thoughts:

- use Lagrangian

* Try to connect the two fcns to $F(x_0, y_0)$

$h(y_0) \leq F(x_0, y_0) \leq g(x_0)$ ← Ayan attempt

$$h(y_0) = \min_{x \in \mathcal{X}} F(x, y_0) \leq F(x_0, y_0)$$

$$g(x_0) = \max_{y \in \mathcal{Y}} F(x_0, y) \geq F(x_0, y_0)$$

$$\Rightarrow \forall x_0, y_0 \quad h(y_0) \leq g(x_0)$$

$$\Rightarrow \max_{y_0 \in \mathcal{Y}} h(y_0) \leq \min_{x_0 \in \mathcal{X}} g(x_0)$$

$$\Rightarrow \max_{y \in \mathcal{Y}} \min_{x \in \mathcal{X}} F(x,y) \leq \min_{x \in \mathcal{X}} \max_{y \in \mathcal{Y}} F(x,y)$$

Connecting to duality :

$p^* = \min f_0(\vec{x})$

$f_i(\vec{x}) \leq 0 \quad 1 \leq i \leq m$

$h_i(\vec{x}) = 0 \quad 1 \leq i \leq p$

$$L(\vec{x}, \vec{\lambda}, \vec{\nu}) = f_0(\vec{x}) + \sum_{i=1}^{m} \lambda_i f_i(\vec{x}) + \sum_{i=1}^{p} \nu_i h_i(\vec{x})$$

$$g(\vec{\lambda}, \vec{\nu}) = \min_{\vec{x}} L(\vec{x}, \vec{\lambda}, \vec{\nu})$$

$$d^* = \max_{\vec{x} \geq 0} g(\vec{\lambda}, \vec{\nu})$$

$$= \max_{\vec{\lambda} \geq 0} \min_{\vec{x}} L(\vec{x}, \vec{\lambda}, \vec{\nu})$$

↳ can we express the primal using this?

<u>consider</u>: $\max L(\vec{x}, \vec{\lambda}, \vec{\nu})$

$$= \max_{\substack{\vec{\lambda} \geq 0 \\ \vec{\nu}}} \underbrace{f_0(\vec{x})}_{\text{const}} + \sum_{i=1}^{m} \underbrace{\lambda_i}_{\geq 0} \underbrace{f_i(\vec{x})}_{\substack{\text{const} \\ \leq 0}} + \sum_{i=1}^{p} \nu_i \underbrace{h_i(\vec{x})}_{=0}$$

think about feasible vs. infeasible x's

↳ ie that $h_i(\vec{x}) = 0$, $f_i(\vec{x}) \leq 0$

↳ max value this can take is 0 (when $\lambda = 0$ or $f_i(x) = 0$)

$$= \begin{cases} f_0(\vec{x}) & \text{if } \vec{x} \text{ is feasible} \\ \infty & \text{if } \vec{x} \text{ is infeasible} \end{cases}$$

↳ eg consider $f_i(\vec{x}) > 0$

$\max \sum_{i=1}^{m} \underbrace{\lambda_i}_{\geq 0} \underbrace{f_i(x)}_{\geq 0}$

→ max value $= \infty$

↳ alternatively $h_i(\vec{x}) \mathrel{!}= 0$

$\sum_{i=1}^{p} \nu_i \underbrace{h_i(\vec{x})}_{!=0}$

$h_i(\vec{x}) > 0$ → choose $\nu = \infty$

$h_i(\vec{x}) < 0$ → choose $\nu = -\infty$

this is why we have a constraint on λ & not ν

$$p^* = \min_{\vec{x}} \max_{\substack{\vec{\lambda} \geq 0 \\ \vec{\nu}}} L(\vec{x}, \vec{\lambda}, \vec{\nu})$$

$$\Rightarrow p^* \geq d^*$$

↳ bc d^* is max min

<u>Strong duality / slater's condition</u>

$p^* = d^*$

- strong duality holds if $\exists \vec{x}_0$ s.t. $f_i(\vec{x}_0) < 0 \ \forall i$ — $\vec{x}_0 \in$ Relative interior (D)

  "strictly feasible"

- will only hold if the problem is convex. (don't need problm to be convex if you're considering weak duality).

Refined Slater's Condition

- convex problem
- say you have $f_1, f_2, \dots, f_k$ that are affine

$\exists \vec{x}_0$ s.t. $f_i(\vec{x}) \le 0 \quad \forall i = 1, \dots, k$ } affine don't have to use strict inequality

AND $f_i(\vec{x}) < 0 \quad \forall i = k+1, \dots, m$ } but other convex fcns do

- if you only have equality constraints, Slater's will hold
- if you have no " " " "

LPs & Duality

Consider

$$\min \vec{c}^T \vec{x}$$
$$A\vec{x} \le \vec{b} \rightarrow A\vec{x} - \vec{b} \le 0$$

$$L(\vec{x}, \vec{\lambda}) = \vec{c}^T \vec{x} + \vec{\lambda}^T (A\vec{x} - \vec{b})$$
$$= (A^T \vec{\lambda} + \vec{c})^T \vec{x} - \vec{b}^T \vec{\lambda}$$

$$g(\vec{\lambda}) = \min_{\vec{x}} L(\vec{x}, \vec{\lambda})$$

$$= \begin{cases} -\infty & \text{if } (A^T \vec{\lambda} + c) \ne 0 \\ -\vec{b}^T \vec{\lambda} & \text{if } (A^T \vec{\lambda} + c) = 0 \end{cases}$$

$$\max_{\substack{\lambda \ge 0 \\ A^T \vec{\lambda} + c = 0}} -\vec{b}^T \vec{\lambda} = d^*$$

} dual problem

↳ dual of an LP is an LP

→ dual of dual is the original

Business

- 200 kilos merlot grapes
- 300 kilos shiraz grapes

Blend 1: 4 kilos merlot, 1 kilo shiraz } \$20 per bottle

Blend 2: 2 kilos merlot, 3 kilos shiraz } \$15/bottle

q_1 bottles of Blend 1

q_2 " " " 2

$$\max \quad 20q_1 + 15q_2$$

$$4q_1 + 2q_2 \le 200$$
$$q_1 + 3q_2 \le 300$$
$$q_1, q_2 \geq 0$$

λ_1 money per kilo merlot (if you sold it off)

λ_2 money per kilo shiraz

$$\max_{q_1, q_2 > 0} \quad 20q_1 + 15q_2 + \lambda_1(200 - 4q_1 - 2q_2) + \lambda_2(300 - q_1 - 3q_2)$$

$$\max_{q_1, q_2 > 0} \quad \underbrace{(20 - 4\lambda_1 - \lambda_2)}\, q_1 + \underbrace{(15 - 2\lambda_1 - 3\lambda_2)}\, q_2 + 200\lambda_1 + 300\lambda_2$$

if negative, don't go into biznes :;
better to just sell the grapes

if $= 0$, we only get $200\lambda_1 + 300\lambda_2$

$$\min \quad 200\lambda_1 + 300\lambda_2$$
$$20 - 4\lambda_1 - \lambda_2 = 0$$
$$15 - 2\lambda_1 - 3\lambda_2 = 0$$

→ shadow prices (how much are you willing to pay to violate / be away from the constraint)