Homework 2

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1. Asymptotic Comparison

a) $f(n) = n^{1/2}$ $g(n) = n^{2/3}$

$$\lim_{n\to\infty} \frac{n^{\frac{1}{2}}}{n^{2/3}} = \lim_{n\to\infty} n^{\frac{1}{2}-\frac{2}{3}}$$

$$= \lim_{n\to\infty} n^{\frac{3}{6}-\frac{4}{6}}$$

$$= \lim_{n\to\infty} n^{-\frac{1}{6}}$$

$$= \lim_{n\to\infty} \frac{1}{n^{1/6}} = 0$$

$$\boxed{\rightarrow f(n) = O(g(n))}$$

b) $f(n) = 330(n + \log n)$ $g(n) = n + (\log n)^2$

$$\lim_{n\to\infty} \frac{330(n+\log n)}{n + (\log n)^2}$$

$$= 330 \lim_{n\to\infty} \frac{1 + \frac{1}{n}}{1 + 2\log n\left(\frac{1}{n}\right)}$$

$$= 330 \lim_{n\to\infty} \frac{\frac{n+1}{n}}{\frac{n+2\log n}{n}}$$

$$= 330 \lim_{n\to\infty} \frac{n+1}{\cancel{n}} \cdot \frac{\cancel{n}}{n+2\log n}$$

$$= 330 \lim_{n\to\infty} \frac{n+1}{n+2\log n}$$

$$= 330 \lim_{n\to\infty} \frac{\cancel{n}(1+\cancel{\tfrac{1}{n}}^{\,0})}{\cancel{n}(1+\cancel{\tfrac{2\log n}{n}}^{\,0})}$$

$$\lim_{n\to\infty} \frac{2\log n}{n} = \lim_{n\to\infty} \frac{\frac{2}{n}}{n}$$

$$= \lim_{n\to\infty} \frac{2}{n^2} = 0$$

$$\boxed{= 330}$$

$330 \neq 0$ and $330 \neq \infty$

$$\boxed{\rightarrow f(n) = \Theta(g(n))}$$

c) $f(n) = 330 \log n$ $g(n) = \log(n^3)$

$$\lim_{n\to\infty} \frac{330\log n}{\log(n^3)} = 330\lim_{n\to\infty} \frac{\frac{1}{n}}{\frac{1}{n^3}(3n^2)}$$

$$= 330\lim_{n\to\infty} \frac{1}{\cancel{n}}\cdot\frac{\cancel{n}}{3}$$

$$= 110$$

$110 \neq 0$ and $110 \neq \infty$

$$\boxed{\Rightarrow f(n) = \Theta(g(n))}$$

d) $f(n) = n^{1.01}$ $\quad g(n) = n\log^2 n$

$$\lim_{n\to\infty} \frac{\cancel{n^1} \times n^{0.01}}{\cancel{n}\log^2 n}$$

$$= \lim_{n\to\infty} \frac{n^{0.01}}{\log^2 n}$$

$$= \lim_{n\to\infty} \frac{0.01n^{-0.99}}{2\log n\left(\frac{1}{n}\right)} = \lim_{n\to\infty} \frac{0.01\, n^{0.01}}{2\log n}$$

$$= \lim_{n\to\infty} \frac{(0.01)^2 n^{-0.99}}{2\left(\frac{1}{n}\right)} = \lim_{n\to\infty} \frac{(0.01)^2 n^{0.01}}{2} = \infty$$

$$\boxed{\Rightarrow f(n) = \Omega(n)}$$

e) $f(n) = \frac{n^2}{\log n}$ $\quad g(n) = n(\log n)^2$

$$\lim_{n\to\infty} \frac{n^{\cancel{2}}}{\cancel{n}(\log n)^3} = \lim_{n\to\infty} \frac{n}{(\log n)^3}$$

$$= \lim_{n\to\infty} \frac{1}{3(\log n)^2\left(\frac{1}{n}\right)}$$

$$= \lim_{n\to\infty} \frac{n}{3(\log n)^2}$$

$$= \frac{1}{3}\lim_{n\to\infty} \frac{1}{2(\log n)\left(\frac{1}{n}\right)}$$

$$= \frac{1}{6}\lim_{n\to\infty} \frac{n}{\log n}$$

$$= \frac{1}{6}\lim_{n\to\infty} \frac{1}{1/n}$$

$$= \frac{1}{6}\lim_{n\to\infty} n$$

$$= \infty$$

$$\Rightarrow \boxed{f(n) = \Omega(g(n))}$$

$$= \infty$$

$$\Rightarrow \boxed{f(n) = \Omega(g(n))}$$

f) $f(n) = (\log n)^{\log n}$ $\quad$ $g(n) = \dfrac{n}{\log n}$

n	f(n)	g(n)
2	0.78	2.88
5	2.15	3.11
10	6.82	4.34
50	207.7	12.78
100	1133	21.71
200	6866	37.7

When $n \geqslant 10$, $f(n)$ grows faster than $g(n)$

$\Rightarrow$ For $n_0 = 10$ and $c = 1$,

$f(n) \geqslant c \cdot g(n)$

$$\boxed{\Rightarrow f(n) = \Omega\, g(n)}$$

g) $f(n) = n2^n$ $\quad$ $g(n) = 3^n$

$$\lim_{n\to\infty} \frac{n2^n}{3^n} = \lim_{n\to\infty} n\left(\frac{2}{3}\right)^n$$

$$= \lim_{n\to\infty} \frac{n}{\left(\frac{3}{2}\right)^n}$$

$$= \lim_{n\to\infty} \frac{n}{e^{n\ln\left(\frac{3}{2}\right)}}$$

$$= \lim_{n\to\infty} \frac{1}{e^{n\ln\left(\frac{3}{2}\right)} \cdot \ln\left(\frac{3}{2}\right)}$$

$$= \lim_{n\to\infty} \frac{1}{\left(\frac{3}{2}\right)^n \cdot \ln\left(\frac{3}{2}\right)}$$

$$= 0$$

$$\boxed{\Rightarrow f(n) = O(g(n))}$$

h) $f(n) = \sum_{i=1}^{n} i^k$ $\quad$ $g(n) = n^{k+1}$

$$f(n) = \underbrace{1^k + 2^k + 3^k + \cdots + (n-1)^k}_{\substack{\text{each one} \\ \text{of these terms} \\ \text{is } < n^k}} + n^k$$

$\Longrightarrow$ $f(n)$ is of order n^k

$g(n)$ is of order n^{k+1}

$$\lim_{n\to\infty} \frac{n^k}{n^{k+1}} = \lim_{n\to\infty} \frac{n^k}{n^k \times n} = \frac{1}{n} = 0$$

$$\boxed{\Longrightarrow \quad f(n) = O(g(n))}$$