1. Sums

a. $\sum_{i=1}^{33}\left(\frac{1}{2}\right)^i = \sum_{i=0}^{33}\left(\frac{1}{2}\right)^i - \left(\frac{1}{2}\right)^0 = \frac{\left(\frac{1}{2}\right)^{34}-1}{\frac{1}{2}-1} - 1 = 2 - 1 \boxed{= 1}$

b. $\sum_{i=0}^{\infty}\left(\frac{2}{5}\right)^i = \lim_{n\to\infty}\sum_{i=0}^{n}\left(\frac{2}{5}\right)^i = \lim_{n\to\infty}\frac{\left(\frac{2}{5}\right)^{n+1}-1}{\frac{2}{5}-1} = \frac{0-1}{\frac{-3}{5}} = \boxed{\frac{5}{3}}$

c. $\sum_{i=1}^{N}(i^3+3i^2-6i+9) = \sum_{i=1}^{N} i^3 + 3\sum_{i=1}^{N} i^2 - 6\sum_{i=1}^{N} i + 9N$

$$= \frac{N^2(N+1)^2}{4} + 3\left(\frac{N(N+1)(2N+1)}{6}\right) - 6\left(\frac{N(N+1)}{2}\right) + 9N$$

$$\boxed{= \frac{N^2(N+1)^2}{4} + \frac{N(N+1)(2N+1)}{2} - 3N(N+1) + 9N}$$

2. Exponents and Logs

a. $x^1 \cdot x^2 \cdot x^3 \cdots x^{330} = x^{1+2+3+\cdots+330}$

$$= x^{\frac{330(330+1)}{2}}$$

$$= x^{\frac{330(331)}{2}}$$

$$\boxed{= x^{54615}}$$

b. $\log_x x^{330x} \boxed{= 330x}$

c. $\log_{330}(330^{330}\cdot 330) = \log_{330}(330^{331})$

$$\boxed{= 331}$$

3. $10-2-1-(-2) = 9 = n \quad , \quad k=3$

$\Longrightarrow$ Solution $= \binom{n+k-1}{k} = \binom{9+3-1}{3} = \binom{11}{3} = \frac{11!}{3!(11-3)!}$

$$\boxed{= 165}$$

4. Induction

$f(x) = x \qquad x = 1,2,3$

$f(x) = f(x-1) + f(x-2) + f(x-3) \quad x \in \mathbb{N}$ and $x > 3$

Inductive hypothesis: $\forall x \in \mathbb{N}$, $f(x) < 2^x$

Base case : $f(1) = 1$

$2^1 = 2$

$\Longrightarrow f(1) < 2^1$

Therefore, the base case is verified.

Inductive step: Assuming $f(n) < 2^n$

We have to show that $f(n+1) < 2^{n+1}$

If $n = 1, 2, 3$:

$f(n+1) = n+1$

We know $f(n) = n < 2^n$

$\implies n+1 < 2^n + 1 < 2^n \times 2^1 = 2^{n+1}$ for $n = 1, 2, 3$

If $n \in \mathbb{N}$ and $n > 3$:

$f(n+1) = f(n+1-1) + f(n+1-2) + f(n+1-3)$

$\implies f(n+1) = f(n) + f(n-1) + f(n-2)$

$f(n) < 2^n$

$f(n-1) < 2^{n-1}$

$f(n-2) < 2^{n-2}$

$\implies f(n) + f(n-1) + f(n-2) < 2^n + 2^{n-1} + 2^{n-2}$

$= 2^n(1 + 2^{-1} + 2^{-2})$

$= 2^n(1 + \frac{1}{2} + \frac{1}{4})$

$= 2^n(\frac{7}{4})$ and $\frac{7}{4} < 2$

$\implies 2^n \cdot \frac{7}{4} < 2^n \cdot 2$

Conclusion:

$\implies f(n+1) < 2^{n+1}$

Therefore, by induction, we can say that $f(n) < 2^n$ for $\forall n \in \mathbb{N}$

5. Program Understanding

a) This function foo returns the power of 2 that is directly smaller or equal to the input n.

b) $i = 0$

$2^0 = 2^i$

sum $= 0 + (00000001)_2$
$= 0 + 1$

$2^1 = 2^i$

$i = 1$

sum $= 1 + (00000010)_2$
$= 1 + 2$
$= 3$

$2^2 = 2^i$

$i = 2$

sum $= 3 + (00000100)_2$
$= 3 + 4$
$= 7$

$2^3 = 2^i$

$sum = 3 + (00000100)_2$
$= 3 + 4$
$= 7$

$2^3 = 2^i$

$i = 3$

$sum = 7 + (00001000)_2$
$= 7 + 8$
$= 14$

$\vdots$

the statement in the loop is executed i ranges between 0 to n-1 (because the loop condition is $i < n$).

Therefore, when the loop exists, we have:

$$sum = 0 + 2^0 + 2^1 + 2^2 + 2^3 + \dots + 2^{n-1}$$
$$= 1 + 2^1 + 2^2 + 2^3 + \dots + 2^{n-1}$$
$$= \sum_{i=0}^{n-1} 2^i = \frac{2^n - 1}{2 - 1} = \boxed{2^n - 1}$$

geometric series