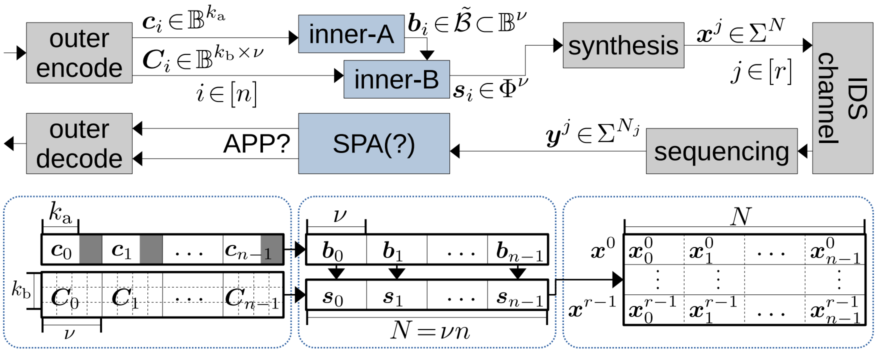


$\mathbb{B} = \{0,1\}, [n] = \{0,1,\ldots,n-1\}$

Alphabet: $\Sigma = \{0,1,2,3\} \leftrightarrow \{\mathrm{A,T,G,C}\}$
Code length: ν symbol (even)
Block length: $N = \nu n$ symbol
(block = n inner codewords)

Constraints (binary):
- run length: RL-ρ
- local balance: LB-(ℓ, δ) $\left|w(\boldsymbol{b}_i^{i+\ell-1}) - \frac{\ell}{2}\right| \le \delta$
- indicator func: $\mathbb{1}_{\rho,\ell,\delta}(\boldsymbol{b})$

IDS channel:
- error prob: $p_\mathrm{i}, p_\mathrm{d}, p_\mathrm{s}$
- drift vector:
  - $\boldsymbol{d}^j = (d_0^j, \ldots, d_N^j)$ (symbol level)
  - $\tilde{\boldsymbol{d}}^j = (\tilde{d}_0^j, \ldots, \tilde{d}_n^j)$ (word level) $\tilde{d}_i^j = d_{i\nu}^j$

Binary constraint vector set [SITA2024]

$\mathcal{B} \subset \mathbb{B}^\nu \quad (|\mathcal{B}| = 2^{k_a})$

$\forall \boldsymbol{b} \in \mathcal{B}, w(\boldsymbol{b}) = \nu/2$	(LB: inside CW)
$\forall \boldsymbol{b} \in \mathcal{B}, \forall i \in [\nu]_\mathrm{even}, \left|\tilde{w}(\boldsymbol{b}_i^{\nu-1})\right| \le 2\delta$	(LB: CW boundary)
$\forall \boldsymbol{b} \in \mathcal{B}, \forall i \in [\nu]_\mathrm{odd}, \left|\tilde{w}(\boldsymbol{b}_i^{\nu-1})\right| \le 2\delta - 1$	(LB: CW boundary)
$\forall \boldsymbol{b} \in \mathcal{B}, \lambda(\boldsymbol{b}) \le \rho$	(RL: inside CW)
$\forall \boldsymbol{b} \in \mathcal{B}, \lambda(\boldsymbol{b}, \nu-1) \le \rho - 1$	(RL: CW boundary)
$\boldsymbol{b} \in \mathcal{B} \to \bar{\boldsymbol{b}} \in \mathcal{B}$	(re-balance)
$(01)^{\frac{\nu}{2}}, (10)^{\frac{\nu}{2}} \notin \mathcal{B}$	(erasure symbol)

$\tilde{\mathcal{B}} = \mathcal{B} \cup \{(01)^{\frac{\nu}{2}}, (10)^{\frac{\nu}{2}}\}$ (re-balance)

Composite symbol sets

$$\Phi_0 \subseteq \left\{(\sigma_0,\sigma_1,\sigma_2,\sigma_3) \in \mathbb{Z}_{k+1}^4 \,\middle|\, \sigma_0+\sigma_1 = k, \sigma_2=\sigma_3=0\right\} \text{ AT}$$
$$\Phi_1 \subseteq \left\{(\sigma_0,\sigma_1,\sigma_2,\sigma_3) \in \mathbb{Z}_{k+1}^4 \,\middle|\, \sigma_0=\sigma_1=0, \sigma_2+\sigma_3 = k\right\} \text{ GC}$$
$$|\Phi_0| = |\Phi_1| = 2^{k_b} \le k+1 \qquad \Phi = \Phi_0 \cup \Phi_1$$

Inner codebook

$$\mathcal{C} = \left\{(c_0, \ldots, c_{\nu-1}) \,\middle|\, c_j \in \Phi_{b_j}, (b_0,\ldots,b_{\nu-1}) \in \mathcal{B}\right\}$$

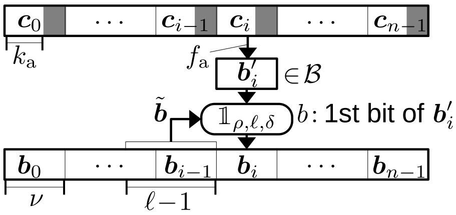
Mapping

$f_\mathrm{a}: \mathbb{B}^{k_a} \to \mathcal{B}$ (bijection)
$f_i: \mathbb{B}^{k_b} \to \Phi_i \ (i \in \mathbb{B})$ (bijection)
$f_\mathrm{d}: \Sigma \to \mathbb{B} \qquad f_\mathrm{d}(x) = \mathbb{1}[x \in \{2,3\}]$

Encoding

input: $\boldsymbol{c} = (\boldsymbol{c}_0, \ldots, \boldsymbol{c}_{n-1}) \quad \boldsymbol{c}_i \in \mathbb{B}^{k_a}$
$\boldsymbol{C} = (\boldsymbol{C}_0, \ldots, \boldsymbol{C}_{n-1}) \quad \boldsymbol{C}_i \in \mathbb{B}^{k_b \times \nu}$
output: $\boldsymbol{s} = (\boldsymbol{s}_0, \ldots, \boldsymbol{s}_{n-1}) \quad \boldsymbol{s}_i \in \Phi^\nu$

(1) encode $\boldsymbol{c}$ to $\boldsymbol{b} = (\boldsymbol{b}_0, \ldots, \boldsymbol{b}_{n-1}) \in \tilde{\mathcal{B}}^n$



$$\boldsymbol{b}_i = \begin{cases} \boldsymbol{b}'_i & (\mathbb{1}_{\rho,\ell,\delta}(\tilde{\boldsymbol{b}}, \boldsymbol{b}'_i) = 1) & \text{(unchanged)} \\ \bar{\boldsymbol{b}}'_i & (\mathbb{1}_{\rho,\ell,\delta}(\tilde{\boldsymbol{b}}, \bar{\boldsymbol{b}}'_i) = 1) & \text{(invert)} \\ (\bar{b}b)^{\frac{\nu}{2}} & (\mathbb{1}_{\rho,\ell,\delta}(\tilde{\boldsymbol{b}}, (\bar{b}b)^{\frac{\nu}{2}}) = 1) & \text{(rebalance)} \\ (b\bar{b})^{\frac{\nu}{2}} & (\text{otherwise}) & \text{(rebalance)} \end{cases}$$

(2) encode $(\boldsymbol{C}_i, \boldsymbol{b}_i)$ to $\boldsymbol{s}_i$ $(i \in [n])$

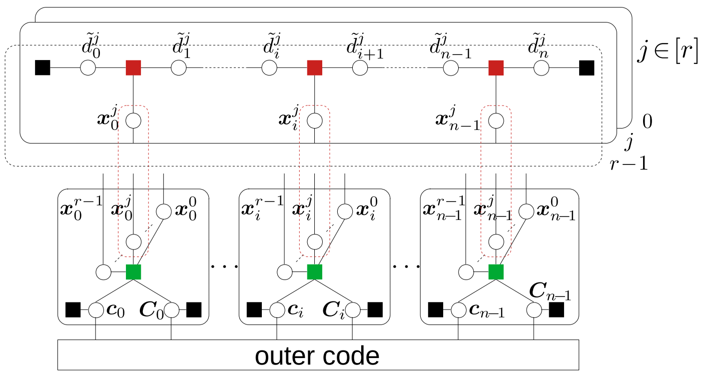
$s_{i,t} = f_{b_{i,t}}(\boldsymbol{C}_{i,t}^\mathsf{T}) \quad (t \in [\nu])$
$\boldsymbol{s}_i = (s_{i,0}, \ldots, s_{i,\nu-1}) \in \Phi^\nu$
$\boldsymbol{b}_i = (b_{i,0}, \ldots, b_{i,\nu-1}) \in \tilde{\mathcal{B}}$
$\boldsymbol{C}_{i,t}$: t th column of $\boldsymbol{C}_i$

(ex) $\nu = 6, |\mathcal{B}| = 16, k = 7, k_a = 4, k_b = 3$

```
B:0:001011  8:110100  Φ0:0:0700  Φ1: 0:0007
  1:001101  9:110010     1:1600      1:0016
  2:001110  A:110001     2:2500      2:0025
  3:010011  B:101100     3:3400      3:0034
  -:010101  -:101010     4:4300      4:0043
  4:010110  C:101001     5:5200      5:0052
  5:011001  D:100110     6:6100      6:0061
  6:011010  E:100101     7:7000      7:0070
  7:011100  F:100011       (Lee distance?)
```

```
c_i :  0110(6)=011010
C_i^T: 011(3)           s_i : (0,3)=3400
       100(4)                 (1,4)=0043
       010(2)                 (1,2)=0025
       011(3)                 (0,3)=3400
       110(6)                 (1,6)=0061
       100(4)                 (0,4)=4300
```

Decoding



$$p\left(\boldsymbol{y}^0, \ldots, \boldsymbol{y}^{r-1}, \boldsymbol{x}^0, \ldots, \boldsymbol{x}^{r-1}, \tilde{\boldsymbol{d}}^0, \ldots, \tilde{\boldsymbol{d}}^{r-1}, \boldsymbol{c}, \boldsymbol{C}\right)$$
$$= p\left(\boldsymbol{y}^0, \ldots, \boldsymbol{y}^{r-1} \middle| \boldsymbol{x}^0, \ldots, \boldsymbol{x}^{r-1}, \tilde{\boldsymbol{d}}^0, \ldots, \tilde{\boldsymbol{d}}^{r-1}, \boldsymbol{c}, \boldsymbol{C}\right) \times$$
$$p\left(\tilde{\boldsymbol{d}}^0, \ldots, \tilde{\boldsymbol{d}}^{r-1}\right) p\left(\boldsymbol{x}^0, \ldots, \boldsymbol{x}^{r-1} \middle| \boldsymbol{c}, \boldsymbol{C}\right) p(\boldsymbol{c}, \boldsymbol{C})$$
$$= p(\boldsymbol{c})\, p(\boldsymbol{C}) \prod_{j=0}^{r-1} p\left(\boldsymbol{y}^j \middle| \boldsymbol{x}^j, \tilde{\boldsymbol{d}}^j\right) p\left(\tilde{\boldsymbol{d}}^j\right) p\left(\boldsymbol{x}^j \middle| \boldsymbol{c}, \boldsymbol{C}\right)$$
$$\simeq \left(\prod_{i=0}^{n-1} p(\boldsymbol{c}_i)\, p(\boldsymbol{C}_i)\right) \times$$
$$\prod_{j=0}^{r-1}\left(p(\tilde{d}_0^j) \prod_{i=0}^{n-1} \underbrace{p\left([\boldsymbol{y}^j]_{i\nu+\tilde{d}_i^j}^{i\nu+\nu-1+\tilde{d}_{i+1}^j} \middle| \boldsymbol{x}_i^j, \tilde{d}_i^j, \tilde{d}_{i+1}^j,\right) p\left(\tilde{d}_{i+1}^j \middle| \tilde{d}_i^j\right)} \underbrace{p\left(\boldsymbol{x}_i^j \middle| \boldsymbol{c}_i, \boldsymbol{C}_i\right)}\right)$$

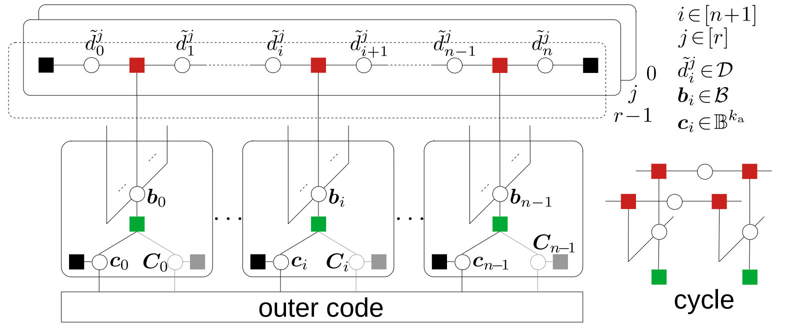
$$\underbrace{\prod_{j=0}^{r-1} p\left(\boldsymbol{x}_i^j \middle| \boldsymbol{c}_i, \boldsymbol{C}_i\right)} = \prod_{j=0}^{r-1} \underbrace{p\left(\boldsymbol{b}_i^j \middle| \boldsymbol{c}_i\right)}_{\text{(sim)}} p\left(\boldsymbol{x}_i^j \middle| \boldsymbol{b}_i^j, \boldsymbol{C}_i\right) = \prod_{j=0}^{r-1} p\left(\boldsymbol{b}_i^j \middle| \boldsymbol{c}_i\right) \prod_{t=0}^{\nu-1} p\left(x_{i,t}^j \middle| b_{i,t}^j, \boldsymbol{C}_{i,t}\right)$$

[approximation?]

$$\boldsymbol{b}_i^j = \left(f_\mathrm{d}(x_{i,0}^j), \ldots, f_\mathrm{d}(x_{i,\nu-1}^j)\right)$$
$$p\left(x_{i,t}^j \middle| b_{i,t}^j, \boldsymbol{C}_{i,t}\right) = \sigma_{x_{i,t}^j}/k \quad \left(x_{i,t}^j \in \Sigma, b_{i,t}^j \in \mathbb{B}, \boldsymbol{C}_{i,t}^\mathsf{T} \in \mathbb{B}^{k_b}\right)$$
$$(\sigma_0, \sigma_1, \sigma_2, \sigma_3) = f_{b_{i,t}^j}\left(\boldsymbol{C}_{i,t}^\mathsf{T}\right)$$

non-uniform k

Decoding (separate)

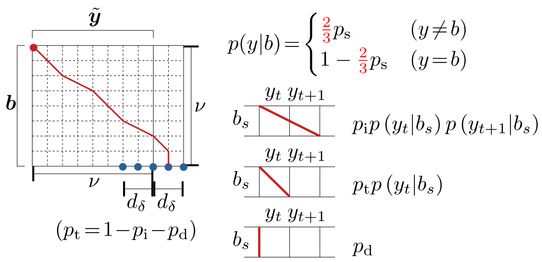


$$
\begin{aligned}
&p\left(\boldsymbol{y}^0,\dots,\boldsymbol{y}^{r-1},\tilde{\boldsymbol{d}}^0,\dots,\tilde{\boldsymbol{d}}^{r-1},\boldsymbol{b},\boldsymbol{c}\right)\\
&= p\left(\boldsymbol{y}^0,\dots,\boldsymbol{y}^{r-1}\middle|\tilde{\boldsymbol{d}}^0,\dots,\tilde{\boldsymbol{d}}^{r-1},\boldsymbol{b},\boldsymbol{c}\right) p\left(\tilde{\boldsymbol{d}}^0,\dots,\tilde{\boldsymbol{d}}^{r-1}\right) p\left(\boldsymbol{b}|\boldsymbol{c}\right) p\left(\boldsymbol{c}\right)\\
&= p\left(\boldsymbol{b}|\boldsymbol{c}\right) p\left(\boldsymbol{c}\right) \prod_{j=0}^{r-1} p\left(\boldsymbol{y}^j\middle|\boldsymbol{b}^j,\tilde{\boldsymbol{d}}^j\right) p\left(\tilde{\boldsymbol{d}}^j\right)\\
&\simeq \left(\prod_{i=0}^{n-1} p\left(\boldsymbol{b}_i|\boldsymbol{c}_i\right) p\left(\boldsymbol{c}_i\right)\right) \prod_{j=0}^{r-1}\left(p(\tilde{d}^j_0)\prod_{i=0}^{n-1} p\left([\boldsymbol{y}^j]_{i\nu+\tilde{d}^j_i}^{i\nu+\nu-1+\tilde{d}^j_{i+1}}\middle|\boldsymbol{b}_i,\tilde{d}^j_i,\tilde{d}^j_{i+1},\right) p\left(\tilde{d}^j_{i+1}\middle|\tilde{d}^j_i\right)\right)\\
&\propto \left(\prod_{i=0}^{n-1} \underbrace{p\left(\boldsymbol{b}_i|\boldsymbol{c}_i\right)}_{\text{(sim)}} p\left(\boldsymbol{c}_i\right)\right) \prod_{j=0}^{r-1}\left(p(\tilde{d}^j_0)\prod_{i=0}^{n-1} p\left([\tilde{\boldsymbol{y}}^j]_{i\nu+\tilde{d}^j_i}^{i\nu+\nu-1+\tilde{d}^j_{i+1}}\middle|\boldsymbol{b}_i,\tilde{d}^j_i,\tilde{d}^j_{i+1},\right) p\left(\tilde{d}^j_{i+1}\middle|\tilde{d}^j_i\right)\right)
\end{aligned}
$$

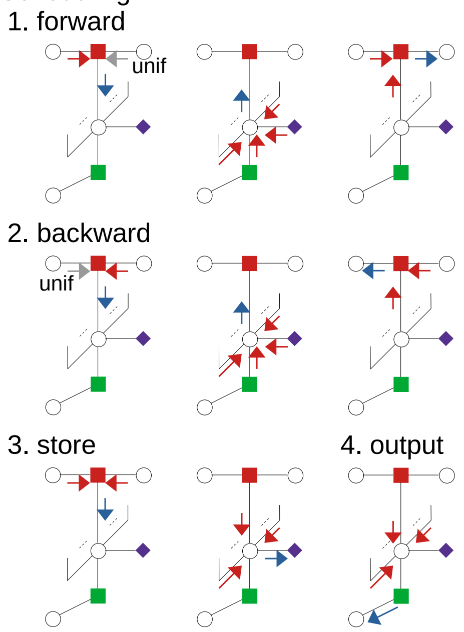
$$\left(\tilde{\boldsymbol{y}}^j = f_{\mathrm{d}}(\boldsymbol{y}^j) \in \mathbb{B}^{N_j}\right)$$

```
PF[Nseq][Ns+1][D]
PB[Nseq][Ns+1][D]
PD[Nseq][Ns][Q]
PU[Nseq][Ns][Q]
PI[Ns][Q]
PO[Ns][Q]
PM[Ns][Q]
PU0[Ns][Q]
```

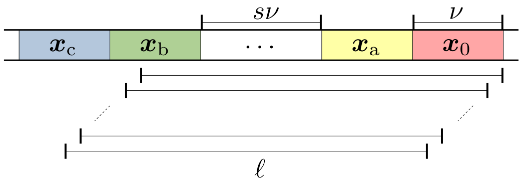
LUT



Scheduling



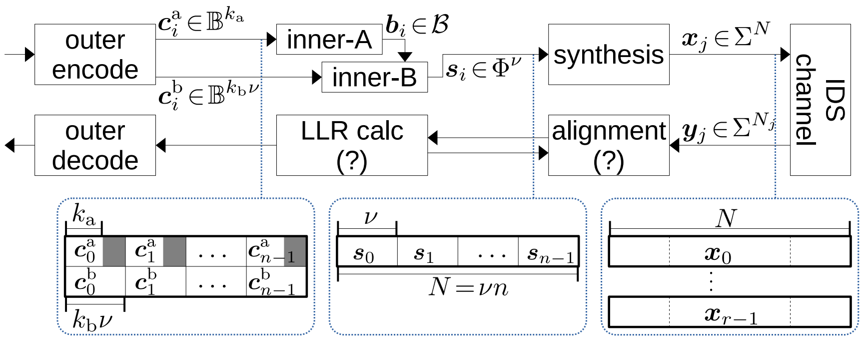
Encoding channel matrix (binary)



(1) $\ell = \nu$

8 10

(A,T,G,C)
(0,1,2,3)



$i \in [n]$
$j \in [r]$
$N = \nu n$

Code length: ν (even)
Binary constraint vector set: $\mathcal{B} \subset \mathbb{B}^\nu$ $\quad (|\mathcal{B}| \geq 2^{k_{\mathrm{a}}})$
Composite symbol set:

$$\Phi_0 \subseteq \left\{(\sigma_0,\sigma_1,\sigma_2,\sigma_3) \in \mathbb{Z}_{k+1}^4 \,\middle|\, \sigma_0+\sigma_1=k, \sigma_2=\sigma_3=0\right\} \text{ AT}$$
$$\Phi_1 \subseteq \left\{(\sigma_0,\sigma_1,\sigma_2,\sigma_3) \in \mathbb{Z}_{k+1}^4 \,\middle|\, \sigma_0=\sigma_1=0, \sigma_2+\sigma_3=k\right\} \text{ GC}$$
$$|\Phi_0|=|\Phi_1|=2^{k_{\mathrm{b}}} \leq k+1 \qquad \Phi=\Phi_0 \cup \Phi_1$$

Inner codebook:

$$\mathcal{C} \subseteq \left\{(c_0,\ldots,c_{\nu-1}) \,\middle|\, c_j \in \Phi_{b_j}, (b_0,\ldots,b_{\nu-1}) \in \mathcal{B}\right\}$$

(example) $\nu=6, |\mathcal{B}|=16, k=7, k_{\mathrm{a}}=4, k_{\mathrm{b}}=3,$

$\mathcal{B}$:		Φ_0:	Φ_1:
0:001011	8:110100	0:0700	0:0007
1:001101	9:110010	1:1600	1:0016
2:001110	A:110001	2:2500	2:0025
3:010011	B:101100	3:3400	3:0034
-:010101	-:101010	4:4300	4:0043
4:010110	C:101001	5:5200	5:0052
5:011001	D:100110	6:6100	6:0061
6:011010	E:100101	7:7000	7:0070
7:011100	F:100011		

Lee dist?

Encode:

$f_{\mathrm{e}}^{\mathrm{a}} : \mathbb{B}^{k_{\mathrm{a}}} \to \mathcal{B}$ (lossy)

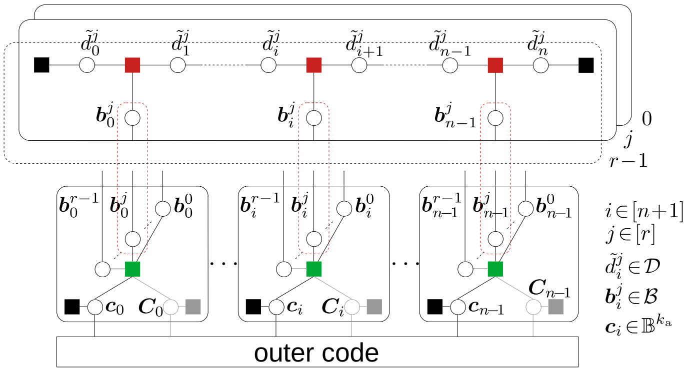
$f_{\mathrm{e}}^{i} : \mathbb{B}^{k_{\mathrm{b}}} \to \Phi_i \ (i \in \mathbb{B})$ (bijection)

$$f_{\mathrm{e}}(c_i^{\mathrm{a}}, c_i^{\mathrm{b}}) = s_i = (s_{i,0},\ldots,s_{i,\nu-1}) : \mathbb{B}^{k_{\mathrm{a}}} \times \mathbb{B}^{k_{\mathrm{b}}\nu} \to \Phi^\nu$$
$$s_{i,j} = f_{\mathrm{e}}^{b_i}(c_{i,j}^{\mathrm{b}}) \qquad (b_0,\ldots,b_{\nu-1}) = f_{\mathrm{e}}^{\mathrm{a}}(c_i^{\mathrm{a}})$$
$$c_{i,j}^{\mathrm{b}} = [c_i^{\mathrm{b}}]_{j\nu}^{j\nu+\nu-1}$$

Decode:

Decoding (separate)

[approximation?]



$$
\begin{aligned}
& p\left(\boldsymbol{y}^0,\dots,\boldsymbol{y}^{r-1},\boldsymbol{b}^0,\dots,\boldsymbol{b}^{r-1},\tilde{\boldsymbol{d}}^0,\dots,\tilde{\boldsymbol{d}}^{r-1},\boldsymbol{c}\right) \\
&= p\left(\boldsymbol{y}^0,\dots,\boldsymbol{y}^{r-1}\middle|\boldsymbol{b}^0,\dots,\boldsymbol{b}^{r-1},\tilde{\boldsymbol{d}}^0,\dots,\tilde{\boldsymbol{d}}^{r-1},\boldsymbol{c}\right) p\left(\tilde{\boldsymbol{d}}^0,\dots,\tilde{\boldsymbol{d}}^{r-1}\right) p\left(\boldsymbol{b}^0,\dots,\boldsymbol{b}^{r-1}\middle|\boldsymbol{c}\right) p\left(\boldsymbol{c}\right) \\
&= p\left(\boldsymbol{c}\right) \prod_{j=0}^{r-1} p\left(\boldsymbol{y}^j\middle|\boldsymbol{b}^j,\tilde{\boldsymbol{d}}^j\right) p\left(\tilde{\boldsymbol{d}}^j\right) p\left(\boldsymbol{b}^j\middle|\boldsymbol{c}\right) \\
&\simeq \left(\prod_{i=0}^{n-1} p\left(\boldsymbol{c}_i\right)\right) \prod_{j=0}^{r-1} \left(p(\tilde{d}_0^j) \prod_{i=0}^{n-1} p\left([\boldsymbol{y}^j]_{i\nu+\tilde{d}_i^j}^{i\nu+\nu-1+\tilde{d}_{i+1}^j} \middle| \boldsymbol{b}_i^j, \tilde{d}_i^j, \tilde{d}_{i+1}^j, \right) p\left(\tilde{d}_{i+1}^j\middle|\tilde{d}_i^j\right) p\left(\boldsymbol{b}_i^j\middle|\boldsymbol{c}_i\right)\right) \\
&\propto \left(\prod_{i=0}^{n-1} p\left(\boldsymbol{c}_i\right)\right) \prod_{j=0}^{r-1} \left(p(\tilde{d}_0^j) \prod_{i=0}^{n-1} p\left([\tilde{\boldsymbol{y}}^j]_{i\nu+\tilde{d}_i^j}^{i\nu+\nu-1+\tilde{d}_{i+1}^j} \middle| \boldsymbol{b}_i^j, \tilde{d}_i^j, \tilde{d}_{i+1}^j, \right) p\left(\tilde{d}_{i+1}^j\middle|\tilde{d}_i^j\right) p\left(\boldsymbol{b}_i^j\middle|\boldsymbol{c}_i\right)\right)
\end{aligned}
$$

$\prod_{j=0}^{r-1} p\left(\boldsymbol{b}_i^j\middle|\boldsymbol{c}_i\right)$:sim $\qquad \tilde{\boldsymbol{y}}^j = f_{\mathrm{d}}(\boldsymbol{y}^j) \in \mathbb{B}^{N_j}$