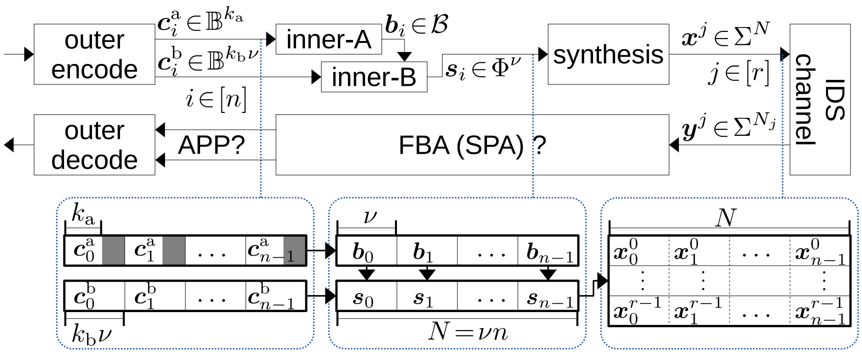
(A,T,G,C)
(0,1,2,3)



Alphabet: $\Sigma = \{0,1,2,3\} \leftrightarrow$ {A, T, G, C}
Code length: ν (even)
Block length: $N = \nu n$ symbol
(n codewords/block)

[ex] $\nu = 6, |\mathcal{B}| = 16, k = 7, k_{\mathrm{a}} = 4, k_{\mathrm{b}} = 3$

$\mathcal{B}$:		Φ_0:	Φ_1:
0:001011	8:110100	0:0700	0:0007
1:001101	9:110010	1:1600	1:0016
2:001110	A:110001	2:2500	2:0025
3:010011	B:101100	3:3400	3:0034
-:010101	-:101010	4:4300	4:0043
4:010110	C:101001	5:5200	5:0052
5:011001	D:100110	6:6100	6:0061
6:011010	E:100101	7:7000	7:0070
7:011100	F:100011	(Lee distance?)	

[Binary constraint vector set]

$\mathcal{B} \subset \mathbb{B}^{\nu} \quad (|\mathcal{B}| \geq 2^{k_{\mathrm{a}}}, \mathcal{B} \cap \{(01)^{\frac{\nu}{2}}, (10)^{\frac{\nu}{2}}\} = \phi)$

$\tilde{\mathcal{B}} = \mathcal{B} \cup \{(01)^{\frac{\nu}{2}}, (10)^{\frac{\nu}{2}}\}$ (re-balance)

[Composite symbol sets]

$\Phi_0 \subseteq \left\{ (\sigma_0, \sigma_1, \sigma_2, \sigma_3) \in \mathbb{Z}_{k+1}^4 \,\middle|\, \sigma_0 + \sigma_1 = k, \sigma_2 = \sigma_3 = 0 \right\}$ AT

$\Phi_1 \subseteq \left\{ (\sigma_0, \sigma_1, \sigma_2, \sigma_3) \in \mathbb{Z}_{k+1}^4 \,\middle|\, \sigma_0 = \sigma_1 = 0, \sigma_2 + \sigma_3 = k \right\}$ GC

$|\Phi_0| = |\Phi_1| = 2^{k_{\mathrm{b}}} \leq k+1 \qquad \Phi = \Phi_0 \cup \Phi_1$

[Inner codebook]

$\mathcal{C} \subseteq \left\{ (c_0, \dots, c_{\nu-1}) \,\middle|\, c_j \in \Phi_{b_j}, (b_0, \dots, b_{\nu-1}) \in \mathcal{B} \right\}$

[Encode]

mapping: $f_{\mathrm{e}}^{\mathrm{a}} : \mathbb{B}^{k_{\mathrm{a}} n} \to \mathcal{B}^n$ (lossy: SITA2024)

$f_{\mathrm{e}}^{i} : \mathbb{B}^{k_{\mathrm{b}}} \to \Phi_i \ (i \in \mathbb{B})$ (bijection)

input: $\boldsymbol{c}^{\mathrm{a}} = (\boldsymbol{c}_0^{\mathrm{a}}, \dots, \boldsymbol{c}_{n-1}^{\mathrm{a}}) \quad \boldsymbol{c}_i^{\mathrm{a}} \in \mathbb{B}^{k_{\mathrm{a}}}$

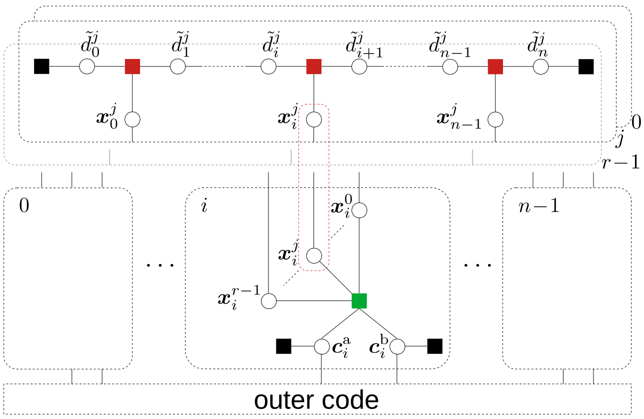
$\boldsymbol{c}^{\mathrm{b}} = (\boldsymbol{c}_0^{\mathrm{b}}, \dots, \boldsymbol{c}_{n-1}^{\mathrm{b}}) \quad \boldsymbol{c}_i^{\mathrm{b}} \in \mathbb{B}^{k_{\mathrm{b}}\nu}, \boldsymbol{c}_{i,j}^{\mathrm{b}} = [\boldsymbol{c}_i^{\mathrm{b}}]_{j\nu}^{j\nu+\nu-1}$

output: $\boldsymbol{s} = (\boldsymbol{s}_0, \dots, \boldsymbol{s}_{n-1}) \quad \boldsymbol{s}_i = (s_{i,0}, \dots, s_{i,\nu-1}) \in \Phi^{\nu}$

(1) $\boldsymbol{b} = (\boldsymbol{b}_0, \dots, \boldsymbol{b}_{n-1}) = f_{\mathrm{e}}^{\mathrm{a}}(\boldsymbol{c}^{\mathrm{a}}) \quad \boldsymbol{b}_i = (b_{i,0}, \dots, b_{i,\nu-1}) \in \tilde{\mathcal{B}}$

(2) $s_{i,j} = f_{\mathrm{e}}^{b_i}(\boldsymbol{c}_{i,j}^{\mathrm{b}}) \quad (i \in [n], j \in [\nu])$

[Decode]



$\tilde{d}_i^j = d_{i\nu}^j$ (symbol level)
$j \in [r]$

$$p\left(\boldsymbol{y}^0, \dots, \boldsymbol{y}^{r-1}, \boldsymbol{x}^0, \dots, \boldsymbol{x}^{r-1}, \tilde{\boldsymbol{d}}^0, \dots, \tilde{\boldsymbol{d}}^{r-1}, \boldsymbol{c}^{\mathrm{a}}, \boldsymbol{c}^{\mathrm{b}}\right)$$

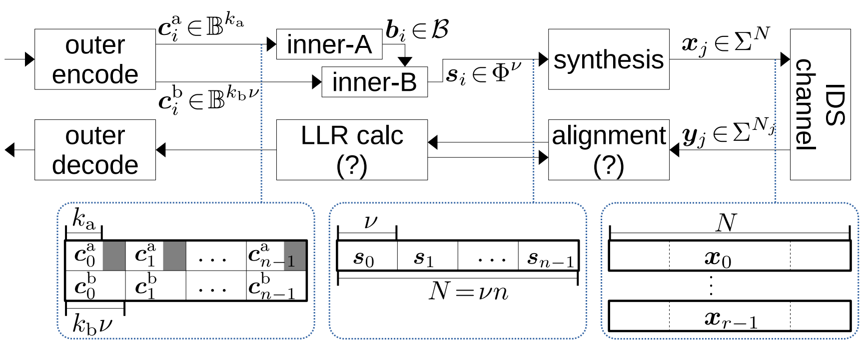
$$= p\left(\boldsymbol{y}^0, \dots, \boldsymbol{y}^{r-1} \middle| \boldsymbol{x}^0, \dots, \boldsymbol{x}^{r-1}, \tilde{\boldsymbol{d}}^0, \dots, \tilde{\boldsymbol{d}}^{r-1}, \boldsymbol{c}^{\mathrm{a}}, \boldsymbol{c}^{\mathrm{b}}\right) p\left(\tilde{\boldsymbol{d}}^0, \dots, \tilde{\boldsymbol{d}}^{r-1}\right) p\left(\boldsymbol{x}^0, \dots, \boldsymbol{x}^{r-1} | \boldsymbol{c}^{\mathrm{a}}, \boldsymbol{c}^{\mathrm{b}}\right) p\left(\boldsymbol{c}^{\mathrm{a}}, \boldsymbol{c}^{\mathrm{b}}\right)$$

$$= p\left(\boldsymbol{c}^{\mathrm{a}}\right) p\left(\boldsymbol{c}^{\mathrm{b}}\right) \prod_{j=0}^{r-1} p\left(\boldsymbol{y}^j \middle| \boldsymbol{x}^j, \tilde{\boldsymbol{d}}^j\right) p\left(\tilde{\boldsymbol{d}}^j\right) p\left(\boldsymbol{x}^j | \boldsymbol{c}^{\mathrm{a}}, \boldsymbol{c}^{\mathrm{b}}\right)$$

$$= \left(\prod_{i=0}^{n-1} p\left(\boldsymbol{c}_i^{\mathrm{a}}\right) p\left(\boldsymbol{c}_i^{\mathrm{b}}\right) \right) \prod_{j=0}^{r-1} \left(p(\tilde{d}_0^j) \prod_{i=0}^{n-1} p\left([\boldsymbol{y}^j]_{i\nu+\tilde{d}_i^j}^{i\nu+\nu-1+\tilde{d}_{i+1}^j} \middle| x_i^j, \tilde{d}_i^j, \tilde{d}_{i+1}^j, \right) p\left(\tilde{d}_{i+1}^j \middle| \tilde{d}_i^j\right) p\left(\boldsymbol{x}_i^j \middle| \boldsymbol{c}_i^{\mathrm{a}}, \boldsymbol{c}_i^{\mathrm{b}}\right) \right)$$

8 10

(A,T,G,C)
(0,1,2,3)



$i \in [n]$
$j \in [r]$
$N = \nu n$

Code length: ν (even)
Binary constraint vector set: $\mathcal{B} \subset \mathbb{B}^\nu$ $\quad (|\mathcal{B}| \geq 2^{k_\mathrm{a}})$
Composite symbol set:

$\Phi_0 \subseteq \left\{(\sigma_0,\sigma_1,\sigma_2,\sigma_3) \in \mathbb{Z}_{k+1}^4 \,\middle|\, \sigma_0+\sigma_1=k, \sigma_2=\sigma_3=0\right\}$ AT

$\Phi_1 \subseteq \left\{(\sigma_0,\sigma_1,\sigma_2,\sigma_3) \in \mathbb{Z}_{k+1}^4 \,\middle|\, \sigma_0=\sigma_1=0, \sigma_2+\sigma_3=k\right\}$ GC

$|\Phi_0| = |\Phi_1| = 2^{k_\mathrm{b}} \leq k+1 \qquad \Phi = \Phi_0 \cup \Phi_1$

Inner codebook:

$\mathcal{C} \subseteq \left\{(c_0,\ldots,c_{\nu-1}) \,\middle|\, c_j \in \Phi_{b_j}, (b_0,\ldots,b_{\nu-1}) \in \mathcal{B}\right\}$

(example) $\nu=6, |\mathcal{B}|=16, k=7, k_\mathrm{a}=4, k_\mathrm{b}=3,$

$\mathcal{B}$:		Φ_0:	Φ_1:
0:001011	8:110100	0:0700	0:0007
1:001101	9:110010	1:1600	1:0016
2:001110	A:110001	2:2500	2:0025
3:010011	B:101100	3:3400	3:0034
-:010101	-:101010	4:4300	4:0043
4:010110	C:101001	5:5200	5:0052
5:011001	D:100110	6:6100	6:0061
6:011010	E:100101	7:7000	7:0070
7:011100	F:100011		

Lee dist?

Encode:

$f_\mathrm{e}^\mathrm{a} : \mathbb{B}^{k_\mathrm{a}} \to \mathcal{B}$ (lossy)

$f_\mathrm{e}^i : \mathbb{B}^{k_\mathrm{b}} \to \Phi_i \ (i \in \mathbb{B})$ (bijection)

$f_\mathrm{e}(c_i^\mathrm{a}, c_i^\mathrm{b}) = s_i = (s_{i,0},\ldots,s_{i,\nu-1}) : \mathbb{B}^{k_\mathrm{a}} \times \mathbb{B}^{k_\mathrm{b}\nu} \to \Phi^\nu$

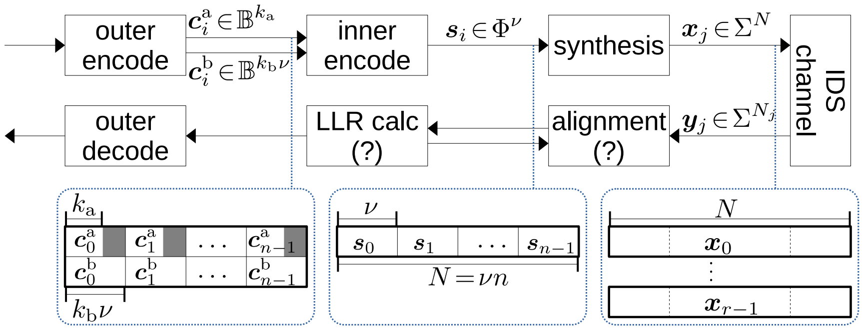
$s_{i,j} = f_\mathrm{e}^{b_i}(c_{i,j}^\mathrm{b}) \qquad (b_0,\ldots,b_{\nu-1}) = f_\mathrm{e}^\mathrm{a}(c_i^\mathrm{a})$

$c_{i,j}^\mathrm{b} = [c_i^\mathrm{b}]_{j\nu}^{j\nu+\nu-1}$

Decode:

8 10

(A,T,G,C)
(0,1,2,3)



$i \in [n]$
$j \in [r]$
$N = \nu n$

Code length: ν (even)
Binary constraint vector set: $\mathcal{B} \subset \mathbb{B}^{\nu}$ $\quad (|\mathcal{B}| \geq 2^{k_{\mathrm{a}}})$
Composite symbol set:

$\Phi_0 \subseteq \left\{ (\sigma_0, \sigma_1, \sigma_2, \sigma_3) \in \mathbb{Z}_{k+1}^4 \,\middle|\, \sigma_0 + \sigma_1 = k, \sigma_2 = \sigma_3 = 0 \right\}$ AT

$\Phi_1 \subseteq \left\{ (\sigma_0, \sigma_1, \sigma_2, \sigma_3) \in \mathbb{Z}_{k+1}^4 \,\middle|\, \sigma_0 = \sigma_1 = 0, \sigma_2 + \sigma_3 = k \right\}$ GC

$|\Phi_0| = |\Phi_1| = 2^{k_{\mathrm{b}}} \leq k+1 \qquad \Phi = \Phi_0 \cup \Phi_1$

Inner codebook:

$\mathcal{C} \subseteq \left\{ (c_0, \ldots, c_{\nu-1}) \,\middle|\, c_j \in \Phi_{b_j}, (b_0, \ldots, b_{\nu-1}) \in \mathcal{B} \right\}$

Encode:

$$\left.\begin{array}{l} f_{\mathrm{e}}^{\mathrm{a}} : \mathbb{B}^{k_{\mathrm{a}}} \to \mathcal{B} \\ f_{\mathrm{e}}^{i} : \mathbb{B}^{k_{\mathrm{b}}} \to \Phi_i \ (i \in \mathbb{B}) \end{array}\right] \text{bijection}$$

$f_{\mathrm{e}}(\boldsymbol{c}_i^{\mathrm{a}}, \boldsymbol{c}_i^{\mathrm{b}}) = \boldsymbol{s}_i = (s_{i,0}, \ldots, s_{i,\nu-1}) : \mathbb{B}^{k_{\mathrm{a}}} \times \mathbb{B}^{k_{\mathrm{b}}\nu} \to \Phi^{\nu}$

$s_{i,j} = f_{\mathrm{e}}^{b_i}(\boldsymbol{c}_{i,j}^{\mathrm{b}}) \qquad (b_0, \ldots, b_{\nu-1}) = f_{\mathrm{e}}^{\mathrm{a}}(\boldsymbol{c}_i^{\mathrm{a}})$

$\boldsymbol{c}_{i,j}^{\mathrm{b}} = [\boldsymbol{c}_i^{\mathrm{b}}]_{j\nu}^{j\nu+\nu-1}$

flip (RL, LB)

Decode:

(example) $\nu = 6, |\mathcal{B}| = 16, k = 7, k_{\mathrm{a}} = 4, k_{\mathrm{b}} = 3,$

$\mathcal{B}$:	001011	110100	Φ_0 : 0700	Φ_1 : 0007	Lee dist?
	001101	110010	1600	0016	
	001110	110001	2500	0025	
	010011	101100	3400	0034	
	010101	101010	4300	0043	
	010110	101001	5200	0052	
	011001	100110	6100	0061	
	011010	100101	7000	0070	
	011100	100011			

```
           001011 001101 001110 ...110100
base       AAGAGG AAGGAG AAGGGA    GGAGAA
           AAGAGC AAGGAC AAGGGT    GGAGAT
           AAGACG AAGGTG AAGGCA    GGAGTA  64
           ...    ...    ...       ...
           TTCTCC TTCCTC TTCCCT    CCTCTT
composite
                                           ??
```

multi-base composite (Manchester coding)

$\nu = 1:$ (A,C,G,T)
$\nu = 2:$ (AA,AC,AG,AT, CA,CC,CG,CT, GA,GC,GG,GT, TA,TC,TG,TT)
$\nu = 3:$ (AAA, AAC, ..., TTT)

```
           001011 001101 001110 ...110100
base       AAGAGG AAGGAG AAGGGA    GGAGAA
           AAGAGC AAGGAC AAGGGT    GGAGAC
           AAGACG AAGGTG AAGGCA    GGAGCA  64
           ...    ...    ...       ...
           CCTCTT CCTTCT CCTTTC
composite
```