Definition of inner code

β : code length
ν : number of codes
$\mathcal{C}_i \subset \Sigma^\beta$: code $(i \in [\nu\rangle)$
$\boldsymbol{b} = (b_0, \ldots, b_{\nu-1})$: bit-width

$|\mathcal{C}_i| = 2^{b_i}$

$R_\mathrm{I} = \frac{1}{2n\beta}\sum_{i=0}^{\nu-1} b_i$: inner code rate

Metric

$q(\mathcal{B}) = \max_{\boldsymbol{x},\boldsymbol{y}\in\mathcal{B}} \{p(\boldsymbol{y}|\boldsymbol{x})\} \quad (\boldsymbol{x} \neq \boldsymbol{y})$

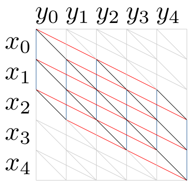
$\boldsymbol{x} = (x_0, \ldots, x_{\beta-1}) \in \mathcal{B}$

$\boldsymbol{y} = (y_0, \ldots, y_{\beta-1}) \in \mathcal{B}$

t: $p_\mathrm{t} p_\mathrm{s}^t (1-p_\mathrm{s})^{1-t} \quad (t \in \{0,1\})$

t: $p_\mathrm{i} p_\mathrm{s}^t (1-p_\mathrm{s})^{2-t} \quad (t \in \{0,1,2\})$

p_d



$p_\mathrm{i}, p_\mathrm{d}, p_\mathrm{s}$: ins,del,sub
$p_\mathrm{t} = 1 - p_\mathrm{i} - p_\mathrm{d}$

Construction from binary code

$\boldsymbol{b}' = (b'_0, \ldots, b'_{\nu-1}) \in [\beta+1\rangle^\nu$

even β : $2^{b'_i} \leq \binom{\beta}{\beta/2}$ $\quad (b_i = b'_i + \beta)$

odd β : $2^{b'_i} \leq \binom{\beta}{(\beta-1)/2} = \binom{\beta}{(\beta+1)/2}$

			b'							
β	t	$\binom{\beta}{t}$	0	1	2	3	4	5	6	7
2	1	2	T	T	–	–	–	–	–	–
3	1, 2	3	T	T	F	–	–	–	–	–
4	2	6	T	T	T	F	–	–	–	–
5	2, 3	10	T	T	T	T	F	–	–	–
6	3	20	T	T	T	T	T	F	–	–
7	3, 4	35	T	T	T	T	T	T	F	–
8	4	70	T	T	T	T	T	T	T	F
9	4, 5	126	T	T	T	T	T	T	T	F

[computer search: candidate codes]

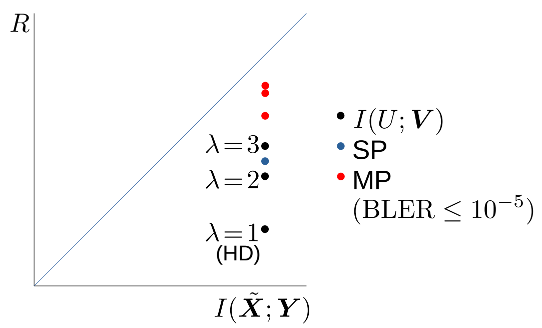
input: β, b', t
output: $\mathcal{L} = \{\mathcal{B}^0, \ldots, \mathcal{B}^{m-1}\} \quad (\mathcal{B}^j \subset \mathbb{B}^\beta)$
(list of best score codes)
constraint: $|\mathcal{B}^j| = 2^{b'}, \forall \boldsymbol{u} \in \mathcal{B}^j, w(\boldsymbol{u}) = t$
objective: minimize $q(\mathcal{B}^j)$

[computer search: code sequence]

input: $(\mathcal{L}_0, \ldots, \mathcal{L}_{\nu-1})$
output: $(\mathcal{B}_0, \ldots, \mathcal{B}_{\nu-1})$
constraint: $\mathcal{B}_i \in \mathcal{L}_i$
objective: minimize $\max_{i\in[\nu\rangle} |\mathcal{B}_i \cap \mathcal{B}_{|i+1|_\nu}|$

Evaluation of inner code

channel input: $u \in \mathbb{F}_{2^{b(i,\nu)}}$
channel output: $\boldsymbol{p} \in \mathbb{P}_{2^{b(i,\nu)}}$ (APP from inner code)
(a) HD+H(p): $\boldsymbol{v} = (v, Q_\lambda(H(\boldsymbol{p})))$
Q_λ : λ-level quantization
(b) list output: (list size λ) $\boldsymbol{v} = (v_0, \ldots, v_{\lambda-1}) = \arg\max(\boldsymbol{p}, \lambda) \in \mathbb{F}^\lambda_{2^{b(i,\nu)}}$
mutual info.: $I(U;\boldsymbol{V})$



$\beta=4, \nu=1$

$\boldsymbol{b}$	R_I
(4)	4/8
(5)	5/8
(6)	6/8

$\beta=4, \nu=2$

$\boldsymbol{b}$	R_I
(4, 4)	8/16
(5, 4)	9/16
(5, 5)	10/16
(6, 4)	10/16
(6, 5)	11/16
(6, 6)	12/16

$\beta=4, \nu=3$

$\boldsymbol{b}$	R_I
(4, 4, 4)	12/24
(5, 4, 4)	13/24
(5, 5, 4)	14/24
(6, 4, 4)	14/24
(5, 5, 5)	15/24
(6, 5, 4)	15/24
(6, 5, 5)	16/24
(6, 6, 5)	17/24
(6, 6, 6)	18/24

$\beta=5, \nu=1$

$\boldsymbol{b}$	R_I
(5)	5/10
(6)	6/10
(7)	7/10
(8)	8/10

$\beta=5, \nu=2$

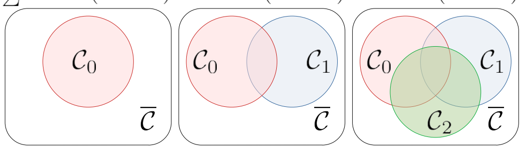
$\boldsymbol{b}$	R_I
(5, 5)	10/20
(6, 5)	11/20
(6, 6)	12/20
(7, 5)	12/20
(7, 6)	13/20
(7, 7)	14/20
(8, 6)	14/20
(8, 7)	15/20
(8, 8)	16/20

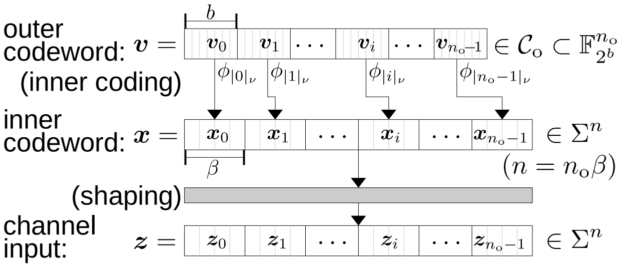
Concatenated coding

channel alphabet: $\Sigma = \{\mathsf{A},\mathsf{T},\mathsf{G},\mathsf{C}\}$
outer code: $\mathcal{C}_o \subset \mathbb{F}_{2^b}^{n_o}$
n_o : code length
b : symbol size (bits)

inner code: $\mathcal{C}_i \subset \Sigma^\beta \quad (i \in [\nu\rangle, b < 2\beta)$
β : code length (symbols)
ν : number of code books
$|\mathcal{C}_i| = 2^b$: number of codewords
$R = b/2\beta$: inner code rate
$\phi_i : \mathbb{F}_{2^b} \to \mathcal{C}_i$: encoding function

Σ^β $(\nu=1)$ $(\nu=2)$ $(\nu=3)$





prior: $p(\boldsymbol{x}_i) = \begin{cases} 1/2^b & (\boldsymbol{x}_i \in \mathcal{C}_{|i|_\nu}) \\ 0 & (\boldsymbol{x}_i \notin \mathcal{C}_{|i|_\nu}) \end{cases}$

params:

β	b
3	4, 5
4	6, 7
5	8, 9
6	10, 11

$\nu = 2$

Constraint

RL-LB constrained coding $(\ell; w, \delta)$:
* run length coding: $f_\mathrm{R}(\boldsymbol{z}) \le \ell$
* local GC-balance: $\left|f_\mathrm{B}\left(\boldsymbol{z}_i^{i+w-1}\right)\right| \le \delta$ $(\forall i \in [n-w\rangle)$
* synchronization: MIR?

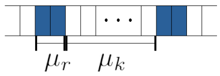
vector over Σ : $\boldsymbol{u} = (u_0, \dots, u_{n-1}) \in \Sigma^n$
weight: $w(\boldsymbol{u})_x = |\{i \in [n\rangle \,|\, u_i = x\}| \quad (x \in \Sigma)$
balance: $f_\mathrm{B}(\boldsymbol{u}) = w_\mathsf{G}(\boldsymbol{u}) + w_\mathsf{C}(\boldsymbol{u}) - w_\mathsf{A}(\boldsymbol{u}) - w_\mathsf{T}(\boldsymbol{u}) \in [-n, n]$
max run length: $f_\mathrm{R}(\boldsymbol{u}) \in [1, n]$

Inner codebook

$\mathcal{C}_i = \left\{(\phi(x_0y_0), \dots, \phi(x_{\beta-1}y_{\beta-1})) \middle| \boldsymbol{x} \in \mathcal{B}_i, \boldsymbol{y} \in \mathbb{B}^\beta\right\}$
$\boldsymbol{x} = (x_0, \dots, x_{\beta-1})$
$\boldsymbol{y} = (y_0, \dots, y_{\beta-1}) \quad (i \in [\nu\rangle)$
$\mathcal{B}_i \subset \mathbb{B}^\beta$
$|\mathcal{C}_i| = 2^\beta |\mathcal{B}_i| = 2^b, \ |\mathcal{B}_i| = 2^{b-\beta}$

$\boldsymbol{v}$	$\phi(\boldsymbol{v})$
00	A
01	T
10	G
11	C

Marker



rate: $R_\mathrm{i} = \frac{\mu_k}{\mu_k + \mu_r}$

Rate

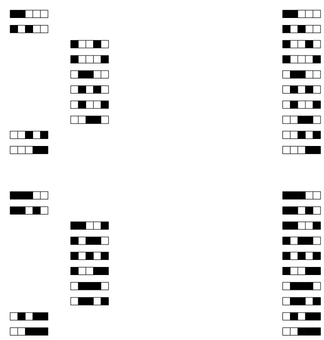
$R = R_\mathrm{o} R_\mathrm{i} R_\mathrm{c}$ [bits/symbol]
$R_\mathrm{o} \le$ 1-cross entropy: inner APP
$R_\mathrm{i} =$ inner code rate
$R_\mathrm{c} \le$ bound of $(\ell; w, \delta)$ constraint
bound for given $P(X)$

$R_\mathrm{o} R_\mathrm{i} \le I(X;Y)$:

(example)

		β	ν	R_I	ℓ	δ	b_i	$\mathcal{B}_i$
1	3.4.2	3	2	4/6	2	2	4	$\mathcal{B}_0$
							4	$\mathcal{B}_1$
2	3.5.1	3	1	5/6	3	*	5	$\mathcal{B}_0$
3	4.5.2	4	2	5/8	2	2	5	$\mathcal{B}_0$
							5	$\mathcal{B}_1$
4	4.6.1a	4	1	6/8	2	2	6	$\mathcal{B}_0$
5	4.6.1b	4	1	6/8	4	*	6	$\mathcal{B}_0$
6	4.7.1	4	1	7/8	4	*	7	$\mathcal{B}_0$
7	5.7.2	5	2	7/10	5	5	7	$\mathcal{B}_0$
							7	$\mathcal{B}_1$
9	5.8.2	5	2	8/10	3	3	8	$\mathcal{B}_0$
							8	$\mathcal{B}_1$
10	5.9.1	5	1	9/10	4	*	9	$\mathcal{B}_0$
8	5.a.2	5	2	15/20	3	3	8	$\mathcal{B}_0$
							7	$\mathcal{B}_1$

□ : 0
■ : 1



(ex2b)

β	3	3	4	4	4	4	5	5	6
b	4	5	5	6	6	7	8	9	10
ν	2	1	2	1	1	1	2	1	1
R_i	0.67	0.83	0.63	0.75	0.75	0.88	0.80	0.90	0.83
				a	b				
$(\ell; w, \delta)$	$(2;10,2)$	$(3;*,*)$	$(2;10,2)$	$(2;10,2)$	$(4;*,*)$	$(4;*,*)$	$(3;10,2)$	$(4;*,*)$	

* (multi-read)

NB-IDS channel

block length: n

alphabet: $\Sigma \quad (|\Sigma| = q)$

error prob.: $p_{\mathrm{id}} \ (< \frac{1}{2})$ (ins/del)

p_{s} (sub)

input: $\boldsymbol{x} = (x_0, \ldots, x_{n-1}) = \Sigma^n$

output: $\boldsymbol{y} = (y_0, \ldots, y_{n'-1}) = \Sigma^{n'}$

$(n{-}D \le n' \le n{+}D)$

[transmission]

1) drift vector: $\boldsymbol{d} = (d_0, \ldots, d_{n-1}, d_n) \in \mathcal{D}^{n+1}$

2) intermediate vector: $\boldsymbol{z} = (z_0, \ldots, z_{n'-1}) \in \Sigma^{n'}$

$$z_j = x_i \ (j \in [i + d_i, i + d_{i+1}]) \quad i \in [n\rangle, \ n' = n + d_n$$

3) output vector:

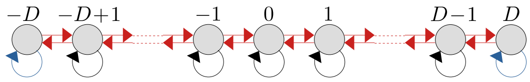
$$p(y_i|z_i) = \begin{cases} 1 - p_{\mathrm{s}} & (y_i = z_i) \\ \frac{p_{\mathrm{s}}}{q-1} & (y_i \neq z_i) \end{cases}$$

max drift: D

set of drift values: $\mathcal{D} = \{-D, \ldots, 0, \ldots, D\}$

drift vector: $\boldsymbol{d} = (d_0 {=} 0, \ldots, d_{n-1}, d_n) \in \mathcal{D}^{n+1}$

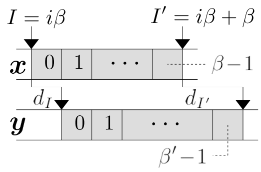
$$p(d_{i+1}|d_i) = \begin{cases} 1 - 2p_{\mathrm{id}} & (d_{i+1} = d_i, |d_i| < D) \\ 1 - p_{\mathrm{id}} & (d_{i+1} = d_i, |d_i| = D) \\ p_{\mathrm{id}} & (|d_{i+1} - d_i| = 1, |d_{i+1}| \le D) \\ 0 & (\text{otherwise}) \end{cases}$$



APP by FBA

$$p(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{d}) = p(\boldsymbol{y}|\boldsymbol{x}, \boldsymbol{d}) p(\boldsymbol{x}, \boldsymbol{d}) = p(\boldsymbol{y}|\boldsymbol{x}, \boldsymbol{d}) p(\boldsymbol{x}) p(\boldsymbol{d})$$

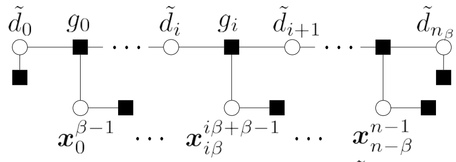
$$= p(d_0) \prod_{i=0}^{n-1} p\left(\boldsymbol{y}_{i+d_i}^{i+d_{i+1}} \middle| x_i, d_i, d_{i+1}\right) p(d_{i+1}|d_i) p(x_i)$$



segment level synchronization: $n = n_\beta \beta$

$$= p(d_0) \prod_{i=0}^{n_\beta - 1} p\left(\boldsymbol{y}_{I+d_I}^{I'-1+d_{I'}} \middle| \boldsymbol{x}_I^{I'-1}, d_I, d_{I'}\right) p\left(d_{I'}|d_I\right) p\left(\boldsymbol{x}_I^{I'-1}\right)$$

$$(I = i\beta, I' = i\beta + \beta)$$



$(\tilde{d}_i = d_{i\beta})$

[LUT] (for a large D)

$G\left(\beta', \boldsymbol{y}_0^{\beta'-1}\right)$ list of $\left[\boldsymbol{x}_0^{\beta-1}, \boldsymbol{p}_0\left(\boldsymbol{y}_0^{\beta'-1} \middle| \boldsymbol{x}_0^{\beta-1}, 0, d'\right) \boldsymbol{p}_0(d'|0) \ (> p_{\mathrm{th}})\right]$

$$\boldsymbol{p}_\alpha\left(\boldsymbol{y}_{\alpha'}^{\beta'-1} \middle| \boldsymbol{x}_\alpha^{\beta-1}, d, d'\right) \boldsymbol{p}_\alpha(d'|d) = \qquad \begin{matrix}(d' = \beta' - \beta) \\ (d = \alpha' - \alpha)\end{matrix}$$

$$\sum_{\delta \in \{-1,0,1\}} p\left(\boldsymbol{y}_{\alpha'}^{\alpha'+\delta} \middle| x_\alpha, d, d + \delta\right) p(d + \delta|d)$$

$$\times \boldsymbol{p}_{\alpha+1}\left(\boldsymbol{y}_{\alpha'+\delta+1}^{\beta'} \middle| \boldsymbol{x}_{\alpha+1}^{\beta-1}, d + \delta, d'\right) \boldsymbol{p}_{\alpha+1}(d'|d + \delta)$$

Gtable:
  int beta
  double Pid
  double Ps
  double Pth
  LIST[len][y]

LIST:
  int num
  XP[num]

XP:
  int x
  float p

[run-length of random sequence]

prob. of run-length ℓ: $p(\ell) = \left(\frac{1}{q}\right)^{\ell-1}\left(\frac{q-1}{q}\right) = \frac{q-1}{q^\ell}$

expectation: $E(\ell) = \sum_{\ell=1}^{n} \ell p(\ell) = \sum_{\ell=1}^{n} \frac{(q-1)\ell}{q^\ell}$

$$= (q-1)\sum_{\ell=1}^{n} \frac{\ell}{q^\ell}$$

$$= \frac{q}{q-1}\left(1 - \frac{1}{q^n}\right) - \frac{n}{q^n}$$

[local GC-balance of random sequence]

window size: w

prob. binary vector having weight t : $p_w(t) = \binom{w}{t}/2^w$

absolute GC-balance: $\sum_{t=0}^{w} p_w(t)\,|w-2t|$

[rate upper bound]

run length: constraint graph(?)

local GCB: $\frac{1}{w}\log_2\left(2^w \sum_{t\in\mathcal{T}}\binom{w}{t}\right) = 1 + \frac{1}{w}\log_2\left(\sum_{t\in\mathcal{T}}\binom{w}{t}\right)$

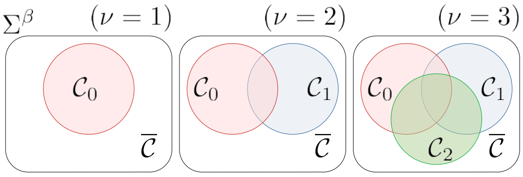
$(\mathcal{T} = \{t \mid |w-2t| \le \delta\})$

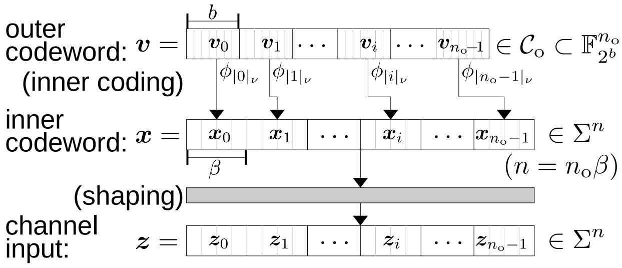
RL+LGCB: $(\ell; w, \delta)$ $\frac{1}{w}\log_2\left|\left\{\boldsymbol{u}\in\Sigma^w \middle| |f_{\mathrm{B}}(\boldsymbol{u})| \le \delta, f_{\mathrm{R}}(\boldsymbol{u}) \le \ell\right\}\right|$

Concatenated coding

channel alphabet: $\Sigma = \{\mathsf{A}, \mathsf{T}, \mathsf{G}, \mathsf{C}\}$
outer code: $\mathcal{C}_o \subset \mathbb{F}_{2^b}^{n_o}$
- n_o : code length
- b : symbol size (bits)

inner code: $\mathcal{C}_i \subset \Sigma^\beta \quad (i \in [\nu\rangle, b < 2\beta)$
- β : code length (symbols)
- ν : number of code books
- $|\mathcal{C}_i| = 2^b$: number of codewords
- $R = b/2\beta$: inner code rate
- $\phi_i : \mathbb{F}_{2^b} \to \mathcal{C}_i$: encoding function





prior: $p(\boldsymbol{x}_i) = \begin{cases} 1/2^b & (\boldsymbol{x}_i \in \mathcal{C}_{|i|_\nu}) \\ 0 & (\boldsymbol{x}_i \notin \mathcal{C}_{|i|_\nu}) \end{cases}$

params:

β	b
3	$4, 5$
4	$6, 7$
5	$8, 9$
6	$10, 11$

vector over Σ : $\boldsymbol{u} = (u_0, \ldots, u_{n-1}) \in \Sigma^n$
weight: $w(\boldsymbol{u})_x = |\{i \in [n\rangle \mid u_i = x\}| \quad (x \in \Sigma)$
balance: $f_{\mathrm{B}}(\boldsymbol{u}) = w_{\mathsf{G}}(\boldsymbol{u}) + w_{\mathsf{C}}(\boldsymbol{u}) - w_{\mathsf{A}}(\boldsymbol{u}) - w_{\mathsf{T}}(\boldsymbol{u}) \in [-n, n]$
max run length: $f_{\mathrm{R}}(\boldsymbol{u}) \in [1, n]$

Constraint

RL-LB constrained coding $(\ell; w, \delta)$:
* run length coding: $f_{\mathrm{R}}(\boldsymbol{z}) \leq \ell$
* local GC-balance: $\left|f_{\mathrm{B}}\left(\boldsymbol{z}_i^{i+w-1}\right)\right| \leq \delta$ $\quad (\forall i \in [n-w\rangle)$
* synchronization: MIR?

[example 1]
forbidden set $\overline{\mathcal{C}} \subseteq \Sigma^\beta \setminus \cup_{i \in [\nu\rangle} \mathcal{C}_i \quad (|\overline{\mathcal{C}}| \leq 2^{2\beta} - 2^b)$

ϕ	(empty)
$\overline{\mathcal{C}}_{*,\lambda} = \left\{\boldsymbol{u} \in \Sigma^\beta \mid f_{\mathrm{R}}(\boldsymbol{u}) \geq \lambda\right\}$	(RL)
$\overline{\mathcal{C}}_{\omega,*} = \left\{\boldsymbol{u} \in \Sigma^\beta \mid \|f_{\mathrm{B}}(\boldsymbol{u})\| \geq \omega\right\}$	(LB)
$\overline{\mathcal{C}}_{\omega,\lambda} = \overline{\mathcal{C}}_{\omega,*} \cup \overline{\mathcal{C}}_{*,\lambda}$	(both)

[example 2a] $\beta = 4, b = 7$

$\mathcal{C}_i = \left\{(\phi(x_0y_0), \ldots, \phi(x_3y_3)) \middle| (x_0, \ldots, x_3) \in \mathcal{B}_i, (y_0, \ldots, y_3) \in \mathbb{B}^4\right\}$

$\mathcal{B}_0$:0011,0101,0110,1001,1010,1100,0010,0100
$\mathcal{B}_1$:0011,0101,0110,1001,1010,1100,1101,1011
$|\mathcal{C}_i| = 8 \times 16 = 128 = 2^b$

$\boldsymbol{v}$	$\phi(\boldsymbol{v})$
00	A
01	T
10	G
11	C

[example 2b] $\beta = 5, b = 8$

$\mathcal{C}_i = \left\{(\phi(x_0y_0), \ldots, \phi(x_4y_4)) \middle| (x_0, \ldots, x_4) \in \mathcal{B}_i, (y_0, \ldots, y_4) \in \mathbb{B}^5\right\}$

$\mathcal{B}_0$:00011,00101,00110,01001,01010,01100,10001,10010,10100,11000
$\mathcal{B}_1$:11100,11010,11001,10110,10101,10011,01110,01101,01011,00111
$|\mathcal{C}_i| = 8 \times 32 = 256 = 2^b$

$\boldsymbol{a} = (a_0, \ldots, a_4), \boldsymbol{c} = (c_0, \ldots, c_4) \in \mathcal{B}_0$
$\boldsymbol{b} = (b_0, \ldots, b_4), \boldsymbol{d} = (d_0, \ldots, d_4) \in \mathcal{B}_1$

RL: max RL in $\boldsymbol{a}$ and $\boldsymbol{b}$: 3 (by definition)
max RL in $(\boldsymbol{a}, \boldsymbol{b})$: 3 (proof 1)
max RL in $(\boldsymbol{b}, \boldsymbol{c})$: 3 (proof 2)

(proof 1) length of the last run of $\boldsymbol{a}$: 1 (run of 1)
2 (run of 0s)
length of the first run of $\boldsymbol{b}$: 2 (run of 1s)
1 (run of 0)

(proof 2) length of the last run of $\boldsymbol{b}$: 1 (run of 0)
2 (run of 1s)
length of the first run of $\boldsymbol{c}$: 2 (run of 0s)
1 (run of 1)

LB: even $w (\geq 10)$
$(\boldsymbol{a}_t^4, \boldsymbol{b}, \boldsymbol{c}_0^{t-1})$:

* (multi-read)

Definition of inner code

β : code length

ν : number of codes

$\mathcal{C}_i \subset \Sigma^\beta$: code $(i \in [\nu\rangle)$

$\boldsymbol{b} = (b_0, \ldots, b_{\nu-1})$: bit-width

$|\mathcal{C}_i| = 2^{b_i}$

$R_{\mathrm{I}} = \frac{1}{2n\beta} \sum_{i=0}^{\nu-1} b_i$: inner code rate

Metric $d(\boldsymbol{x}, \boldsymbol{y})$:

$\boldsymbol{x} = (x_0, \ldots, x_{\beta-1}) \in \mathbb{B}^\beta$

$\boldsymbol{y} = (y_0, \ldots, y_{\beta-1}) \in \mathbb{B}^\beta$

$p_{\mathrm{i}}, p_{\mathrm{d}}, p_{\mathrm{s}}$: ins,del,sub

$p_{\mathrm{t}} = 1 - p_{\mathrm{i}} - p_{\mathrm{d}}$

t $\quad -\ln p_{\mathrm{t}} p_{\mathrm{s}}^t (1-p_{\mathrm{s}})^{1-t}$ $(t \in \{0,1\})$

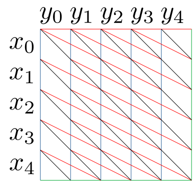
t $\quad -\ln p_{\mathrm{i}} p_{\mathrm{s}}^t (1-p_{\mathrm{s}})^{2-t}$ $(t \in \{0,1,2\})$

$-\ln p_{\mathrm{d}}$

$-\ln p_{\mathrm{i}}$

$-\ln 1/2$

0



Construction from binary code

$\boldsymbol{b}' = (b'_0, \ldots, b'_{\nu-1}) \in [\beta+1\rangle^\nu$

even β : $2^{b'_i} \le \binom{\beta}{\beta/2}$ $\quad (b_i = b'_i + \beta)$

odd β : $2^{b'_i} \le \binom{\beta}{(\beta-1)/2} = \binom{\beta}{(\beta+1)/2}$

			b'							
β	t	$\binom{\beta}{t}$	0	1	2	3	4	5	6	7
2	1	2	T	T	$-$	$-$	$-$	$-$	$-$	$-$
3	1, 2	3	T	T	F	$-$	$-$	$-$	$-$	$-$
4	2	6	T	T	T	F	$-$	$-$	$-$	$-$
5	2, 3	10	T	T	T	T	F	$-$	$-$	$-$
6	3	20	T	T	T	T	T	F	$-$	$-$
7	3, 4	35	T	T	T	T	T	T	F	$-$
8	4	70	T	T	T	T	T	T	T	F
9	4, 5	126	T	T	T	T	T	T	T	F

[computer search: candidate codes]

input: β, b', t

output: $\mathcal{L} = \{\mathcal{B}^0, \ldots, \mathcal{B}^{m-1}\}$ $\quad (\mathcal{B}^j \subset \mathbb{B}^\beta)$
(list of highest score codes)

constraint: $|\mathcal{B}^j| = 2^{b'}, \forall \boldsymbol{u} \in \mathcal{B}^j, w(\boldsymbol{u}) = t$

objective: maximize
$$\min_{\boldsymbol{x},\boldsymbol{y} \in \mathcal{B}^j} d(\boldsymbol{x}, \boldsymbol{y}) \quad (\boldsymbol{x} \ne \boldsymbol{y})$$

[computer search: code sequence]

input: $(\mathcal{L}_0, \ldots, \mathcal{L}_{\nu-1})$

output: $(\mathcal{B}_0, \ldots, \mathcal{B}_{\nu-1})$

constraint: $\mathcal{B}_i \in \mathcal{L}_i$

objective: minimize $\max_{i \in [\nu\rangle} \left|\mathcal{B}_i \cap \mathcal{B}_{|i+1|_\nu}\right|$

Evaluation of inner code

channel input: $u \in \mathbb{F}_{2^{b(i,\nu)}}$

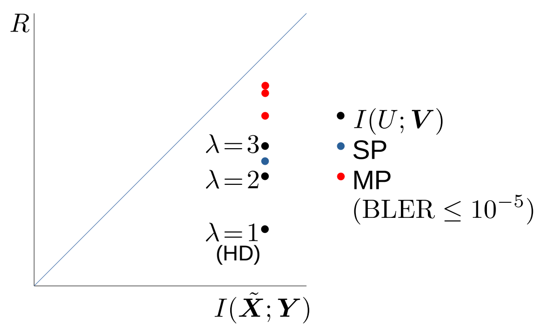
channel output: $\boldsymbol{p} \in \mathbb{P}_{2^{b(i,\nu)}}$ (APP from inner code)

(a) HD+H(p): $\boldsymbol{v} = (v, Q_\lambda(H(\boldsymbol{p})))$

Q_λ : λ-level quantization

(b) list output: (list size λ) $\boldsymbol{v} = (v_0, \ldots, v_{\lambda-1})$
$= \arg\max(\boldsymbol{p}, \lambda) \in \mathbb{F}^\lambda_{2^{b(i,\nu)}}$

mutual info.: $I(U; \boldsymbol{V})$



$\beta = 4, \nu = 1$	
$\boldsymbol{b}$	R_{I}
(4)	4/8
(5)	5/8
(6)	6/8

$\beta = 4, \nu = 2$	
$\boldsymbol{b}$	R_{I}
(4, 4)	8/16
(5, 4)	9/16
(5, 5)	10/16
(6, 4)	10/16
(6, 5)	11/16
(6, 6)	12/16

$\beta = 4, \nu = 3$	
$\boldsymbol{b}$	R_{I}
(4, 4, 4)	12/24
(5, 4, 4)	13/24
(5, 5, 4)	14/24
(6, 4, 4)	14/24
(5, 5, 5)	15/24
(6, 5, 4)	15/24
(6, 5, 5)	16/24
(6, 6, 5)	17/24
(6, 6, 6)	18/24

$\beta = 5, \nu = 1$	
$\boldsymbol{b}$	R_{I}
(5)	5/10
(6)	6/10
(7)	7/10
(8)	8/10

$\beta = 5, \nu = 2$	
$\boldsymbol{b}$	R_{I}
(5, 5)	10/20
(6, 5)	11/20
(6, 6)	12/20
(7, 5)	12/20
(7, 6)	13/20
(7, 7)	14/20
(8, 6)	14/20
(8, 7)	15/20
(8, 8)	16/20