Setting priors based using historical data

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1 Beta-Binomial Model

A beta prior can be parameterized in terms of an expected conversion rate m and a shape parameter β. The former can be estimated from historical data. The mode of the beta distribution is

$$m = \frac{\alpha - 1}{\alpha + \beta - 2}$$

If α is fixed, we can easily find a value of β that satisfies this equation:

$$\begin{aligned} \alpha - 1 &= m(\alpha + \beta - 2) && (1) \\ &= m\alpha + m\beta - 2m && (2) \end{aligned}$$

Solving for β:

$$\begin{aligned} \beta &= \frac{\alpha - 1 - m\alpha + 2m}{m} && (3) \\ &= 2 - \alpha + \frac{1}{m}(\alpha - 1) && (4) \end{aligned}$$

Thus, the prior is specified by first learning the expected conversion rate from historical data, and then mapping it to β using equation 4. α can be chosen to give the desired variance of the, which can be determined graphically. The variance of the prior increases with α.

2 Zero-Inflated Log-normal model

The conversion component of this model is just another beta-binomial model, so we can specify (α, β) as per section 1. The second component of the model describes the revenue amongst spenders. The prior distribution is specified as a product of a conditional and marginal distribution:

$$\sigma^2 \sim \text{Inv}\chi^2(\nu_0, \sigma_0^2) \Leftrightarrow \text{Inv-Gamma}\left(\frac{\nu_0}{2}, \frac{\nu_0 \sigma_0^2}{2}\right) \quad (5)$$

$$\mu | \sigma^2, \text{data} \sim \mathcal{N}\left(\mu_0, \frac{\sigma^2}{\kappa_0}\right) \quad (6)$$

The prior is specified is defined over $\sigma^2, \mu|\sigma^2$. We can summarize the prior over σ^2 by it's mode. The mode of the inverse gamma prior (Equation 5) is given by:

$$\begin{aligned} \sigma_m^2 &= \frac{\beta}{\alpha + 1}, \text{ where } \alpha = \nu_0/2, \beta = \nu_0 \sigma_0^2/2 && (7) \\ &= \frac{\nu_0 \sigma^2}{\nu_0/2 + 1} && (8) \\ &= \frac{\nu_0 \sigma_0^2}{\nu_0 + 2} && (9) \end{aligned}$$

Next, set the expected value of the prior to the average revenue amongst spenders from historical data, call it μ_{obs}:

$$\begin{aligned}
\log \mu_{obs} &= \mathbb{E}[\mu|\sigma^2, \log(\text{data})] &(10)\\
&= \mu_0 + \frac{\sigma_m^2}{2} &(11)\\
\mu_0 &= \log \mu_{obs} - \frac{\sigma_m^2}{2} &(12)\\
&= \log \mu_{obs} - \frac{1}{2}\left[\frac{\nu_0 \sigma_0^2}{\nu_0 + 2}\right] &(13)
\end{aligned}$$

Thus, using equation 13 we can parameterize the hyperparameter μ_0 using the historical revenue amongst spenders μ_{obs}.