STA2201H Methods of Applied Statistics II

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Week 2: Generalized Linear Models

Overview

Lecture (assuming this is review):

- General Linear Models
- Generalized Linear Models
- Exponential family
- Likelihood-based estimation and inference
- Poisson
- Binomial
- Practicalities of doing / thinking about GLMs
- Extra notes: multinomial (but not covered in lecture)

Lab: EDA

The model fitting process

What are we actually trying to achieve? From last week, applied statistics is:

Using statistical methods to answer questions and draw reasonable conclusions from data that have uncertainty and randomness.

The model fitting process

Overview of process

1. Look at the data (EDA, today's lab)
2. Decide on a model
   - Probability distribution for response Y e.g. $Y \sim N(\mu, \sigma^2)$
   - (This is deciding on the likelihood)
   - Equation involving explanatory variables (we are trying to explain $E[Y|X]$)
3. Estimate the parameters
4. Check the model and residuals
5. Inference, interpretation
6. Communication

Motivating examples

Outcomes we may be interested in investigating (in relation to other explanatory variables):

- Police stop and frisks in NYC
- Infant deaths in the US
- Who voted for the Liberal party v other party
- Who voted Liberal, Conservatives, LDP
- Concentration of drug at particular times after ingestion

The take-away: none of these are Normal.

General linear models

Let's start with a recap of general linear models. We observe $y_1, y_2, \ldots y_n$ which are realizations of the random variables $Y_1, Y_2, \ldots, Y_n$

In linear models the y_i's have two pieces:

1. A **systematic part**, with the form

$$E(\mathbf{Y}|\mathbf{X}) = \mu = \mathbf{X}\beta$$

2. A **random part**, where errors are assumed to be i.i.d such that $E[\epsilon] = 0$ and $var[\epsilon] = \sigma^2$. We usually further assume that errors are Normal with constant variance σ^2.

Multiple linear regression

One of the most common examples of a general linear model.

Goal: we are trying to measure the association between response/outcome/dependent variable Y_i and one or more explanatory variables/covariates $X_{i,1}, X_{i,2}, \ldots, X_{i,k}$

- The conditional expectation function (CEF) $E(Y_i|X_{i,1}, X_{i,2}, \ldots, X_{i,k})$ describes the expected value (population mean) of Y_i given values of the variables $X_{i,1}, X_{i,2}, \ldots, X_{i,k}$.

Multiple linear regression

MLR is a model for the CEF:

$$
\begin{aligned}
Y_i &= E\left(Y_i \mid X_{i,1}, X_{i,2}, \ldots, X_{i,k}\right) + \varepsilon_i \\
&= \beta_0 + \beta_1 X_{i1} + \beta_2 X_{i2} + \cdots + \beta_k X_{ik} + \varepsilon_i
\end{aligned}
$$

Specifically, the most basic MLR model is a simple linear function of the X's and associated parameters β.

Estimation

Minimizing the sum of squared residuals

$$S(\beta) = \sum_{i=1}^{n} \left(y_i - x_i^{\mathrm{T}} \beta\right)^2 = (y - X\beta)^{\mathrm{T}} (y - X\beta)$$

Leads to the MLR-OLS estimator

$$\hat{\beta} = \left(X^{\mathrm{T}} X\right)^{-1} X^{\mathrm{T}} y$$

Sampling distribution of the MLR-OLS estimator

- Under the Gauss Markov and normality assumptions, the OLS estimator, $\hat{\beta}_k$ is normally distributed with a mean equal to

$$E\left(\hat{\beta}_k\right) = \beta_k$$

and variance

$$\text{Var}\left(\hat{\beta}_k\right) = \frac{\sigma^2}{\sum_i \left(X_{ik} - \bar{X}_{ik}\right)^2 (1 - R_k^2)}$$

We can use this property for inference: The sampling distribution of standard error standardized estimator follows a t-distribution with $n - (k+1)$ degrees of freedom.

General linear models

$$
\begin{aligned}
Y_i &\sim N(\mu_i, \sigma^2) \\
\mu_i &= \mathbf{X}_i^T \beta
\end{aligned}
$$

General linear models are not appropriate when

- The range of Y is restricted
- The variance of Y depends on the mean

Generalized Linear Models extend the classical set-up to allow for a wider range of distributions. Introduced by Nelder and Wedderburn (1972) [Later, GAMs in 1990].

Generalized linear models

Generalized linear models

GLMs have an additional piece on top of the classical linear models:

1. **random component**: $Y_i \sim$ some distribution with $E[Y_i|\mathbf{X}_i] = \mu_i$
2. **systematic component**: $\mathbf{X}_i^T\beta$
3. The **link function** that links the random and systematic components $g(u_i) = \mathbf{X}_i^T\beta$

- Set-up is almost the same, particularly in terms of specifying a good linear predictor $\mathbf{X}_i^T\beta$
- Just need to think about the link and the distribution of the outcome

GLMs

$$\begin{aligned} Y_i &\sim G(\mu_i, \phi) \\ E[Y_i|\mathbf{X}_i] &= \mu_i \\ g(\mu_i) &= \mathbf{X}_i^T \beta \end{aligned}$$

- ϕ is the scale parameter.

What can Y be distributed as? In principle, anything. In practice (and original formulation), distributions come from the **exponential family**.

Exponential Family

Exponential Family

The random variable Y belongs to the exponential family of distributions if its support does not depend upon any unknown parameters and its density or probability mass function takes the form

$$p(y|\theta, \phi) = \exp\left(\frac{y\theta - b(\theta)}{\phi} + c(y, \phi)\right)$$

- $\theta = h(\mu)$ depends on the expected value of y and is the **canonical parameter**
- ϕ is the scale parameter (if known: one-parameter family)
- b and c are arbitrary functions

Example: Poisson distribution

Example: Normal distribution

$$p(y|\theta, \phi) = \exp\left(\frac{y\theta - b(\theta)}{\phi} + c(y, \phi)\right)$$

Normal:

$$p(y|\mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{(y-\mu)^2}{2\sigma^2}\right\}$$

Write as

$$p(y|\mu, \sigma^2) = \exp\left\{\frac{y\mu - \frac{1}{2}\mu^2}{\sigma^2} - \frac{1}{2}\left[\frac{y^2}{\sigma^2} + \log(2\pi\sigma^2)\right]\right\}$$

- $\theta = \mu$
- $b(\theta) = \frac{1}{2}\theta^2$
- $\phi = \sigma^2$
- $c(y, \phi) = -\frac{1}{2}\left[\frac{y^2}{\sigma^2} + \log(2\pi\sigma^2)\right]$

Other examples

Other common examples:

- Binomial
- Gamma
- Negative binomial
- Inverse Gaussian

Propertites of exponential families

Mean and variance for exponential families

$$p(y|\theta, \phi) = \exp\left(\frac{y\theta - b(\theta)}{\phi} + c(y, \phi)\right)$$

It can be shown that

$$E(Y|\theta, \phi) = b'(\theta) = \mu$$

and

$$Var(Y|\theta, \phi) = \phi b''(\theta) = \phi V(\mu)$$

Note the variance of Y depends not only on the scale parameter but also on a function of the mean.

Examples:

$$E(Y|\theta,\phi) = b'(\theta)$$

and

$$Var(Y|\theta,\phi) = \phi b''(\theta)$$

- Poisson: $E(Y|\theta,\phi) = e^{\theta} = \mu$, $Var(Y|\theta,\phi) = 1 \times e^{\theta} = \mu$
- Normal: $E(Y|\theta,\phi) = \theta = \mu$, $Var(Y|\theta,\phi) = \sigma^2 \times 1 = \sigma^2$

The canonical link

The link function $\eta_i = g(\mu)$ could in theory be any function linking the linear predictor to the distribution of the outcome variable, which is also is **monotonic** and **smooth**.

Recall $\theta = h(\mu)$. If we choose $g = h$, then

$$\theta_i = h(\mu_i) = h(h^{-1}(\eta_i)) = \eta_i = \mathbf{x}_i^T \beta$$

In other words, it ensures that the systematic component of our model is modeling the parameter of interest.

Canonical links

- Normal: identity $\theta = h(\mu) = \mu$
- Poisson: $\theta = h(\mu) = \log \mu$
- Binomial: $\theta = h(\mu) = \log(\frac{\mu}{1-\mu})$
- Exponential/Gamma: $\theta = h(\mu) = -\mu^{-1}$
- Inverse Gaussian: $\theta = h(\mu) = \mu^{-2}$

Likelihood-based estimation and inference

Estimation

Estimation

- Inference is based on MLE, but cannot derive closed form solutions for regression coefficients
- Note we are assuming independence $cov(Y_i, Y_j|\theta_i, \theta_j, \phi) = 0$ for $i \neq j$. (more on dependence later)

The log-likelihood function is:

$$\ell(\theta) = \sum_i \ell(\theta_i) = \sum_i \frac{Y_i\theta_i - b(\theta_i)}{\phi} + c(Y_i, \phi)$$

What's our usual approach here?

Score function

Score function

Information matrix

Score function and Information matrix

$$\mathbf{S}(\beta) = \mathbf{D^T V^{-1}} \frac{\mathbf{Y} - \mu(\beta)}{\phi}$$

where D^T is a matrix of the $\partial\mu_i/\partial\beta_j$ and $\mathbf{V}$ is diagonal with ith element $b''(\theta_i)$.

$$\mathbf{I}(\beta) = \mathbf{x^T W}(\beta)\mathbf{x}$$

where $\mathbf{W}$ is diagonal with $w_i = (\frac{\partial\mu_i}{\partial\eta_i})^2/\phi b''(\theta_i)$.

What about ϕ?

When ϕ is unknown, can estimate it using

$$\hat{\phi} = \frac{1}{n-k-1} \sum_i \frac{(Y_i - \hat{\mu}_i)^2}{V(\hat{\mu})}$$

where $\hat{\mu} = \hat{\mu}(\hat{\beta})$.

Newton-Raphson

Want to find roots such that $\mathbf{S}(\beta) = 0$. First order TS approximation:

$$\mathbf{S}(\beta) \approx \mathbf{S}(\beta^{(\mathbf{0})}) + (\beta - \beta^{(\mathbf{0})})^T \mathbf{S}'(\beta^{(\mathbf{0})})$$

Newton-Raphson iterates the step:

$$\beta^{(\mathbf{t+1})} = \beta^{(\mathbf{t})} - \mathbf{S}'(\beta^{(\mathbf{t})})^{-1} \mathbf{S}(\beta^{(\mathbf{t})})$$

Method of scoring replaces observed information with its expectation $\mathbf{E}[\mathbf{S}'(\beta)] = -\mathbf{I}(\beta)$.

$$\beta^{(\mathbf{t+1})} = \beta^{(\mathbf{t})} + \mathbf{I}(\beta^{(\mathbf{t})})^{-1} \mathbf{S}(\beta^{(\mathbf{t})})$$

Method of scoring

Estimation

Can be rewritten in the form:

$$\widehat{\beta}^{(\mathbf{t+1})} = (\mathbf{x}^{\mathbf{T}}\mathbf{W}\mathbf{x})^{-\mathbf{1}}\mathbf{x}^{\mathbf{T}}\mathbf{W}\mathbf{z}$$

where:

$$z_i = x_i\beta + (Y_i - \mu_i)\frac{\partial\eta_i}{\partial\mu_i}$$

- **W** and **z** change depending on $\hat{\beta}$ and vice versa
- Use iteratively weighted least squares (IWLS)
    1. Choose initial value $\hat{\beta}^{(0)}$
    2. Calculate **W** and **z**
    3. Repeat until convergence

Inference

Inference

We know that for the MLE, the limiting distribution is

$$\hat{\beta} \sim N(\beta, (\mathbf{x^T W x})^{-1})$$

$$\hat{\beta} \sim N(\beta, I(\hat{\beta})^{-1})$$

Standard errors are the square roots of the inverse of the information matrix.

- Use this for the classic Wald Tests e.g. $\sqrt{W} = \frac{\hat{\beta}-\beta_0}{se(\hat{\beta})}$ follows z distribution.

Likelihood ratio test

Testing nested models, ω_1 and ω_2, $\omega_1 \in \omega_2$ and number of parameters $p_2 > p1$

$$2[\log \ell(\widehat{\beta_{\mathbf{1}}}|\mathbf{y}) - \log \ell(\widehat{\beta_{\mathbf{2}}}|\mathbf{y})] \sim \chi_{p_1 - p_2}$$

- Comparing fit of two models
- Model with more predictors will almost always fit better, but is the difference signficiant?

GLM in R

- `glm()`
- same set up as `lm()`; additional `family` argument with a link
- e.g. `glm(y~x, family = binomial(link = 'logit')`

Poisson regression

Review

- mean ?
- variance ?
- link: ?

What's a problem with just looking at counts?

Offsets

$$
\begin{aligned}
Y_i &\sim \text{Poisson}(\lambda_i) \\
\text{or } Y_i &\sim \text{Poisson}(\mu_i O_i) \\
\log \mu_i &= \mathbf{x_i}^T \beta
\end{aligned}
$$

Offset controls for exposure to risk/making inferences to some baseline. e.g.

- population size
- age
- time since exposed

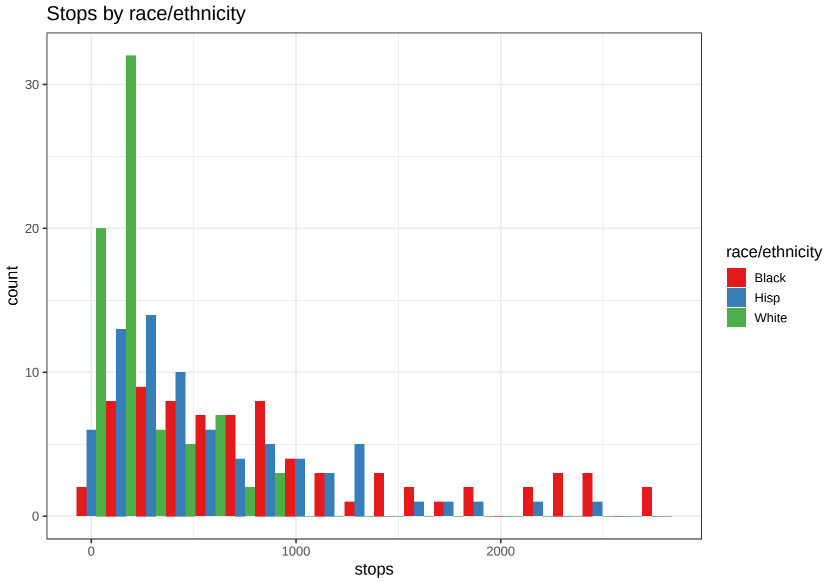
Example: Police stops

Police stop and frisks in NYC (Gelman Hill Chapter 6). Is there a difference in the number of stops by race/ethnicity?

The data look like:

precinct	stops	arrests	race_eth
1	202	980	Black
1	102	295	Hisp
1	81	381	White
2	132	753	Black
2	144	557	Hisp
2	71	431	White
3	752	2188	Black
3	441	627	Hisp
3	410	1238	White
4	385	471	Black

Distribution



GLM

Use arrests as exposure

```
mod1 <- glm(stops~race_eth,family=poisson,offset=log(arrests),data=d)
summary(mod1)
```

```
##
## Call:
## glm(formula = stops ~ race_eth, family = poisson, data = d, offset = log(arrests))
##
## Deviance Residuals:
##     Min       1Q   Median       3Q      Max
## -47.327   -7.740   -0.182   10.241   39.140
##
## Coefficients:
##               Estimate Std. Error z value Pr(>|z|)
## (Intercept)   -0.588086   0.003784 -155.40   <2e-16 ***
## race_ethHisp   0.070208   0.006061   11.58   <2e-16 ***
## race_ethWhite -0.161581   0.008558  -18.88   <2e-16 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## (Dispersion parameter for poisson family taken to be 1)
##
##     Null deviance: 46120  on 224  degrees of freedom
## Residual deviance: 45437  on 222  degrees of freedom
## AIC: 47150
##
## Number of Fisher Scoring iterations: 5
```

GLM

Add in factors for precinct

```
mod2 <- glm(stops~race_eth + factor(precinct), family=poisson,offset=log(arrests),data=d)
summary(mod2)[["coefficients"]][1:10,]
```

```
##                       Estimate  Std. Error    z value      Pr(>|z|)
## (Intercept)        -1.37886803 0.051019006 -27.026556 7.205634e-161
## race_ethHisp        0.01018798 0.006802045   1.497782  1.341899e-01
## race_ethWhite      -0.41900122 0.009434996 -44.409261  0.000000e+00
## factor(precinct)2 -0.14904964 0.074030344  -2.013359  4.407691e-02
## factor(precinct)3  0.55995498 0.056758425   9.865583  5.869222e-23
## factor(precinct)4  1.21063605 0.057548994  21.036615  3.032678e-98
## factor(precinct)5  0.28286532 0.056794015   4.980548  6.340447e-07
## factor(precinct)6  1.14420375 0.058047383  19.711547  1.716374e-86
## factor(precinct)7  0.21817307 0.064335032   3.391202  6.958688e-04
## factor(precinct)8 -0.39056473 0.056867814  -6.867940  6.513564e-12
```

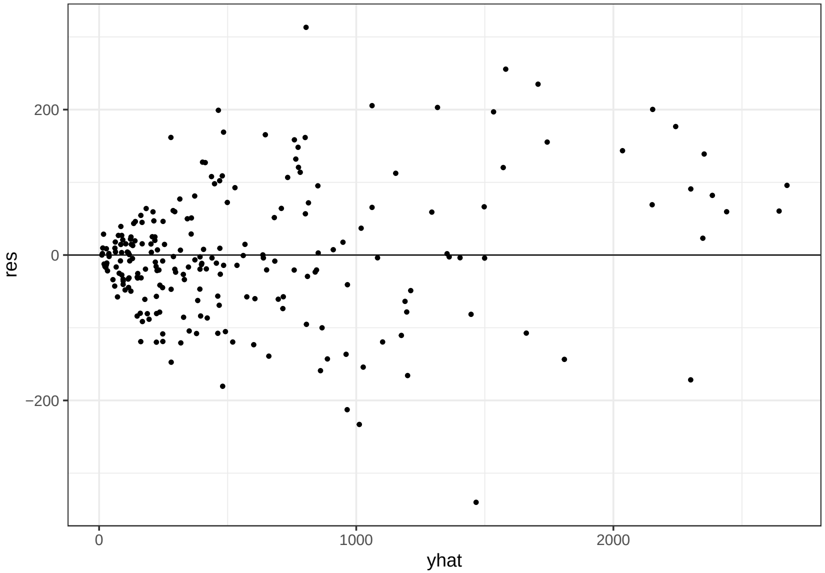
Coefficient interpretation

- e.g. after controlling for precinct, compared to blacks, whites have $1 - exp(-0.42) = 34\%$ less chance of being stopped.
- be wary of exposure variable: stops are compared to the number of arrests in the previous year
- so that the coefficient 'whites' will be less than 1 if the people in that group are stopped disproportionately less than their rates of arrest, as compared to blacks.
- would be different if we had population as exposure variable

Is this a reasonable model?

Look at predicted values versus residuals $(y_i - \hat{y}_i)$. What do we expect?

Predicted values versus residuals



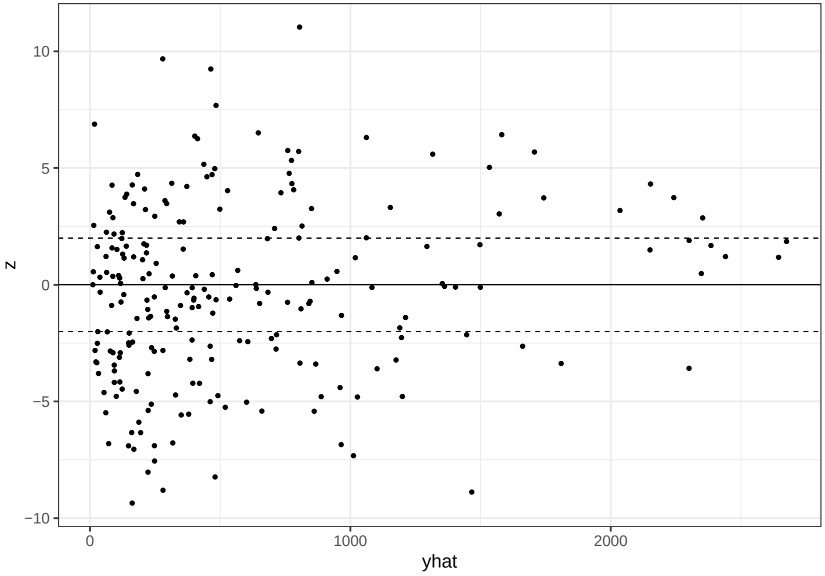
Is this a reasonable model?

Consider standardized residuals

$$z_i = \frac{y_i - \hat{y}_i}{sd(\hat{y}_i)}$$

If Poisson is a good model then these should have mean 0 and sd 1.

Predicted values versus standardized residuals



Overdispersion

- Extra variation in the data beyond what is allowed for in statistical model
- Poisson does not have independent variance parameter

Test for overdispersion: compare sum of squares of standardized residuals to χ^2_{n-k} distribution.

Estimated overdispersion factor is

$$\frac{1}{n-k}\sum_i z_i^2$$

Overdispersion

overdispersion factor is

```r
sum(res_df$z^2)/(n-k)
```

```
## [1] 21.88505
```

P-value of test is

```r
pchisq(sum(res_df$z^2), n-k, lower.tail = FALSE)
```

```
## [1] 0
```

But what's a problem here?

Fit overdispersed Poisson

- General form includes extra dispersion parameter θ
- Assume variance is proportion to the mean, rather than equal to the mean $E[Y] = \mu\theta$

```
mod3 <- glm(stops~race_eth + factor(precinct), family=quasipoisson,offset=log(arrests),data=d)
summary(mod3)[["coefficients"]][1:10,]
```

```
##                        Estimate Std. Error     t value     Pr(>|t|)
## (Intercept)         -1.37886803 0.23867441 -5.7771925 4.326149e-08
## race_ethHisp         0.01018798 0.03182097  0.3201657 7.492943e-01
## race_ethWhite       -0.41900122 0.04413830 -9.4929170 5.489337e-17
## factor(precinct)2 -0.14904964 0.34632483 -0.4303753 6.675488e-01
## factor(precinct)3  0.55995498 0.26552425  2.1088656 3.664011e-02
## factor(precinct)4  1.21063605 0.26922265  4.4967837 1.384310e-05
## factor(precinct)5  0.28286532 0.26569075  1.0646412 2.887722e-01
## factor(precinct)6  1.14420375 0.27155419  4.2135374 4.352372e-05
## factor(precinct)7  0.21817307 0.30096874  0.7249028 4.696562e-01
## factor(precinct)8 -0.39056473 0.26603599 -1.4680898 1.442019e-01
```

Notice

```
summary(mod3)[["dispersion"]]
```

```
## [1] 21.88506
```

...and the SEs are inflated $\sim \sqrt{21.9}$.

Overdisperson

Downside to quasi-Poisson it's not true MLE so you don't get likelihood etc to compare models.

Alternative:

- Could also add a multiplicative random effect θ to represent unobserved heterogeneity.
- Conditional distribution is Poisson $E[Y|\theta] \sim Pois(\mu\theta)$
- Leads to unconditional distribution being Negative Binomial distribution
- Can choose parameters so $E(Y) = \mu$ and $Var(Y) = \mu(1 + \sigma^2\mu)$

Overdispersion

Fit Negative Binomial

```
library(MASS)
mod4 <- glm.nb(stops~race_eth + factor(precinct), data = d)
summary(mod3)[["coefficients"]][1:10,]
```

```
##                       Estimate Std. Error    t value     Pr(>|t|)
## (Intercept)       -1.37886803 0.23867441 -5.7771925 4.326149e-08
## race_ethHisp       0.01018798 0.03182097  0.3201657 7.492943e-01
## race_ethWhite     -0.41900122 0.04413830 -9.4929170 5.489337e-17
## factor(precinct)2 -0.14904964 0.34632483 -0.4303753 6.675488e-01
## factor(precinct)3  0.55995498 0.26552425  2.1088656 3.664011e-02
## factor(precinct)4  1.21063605 0.26922265  4.4967837 1.384310e-05
## factor(precinct)5  0.28286532 0.26569075  1.0646412 2.887722e-01
## factor(precinct)6  1.14420375 0.27155419  4.2135374 4.352372e-05
## factor(precinct)7  0.21817307 0.30096874  0.7249028 4.696562e-01
## factor(precinct)8 -0.39056473 0.26603599 -1.4680898 1.442019e-01
```

Binary data

Binary Responses

We have n random variables $Z_1, \ldots, Z_n$ that are binary

$$Z_i = \begin{cases} 1 \text{ if outcome is a success} \\ 0 \text{ if outcome is a failure} \end{cases}$$

with

$$Pr(Z_1 = 1) = \pi_i$$

so

$$Pr(Z_1 = 0) = 1 - \pi_i$$

Logistic regression

We are interested in describing the probability of success π_i with a linear model

$$g(\pi_i) = \mathbf{x}^\mathbf{T}\beta$$

The **canonical link** is the logistic function, so

$$\text{logit } \pi_i = \log \frac{\pi_i}{1 - \pi_i} = \mathbf{x}^\mathbf{T}\beta$$

Binomial distribution

Suppose now we are interested in groups of binary outcomes, where groups are defined in such a way that all individuals in a group have identical values of all covariates.

We are interested in the number of successes within that group $\sum_{i=1}^{n_i} Z_i = Y_i$ with group size n_i. This outcome follows a binomial distribution

$$Y_i \sim \text{Binomial}(n_i, \pi_i)$$

Logistic-binary regression

We can model this in the same way as before

$$\begin{aligned} Y_i &\sim \text{Binomial}(n_i, \pi_i) \\ \text{logit}\ \pi_i &= \mathbf{x}^\mathbf{T}\beta \end{aligned}$$

- Binary data can be thought of as a special case of the count data
- Count data can be thought of a special case of the binary data

Latent variable formulation

$$
\begin{aligned}
y_i &= \begin{cases} 1 \text{ if } z_i > 0 \\ 0 \text{ if } z_i < 0 \end{cases} \\
z_i &= X_i\beta + \epsilon_i \\
\epsilon_i &\sim f(.)
\end{aligned}
$$

Latent variable formulation

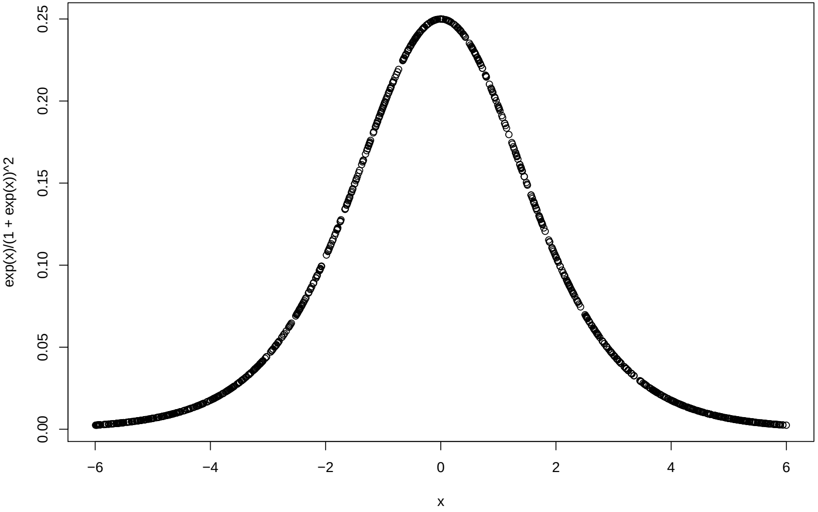
$$
\begin{aligned}
y_i &= \begin{cases} 1 \text{ if } z_i > 0 \\ 0 \text{ if } z_i < 0 \end{cases} \\
z_i &= X_i \beta + \epsilon_i \\
\epsilon_i &\sim f(.)
\end{aligned}
$$

For logistic regression, the errors ϵ have a *logistic* probability distribution

$$p(x) = \frac{e^x}{(1 + e^x)^2}$$

Latent variable formulation

The logistic pdf looks like



Latent variable formulation

Write $\eta_i = X_i\beta$.

Note that

$$\begin{aligned}\pi_i &= Pr(z_i > 0) \\ &= Pr(\epsilon_i > -\eta_i) \\ &= 1 - F(-\eta_i) \\ &= F(\eta_i)\end{aligned}$$

For the logistic, $F(\eta_i) = \frac{e^x}{(1+e^x)}$ so $\eta_i = F^{-1}(\pi_i) = \frac{\pi_i}{1-\pi_i}$ as before.

Probit regression

Any transformation that maps probabilities into the real line could be used to produce a generalized linear model, as long as the transformation is one-to-one, continuous and differentiable.

- We could also make errors Normal

$$\epsilon \sim N(0,1)$$

This implies

$$\pi_i = \Phi(\eta_i)$$

or

$$\Phi^{-1}(\pi_i) = \mathbf{X_i}\beta$$

where Φ is the standard normal cdf. This form is called **probit**. What's the interpretation of the β's?

Example: contracpetive use

Data set on contraceptive use in Fiji (source)

What the data look like:

age	education	wantsMore	notUsing	using
<25	low	yes	53	6
<25	low	no	10	4
<25	high	yes	212	52
<25	high	no	50	10
25-29	low	yes	60	14
25-29	low	no	19	10

Try a simple model: Using $\sim$ Age $+$ Desire

Example

Logit link:

```
##
## Call:
## glm(formula = cbind(using, notUsing) ~ age + wantsMore, family = binomial(link = "logit"),
##     data = d)
##
## Deviance Residuals:
##     Min       1Q   Median       3Q      Max
## -2.7870  -1.3208  -0.3417   1.2346   2.4577
##
## Coefficients:
##              Estimate Std. Error z value Pr(>|z|)
## (Intercept)   -0.8698     0.1571  -5.536 3.10e-08 ***
## age25-29       0.3678     0.1754   2.097    0.036 *
## age30-39       0.8078     0.1598   5.056 4.27e-07 ***
## age40-49       1.0226     0.2039   5.014 5.32e-07 ***
## wantsMoreyes  -0.8241     0.1171  -7.037 1.97e-12 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## (Dispersion parameter for binomial family taken to be 1)
##
##     Null deviance: 165.772  on 15  degrees of freedom
## Residual deviance:  36.888  on 11  degrees of freedom
## AIC: 118.4
##
## Number of Fisher Scoring iterations: 4
```

What's the interpretation of the `wantsMore` coefficient?

Example

Probit link:

```
##
## Call:
## glm(formula = cbind(using, notUsing) ~ age + wantsMore, family = binomial(link = "probit"),
##     data = d)
##
## Deviance Residuals:
##     Min       1Q   Median       3Q      Max
## -2.8352  -1.3411  -0.3773   1.2834   2.4893
##
## Coefficients:
##              Estimate Std. Error z value Pr(>|z|)
## (Intercept)  -0.51535    0.09178  -5.615 1.97e-08 ***
## age25-29      0.20861    0.10071   2.071   0.0383 *
## age30-39      0.46856    0.09267   5.056 4.27e-07 ***
## age40-49      0.60487    0.12207   4.955 7.23e-07 ***
## wantsMoreyes -0.49646    0.07102  -6.991 2.73e-12 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## (Dispersion parameter for binomial family taken to be 1)
##
##     Null deviance: 165.772  on 15  degrees of freedom
## Residual deviance:  38.261  on 11  degrees of freedom
## AIC: 119.77
##
## Number of Fisher Scoring iterations: 4
```

What's the interpretation of the `wantsMore` coefficient?

Comparison

- $\mathbf{X}\beta$ refers to change in z-score
- Not overly intuitive, but then again what are odds ratios
- Can convert between the two: divide by $\pi/\sqrt{3}$
- ... in both cases might be better off converting to the original (probability) scale
- Probit common in economics, but then again so are linear probability models...

Multinomial

- Additional notes on GitHub looking at models for categorical outcomes
- Natural extension of binary, but can be ordinal or not

Lab

- Using data from Open Data Portal in Toronto
  - `opendatatoronto` package
- EDA
- Questions at end need to be handed in via GitHub