CALCULUS III
DOUBLE & TRIPLE INTEGRALS STEP-BY-STEP

A Manual For Self-Study

prepared by

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Introduction

In this discussion we will extend the concept of a definite integral to functions of two and three variables. Whereas functions of a single variable are usually integrated over intervals, functions of two variables are usually integrated over regions in 2-space and functions of three variables are usually integrated over regions in 3-space. Calculating such integrals will require some new techniques that will be a central focus in our discussion. Once we have developed the basic methods for integrating functions of two and three variables, we will show how such integrals can be used to calculate **surface areas** and **volumes of solids**; and we will aso show how they can be used to find **masses and centers of gravity of flat plates** and **three dimensional solids**. In addition to our study of integration, we will generalize the concept of a parametric curve in 2-space to a parametric surface in 3-space. This will allow us to work with a wider variety of surfaces than previously possible and will provide a powerful tool for generating surfaces using computers an other graphing utilities such as matlab.

DOUBLE INTEGRALS

The notion of a definite integral can be extended to functions of two or more variables. In our discussion we will discuss the double integral, which is the extension to functions of two variables.

Recall that definite integral of a function of any single variable say x, arose from the area problem which we state below.

THE AREA PROBLEM. *Given a function $f : [a, \ b] \subseteq R \to R$ of any single variable, say x that is continuous and nonnegative on a closed bounded interval $[a, \ b]$ on the x-axis, find the area of the plane region enclosed between the graph of or the curve $y = f(x)$ and the interval $[a, \ b]$.*

$$\int_a^b f(x)\, dx = \lim_{\max \Delta\, x_k \to 0} \sum_{k=1}^N f(x_k^*)\, \Delta\, x_k = \lim_{N \to \infty} \sum_{k=1}^N f(x_k^*)\, \Delta\, x_k \tag{1}$$

[In the rightmost expression in (1), we use the "limit as $N \to \infty$" to encapsulate the process by which we increase the number of subintervals of $[a, \ b]$ in such a way that the lengths of the subintervals approach zero.] Integrals of functions of two variables arise from the problem of finding volumes under surfaces.

VOLUME CALCULATED AS A DOUBLE INTEGRAL

THE VOLUME PROBLEM: Let $f : D \subseteq R^2 \to R$ be a function of any two variables, say x and y. Let it be that f is continuous and is nonnegative on a bounded region D in the xy-plane, find the volume of the solid E in space enclosed between the graph of the surface $z = f(x,y)$ and the xy-planar region D.

The restriction to a bounded region ensures that D does not extend indefinitely in any direction, thus the region D can can be enclosed within some suitably large rectangle R whose sides are parallel to the coordinate axes. The procedure for finding the volume of a solid E in space will be similar to the limiting process used for finding areas, except that now the approximating elements will be rectangular parallelepipeds rather than rectangles. We proceed as follows:

1. Using lines parallel to the coordinate axes, divide the rectangle R enclosing the region D into subrectangles, and exclude from consideration all those subrectangles that contain any points outside of D. This leaves only rectangles that are subsets of D. Assume that there are N such subrectangles, and denote the area of the kth subrectangle by $\Delta\, A_k$.

2. Choose any arbitrary point in each of the N subrectangles, and denote the point in the kth subrectangle by $(x_k^*, \ y_k^*)$. The product $f(x_k^*, y_k^*)\, \Delta A_k$ is the volume of a rectangular parallelepiped with base area ΔA_k and height $f(x_k^*, y_k^*)$, so the sum

$$\sum_{k=1}^N f(x_k^*, y_k^*)\, \Delta A_k$$

can be viewed as an approximation to the volume $V(E)$ of the entire solid E.

3. There are two sources of error in the volume approximation: First, the parallelepipeds have flat tops, whereas the surface $z = f(x, y)$ may be curved; the second, the rectangles that form the bases of the parallelepipeds may not completely cover the region D. However, if we repeat the above process with more and more subdivisions in such a way that both the lengths and the widths of the parallelepipeds approach zero, then it is plausible that the errors of both types approach zero, and the exact volume of the solid will be

$$\lim_{N \to \infty} \sum_{k=1}^{N} f(x_k^*, y_k^*) \, \Delta A_k.$$

This suggests the following definition.

Definition 1 (Volume Under a Surface). *If $f : D \subseteq R^2 \to R$ is a function of any two variables say x and y and is such that f is continuous and nonnegative on a region D in the xy-plane, then the volume of the solid E enlosed between the surface $z = f(x, y)$ and the region D is defined by*

$$V(E) = \lim_{N \to \infty} \sum_{k=1}^{N} f(x_k^*, y_k^*) \, \Delta A_k \tag{2}$$

Here, $N \to \infty$ indicates the process of increasing the number of subrectangle of the rectangle R that encloses D in such a way that both the length and the width of the subrectangles approach zero.

It is assumed in the Definition #1 that f is nonnegative on the region D. If f is continuous on D and has both positive and negative values, then the limit

$$\lim_{N \to \infty} \sum_{k=1}^{N} f(x_k^*, x_k^*) \, \Delta A_k \tag{3}$$

no longer represents the volume between D and the surface $z = f(x, y)$; rather, it represents *a difference of volumes—* the volume between D and the portion of the surface below the xy-plane. We call this the **net signed volume** between the region D and the surface $z = f(x, y)$.

DEFINITION OF A DOUBLE INTEGRAL

As in Definition # 1, the notation $N \to \infty$ in (3) encapsulates a process in which the enclosing rectangle R for D is repeatedly subdivided in such a way that both lengths and the widths of the subrectangles approach zero. Note that subdividing so that the subrectangle lengths approach zero forces the mesh of the partition of the length of the enclosing rectangle R to approach zero. Similarly, subdividing so that the subrectangle widths approach zero forces the mesh of the paritition of the widths of the enclosing rectangle R to approach zero. Thus, we have extended the notion conveyed by Formula (1) where the definite integral of a single variable function is expressed as a limit of Riemann sums. By extension, the sums in (3) are also called **Riemann sums**, and the limit of the Riemann sums (when it exists) is denoted by

$$\int \int_D f(x, y) \, dA = \lim_{N \to \infty} \sum_{k=1}^{N} f(x_k^*, y_k^*) \, \Delta A_k \tag{4}$$

which is called the **double integral of $f(x, y)$ over** D.

If f is continuous and nonnegative on the region D, then volume formula in (2) can be expressed as

$$\text{Volume of the solid E} = V(E) = \int \int_D f(x, y) \, dA \tag{5}$$

If f has both positive and negative values on D, then a positive value for the double integral of f over D means that there is more volume bove D than below, a negative value for the double integral means that there is more volume below D than above. and a value of zero means that the volume above D is the same as the volume below D.

EVALUATING DOUBLE INTEGRALS

Except in the simplest ases, it is impractical to obtain the value of a double integral from the limit in (4). However, we will now show how to evaluate double integrals by calculating two successive single integrals. For the rest of this section we will limit our discussion to the case where D is a rectangle; in the next section we will consider double integrals over more complicated regions.

The partial derivatives of a function $f(x, y)$ are calculated by holding one of the variables fixed and differentiating with respect to the other variable. Let us consider the reverse of this process, **partial integration**. The symbols

$$\int_{\alpha}^{\beta} f(x, y)\, dx \ \text{ and } \ \int_{\gamma}^{\delta} f(x, y)\, dy$$

denote **partial definite integrals**; the first integral, called the **partial definite integral with respect to** x, is evaluated by holding y fixed and integrating with respect to x, and the second integral, called the **partial definite integral with respect to** y, is evaluated by holding x fixed and integrating with respect to y. As the following example shows, the partial definite integral with respect to x is a function of y, and the partial definite integral with respect to y is a function x.

Example 1.1

$$\int_0^1 xy^2\, dx = y^2 \int_0^1 dx = \frac{y^2 x^2}{2}\Big]_{x=0}^{1} = \frac{y^2}{2}$$

$$\int_0^1 xy^2\, dy = x \int_0^1 y^2\, dy = \frac{xy^3}{3}\Big]_{x=0}^{1} = \frac{x}{3}$$

A partial definite integral with respect to x is a function of y and hence can be integrated with respect to y; similarly, a partial definite integral with respect to y can be integrated with respect to x because it is a function of x. This two-stage integration process is called **iterated** (or **repeated**) **integration**.

We introduce the following notation:

$$\int_{\gamma}^{\delta} \int_{\alpha}^{\beta} f(x, y)\, dx\, dy = \int_{\gamma}^{\delta} \left[\int_{\alpha}^{\beta} f(x, y)\, dx \right] dy \tag{6}$$

$$\int_{\alpha}^{\beta} \int_{\gamma}^{\delta} f(x, y)\, dy\, dx = \int_{\alpha}^{\beta} \left[\int_{\gamma}^{\delta} f(x, y)\, dy \right] dx \tag{7}$$

These integrals are called **iterated integrals**.

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Example 1.2   Evaluate the iterated integrals

$$(a) \quad \int_1^3 \int_2^4 (40 - 2xy)\, dy\, dx \qquad\qquad (b) \quad \int_2^4 \int_1^3 (40 - 2xy)\, dx\, dy$$

Solution (a)

$$\int_1^3 \int_2^4 (40 - 2xy)\, dy\, dx = \int_1^3 \left[\int_2^4 (40 - 2xy)\, dy \right] dx$$

$$= \int_1^3 (40y - xy^2)\Big]_{y=2}^4 dx$$

$$= \int_1^3 [(160 - 16x) - (80 - 4x)]\, dx$$

$$= \int_1^3 (80 - 12x)\, dx$$

$$= (80x - 6x^2)\Big]_1^3 = 112$$

Solution (b)

$$\int_2^4 \int_1^3 (40 - 2xy)\, dx\, dy = \int_2^4 \left[\int_1^3 (40 - 2xy)\, dx \right] dy$$

$$= \int_2^4 (40x - x^2 y)\Big]_{x=1}^3 dy$$

$$= \int_2^4 [(120 - 9y) - (40 - y)]\, dy$$

$$= \int_2^4 (80 - 8y)\, dy$$

$$= (80y - 4y^2)\Big]_2^4 = 112$$

It is no accident that both parts of Example # 2 produced the same answer.

PROBLEM: Consider the solid E in 3-space bounded above by the surface $z = 40 - 2xy$ and bounded below by the rectangular region D in the xy-plane ($z = 0$) defined by the set

$$D = \{(x, y) : 1 \leq x \leq 3,\ 2 \leq y \leq 4\}.$$

Suppose that we wished to calculate the **volume of the solid** E, which in these discussion will be denoted by $V(E)$. If we apply the method of slicing (disks, washers, cylindrical shell, etc) that you learned in working with volumes in your studies of calculus, using slices perpendicular to the x-axis, then volume of E is given by

$$V(E) = \int_1^3 A(x)\, dx$$

where $A(x)$ is the area of a vertical cross section of the solid E at x. For a fixed value of x where $1 \leq x \leq 3$, then $z = 40 - 2xy$ is a function of y, so the integral

$$A(x) = \int_2^4 (40 - 2xy)\, dy$$

represents the area under the graph of this function of y. Thus,

$$V(E) = \int_1^3 A(x)\, dx = \int_1^3 \left[\int_2^4 (40 - 2xy)\, dy \right] dx = \int_1^3 \int_2^4 (40 - 2xy)\, dy\, dx$$

is the volume of the solid E.

Similarly, by the method of slicing with cross section of E taken perpendicular to the y-axis, the volume of E is given by

$$V(E) = \int_2^4 A(y)\, dy = \int_2^4 \left[\int_1^3 (40 - 2xy)\, dx \right] dy = \int_2^4 \int_1^3 (40 - 2xy)\, dx\, dy.$$

Thus, the iterated integrals in parts (a) and (b) of Example #2 both measure the volume of S, which by Formula (5) is the double integral of $z = 40 - 2xy$ over D. That is,

$$\iint_D (40 - 2\, x\, y)\, dA = \int_1^3 \int_2^4 (40 - 2\, x\, y)\, dy\, dx = \int_2^4 \int_1^3 (40 - 2\, x\, y)\, dx\, dy.$$

The geometric argument above applies to any continuous function $f(x, y)$ of any two variables, say x and y that is nonnegative on a rectangular planar region D described by the set

$$D = [\alpha,\ \beta] \times [\gamma,\ \delta] = \{(x, y) : \alpha \leq x \leq \beta,\ \gamma \leq y \leq \delta\},$$

as is the case for $f(x, y) = 40 - 2\, x\, y$ on $[1,\ 3] \times [2,\ 4]$. The conclusion that the double integral of $f(x, y)$ over D has the same value as either of two possible iterated integrals is true even when f is negative at some points in D. We state this result in the following theorem without proof. The ever so popular Theorem taught in advanced analysis courses and calculus classrooms all over the world known as **Fubini's Theorem**.

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Fubini's Theorem (WEAK FORM). *Let D be a planar region described by the set*

$$D = [\alpha,\ \beta] \times [\gamma,\ \delta] = \{(x,y) : \alpha \leq x \leq \beta,\ \gamma \leq x \leq \delta\}$$

If $f(x,y)$ is continuous throughout D, then

$$\int\int_D f(x,y)\,dA = \int_\gamma^\delta \int_\alpha^\beta f(x,y)\,dx\,dy = \int_\alpha^\beta \int_\gamma^\delta f(x,y)\,dy\,dx.$$

Fubini's Theorem says that: *Double integrals over rectangles can be calculated as iterated integrals.* This can be done in two ways, both of which produce the value of the double integral. Thus, we can evaluate a double integral by integrating with respect to one variable at a time. Fubini's Theorem also says that: *We may calculate the double integral by integrating in either order.*

Historical Note—Guido Fubini(January 19, 1879-June 6, 1943) was an Italian mathematician, known for Fubini's Theorem and the Fubini-Study metric.

Born in Venice, he was steered towards mathematics at an early age by his teachers and his father, who was himself a teacher of mathematics. In 1896 he entered the Sicuola Normale Superiore di Pisa, where he studied under the notable mathematicians Ulisse Dini and Luigi Bianchi. He gained some early fame when his 1900 doctoral thesis, entitle Clifford's parallelism in elliptic spaces, was discussed in a widely-read work on differential geometry published by Bianchi in 1902.

After earning his doctorate, he took up a series of professorships. In 1901 he began teaching at the University of Catania in Sicily; shortly afterwards he moved to the University of Genoa; and in 1908 he moved to the Politecnico in Turin and the the University of Turin, where he would stay for some decades. During this time his research focused primarily on topics in mathematical analysis, especially differential equations, functional analysis, and complex analysis; but he also studied calculus of variations, group theory, non-Euclidean geometry, and projective geometry, among other topics. With the outbreak of World War I, he shifted his work towards more applied topics, studying the accuracy of artillery fire; ater the war, he continued in an applied direction, applying result from his work to problems n electrical circuits and acoustics.

In 1939, when Fubini at the age of 60 was nearing retirement, Benito Mussolini's Fascists adopted the anti-Jewish polices advocated for several years by Adolf Hitler's Nazis. As a Jew, Fubini feared for the safety of his family, and so accepted an invitation by Princeton University to teach there; he died in New York City four years later.

Example 1.3 Use a double integral to find the volume of the solid E thatis bounded above by the plane $z = 4 - x - y$ and below by the rectangle $D = [0,\ 1] \times [0,\ 2]$.

Solution The volume of the solid E is the double integral of $z = 4 - x - y$ over D. Using Fubini's Theorem, this can be obtained from either of the iterated integrals

$$\int_0^2 \int_0^1 (4 - x - y)\, dx\, dy \ \text{ or } \ \int_0^1 \int_0^2 (4 - x - y)\, dy\, dx \tag{8}$$

Using the first of these, we obtain

$$\begin{aligned} V(E) &= \int\!\!\int_D (4 - x - y)\, dA = \int_0^2 \int_0^1 (4 - x - y)\, dx\, dy \\ &= \int_0^2 \left[4x - \frac{x^2}{2} - x\,y \right]_{x=0}^1 dy = \int_0^2 \left(\frac{7}{2} - y \right) dy \\ &= \left[\frac{7}{2}y - \frac{y^2}{2} \right]_0^2 = 5 \end{aligned}$$

You can check this result by evaluating the second integral in (8). evaluating the second in (8) gives

$$\begin{aligned} \int_0^1 \int_0^2 (4 - x - y)\, dy\, dx &= \int_0^1 \left[4y - x\,y - \frac{y^2}{2} \right]_{y=0}^2 dx \\ &= \int_0^1 (6 - 2\,x)\, dx \\ &= \left[6x - x^2 \right]_0^1 = 5 \end{aligned}$$

Fubini's Theorem says that

$$\int_0^2 \int_0^1 (4 - x - y)\, dx\, dy = \int_0^1 \int_0^2 (4 - x - y)\, dy\, dx = 5$$

which is indeed true. Thus the volume of the solid E is 5 $cu.\ units$.

Example 1.4 Evaluate the double integral

$$\int\int_D xy^2\, dA$$

over the rectangle $D = \{(x, y) : -3 \le x \le 2,\ 0 \le y \le 1\}$.

Solution In view of Fubini's Theorem, the value of the double integral can be obtained by evaluating one of two possible iterated double integrals. We choose to integrate first with respect to x and then with respect y.

$$\int\int_D xy^2\, dA = \int_0^1 \int_{-3}^2 xy^2\, dx\, dy$$

$$= \int_0^1 \left[\frac{1}{2} x^2\, y^2\right]_{x=-3}^2 dy$$

$$= \int_0^1 \left(-\frac{5}{2} y^2\right) dy = -\frac{5}{6} y^3\Big]_0^1 = -\frac{5}{6}$$

The integral in Example #4 can be interpreted as the net signed volume between the rectangle $[-3,\ 2] \times [0,\ 1]$ and the surface $z = xy^2$. That is, it is the volume below $z = xy^2$ and above $[0,\ 2] \times [0,\ 1]$ minus the volume above $z = xy^2$ and below $[-3,\ 0] \times [0,\ 1]$.

Definition 2 (Average value or mean value of a function). *If $f(x, y)$ is continuous over the rectangle*

$$D = [\alpha,\ \beta] \times [\gamma,\ \delta] = \{(x, y) : \alpha \le x \le \beta,\ \gamma \le y \le \delta\},$$

then the **average value** *or* **mean value** *of $f(x, y)$ on D is given as:*

$$f_{\text{ave}} = \frac{1}{A(D)} \int\int_D f(x, y)\, dA$$

where $A(D) = (\beta - \alpha)(\delta - \gamma)$ is the area of the rectangle D.

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Example 1.5 *Find the average value of $f(x, y) = 1 - 6x^2y$ on the rectangle*

$$D = \{(x, y) : 0 \le x \le 2, \; -1 \le y \le 1\}.$$

Solution The area of the rectangle D is $(2 - 0)(1 - (-1)) = 4$ and so $A(D) = 4$ *sq. units* and $\int \int_D f(x, y) \, dA = 4$ (from Example 3). Therefore

$$f_{\text{ave}} = \frac{1}{A(D)} \int \int_D f(x, y) \, dA = \frac{1}{4} \cdot 4 = 1.$$

Thus the mean value of the given function over the rectangle is 1.

Example 1.6 Suppose that the temperature (in degrees Celcius) at a point (x, y) on a flat metal plate is $T(x, y) = 10 - 8\,x^2 - 2\,y^2$, where x and y are in meters. Find the average temperature of the rectangular portion of the plate for which $0 \le x \le 1$ and $0 \le y \le 2$.

SolutionThe rectangle D over which we must find T_{ave} has area $A(D) = 2$. By choice, we choose to integrate, first with respect to y and then with respect to x, thus

$$
\begin{aligned}
\int \int_D T(x, y) \, dA &= \int_0^1 \int_0^2 (10 - 8\,x^2 - 2\,y^2) \, dy \, dx \\
&= \int_0^1 \left[\int_0^2 (10 - 8\,x^2 - 2\,y^2) \, dy \right] dx \\
&= \int_0^1 \left[10\,y - 8\,x^2\,y - \frac{2}{3}\,y^3 \right]_{y=0}^{2} dx \\
&= \int_0^1 \left(\frac{44}{3} - 16\,x^2 \right) dx \\
&= \left[\frac{44}{3}\,x - \frac{16}{3}\,x^3 \right]_0^1 = \frac{28}{3}.
\end{aligned}
$$

Now

$$T_{\text{ave}} = \frac{1}{A(D)} \int \int_D T(x, y) \, dA = \frac{1}{2} \cdot \frac{28}{3} = \frac{14}{3}$$

and so the average temperature over the metal plate is 14/3 °C.

Theorem 2 (SEPARATION OF VARIABLES FOR ITERATED INTEGRALS). *Let $g(x)$ be a continuous function on the interval $[\alpha,\ \beta]$ on the x-axis and $h(y)$ be a continuous function on the interval $[\gamma,\ \delta]$ on the y-axis. Then $f(x,y) = g(x)\,h(y)$ is a continuous function on the rectangle $D = [\alpha,\ \beta] \times [\gamma,\ \delta]$ and*

$$\int\int_D f(x,y)\,dA = \left(\int_\alpha^\beta g(x)\,dx\right)\left(\int_\gamma^\delta h(y)\,dy\right)$$

Proof. *By Fubini's Theorem we can write*

$$\int\int_D f(x,y)\,dA = \int_\delta^\gamma \int_\alpha^\beta f(x,y)\,dx\,dy$$

$$= \int_\delta^\gamma \int_\alpha^\beta g(x)\,h(y)\,dx\,dy$$

$$= \int_\delta^\gamma \int_\alpha^\beta h(y)\,g(x)\,dx\,dy$$

$$= \int_\delta^\gamma h(y)\left[\int_\alpha^\beta g(x)\,dx\right]dy$$

$$= \left[\int_\alpha^\beta g(x)\,dx\right]\left[\int_\delta^\gamma h(y)\,dy\right]$$

Q.E.D

Example 1.7   Evaluate the iterated integral

$$\int_0^{\ln 2} \int_{-1}^1 \sqrt{e^y + 1}\, \tan x\, dx\, dy$$

Solution

$$\int_0^{\ln 2} \int_{-1}^1 \sqrt{e^y + 1}\, \tan x\, dx\, dy \;=\; \left(\int_0^{\ln 2} \sqrt{e^y + 1}\, dy\right)\left(\int_{-1}^1 \tan x\, dx\right) \;=\; 0$$

Since $\tan x$ is an odd function on $[-1, 1]$, then $\int_{-1}^1 \tan x\, dx = 0$. On the other hand, if you did not make the observation, then

$$\int_{-1}^1 \tan x\, dx = \ln|\sec x|\Big]_{-1}^1 = \ln|\sec 1| - \ln|\sec -1| = \ln|\sec 1| - \ln|\sec 1| = 0$$

or you chose to compute first, the definite integral

$$\int_0^{\ln 2} \sqrt{e^y + 1}\, dy$$

To evaluate this integral, make the substitution $u = \sqrt{e^y + 1} \Leftrightarrow y = \ln(u^2 - 1)$ and so $dy = \frac{2\,u\,du}{u^2 - 1}$. Then the original integral becomes

$$\int_0^{\ln 2} \sqrt{e^y + 1}\, dy = \int_{\sqrt 2}^{\sqrt 3} \frac{2\,u^2\,du}{u^2 - 1} = \int_{\sqrt 2}^{\sqrt 3}\left(2 - \frac{1}{u+1} + \frac{1}{u-1}\right) du = 2u + \ln\left|\frac{u-1}{u+1}\right|\Big]_{\sqrt 2}^{\sqrt 3}$$

$$= 2(\sqrt 3 - \sqrt 2) + \ln\left|\frac{\sqrt 3 - 1}{\sqrt 2 - 1} \cdot \frac{\sqrt 2 + 1}{\sqrt 3 + 1}\right| \quad \text{(which was a much harder integral)}$$

Using computer software packages such as **matlab 7**

One can use matlab to evaluate iterated integrals such as the ones above. Suppose we wanted matlab to evaluate the double integral

$$\int_{-1}^{1} \int_{0}^{2} (1 - 6x^2 y) \, dx \, dy.$$

Then in a special file called an **m-file** or at the command line prompt you would type the following matlab commands in a matlab editor window or in an editor of your choice.

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% This is a simple matlab m-file                                 %
% to evaluate a double integral ∫_c^d ∫_a^b (1 − 6 x² y) dx dy    %
% over a rectangle R = {(x,y) : a ≤ x ≤ b,  c ≤ y ≤ d}.          %
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%
clear all; % clear all variables

close all; % close all figure windows

clc; % clear screen

a = 0; b = 2; c = -1; d = 1; % x and y limits of integration
F = @(x,y) 1 − 6 * x.^2 * y; % anonymous definition of f
dblquad(F, a, b, c, d)
```

When you run the above m-file in matlab, the following output appears in the matlab command window as:

```
>>
ans =

      4.0000
```

To distinguish between double integrals of functions of two variables and definite integrals of functions of one variable, we will refer to the latter as **single integrals**. Because double integrals, like single integrals, are defined as limits, they inherit many of the properties of limits. The following results, which we state without proof, are analogs of those properties of the single integrals you studied in calculus. Double integrals of continuous functions have algebraic properties that are useful in computations and applications.

If $f(x, y)$ and $g(x, y)$ are continuous function on a planar region D, then

1. CONSTANT MULTIPLE RULE:

$$\int \int_D \lambda\, f(x, y)\, dA \;=\; \lambda \int \int_D f(x, y)\, dA \;\text{ for any constant } \lambda.$$

2. SUM AND DIFFERENCE RULE:

$$\int \int_D (f(x, y) \pm g(x, y))\, dA \;=\; \int \int_D f(x, y)\, dA \pm \int \int_D g(x, y)\, dA.$$

3. DOMINATION LAW

    A)POSITIVE DEFINITE

$$\int\int_D f(x,y)\,dA \geq 0 \ \ \text{if} \ \ f(x,y) \geq 0 \text{ on } D.$$

    B)MONOTONICITY

$$\int\int_D f(x,y)\,dA \geq \int\int_D g(x,y)\,dA \ \ \text{if} \ \ f(x,y) \geq g(x,y) \ \ \text{on } D.$$

4. ADDITIVITY LAW:

$$\int\int_D f(x,y)\,dA = \int\int_{D_1} f(x,y)\,dA + \int\int_{D_2} g(x,y)\,dA$$

if $D = D_1 \cup D_2$ is the union of two nonoverlapping regions D_1 and D_2.

DOUBLE INTEGRALS OVER NONRECTANGULAR REGIONS

In this section we will show how to evaluate double integrals over regions other than rectangles.

ITERATED INTEGRALS WITH NONCONSTANT LIMITS OF INTEGRATION

Later in this section we will see that double integrals over nonrectangular regions can often be evaluated as iterated integrals of the following types:

$$\int_{\alpha}^{\beta} \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx \;=\; \int_{\alpha}^{\beta} \left[\int_{g_1(x)}^{g_2(x)} f(x,y)\,dy \right] dx \tag{1}$$

$$\int_{\gamma}^{\delta} \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy \;=\; \int_{\gamma}^{\delta} \left[\int_{h_1(y)}^{h_2(y)} f(x,y)\,dx \right] dy \tag{2}$$

We begin with an example that illustrates how to evaluate such integrals.

Example 2.1 Evaluate the iterated integrals with nonconstant limits of integration

$$(a) \;\; \int_0^1 \int_{-x}^{x^2} y^2\,x\,dy\,dx \qquad\qquad (b) \;\; \int_0^{\pi/3} \int_0^{\cos y} x\,\sin y\,dx\,dy$$

Solution (a)

$$\int_0^1 \int_{-x}^{x^2} y^2\,x\,dy\,dx \;=\; \int_0^1 \left[\int_{-x}^{x^2} y^2\,x\,dy \right] dx \;=\; \int_0^1 \frac{y^3\,x}{3}\Big]_{y=-x}^{x^2} dx$$

$$= \int_0^1 \left[\frac{x^7}{3} + \frac{x^4}{3} \right] dx \;=\; \left[\frac{x^8}{24} + \frac{x^5}{15} \right]_0^1 \;=\; \frac{13}{120}$$

Solution (b)

$$\int_0^{\pi/3} \int_0^{\cos y} x\,\sin y\,dx\,dy \;=\; \int_0^{\pi/3} \left[\int_0^{\cos y} x\,\sin y\,dx \right] dy \;=\; \int_0^{\pi/3} \left[\frac{1}{2} x^2\,\sin y \right]_{x=0}^{\cos y} dx$$

$$= \int_0^{\pi/3} \left[\frac{1}{2} \cos^2 y\,\sin y \right] dy \;=\; \left[-\frac{1}{6} \cos^3 y \right]_0^{\pi/3} \;=\; \frac{7}{48}$$

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DOUBLE INTEGRALS OVER NONRECTANGULAR REGIONS

Plane regions can be extremely complex, and the theory of double integrals over very general regions is a topic for advanced courses in mathematics. We will limit our study of double integrals to two basic types of regions, which we will call X-**simple** and Y-**simple**; they are defined as follows:

Definition 3 (X-SIMPLE REGION). *Let $y = g_1(x)$ and $y = g_2(x)$ be functions whose graphs are continuous curves such that $g_1(x) \leq g_2(x)$ for $\alpha \leq x \leq \beta$. Then a planar region D parallel to the xy-plane is called an X-**simple region** if it is **bounded below** by the graph of $y = g_1(x)$ and **bounded above** by the graph $y = g_2(x)$ and also bounded on the sides by vertical lines $x = \alpha$ and $x = \beta$ where $\alpha < \beta$.*

Definition 4 (Y-SIMPLE REGION). *Let $x = h_1(y)$ and $x = h_2(y)$ be functions whose graphs are continuous plane curves such that $h_1(y) \leq h_2(y)$ for $\gamma \leq y \leq \delta$. Then a planar region D parallel to the xy-plane is called a Y-**simple region** if it is **bounded on the left side** by the graph of $x = h_1(y)$ and **bounded on the right side** by the graph $x = h_2(y)$ and also bounded below by horizontal lines $y = \gamma$ and $y = \delta$ where $\gamma < \delta$.*

Some books refer to these regions as types I and II regions respectively; however, we will reserve this terminology for later when we discuss Triple integrals. So we will break tradition and such regions will be called X-simple and Y-simple planar regions or sets respectively throughout our discussion of double integrals.

The following theorem will enable us to evaluate double integrals over X-simple and Y-simple regions using iterated integrals.

Fubini's Theorem (STRONG FORM). *If D is an X-simple region over which $f(x,y)$ is continuous, then*

$$\int\int_D f(x,y)\, dA = \int_\alpha^\beta \int_{g_1(x)}^{g_2(x)} f(x,y)\, dy\, dx \tag{3}$$

If D is a Y-simple region over which $f(x,y)$ is continuous, then

$$\int\int_D f(x,y)\, dA = \int_\gamma^\delta \int_{h_1(y)}^{h_2(y)} f(x,y)\, dx\, dy \tag{4}$$

Using Fubini's Theorem, the integral in Example 1(a) is the double integral of the function $f(x,y) = y^2 x$ over the X-simple region bounded on the left and right by vertical lines $x = 0$ and $x = 1$ and bounded below and above by the curves $y = -x$ and $y = x^2$. Also, the integral in Example 1(b) is the double integral of $f(x,y) = x \sin y$ over the Y-simple region bounded below and above by the horizontal lines $y = 0$ and $y = \pi/3$ and bounded on the left and right by the curves $x = 0$ and $x = \cos y$.

We wll not prove the theorem, but for the case where $f(x,y)$ is nonnegative on the region D, it can be made plausible by a geometric argument that is similar to that given for Fubini's Theorem (weak form). Since $f(x,y)$ is nonnegative,

the double integral can be interpreted as the volume of the solid E that is bounded above by the surface $z = f(x, y)$ and below by the region D, so it suffices to show that the iterated integrals also represent this volume.

Consider the iterated in (3), for example. For a fixed value of x, the function $f(x, y)$ is a function of y, and hence the integral

$$A(x) = \int_{g_1(x)}^{g_2(x)} f(x, y) \, dy$$

represents the area under the graph of this function of y between $y = g_1(x)$ and $y = g_2(x)$. This area, is the cross-sectional area at x of the solid E, and hence by the method of slicing, the volume $V(E)$ of the solid E is

$$V(E) = \int_{\alpha}^{\beta} \int_{g_1(x)}^{g_2(x)} f(x, y) \, dy \, dx$$

which shows that in (3) the iterated integral is equall to the double integral. Similarly for (4).

HOW TO FIND THE LIMITS OF INTEGRATION WHEN FACED WITH EVALUATING A DOUBLE INTEGRAL

To apply Fubini's Theorem, it is helpful to start with a two-dimensional sketch of the region R. [It is not necessary to sketch a graph of $f(x, y)$.]. For an X-simple region, the limits of integration in Formula (3) of the theorem can be obtained as follows:

We now give a procedure for finding limits of integration that applies for many regions in the plane. Regions that are more complicated, and for which this procedure fails, can often be split up into pieces on which the procedure works.

When faced with the task of evaluating the double integral $\int\int_D f(x,y)\,dA$ where D is an X-simple planar region, that is, integrating first with respect to y and then with respect to x, you should take the following steps:

STEP 1. Sketch the region D of integration and its bounding curves.

STEP 2. Since x is held fixed for the first integration

$$A(x) \;=\; \int_{y=?}^{y=?} f(x,y)\,dy,$$

we find the y-limits of integration by drawing a vertical line, say L passing through the interior of the region D drawn in STEP 1. Draw L at an arbitrary fixed value x. The line L intersects the boundary of D at two points. The lower point of intersection is $(x,\ g_1(x))$ on the curve $g_1(x)$ and the higher point of intersection is $(x,\ g_2(x))$ on the curve $g_2(x)$. The y coordinates of these two points of intersection are your lower and higher y-limits of integration respectively and are usually functions of x (instead of constants, unless the region D is a rectangle). This process yields

$$A(x) \;=\; \int_{y=g_1(x)}^{y=g_2(x)} f(x,y)\,dy$$

STEP 3. To find the x-limits of the second integration

$$\int_{x=?}^{x=?} A(x)\,dx$$

by imagining that the vertical line L drawn in STEP 2, can move freely (from left to right and from right to left). First move L to the leftmost part of the region and then to the rightmost part of the region. Then L intersects the region at $x = \alpha$ (to the left) and $x = \beta$ (to the right). Now the x coordinates of these two points of intersections are $x = \alpha$ and $x = \beta$ are your lower and higher x-limits of integration respectively. This process yields

$$\int_{x=\alpha}^{x=\beta} A(x)\,dx$$

and the procedure is now complete with

$$\int\int_D f(x,y)\,dA \;=\; \int_{x=\alpha}^{x=\beta} A(x)\,dx \;=\; \int_{x=\alpha}^{x=\beta}\left[\int_{y=g_1(x)}^{y=g_2(x)} f(x,y)\,dy\right] dx$$

Example 2.2   Evaluate

$$\int\int_D x\,y\,dA$$

over the region D enclosed between $y = \frac{1}{2}x$, $y = \sqrt{x}$, $x = 2$, and $x = 4$.

Solution We view D as an X-simple region. If we draw the region D together with a vertical line corresponding to a fixed x. This line meets the region D at the lower boundary $y = \frac{1}{2}x$ and the upper boundary $y = \sqrt{x}$. These are the y-limits of integration. Moving this line first left and then right yields the x-limits of integration, $x = 2$ and $x = 4$. Thus,

$$\int\int_D x\,y\,dA = \int_2^4 \int_{x/2}^{\sqrt{x}} x\,y\,dy\,dx = \int_2^4 \left[\frac{xy^2}{2}\right]_{y=x/2}^{\sqrt{x}} dx = \int_2^4 \left(\frac{x^2}{2} - \frac{x^3}{8}\right) dx$$

$$= \left[\frac{x^3}{6} - \frac{x^4}{32}\right]_2^4 = \left(\frac{64}{6} - \frac{256}{32}\right) - \left(\frac{8}{6} - \frac{16}{32}\right) = \frac{11}{6}$$

If D is a Y-simple region, then the limits of integration in Formula (4) of Fubini's Theorem can be obtained as follows:

We now give a procedure for finding limits of integration that applies for many regions in the plane. Regions that are more complicated, and for which this procedure fails, can often be split up into pieces on which the procedure works.

When faced with the task of evaluating the double integral $\int\int_D f(x,y)\,dA$ where D is a Y-simple planar region, that is, integrating first with respect to x and then with respect to y, you should take the following steps:

    STEP 1. Sketch the region D of integration and its bounding curves.

    STEP 2. Since y is held fixed for the first integration

$$A(y) = \int_{x=?}^{x=?} f(x,y)\,dx,$$

    we find the x-limits of integration by drawing a horizontal line, say M passing through the interior of the region D drawn in STEP 1. Draw M at an arbitrary fixed value y. The line M intersects the boundary of D at two points. The leftmost point of intersection is $(h_1(y),\ y)$ on the curve $h_1(y)$ and the rightmost point of intersection is $(h_2(y),\ y)$ on the curve $h_2(y)$. The x coordinates of these two points of intersection are your lower and higher x-limits of integration respectively and are usually functions of y (instead of constants, unless the region D is a rectangle). This process yields

$$A(y) = \int_{x=h_1(y)}^{x=h_2(y)} f(x,y)\,dx$$

    STEP 3. To find the y-limits of the second integration

$$\int_{y=?}^{y=?} A(y)\,dy$$

    by imagining that the horizontal line M drawn in STEP 2, can move freely (from top to bottom and from bottom to top). First move M to the bottom of the region and then to the top of the region. The line intersects the region at $y = \gamma$ and at the top the line intersects the region at $y = \delta$. Now the x coordinates of these two points of

intersections are $y = \gamma$ and $y = \delta$ are your lower and higher x-limits of integration respectively. This process yields

$$\int_{y=\gamma}^{y=\delta} A(y)\, dy$$

and the procedure is now complete with

$$\int\int_D f(x,y)\, dA = \int_{y=\gamma}^{y=\delta} A(y)\, dy = \int_{y=\gamma}^{y=\delta} \left[\int_{x=h_1(y)}^{x=h_2(y)} f(x,y)\, dx\right] dy$$

Example 2.3   Use the above limit finding procedure for Y-simple region to find the limits of integration for

$$\int\int_D (2\,x - y^2)\, dA$$

over the triangular region D enclosed between the lines $y = -x + 1$, $y = x + 1$, and $y = 3$ and then evaluate the resulting double integral.

Solution   By choice we shall view D as a Y-simple planar region, that is, we shall integrate first with respect to x and then integrate with respect to y. STEP 1 (not shown here). The horizontal line M which passes through D meets the left boundary at $x = h_1(y) = 1 - y$ and the right boundary at $x = h_2(y) = y - 1$ for an arbitrary fixed value y. Thus

$$A(y) = \int_{x=1-y}^{x=y-1} (2\,x - y^2)\, dy$$

$$= \left[x^2 - y^2\,x\right]_{x=1-y}^{y-1} = (y^2 - 2y + 1 - y^3 + y^2 - (1 - 2y + y^2) + y^2 - y^3)$$

$$= 2\,y^2 - 2\,y^3.$$

Next, the line M when at the bottom meets the region at $y = 1$ and at the top the line M meets the region at $y = 3$. Thus

$$\int_{y=1}^{y=3} A(y)\, dy = \int_1^3 (2y^2 - 2y^3)\, dy = \left[\frac{2}{3}y^3 - \frac{1}{2}y^4\right]_1^3 = 18 - \frac{81}{2} - \frac{2}{3} + \frac{1}{2} = -\frac{68}{3}.$$

In Example 3 we could have treated D as an X-simple region, but with an added complication. Viewed as an X-simple region, the upper boundary of D is the line $y = 3$ and the lower boundary consists of two parts, the line $y = -x + 1$ to the left of the y-axis and the line $y = x + 1$ to the right of the y-axis. To carry out the integration it is necessary to decompose D into two parts, D_1 and D_2, and write

$$\int\int_D (2\,x - y^2)\, dA = \int\int_{D_1} (2\,x - y^2)\, dA + \int\int_{D_2} (2\,x - y^2)\, dA$$

$$= \int_{-2}^0 \int_{-x+1}^3 (2\,x - y^2)\, dy\, dx + \int_0^2 \int_{x+1}^3 (2\,x - y^2)\, dy\, dx.$$

This will yield the same result that was obtained in Example 3. (Verify)

▷ **Exercise 2.1**   Verify the two double integrals above.

Example 2.4   Use a double integral to find the volume of the tetrahedron E bounded by the coordinate planes and the plane

$$z = 4 - 4x - 2y \tag{5}$$

.

Solution The tetrahedron E in question is bounded above by the plane

$$z = 4 - 4x - 2y$$

and below by the triangular region $D = \{(x, y) : 0 \leq x \leq 1,\ 0 \leq y \leq 2 - 2x\}$. Thus the volume of the tetrahedron is given by:

$$V(E) = \int\int_D (4 - 4x - 2y)\, dA$$

The region D is bounded by the x-axis$(y = 0)$, the y-axis $(x = 0)$, and the line $y = 2 - 2x$ [obtained by setting $z = 0$ in (5)], so that treating D as an X-simple region yields

$$V(E) = \int\int_D (4 - 4x - 2y)\, dA = \int_0^1 \int_0^{2-2x} (4 - 4x - 2y)\, dy\, dx$$

$$= \int_0^1 (4y - 4xy - y^2)\Big]_{y=0}^{2-2x} dx = \int_0^1 (4 - 8x + 4x^2)\, dx = \frac{4}{3}$$

Example 2.5   Find the volume of the solid E bounded above by the plane $z = 4 - y$ and below by the region D enclosed within the circle $x^2 + y^2 = 4$. The volume of E is given by

$$V(E) = \int\int_D (4 - y)\, dA$$

Treating D as an X-simple we obtain

$$V(E) = \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (4 - y)\, dy\, dx = \int_{-2}^{2} \left[4y - \frac{1}{2}y^2 \right]_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dx$$

$$= \int_{-2}^{2} 8\sqrt{4 - x^2}\, dx = 8(2\pi) = 16\pi. \qquad \text{HINT: } \int_{-a}^{a} \sqrt{a^2 - x^2}\, dx = \frac{1}{2}\pi a^2.$$

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REVERSING THE ORDER OF INTEGRATION ('**Doing the Fubini** ')

Although Fubini's Theorem assures us that: *A double integral may be calculated as an iterated integral in either order of integration, the value of one of the double integrals may be easier to find than the value of the other.*

The next example shows how this can happen.

Example 2.6   Since there is no elementary antiderivative of e^{x^2}, the integral

$$\int_0^2 \int_{y/2}^1 e^{x^2} \, dx \, dy$$

cannot be evaluated by performing the x-integration first. Evaluate this integral by expressing it as an equivalent iterated integral with the order of integration reversed.

Solution   For the inside integration, y is fixed and x varies from the line $x = y/2$ to the line $x = 1$. For the outside integration, y varies from 0 to 2, so the given iterated integral is equal to a double integral over the triangular region $D = \{y/2 \le x \le 1, \ 0 \le y \le 2\}$ which is a Y-simpson set. To reverse the order of integration ('Doing the Fubini'), we treat D as an X-simple region, which enables us to write the given integral as

$$\int_0^2 \int_{y/2}^1 e^{x^2} \, dx \, dy \ = \ \int \int_D e^{x^2} \, dA \ = \ \int_0^1 \int_0^{2x} e^{x^2} \, dy \, dx \ = \ \int_0^1 \left[e^{x^2} y \right]_{y=0}^{2x} dx$$

$$= \ \int_0^1 2 \, x \, e^{x^2} \, dx \ = \ e^{x^2} \Big]_0^1 \ = \ e - 1$$

Example 2.7   Evaluate the double integral

$$\int\int_{D} \frac{\sin x}{x}\, dA$$

over D where $D = \{(x, y) : 0 \le x \le 1,\ 0 \le y \le x\}$ is the triangular region in the xy-plane bounded by the x-axis, the line $y = x$ and the line $x = 1$.

Solution   If we view D an X-simple region, that is, we first integrate with respect to y and then with respect to x, we find

$$\int_{0}^{1}\left[\int_{0}^{x}\frac{\sin x}{x}\, dy\right]\, dx = \int_{0}^{1}\left[y\,\frac{\sin x}{x}\right]_{y=0}^{y=x}\, dx$$

$$= \int_{0}^{1} \sin x\, dx = -\cos x\Big|_{x=0}^{x=1}$$

$$= 1 - cos1 \approx 0.46.$$

If we reversed the order of integration and attempt to calculate

$$\int_{0}^{1}\int_{y}^{1}\frac{\sin x}{x}\, dx\, dy,$$

we run into a problem because the integral

$$\int \frac{\sin x}{x}\, dx$$

cannot be expressed in terms of elementary functions (there is no simple antiderivative!) When calculating double Integrals over bounded nonrectangular regions, there is no general rule for predicting which order of integration will be the easiest or the good one in circumstances like these. If the order you choose first, doesn't work, then try the other. Sometimes neither order will work, and then we need to use **numerical approximation methods**.

▷ **Exercise 2.2**   (Do it now!) Use a double integral to find the volume of the solid E bounded above the cylindrical surface $x^2 + z^2 = 4$ and below by the region D enclosed within the portion of the circle $x^2 + y^2 = 4$ residing in the first quadrant of the xy-plane

Example 2.8   [**Doing the fubini**]. That is, reversing the order of integration for the double integral

$$\int\int_D f(x,y)\, dA = \int_0^2 \int_{x^2}^{2x} (4x + 2)\, dy\, dx$$

using the 3-step procedure described above.

Solution The region D of integration as given is an X-simple region. We are about to **Do the fubini**, this means that we should now view the given X-simple region of integration as a Y-simple one instead. To do this, we used your 3-step procedure you laid out from doing Exercise #1.

1.   Make a sketch of the region D where $D = \{(x,y) \mid 0 \le x \le 2,\ x^2 \le y \le 2x\}$ (Not shown here!).

Calculating

$$\int\int_D f(x,y)\, dA$$

by integrating first with respect to x and then with respect to y.

2.   Finding the x-limits of integration: Since y is held fixed. Any horizontal line L passing through D in the direction of increasing x must intersect D twice. Leftmost at the point $(y/2,\ y)$ on the curve $x = y/2$ and rightmost at the point $(\sqrt{y},\ y)$ is on the cure $x = \sqrt{y}$. Therefore the lower x limit is $x = y/2$ and the higher one is $x = \sqrt{y}$ and so the process would yield

$$A(y) = \int_{x=y/2}^{x=\sqrt{y}} (4x + 2)dx.$$

3.   Finding the y-limits of integration: We include all horizontal lines cutting through D in the direction of increasing y starting with the horizontal line $y = 0$ at the bottom of D and ending with the horizontal line $y = 4$ at the top of D. Now $y = 0$ is the lower y-limit and $y = 4$ is the higher one. The process would yield

$$\int_{y=0}^{y=4} A(y)\, dy.$$

The completed procedure yields

$$\int_{y=0}^{y=4} \int_{x=y/2}^{x=\sqrt{y}} (4x + 2)\, dx\, dy.$$

Then Fubini's Theorem tells us that

$$\int_0^2 \int_{x^2}^{2x} (4x + 2)\, dy\, dx = \int_0^4 \int_{y/2}^{\sqrt{y}} (4x + 2)\, dx\, dy.$$

Of course, you still have to evaluate the double integral; fortunately, you only have to evaluate only one of your choice (preferably the easiest one if it exists!) to show that it has the value 8.

▷ **Exercise 2.3**   Verify the value of both double integrals above.

THAT IS HOW WE DO THE FUBINI !!!

AREA CALCULATED AS A DOUBLE INTEGRAL

Although double integrals arose in the context of calculating volumes, they can also be used to calculate areas. To see why this is so, recall that a right cylinder is a solid that is generated when a plane region is translated along a line that is perpendicular to the region. We know from earlier studies that the volume V of a right cylinder with cross-sectional area A and height h is

$$V = A \cdot h \tag{6}$$

Now suppose that we are interested in finding the area A of a region D in the xy-plane. If we translated the region D upward 1 unit, then the resulting solid will be a right cylinder that has cross-sectional A, base D, and the plane $z = 1$ as its top. Thus, it follows from (6) that

$$\int\int_D 1 \, dA = (\text{area of D}) \cdot 1$$

which we can rewrite as

$$\text{area of D} = A(D) = \int\int_D 1 \, dA = \int\int_D dA \tag{7}$$

Example 2.9 Use a double integral to find the area of the region D enclosed between the parabola $y = \frac{1}{2}x^2$ and the line $y = 2x$.

Solution The region D may be treated equally well as an X-simple or a Y-simple region. Treating D as an X-simple region yields

$$\text{area of D} = \int\int_D dA = \int_0^4 \int_{x^2/2}^{2x} dy \, dx = \int_0^4 [y]_{y=x^2/2}^{2x} \, dx$$

$$= \int_0^4 \left(2x - \frac{1}{2}x^2 \right) dx = \left[x^2 - \frac{x^3}{6} \right]_0^4 = \frac{16}{3}$$

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AVERAGE VALUE OF $f(x, y)$ OVER NONRECTANGULAR REGION D

The average value of an integrable function of one variable on a closed interval is the integral of the function over the interval divided by the length of the interval. For an integrable function of two variables defined on a bounded region D in the plane, the average value is **the integral of f over the region D divided by the area of the region D**. This can be visualized by thinking of the function as giving the height at one instant of some water sloshing around in a tank whose vertical walls lie over the boundary of the region. The average height of the water in the tank can be found by letting the water settle down to a constant height. The height is then equal to the volume of water in the tank divided by the area of D. We are led to define the average value of an integrable function f over a region D to be

$$f_{\text{ave}} = \frac{1}{A(D)} \int\int_D f(x, y)\, dA.$$

DOUBLE INTEGRALS IN POLAR COORDINATES

In this section we will study double integrals in which the integrand and the region of integration are expressed in polar coordinates. Such integrals are important for two reasons: first, they arise naturally in many applications, and second, many double integrals in rectangular coordinates can be evaluated more easily if they are converted to polar coordinates.

θ-SIMPLE POLAR REGIONS

Some double integrals are easier to evaluate if the region of integration is expressed in polar coordinates. This is usually true if the region is bounded by a cardioid, a rose curve, a spiral, or, more generally, by any curve whose equation is simpler in polar coordinates than in rectangular coordinates. Moreover, double integrals whose integrands involve $x^2 + y^2$ also tend to be easier to evaluate in polar coordinates because this sum simplifies to r^2 when the conversion formulas $x = r\cos\theta$ and $y = r\sin\theta$ are applied.

A region D_p in a polar coordinate system that is enclosed between two rays, $\theta = \alpha$ and $\theta = \beta$, and two polar curves, $r = g_1(\theta)$ and $r = g_2(\theta)$. The functions $g_1(\theta)$ and $g_2(\theta)$ are continuous and their graphs do not cross, then the region D_p is called a θ-simple polar region. If $g_1(\theta)$ is identically zero, then the boundary $r = g_1(\theta)$ reduces to a point (the origin). If, in addition, $\beta = \alpha + 2\pi$, then the rays coincide. The following definition expresses these geometric ideas algebraically.

Definition 3.1 [θ-SIMPLE POLAR REGION]. A θ-**simple polar region** in a polar coordinate system is a region that is enclosed between two rays $\theta = \alpha$ and $\theta = \beta$, and two continuous polar curves, $r = g_1(\theta)$ and $r = g_2(\theta)$, where the equations of the rays and the polar curves satisfy the following conditions:

$$\alpha \leq \beta \qquad\qquad (ii)\ \beta - \alpha \leq 2\pi \qquad\qquad (iii)\ 0 \leq g_1(\theta) \leq g_2(\theta)$$

Conditions (i) and (ii) together imply that the ray $\theta = \beta$ can be obtained by rotating the ray $\theta = \alpha$ counterclockwise through and angle that is at most 2π radians. Conditions (iii) implies that the boundary curves $r = g_1(\theta)$ and $r = g_2(\theta)$ can touch but cannot actually cross over one another (why?). Thus, it is appropriate to describe $r = g_1(\theta)$ as the **inner boundary** of the region and $r = g_2(\theta)$ as the **outer boundary**.

A **polar rectangle** is a θ-simple polar region for which the bounding polar curves are circular arcs. For example, the polar rectangle D_p described by

$$D_p = \{(r,\theta) : 1.5 \leq r \leq 2,\ \pi/6 \leq \theta \leq \pi/4\}$$

DOUBLE INTEGRALS IN POLAR COORDINATES

Next we will consider the polar version of the volume problem discussed earlier.

THE VOLUME PROBLEM IN POLAR COORDINATES. *Given a function $F(r,\theta)$ that is continuous and nonnegative on a simple polar region D_p, find the volume of the solid E that is enclosed between the region D_p and the surface whose equation in cylindrical coordinates is $F(r,\theta)$.*

To motivate a formula for the volume $V_p(E)$ of the solid E, we will use a limit process similar to that used to obtain Formula (2) of Section 1, except that here will use circular arcs and rays to subdivide the region D_p into polar rectangle. We will exclude from consideration all polar rectangles that contain any points outside of D_p, leaving only polar rectangles that are subsets of D_p.

When we defined the double integral of a function over a planar region D of the xy-plane in rectangular coordinates, we began by cutting D into rectangles whose sides were parallel to the x and y coordinate axes. These were the natural shapes to use because their sides have either constant x-values or constant y-values.

In polar coordinates, the natural shape is a "polar rectangle" whose sides have constant r- and θ-values. Suppose that a function $F(r,\theta)$ is defined over a region D_p that is bounded by the rays $\theta = \alpha$ and $\theta = \beta$ and by the continuous curves $r = g_1(\theta)$ and $r = g_2(\theta)$. Suppose also that $0 \leq g_1(\theta) \leq g_2(\theta)$ for every value of θ between α and β. Then D_p lies in a fan-shaped region of the xy-plane defined by the inequalities $0 \leq r \leq a$ and $\alpha \leq \theta \leq \beta$.

We cover D_p by a grid of *circular arcs and rays*. The **arcs** are cut from circles centered at the origin, with radii,

$$\Delta r_k, \ 2\,\Delta r_k, \ \ldots, \ m\,\Delta r_k \ \text{ where } \Delta r_k = \frac{a}{m}.$$

The **rays** are given by:

$$\alpha, \ \alpha + \Delta\theta_k, \ \alpha + 2\Delta\theta_k, \ \ldots, \ \alpha + \tilde{m}\Delta\theta_k$$

where $\Delta\theta_k = \frac{(\beta - \alpha)}{\tilde{m}}$. The arcs and rays partition D_p into small patches called "polar rectangles."

We number the polar rectangles that lie inside D_p (the order does not matter), calling their areas:

$$\Delta A_1, \ \Delta A_2, \ \ldots, \ \Delta A_N.$$

We let $(r_k^*, \ \theta_k)$ be any sample point in the polar rectangle whose area is ΔA_k. The product $F(r_k^*, \theta_k^*)\,\Delta A_k$ is the volume of a solid E_k with base area ΔA_k and height $F(r_k^*, \theta_k^*)$, so the sum

$$\sum_{k=1}^{N} F(r_k^*, \theta_k^*)\,\Delta A_k$$

can be viewed as an approximation to the volume $V_p(E)$ of the entire solid. If we now increase the number of subdivisions in such a way that the dimensions of the polar rectangles approach zero, then it seems plausible that the errors in the approximations approach zero, and the exact volume of the solid E is

$$V_p(E) = \lim_{N \to \infty} \sum_{k=1}^{N} F(r_k^*, \theta_k^*)\,\Delta A_k \tag{1}$$

If $F(r, \theta)$ is continuous throughout D_p and has both positive and negative values, then the limit

$$\lim_{N \to \infty} \sum_{k=1}^{N} F(r_k^*, \theta_k^*) \, \Delta A_k \tag{2}$$

represents the net signed volume between the region D_p and the surface $z = f(r, 0)$ (as with double integrals in retangular coordinates). The sums in (2) are called **polar Riemann sums**, and the limit of the polar Riemann sums is denoted by

$$\int \int_{D_p} F(r, \theta) \, dA \;=\; \lim_{N \to \infty} \sum_{k=1}^{N} F(r_k^*, \theta_k^*) \, \Delta A_k \tag{3}$$

which is called the **polar double integral** of $F(r, \theta)$ over D_p. If $F(r, \theta)$ is continuous and nonnegative on D_p, then the volume formula (1) can be expressed as

$$V_p(E) \;=\; \int \int_{D_p} F(r, \theta) \, dA \tag{4}$$

EVALUATING POLAR DOUBLE INTEGRALS

To evaluate the above limit, we first have to write the sum S_N in a way that expresses ΔA_k in terms of Δr_k and $\Delta \theta_k$. For convenience we choose r_k^* to be the average of the radii of the inner and outer arcs bounding the k-th polar rectangle ΔA_k. The radius of the inner arc bounding is then $r_k^* - \frac{1}{2} \Delta r_k$. The radius of the outer arc is $r_k^* + \frac{1}{2} \Delta r_k$.

The area of a wedge-shaped sector of a circle having radius r and angle θ is

$$\text{Area of Sector} = \frac{1}{2} r^2 \theta \quad \text{(A formula you can derive using integration)},$$

as can be seen by multiplying πr^2, the area of the circle, by $\theta / 2\pi$, the fraction of the circle's area contained in the wedge. So the areas of the circular sectors subtended by these arcs at the origin are:

$$\textbf{Inner Radius:} = \frac{1}{2} \left(r_k^* - \frac{1}{2} \Delta r_k \right) \Delta \theta_k$$

$$\textbf{Outer Radius:} = \frac{1}{2} \left(r_k^* + \frac{1}{2} \Delta r_k \right) \Delta \theta_k$$

Therefore,

$$\Delta A_k = \textbf{area of large sector} - \textbf{area of small sector}$$

$$= \frac{1}{2} \Delta \theta_k \left[\left(r_k^* + \frac{1}{2} \Delta r_k \right)^2 - \left(r_k^* - \frac{1}{2} \Delta r_k \right)^2 \right] = \frac{1}{2} \Delta \theta_k \left(2 \, r_k^* \, \Delta r_k \right) = r_k^* \, \Delta r_k \, \Delta \theta_k \tag{5}$$

Combining this result with the sum defining S_N gives

$$S_N = \sum_{k=1}^{N} F(r_k^*, \theta_k) \, \Delta A_k = \sum_{k=1}^{N} F(r_k^*, \theta_k) \, r_k^* \, \Delta r_k \, \Delta \theta_k$$

As $N \to \infty$, and the values Δr_k and $\Delta \theta_k$ approach zero, these sums converge to the double integral

$$V_p(E) = \lim_{N \to \infty} \sum_{k=1}^{N} F(r_k^*, \theta_k) \, r_k^* \, \Delta r_k \, \Delta \theta_k = \iint\limits_{D_p} F(r, \theta) \, r \, dr \, d\theta$$

A version of Fubini's Theorem says that the limit approached by these sums can be evaluated by repeated single integrations with respect to r and θ as

$$\iint\limits_{D_p} F(r, \theta) \, dA = \int_{\theta = \alpha}^{\theta = \beta} \int_{r = g_1(\theta)}^{r = g_2(\theta)} F(r, \theta) \, r \, dr \, d\theta \tag{6}$$

THEOREM. If D_p is a θ-simple polar region whose boundaries are the rays $\theta = \alpha$ and $\theta = \beta$ and the curves $r = g_1(\theta)$ and $r = g_2(\theta)$, and if $F(r, \theta)$ is continuous on D_p, then

$$\iint\limits_{D_p} F(r, \theta) \, dA = \int_{\alpha}^{\beta} \int_{g_1(\theta)}^{g_2(\theta)} F(r, \theta) \, r \, dr \, d\theta \tag{7}$$

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To apply this theorem you will need to be able to find the rays and the curves that form the boundary of the region D_p, since these determine the limits of integration in the iterated integral. This can be done as follows:

HOW TO FIND THE LIMITS OF INTEGRATION FOR DOUBLE INTEGRALS IN POLAR COORDINATES

The procedure for finding limits of integration in rectangular coordinates also works for polar coordinates. When faced with evaluating $\int \int_{D_p} F(r,\theta)\, dA$ over a region D_p in polar coordinates, integrating first with respect to r and then with respect to θ, take the following steps.

STEP 1. Sketch the region D_p of integration and its bounding curves.

STEP 2. Find the r-limits of integration imagining a ray, say M passing through the region D_p in the direction of increasing r. Mark the r values where M enters and leaves D_p. These are your r-limits of integration. They usually depend on the angle θ that the ray M makes with the positive x-axis (that is, the r-limits are usually functions of θ (instead of constants, unless D_p is a circle))

STEP 3. we find the θ-limits of integration by choosing θ-limits that include all rays passing through D_p.

Example 3.1 Evaluate

$$\int\int_{D_p} \sin\theta \, dA$$

where D_p is the region in the first quadrant that is outside the circle $r = 2$ and inside the cardioid $r = 2(1 + \cos\theta)$.

Solution The region D_p is viewed as a θ-simple polar region. Following the last 2 steps outlined above we otain

$$\int\int_{D_p} \sin\theta \, dA = \int_0^{\pi/2} \int_2^{2(1+\cos\theta)} (\sin\theta) r \, dr \, d\theta$$

$$= \int_0^{\pi/2} \left[\frac{1}{2} r^2 \sin\theta \right]_{r=2}^{2(1+\cos\theta)} d\theta$$

$$= 2 \int_0^{\pi/2} [(1 + \cos\theta)^2 \sin\theta - \sin\theta] \, dr$$

$$= 2 \left[-\frac{1}{3}(1 + \cos\theta)^3 + \cos\theta \right]_0^{\pi/2}$$

$$= 2 \left[-\frac{1}{3} - \left(-\frac{5}{3} \right) \right] = \frac{8}{3}$$

Example 3.2 The sphere of radius R centered at the origin (in space) is expressed in rectangular coordinates as $x^2 + y^2 + z^2 = R^2$, and hence its equation in cylindrical coordinates is $r^2 + z^2 = R^2$. Use this equation and a polar double integral to find the volume of the sphere.

Solution In cylindrical coordinates the upper hemisphere is given by the equation

$$z = \sqrt{R^2 - r^2}$$

so the volume enclosed by the entire sphere is

$$V_p(E) = 2 \int\int_{D_p} \sqrt{R^2 - r^2} \, dA$$

where R is the circular region of radius R. Thus,

$$V_p(E) = 2 \int\int_{D_p} \sqrt{R^2 - r^2} \, dA = \int_0^{2\pi} \int_0^R \sqrt{R^2 - r^2}(2r) \, dr \, d\theta$$

$$= \int_0^{2\pi} \left[-\frac{2}{3}(R^2 - r^2)^{3/2} \right]_{r=0}^R d\theta = \int_0^{2\pi} \frac{2}{3} R^3 \, d\theta$$

$$= \left[\frac{2}{3} R^3 \theta \right]_0^{2\pi} = \frac{4}{3} \pi R^3$$

FINDING AREAS USING POLAR DOUBLE INTEGRALS

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Recall from Formula (7) of Section 2 that the area of a region D in the xy-plane can be expressed as

$$\text{area of D} = A(D) = \int\int_{D} dA = \int\int_{D} dA \tag{8}$$

The argument used to derive this result can also be used to show that the formula applies to polar double integrals over regions D_p in polar coordinates.

Example 3.3   Use a polar double integral to find the ear enclosed by the three-petaled rose $r = \sin 3\theta$.

Solution   We will use Formula (8) to calculate the area of the petal D_p in the first quadrant and multiply by three (because the polar region D is made up of 3 petals, each with the same area).

$$A(D) = 3 \cdot A(D_p)$$

$$= 3 \int\int_{D_p} dA = 3 \int_0^{\pi/3} \int_0^{\sin 3\theta} r \, dr \, d\theta$$

$$= \frac{3}{2} \int_0^{\pi/3} \sin^2 3\theta \, d\theta = \frac{3}{4} \int_0^{\pi/3} (1 - \cos 6\theta) \, d\theta$$

$$= \frac{3}{4} \left[\theta - \frac{1}{6} \sin 6\theta\right]_0^{\pi/3} = \frac{1}{4} \pi$$

CONVERTING DOUBLE INTEGRALS FROM RECTANGULAR TO POLAR COORDINATES

Sometimes a double integral that is difficult to evaluate in rectangular coordinates can be evaluated more easily in polar coordinates by making the substitution $x = r \cos\theta$, $y = r \sin\theta$ and expressing the region of integration in polar form; that is, we rewrite the double integral in rectangular coordinates as

$$\int\int_{D} f(x,y) \, dA = \int\int_{D} f(r \cos\theta, r \sin\theta) \, dA = \int\int_{D_p} f(r \cos\theta, r \sin\theta) \, r \, dr \, d\theta \tag{9}$$

Example 3.4   Use polar coordinates to evaluate

$$\int_{-1}^{1} \int_{0}^{\sqrt{1-x^2}} (x^2 + y^2)^{3/2} \, dy \, dx.$$

Solution   In this problem we are starting with an iterated integral in rectangular coordinates rather than a double integral, so before we can make the conversion to polar coordinates we will have to identify the reion of integration. To do this, we observe that for fixed x the y-integration runs from $y = 0$ to $y = \sqrt{1 - x^2}$, which tells us that the lower boundary of the region is the x-axis and the upper boundary is a semicircle of radius 1 centered at the origin. From the x-integration we see that x varies from -1 to 1. In polar coordinates, this is the region swept out as r varies between 0 and 1 and θ varies between 0 and π. Thus,

$$\int_{-1}^{1} \int_{0}^{\sqrt{1-x^2}} (x^2 + y^2)^{3/2} \, dy \, dx = \int\int_{D} (x^2 + y^2)^{3/2} \, dA$$

$$= \int_{0}^{\pi} \int_{0}^{1} (r^3) r \, dr \, d\theta = \int_{0}^{\pi} \frac{1}{5} \, d\theta = \frac{\pi}{5}$$

Tip—*The conversion to polar coordinates worked so nicely in Example 3.4 because the substitution $x = r\cos\theta$, $y = r\sin\theta$ collapsed the $x^2 + y^2$ into the single term r^2, thereby simplifying the integrand. Whenever you see an expression involving $x^2 + y^2$ in the integrand, you should consider the possibility of converting to polar coordinates.*

▷ **Exercise 3.1**   Let $V(E)$ be the volume of the solid E bounded above by the hemisphere $z = \sqrt{1 - r^2}$ and bounded below by the disk enclosed within the circle $r = \sin\theta$. Expressed $V(E)$ as a double integral in polar coordinates.

▷ **Exercise 3.2**   Express the iterated integral as a double integral in polar coordinates

$$\int_{1/\sqrt{2}}^{1} \int_{\sqrt{1-x^2}}^{x} \left(\frac{1}{x^2 + y^2} \right) dy \, dx \text{ (\textbf{Do not evaluate it})}$$

and make a sketch of the region of integration

PARAMETRIC SURFACES; SURFACE AREA

We are familiar with parametric curves in 2-space and 3-space in our earlier studies. In this section we will discuss parametric surfaces in 3-space. As we will see, parametric representations of surfaces are not only important in computer graphics but also allow us to study more general kinds of surfaces than those encountered so far. In your earlier studies, you learned how to find the surface area of a surface of revolution. Our work on parametric surfaces will enable us to derive area formulas for more general kinds surfaces.

PARAMETRIC REPRESENTATION OF SURFACES

We have seen that curves in 3-space can be represnted by three equations involving one parameter, say

$$x = x(t), \quad y = y(t), \quad z = z(t)$$

Surfaces in 3-space can be represented parametrically by three equations involving two parameters, say

$$x = x(u,v), \quad y = y(u,v), \quad z = z(u,v) \tag{1}$$

To understand why such equations represent a surface, think of (u,v) as a point that varies over some region in a uv-plane. If u is held constant, then v is the only varying parameter in (1), and hence these equations represent a curve in 3-space. We call this a **constant u-curve**. Similarly, if v is held constant, then u is the only varying parameter in (1), so again these equations represent a curve in 3-space. We call this a **constant v-curve**. By varying the constants we generate a family of u-curves and a family of v-curves that together form a surface.

Example 4.1 Consider the paraboloid $z = 4 - x^2 - y^2$. One way to parametrize this surface is to take $x = u$ and $y = v$ as the parameters, in which case the surface is represented by the parametric equations

$$x = u, \quad y = v, \quad z = 4 - u^2 - v^2 \tag{2}$$

The constant u-curves correspond to constant x-values and hence appear on the surface as traces (parabolas) parallel to the yz-plane. Similarly, the constant v-curves correspond to constant y-values and hence appears on the surface as traces (parabolas) parallel to the xz-plane.

Example 4.2   The paraboloid $z = 4 - x^2 - y^2$ that was considered in Example 4.1 can also be parameterized by first expressing the equation in cylindrical coordinates. For this purpose, we make the substitution $x = r\cos\theta,\; y = r\sin\theta$, which yields $z = 4 - r^2$. Thus, the paraboloid can be represented parametrically in terms of r and θ as

$$x = r\cos\theta,\;\; y = r\sin\theta,\;\; z = 4 - r^2 \tag{3}$$

The constant r-curves corresond to constant z-values and hence appear on the surface as traces parallel to the xy-plane. The constant θ-curves appears on the surface as traces from vertical planes through the origin at varying angles wth the x-axis.

Using a graphing utility such as matlab 7 the graphs of the paraboloid in Example 4.1 and 4.2 can be generated by running the following m-file:

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% This is a simple matlab m-file to produce            %
% parametric plots of the paraboloid z = 4 - x^2 - y^2   %
% over a rectangle R = {(x,y): -2 <= x <= 2, -2 <= y <= 2}. %
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%
clear all;
close all;
clc;
s = -2:0.01:2; t = -2:0.01:2;
r = linspace(0,2,35); theta = linspace(0, 2*pi,35);
x = @(r,theta) r.*cos(theta); y = @(r,theta) r.*sin(theta); z = @(r,theta) 4-r.^2;
F = @(x,y) 4-x.^2-y.^2;
[X,Y] = meshgrid(s,t);
[R,THETA] = meshgrid(r,theta);
Z = F(X,Y);
subplot(1,2,1)
surf(X,Y,Z);
xlabel('x-axis'), ylabel('y-axis'), zlabel('z-axis');
axis square
title('Example 1');
rotate3d on
subplot(1,2,2);
surf(x(R,THETA),y(R,THETA),z(R,THETA));
xlabel('x-axis'), ylabel('y-axis'), zlabel('z-axis');
axis square
title('Example 2')
rotate3d on
```

Example 4.3 One way to generate the sphere $x^2 + y^2 + z^2 = 1$ with a graphing utility (like matlab) is to graph the upper and lower hemispheres

$$z = \sqrt{1 - x^2 - y^2} \quad \text{and} \quad z = -\sqrt{1 - x^2 - y^2}$$

on the same screen. However, this usually produces a fragmented sphere because roundoff error sporadically produces negatives values inside the radical when $1 - x^2 - y^2$ is near zero. A better graph can be generated by first expressing the sphere in spherical coordinates as $\rho = 1$. Parametric equations of the sphere is given as

$$x = \sin\phi\cos\theta, \quad y = \sin\phi\sin\theta, \quad z = \cos\phi$$

with parameters θ and ϕ where $0 \leq \theta \leq 2\pi$ and $0 \leq \phi \leq \pi$. In the language of cartographers, the constant ϕ-curves are the **lines of latitude** and the constant *theta*-curves are the **lines of longitude**.

To generate a sphere in matlab you can type the following commands at the command prompt

```
>> clear all;
>> close all;
>> clc;
>> phi = linspace(0,pi,40);
>> theta = linspace(0,2*pi,40);
>> x = @(phi,theta) sin(phi).* cos(theta);
>> y = @(phi,theta) sin(phi).* sin(theta);
>> z = @(phi,theta) cos(phi);
>> [Phi,Theta] = meshgrid(phi,theta);
>> surf(x(Phi,Theta),y(Phi,Theta),z(Phi,Theta));
>> colormap gray
>> axis square
>> rotate3d on
>> xlabel('x-axis'), ylabel('y-axis'), zlabel('z-axis');
```

Example 4.4 Find parametric equations for the portion of the right circular cylinder

$$x^2 + z^2 = 9 \ \text{ and } \ 0 \leq y \leq 5$$

in terms of the parameters u and v. The parameter u is the y-coordinate of a point $P(x, y, z)$ on the surface, and v is the angle.

Solution The radius of the cylinder is 3 units, so it is evident that $y = u, \quad x = 3\cos v, \quad z = 3\sin v$. Thus, the surface can be represented parametrically as

$$x = 3\cos v, \quad y = u, \quad z = 3\sin v.$$

To obtain the portion of the surface from $y = 0$ to $y = 5$, we let the parameter u vary over the interval $0 \leq u \leq 5$, and to ensure that the entire lateral surface is covered, we let the parameter v vary over the interval $0 \leq v \leq 2\pi$. A computer generated graph of the surface in which u and v vary over these intervals will show that the constant u-curves appear as circular traces parallel to the xz-plane, and the constant r-curves appear as lines segments parallel to the y-axis.

Representing surfaces of revolution parametrically

The basic idea of Example 4 above can be adapted to obtain parametric equations for surfaces revolution. For example, suppose that we wanted to find parametric equations for the surface generated by revolving the plane curve $y = f(x)$ about the x-axis. Such a surface can be represented parametrically as

$$x = u, \quad y = f(u) \cos v, \quad z = f(u) \sin v \tag{4}$$

where v is the angle.

Example 4.5   Find parametric equations for the surface generated by revolving the curve $y = \frac{1}{x}$ about the x-axis.

Solution   From (4) this surface can be represented parametrically as

$$x = u, \quad y = \frac{1}{u} \cos v, \quad z = \frac{1}{u} \sin v.$$

A portion graph of this surface of revolution can generated using matlab 7 by typing the following matlab commands at the commandline or typing the commands in an m-file and then running the m-file.

```
clear all;
close all;
clc;
u = linspace(0.5,10,40);
v = linspace(0, 2*pi,40);
f = @(u)1./u;
x = @(u,v)u;
y = @(u,v)f(u).*cos(v);
z = @(u,v)f(u).*sin(v);
[U,V] = meshgrid(u,v);
surf(x(U,V),y(U,V),z(U,V));
colormap gray;
axis square;
xlabel('x-axis'), ylabel('y-axis'), zlabel('z-axis')
```

VECTOR-VALUED FUNCTIONS OF TWO VARIABLES

Recall that the parametric equations

$$x = x(t), \quad y = y(t), \quad z = z(t)$$

can be expressed in vector form as

$$\mathbf{r} = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} + z(t)\,\mathbf{k}$$

where $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$ is the radius vector and $\mathbf{r}(t) = x(t)\,\mathbf{i} + z(t)\,\mathbf{j} + z(t)\,\mathbf{k}$ is a vector valued function of one variable. Similarly, the parametric equations

$$x = x(u, v), \quad y = y(u, v), \quad z = z(u, v)$$

can be expressed in vector form as

$$\mathbf{r} = x(u, v)\,\mathbf{i} + y(u, v)\,\mathbf{j} + z(u, v)\,\mathbf{k}.$$

Here the function $\mathbf{r}(u, v) = x(u, v)\,\mathbf{i} + y(u, v)\,\mathbf{j} + z(u, v)\,\mathbf{k}$ is a **vector-valued function of two variables**. We define the **graph of** $\mathbf{r}(u, v)$ to be the graph of the corresponding parametric equations. Geometrically, we can view $\mathbf{r}$ as a vector from the origin to a point (x, y, z) that moves over the surface $\mathbf{r} = \mathbf{r}(u, v)$ as u and v vary. As with vector-valued functions of one variables, we say that $\mathbf{r}(u, v)$ is **continuous** if each component is continuous.

Example 4.6   The paraboloid of Example 1 was expressed parametrically as

$$x = u, \quad y = v, \quad z = 4 - u^2 - v^2.$$

These equations can be expressed in vector form as

$$\mathbf{r} = \mathbf{r}(u, v) = u\,\mathbf{i} + v\,\mathbf{j} + (4 - u^2 - v^2)\,\mathbf{k}$$

PARTIAL DERIVATIVES OF VECTOR-VALUED FUNCTIONS

Partial derivatives of vector-valued functions of two variables are obtained by taking partial derivatives of the components. For example, if

$$\mathbf{r}(u, v) = x(u, v)\,\mathbf{i} + y(u, v)\,\mathbf{j} + z(u, v)\,\mathbf{k}$$

then

$$\frac{\partial \mathbf{r}}{\partial u} = \frac{\partial x}{\partial u}\,\mathbf{i} + \frac{\partial y}{\partial u}\,\mathbf{j} + \frac{\partial z}{\partial u}\,\mathbf{k}$$
$$\frac{\partial \mathbf{r}}{\partial v} = \frac{\partial x}{\partial v}\,\mathbf{i} + \frac{\partial y}{\partial v}\,\mathbf{j} + \frac{\partial z}{\partial v}\,\mathbf{k}$$

These derivatives can also be written as $\mathbf{r}_u$ and $\mathbf{r}_v$ or $\mathbf{r}_u(u, v)$ and $\mathbf{r}_v(u, v)$ and can be expressed as limits

$$\frac{\partial \mathbf{r}}{\partial u} = \lim_{\Delta u \to 0} \frac{\mathbf{r}(u + \Delta u, \ v) - \mathbf{r}(u,v)}{\Delta u} = \lim_{w \to u} \frac{\mathbf{r}(w,v) - \mathbf{r}(u,v)}{w - u} \tag{5}$$

$$\frac{\partial \mathbf{r}}{\partial v} = \lim_{\Delta v \to 0} \frac{\mathbf{r}(u, v + \Delta v) - \mathbf{r}(u,v)}{\Delta v} = \lim_{w \to v} \frac{\mathbf{r}(u,w) - \mathbf{r}(u,v)}{w - v} \tag{6}$$

Example 4.7   Find the partial derivatives of the vector-valued function $\mathbf{r}$ in Example 6.

Solution

$$\frac{\partial \mathbf{r}}{\partial u} = \frac{\partial}{\partial u} \left[u\,\mathbf{i} + v\,\mathbf{j} + (4 - u^2 - v^2)\,\mathbf{k} \right] = \mathbf{i} - 2u\,\mathbf{k}$$

$$\frac{\partial \mathbf{r}}{\partial v} = \frac{\partial}{\partial v} \left[u\,\mathbf{i} + v\,\mathbf{j} + (4 - u^2 - v^2)\,\mathbf{k} \right] = \mathbf{j} - 2v\,\mathbf{k}$$

TANGENT PLANES TO PARAMETRIC SURFACES

Our next objective is to show how to find tangent planes to parametric surfaces. Let σ denote a parametric surface in 3-space, with $P_0(x_0, y_0, z_0)$ a point on σ. We will say that a plane is **tangent** to σ at P_0 provided a line through P_0 lies in the plane if and only if it is a tangent line at P_0 to a curve on σ. We know that if $z = f(x,y)$, then the graph of f has a tangent plane at a point if f is differentiable at that point (a proof of this fact is beyond the scope of this discussion). So we will simply assume the existence of tangent planes at points of interest and focus on finding their equations.

Suppose that the parametric surface σ is the graph of the vector-valued function $\mathbf{r}(u,v)$ and that we are interested in the tangent plane at the point (x_0, y_0, z_0) on the surface that corresponds to the parameter $u = u_0$ and $v = v_0$; that is,

$$\mathbf{r}(u_0, v_0) = x_0\,\mathbf{i} + y_0\,\mathbf{j} + z_0\,\mathbf{k}$$

If $v = v_0$ is kept fixed and u is allowed to vary, then $\mathbf{r}(u, v_0)$ is a vector-valued function of one variable whose graph is the constant v-curve through the point (u_0, v_0); similarly, if $u = u_0$ is kept fixed and v is allowed to vary, then $\mathbf{r}(u_0, v)$ is a vector-valued function of one variable whose graph is the constant u-curve through the point (u_0, v_0). Moreover, it follows from the geometric interpretation of the derivative that if $\partial \mathbf{r}/\partial u \neq \mathbf{0}$ at (u_0, v_0), then this vector is tangent to the constant v-curve through (u_0, v_0); and if $\partial \mathbf{r}/\partial v \neq \mathbf{0}$ at (u_0, v_0), then this vector is tangent to the constant u-curve through (u_0, v_0). Thus, if $\partial \mathbf{r}/\partial u \times \partial \mathbf{r}/\partial v \neq \mathbf{0}$ at (u_0, v_0), then the vector

$$\mathbf{r}_v \times \mathbf{r}_v = \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \end{vmatrix} \tag{7}$$

is orthogonal to both tangent vectors at the point (u_0, v_0) and hence is normal to the tangent plane and the surface at this. Accordingly, we make the following definition.

Definition. *If a parametric surface S is the graph of vector-valued function of two variables $\mathbf{r}(u, v)$, and if $\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \neq \mathbf{0}$ at a point P_0 on S, then the* **principal unit normal vector** *to the surface at P_0 is denoted by $\mathbf{n}$ or $\mathbf{n}(u, v)$ and is defined as*

$$\mathbf{n} = \mathbf{n}(u, v) = \frac{\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}}{\left\| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right\|} \tag{8}$$

Example 4.8   Find an equation of the tangent plane to the parametric surface

$$x = uv, \quad y = u, \quad z = v^2$$

at the point where $u = 2$ and $v = -1$. This surface, called *Whitney's umbrella*, is an example of a self-intersecting parametric surface.

Solution   We start by writing the equations in the vector form

$$\mathbf{r} = \mathbf{r}(u, v) = uv\,\mathbf{i} + u\,\mathbf{j} + v^2\,\mathbf{k}.$$

The first-partial derivatives of $\mathbf{r}$ are

$$\frac{\partial \mathbf{r}}{\partial u}(u, v) = v\,\mathbf{i} + \mathbf{j}$$

$$\frac{\partial \mathbf{r}}{\partial v}(u, v) = u\,\mathbf{i} + 2v\,\mathbf{k}$$

and at $u = 2$ and $v = -1$ these partial derivatives are

$$\frac{\partial \mathbf{r}}{\partial u}(2, -1) = -\mathbf{i} + \mathbf{j}$$

$$\frac{\partial \mathbf{r}}{\partial v}(2, -1) = 2\,\mathbf{i} - 2\,\mathbf{k}.$$

Thus, from (7) and (8)

$$\mathbf{r}_u(2, -1) \times \mathbf{r}_v(2, -1) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 1 & 0 \\ 2 & 0 & -2 \end{vmatrix} = -2\,\mathbf{i} - 2\,\mathbf{j} - 2\,\mathbf{k} = -2\,(\mathbf{i} + \mathbf{j} + \mathbf{k}).$$

since any normal will suffice to find the tangent plane, it makes sense to multiply this vector by $-\frac{1}{2}$ and use the simpler normal $\mathbf{i} + \mathbf{j} + \mathbf{k}$. It follows from the given parametric equations that the point on the surface corresponding to $u = 2$ and $v = -1$ is $(-2, 2, 1)$, so the tangent plane at this point can be expressed in point-normal form as

$$(x + 2) + (y - 2) + (z - 1) = 0 \quad \text{or} \quad x + y + z = 1$$

Example 4.9   The sphere $x^2 + y^2 + z^2 = a^2$ with radius a can be expressed in spherical coordinates as $\rho = a$, and the spherical-to-rectangular conversion formulas can then be used to express the sphere as the graph of the vector-valued function

$$\mathbf{r} = \mathbf{r}(\phi, \theta) = a \sin\phi \cos\theta\,\mathbf{i} + a \sin\phi \sin\theta\,\mathbf{j} + a \cos\phi\,\mathbf{k}$$

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where $0 \leq \phi \leq \pi$ and $0 \leq \theta \leq 2\pi$ (verify). Use this function to show that the radius vector is normal to the tangent plane at each point on the sphere.

Solution We will show that at each point of the sphere the unit normal vector $\mathbf{n}$ is a scalar multiple of $\mathbf{r}$ (and hence is parallel to $\mathbf{r}$). We have

$$\frac{\partial \mathbf{r}}{\partial \phi} \times \frac{\partial \mathbf{r}}{\partial \theta} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial x}{\partial \phi} & \frac{\partial y}{\partial \phi} & \frac{\partial z}{\partial \phi} \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} & \frac{\partial z}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a \cos\phi \cos\theta & a \cos\phi \sin\theta & -a \sin\phi \\ -a \sin\phi \sin\theta & a \sin\phi \cos\theta & 0 \end{vmatrix}$$

$$= a^2 \sin^2\phi \cos\theta\, \mathbf{i} + a^2 \sin^2\phi \sin\theta\, \mathbf{j} + a^2 \sin\phi \cos\phi\, \mathbf{k}$$

and hence

$$\left\| \frac{\partial \mathbf{r}}{\partial \phi} \times \frac{\partial \mathbf{r}}{\partial \theta} \right\| = \sqrt{a^4 \sin^4\phi \cos^2\theta + a^4 \sin^4\phi \sin^2\theta + a^4 \sin^4\phi \cos^2\phi}$$

$$= \sqrt{a^4 \sin^4\phi + a^4 \sin^2\phi \cos^2\phi}$$

$$= a^2 \sqrt{\sin^2\phi} = a^2 |\sin\phi| = a^2 \sin\phi.$$

For $\phi \neq 0$ or π, it follows from (8) that

$$\mathbf{n} = \mathbf{n}(\phi,\theta) = \sin\phi \cos\theta\, \mathbf{i} + \sin\phi \sin\theta\, \mathbf{j} + \cos\phi\, \mathbf{k} = \frac{1}{a} \mathbf{r}.$$

Furthermore, the tangent planes at $\phi = 0$ or $\phi = \pi$ are horizontal to which $\mathbf{r} = \pm a\,\mathbf{k}$ is clearly normal.

SURFACE AREA OF PARAMETRIC SURFACES

In your earlier studies of calculus you learned formulas for finding the surface area of a surface of revolution. We will now obtain a formula for the surface area $A(S)$ of a parametric surface S and from that formula we will then derive a formula for the surface area of a surface of the form $z = f(x, y)$.

Let S be a parametric surface whose vector equation is

$$\mathbf{r} = \mathbf{r}(u, v) = x(u, v)\mathbf{i} + y(u, v)\mathbf{j} + z(u, v)\mathbf{k}.$$

We will say that S is a **smooth parametric surface** on a region R of the uv-plane if $\partial\mathbf{r}/\partial u$ and $\partial\mathbf{r}/\partial v$ are continuous on R and $\partial\mathbf{r}/\partial u \times \partial\mathbf{r}/\partial v \neq \mathbf{0}$ on R. Geometrically, this means that S has a principal unit normal vector (and hence a tangent plane) for all (u, v) in R and $\mathbf{n} = \mathbf{n}(u, v)$ is a continuous function on R. Thus, on a smooth parametric surface the unit normal vector $\mathbf{n}$ varies continuously and has no abrupt changes in direction. We will derive a surface area formula for parametric surfaces that have no self-intersections and are smooth on region R, with the possible exception that $\partial\mathbf{r}/\partial u \times \partial\mathbf{r}/\partial v$ may equal $\mathbf{0}$ on the boundary of R.

We begin by subdividing R into rectangular regions by lines parallel to the u- and v-axes and discarding any nonrectangular portions that contain points of the boundary.

Assume that there are N rectangles, and let R_k denote the kth rectangle. Let (u_k, v_k) be the lower left corner of R_k, and assume that R_k has area $\Delta A_k = \Delta u_k \Delta v_k$, where Δu_k and Δv_k are the dimensions of R_k. The image of R_k will be some *curvilinear patch* S_k on the surface S that has a corner at $\mathbf{r}(u_k, v_k)$; denote the area of this patch by ΔS_k. The two edges of the patch S_k that meet at $\mathbf{r}(u_k, v_k)$ can be approximated by the "secant" vectors

$$\mathbf{r}(u_k + \Delta u_k, v_k) - \mathbf{r}(u_k, v_k)$$

$$\mathbf{r}(u_k, v_k + \Delta v_k) - \mathbf{r}(u_k, v_k)$$

and hence the area of S_k can be approximated by the area of the parallelogram determined by these vectors. However, it follows from Formulas (5) and (6) that if Δu_k and Δv_k are small, then these secant vectors can in turn be approximated by the tangent vectors

$$\frac{\partial\mathbf{r}}{\partial u}\Delta u_k \quad \text{and} \quad \frac{\partial\mathbf{r}}{\partial v}\Delta u_k$$

where the partial derivatives are evaluated at (u_k, v_k). Thus, the area of the patch S_k can be approximated by the area of the parallelogram determined these vectors; that is,

$$\Delta S_k \approx \left\|\frac{\partial\mathbf{r}}{\partial u}\Delta u_k \times \frac{\partial\mathbf{r}}{\partial v}\Delta v_k\right\| = \left\|\frac{\partial\mathbf{r}}{\partial u} \times \frac{\partial\mathbf{r}}{\partial v}\right\|\Delta u_k \Delta v_k = \left\|\frac{\partial\mathbf{r}}{\partial u} \times \frac{\partial\mathbf{r}}{\partial v}\right\|\Delta A_k \tag{9}$$

It follows that the surface area $A(S)$ of the entire surface S can be approximated as

$$A(S) \approx \sum_{k=1}^{N}\Delta S_k = \sum_{k=1}^{N}\left\|\frac{\partial\mathbf{r}}{\partial u} \times \frac{\partial\mathbf{r}}{\partial v}\right\|\Delta A_k$$

Thus, if we assume that the errors in the approximations approach zero as N increases in such a way that the dimensions of the rectangles approach zero, the it is plausible that the exact value of $A(S)$ is

$$A(S) = \lim_{N\to\infty}\sum_{k=1}^{N}\left\|\frac{\partial\mathbf{r}}{\partial u} \times \frac{\partial\mathbf{r}}{\partial v}\right\|\Delta A_k \quad \text{or equivalently,} \quad \int\!\!\int_{R}\left\|\frac{\partial\mathbf{r}}{\partial u} \times \frac{\partial\mathbf{r}}{\partial v}\right\|dA \tag{10}$$

Example 4.10   It follows from (4) that the parametric equations

$$x = x(u,v) = u, \ y = y(u,v) = u \cos v, \ \ z = z(u,v) = u \sin v$$

represent the cone that results when the line $y = x$ in the xy-plane is revolved about the x-axis. Use Formula (10) to find the surface area of that portion of the cone for which $0 \le u \le 2$ and $0 \le v \le 2\pi$.

Solution   The surface can be expressed in vector form as

$$\mathbf{r} = \mathbf{r}(u,v) = u\,\mathbf{i} + u\,\cos v\,\mathbf{j} + u\,\sin v\mathbf{k} \ \ (0 \le u \le 2, \ 0 \le v \le 2\pi)$$

Thus,

$$\frac{\partial \mathbf{r}}{\partial u} = \mathbf{i} + \cos v\mathbf{j} + \sin v\mathbf{k}$$

$$\frac{\partial \mathbf{r}}{\partial v} = -u \sin v\mathbf{j} + u \cos v\mathbf{k}$$

$$\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & \cos v & \sin v \\ 0 & -u \sin v & u \cos v \end{vmatrix} = u\,\mathbf{i} - u \cos v\mathbf{j} - u \sin v\mathbf{k}$$

$$\left\| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right\| = \sqrt{u^2 + (-u \cos v)^2 + (-u \sin v)^2} = |u|\,\sqrt{2} = u\,\sqrt{2}$$

Thus, from (10)

$$A(S) = \iint\limits_{R} \left\| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right\| dA = \int_0^{2\pi} \int_0^2 \sqrt{2}\,u\,du\,dv = 2\sqrt{2} \int_0^{2\pi} dv = 4\,\pi\,\sqrt{2} \ \text{(unit of area)}$$

SURFACE AREA OF SURFACES OF THE FORM $z = f(x, y)$

In the case where S is a surface of the form $z = f(x, y)$, we can take $x = u$ and $y = v$ as parameters and express the surface parametrically as

$$x = u, \quad y = v, \quad z = f(u, v)$$

or in vector form as

$$\mathbf{r} = \mathbf{r}(u, v) = u\,\mathbf{i} + v\,\mathbf{j} + f(u, v)\,\mathbf{k}$$

Thus,

$$\frac{\partial \mathbf{r}}{\partial u} = \mathbf{i} + \frac{\partial f}{\partial u}\mathbf{k} = \mathbf{i} + \frac{\partial z}{\partial x}\mathbf{k}$$

$$\frac{\partial \mathbf{r}}{\partial v} = \mathbf{j} + \frac{\partial z}{\partial v}\mathbf{k} = \mathbf{j} + \frac{\partial z}{\partial y}\mathbf{k}$$

$$\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & \frac{\partial z}{\partial x} \\ 0 & 1 & \frac{\partial z}{\partial y} \end{vmatrix} = -\frac{\partial z}{\partial x}\mathbf{i} - \frac{\partial z}{\partial y}\mathbf{j} + \mathbf{k}$$

$$\left\| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right\| = \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}$$

Thus, it follows from (10) that

$$A(S) = \int\int_R \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}\; dA \tag{11}$$

Example 4.11   Find the surface area of that portion of the surface $z = \sqrt{4 - x^2}$ that lies above the rectangle R in the xy-plane whose coordinates satisfy $0 \leq x \leq 1$ and $0 \leq y \leq 4$.

Solution   The surface is a portion of the cylinder $x^2 + z^2 = 4$. It follows from (11) that the surface area is

$$A(S) = \int\int_R \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}\; dA$$

$$= \int\int_R \sqrt{\left(-\frac{x}{\sqrt{4 - x^2}}\right)^2 + 0 + 1}\; dA = \int_0^4 \int_0^1 \frac{2}{\sqrt{4 - x^2}}\; dx\, dy$$

$$= 2\int_0^4 \left[\sin^{-1}\left(\frac{1}{2}x\right)\right]_{x=0}^1 dy = 2\int_0^4 \frac{\pi}{6}\, dy = \frac{4}{3}\pi \text{ (unit of area)}$$

Example 4.12   Find the surface area of the portion of the paraboloid $z = x^2 + y^2$ below the plane $z = 1$.

Solution   The surface $z = x^2 + y^2$ is a circular paraboloid. The trace of the paraboloid in the plane $z = 1$ projects onto the circle $x^2 + y^2 = 1$ in the xy-plane, and the portion of the paraboloid that lies below the plane $z = 1$ project onto the region R that is enclosed by this circle. Thus, it follows from (11) that the surface area is

$$A(S) = \int\int_R \sqrt{4x^2 + 4y^2 + 1}\, dA$$

The expression $4x^2 + 4y^2 + 1 = 4(x^2 + y^2) + 1$ in the integrand suggests that we evaluate the integral in polar coordinates. In accordance with Formula (9) of the previous section, we substitute $x = r\cos\theta$ and $y = r\sin\theta$ in the integrand, replace dA by $r\,dr\,d\theta$, and find the limits of integration by expressing the region R in polar coordinates. This yields

$$S = \int_0^{2\pi}\int_0^1 \sqrt{4\,r^2 + 1}\, r\,dr\,d\theta = \int_0^{2\pi}\left[\frac{1}{12}\,(4\,r^2 + 1)^{3/2}\right]_{r=0}^1 d\theta$$

$$= \int_0^{2\pi}\frac{1}{12}\,(5\sqrt{5} - 1)\,d\theta = \frac{1}{6}\,\pi\,(5\sqrt{5} - 1)\ (\text{unit of area})$$

TRIPLE INTEGRALS

In the preceding sections we defineed and discussed properties of double integrals for functions of two variables. In this section we will define triple integrals for functions of three variables.

Definition of a Triple Integral

A single integral of a function $f(x)$ where x is confined to a finite closed interval on the x-axis, and a double integral of a function $f(x, y)$ where (x, y) is confined to a finite closed region R in the xy-plane.

Our first goal in this section is to define what is meant by a *triple integral of $f(x, y, z)$* where (x, y, z) is confined to a closed solid region E in an xyz-coordinate system. To ensure that E does not extend indefinitely in some direction, we will assume that it can be enclosed in a suitably large box B whose sides are parallel to the coordinate planes. In this case we say that E is a **finite solid**.

To define the triple integral of $f(x, y, z)$ over E, we extended the notion of a double integral over a region $D \subseteq R^2$ to 3-dimensional real space (R^3). Let us consider the following simple set $B \subseteq R^3$ defined as:

$$B = \text{Box} = \{(x, y, z) \in R^3 |\ \alpha \leq x \leq \beta,\ \gamma \leq y \leq \delta,\ \sigma \leq z \leq \tau\} \ \text{\{called a \textbf{solid rectangular box}\}}.$$

We next form a partition of the box B by dividing it into little subboxes by partitioning the interval $[\alpha, \beta]$ into L sub-intervals of equal width Δx, and dividing the interval $[\gamma, \delta]$ into M sub-intervals of equal width Δy and similarly dividing the interval $[\sigma, \tau]$ into N sub-interval of equal width Δz. The planes through the endpoints of these sub-intervals parallel to the coordinate planes divides B into LMN sub-boxes

$$B_{ijk} = [x_{i-1}, x_i] \times [y_{j-1}, y_j] \times [z_{k-1}, z_k] \ \text{ with } 1 \leq i \leq L,\ 1 \leq j \leq M, 1 \leq k \leq N.$$

Each sub-box B_{ijk} has a volume $\Delta V_{ijk} = \Delta x_i\, \Delta y_j\, \Delta z_k$. If $f(x, y, z)$ is a function of three variables defined on B, then we form the **Riemann triple sum**

$$\sum_{k=1}^{N} \sum_{j=1}^{M} \sum_{i=1}^{L} f(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*)\, \Delta V_{ijk}$$

where $(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*)$ is a sample point in the sub-box B_{ijk} in the ijk-th position of the partition of B. Now we define the **Triple integral over the box** B as the limiting value (if it exists) of the Riemann triple sums above; i.e., The **triple integral** of f over the box B is given by

$$\iiint\limits_{B} f(x, y, z)\ dV = \lim_{L, M, N \to \infty} \sum_{k=1}^{N} \sum_{j=1}^{M} \sum_{i=1}^{L} f(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*)\, \Delta V_{ijk}.$$

If $f(x, y, z)$ is continuous on the box B then the sample point can be chosen to be any point (x_k^*, y_k^*, z_k^*) in the k-th sub-box B_k (assuming we have N sub-boxes) and so the integral above can be simply written as

$$\iiint\limits_{B} f(x, y, z)\ dV = \lim_{N \to \infty} \sum_{k=1}^{N} f(x_k^*, y_k^*, z_k^*)\, \Delta V_k \tag{1}.$$

Properties Of Triple Integrals

Triple integrals enjoy many properties of single and double integerals:

$$\int\int\int_E \alpha\, f(x,y,z)\; dV \;=\; \alpha \int\int\int_E f(x,y,z)\; dV \quad (\alpha \text{ is a constant})$$

$$\int\int\int_E (f(x,y,z)\; dV \pm g(x,y,z))\; dV \;=\; \int\int\int_E f(x,y,z)\; dV \pm \int\int\int_E g(x,y,z)\; dV$$

Moreover, if the region E is subdivided into two subregions E_1 and E_2, then

$$\int\int\int_E f(x,y,z)\; dV \;=\; \int\int\int_{E_1} f(x,y,z)\; dV \;+\; \int\int\int_{E_2} f(x,y,z)\; dV$$

We omit the proofs.

Evaluating Triple Integrals Over Rectangular Boxes

Just as a double integral can be evaluated by two successive single integrations, so a triple integral can be evaluated by three successive integrations. The following theorem, which we state without proof.

Theorem (Fubini's Theorem Extended to R^3). *If $f(x,y,z)$ is continuous on the solid rectangular box B defined by*
$$B \;=\; \text{Box} \;=\; \{(x,y,z) \in R^3 \mid \alpha \leq x \leq \beta,\; \gamma \leq y \leq \delta,\; \sigma \leq z \leq \tau\},$$
then

$$\int\int\int_B f(x,y,z)\; dV \;=\; \int_\alpha^\beta \int_\gamma^\delta \int_\sigma^\tau f(x,y,z)\, dz\, dy\, dx \;=\; \int_\gamma^\delta \int_\alpha^\beta \int_\sigma^\tau f(x,y,z)\, dz\, dx\, dy$$

$$=\; \int_\alpha^\beta \int_\sigma^\tau \int_\gamma^\delta f(x,y,z)\, dy\, dz\, dx \;=\; \int_\sigma^\tau \int_\alpha^\beta \int_\gamma^\delta f(x,y,z)\, dy\, dx\, dz \qquad (2)$$

$$=\; \int_\sigma^\tau \int_\gamma^\delta \int_\alpha^\beta f(x,y,z)\, dx\, dy\, dz \;=\; \int_\gamma^\delta \int_\sigma^\tau \int_\alpha^\beta f(x,y,z)\, dx\, dz\, dy$$

There is a total of six possible orders of the above iterated triple integral.

Example 5.1   *Evaluate the triple integral*

$$\int \int_B \int 12\, x\, y^2\, z^3\; dV$$

over the rectangular box B given by

$$B \;=\; \{(x, y, z) \in R^3 \,|\, -1 \le x \le 2,\; 0 \le y \le 3,\; 0 \le z \le 2\}$$

Solution   Of the six possible iterated integrals we might use, we will choose the first one in (2). Thus, we will first integrate with respect to z, holding x and y fixed, then with respect to y, holding x fixed, and finally with respect to x.

$$\int \int_B \int 12\, x\, y^2\, z^3\; dV \;=\; \int_{-1}^{2} \int_{0}^{3} \int_{0}^{2} 12\, x\, y^2\, z^3\, dz\, dy\, dx$$

$$= \int_{-1}^{2} \int_{0}^{3} [3xy^2 z^4]_{z=0}^{2}\, dy\, dx \;=\; \int_{-1}^{2} \int_{0}^{3} 48xy^2\, dy\, dx$$

$$= \int_{-1}^{2} [16xy^3]_{y=0}^{3}\, dx \;=\; \int_{-1}^{2} 432x\, dx$$

$$= 216\, x^2]_{-1}^{2} \;=\; 648$$

▷ **Exercise 5.1**   Try the five other orders to see if you obtain the same result.

TRIPLE INTEGRALS OVER MORE GENERAL BOUNDED REGIONS (SOLIDS) E OF R^3

We proceed in much the same way as we did above by enclosing the solid region E inside the solid box B, thus $E \subseteq B$. We then define a new function $g(x, y, z)$ which agrees with $f(x, y, z)$ over the solid region B but is 0 for points over B that are not on E; that is,

$$g(x, y, z) = \begin{cases} f(x, y, z) & \text{if} (x, y, z) \in E \\ \\ 0 & \text{if} (x, y, z) \in B \setminus E \end{cases}$$

The new function $g(x, y, z)$ is called the extension of $f(x, y, z)$ on E to B. Thus, we have

$$\iiint\limits_{E} f(x, y, z) \, dV = \iiint\limits_{B} g(x, y, z) \, dV.$$

Such an integral exists only if $f(x, y, z)$ is a continuous function and ∂E (the boundary of E) is "reasonably smooth." By this, we mean that we can parametrically represent the boundary surface of E. The triple integral above now have the usual properties similar to those of double integrals.

Naturally, of course, these solid regions E of 3-space can have a variety of shapes, so we restrict our attention to continuous functions $f(x, y, z)$ and to certain types of simple regions.

TRIPLE INTEGRALS OVER **Type I solid Regions** IN SPACE

Definition 5.1 [TYPE I SOLID REGION] We shall say that a finite solid region E_1 of space is a **Type I solid** if it lies between the graphs of two continuous functions of two variables x and y, that is,

$$E_1 = \{(x, y, z) | \ (x, y) \in D_{xy}, \ u_1(x, y) \leq z \leq u_2(x, y)\}$$

where D_{xy} is the projection of E_1 in the xy-plane.

Theorem 5.1 Let E be a **Type I solid region** in space, then we can express the triple integral $\int \int \int_E f(x, y, z) \; dV$ as

$$\int \int \int_E f(x, y, z) \; dV = \int \int_{D_{xy}} \left[\int_{u_1(x,y)}^{u_2(x,y)} f(x, y, z) \; dz \right] dA \tag{3}$$

where D_{xy} is the projection of E onto the xy-plane.

In particular, if D_{xy} is an X-**Simple** plane region, then we describe E as

$$E_1 = \{(x, y, z) \mid \alpha \leq x \leq \beta, \; g_1(x) \leq y \leq g_2(x), \; u_1(x, y) \leq z \leq u_2(x, y)\}$$

and the above integral in (3) becomes

$$\int \int \int_E f(x, y, z) \; dV = \int_\alpha^\beta \int_{g_1(x)}^{g_2(x)} \left[\int_{u_1(x,y)}^{u_2(x,y)} f(x, y, z) \; dz \right] dy \, dx$$

On the other hand if D_{xy} is a Y-**Simple** planar region, then we describe E as

$$E = \{(x, y, z) \in R^3 \mid h_1(y) \leq x \leq h_2(y), \; \gamma \leq y \leq \delta, \; u_1(x, y) \leq z \leq u_2(x, y)\}$$

and the integral in (3) becomes

$$\int \int \int_E f(x, y, z) \; dV = \int_\gamma^\delta \int_{h_1(y)}^{h_2(y)} \left[\int_{u_1(x,y)}^{u_2(x,y)} f(x, y, z) \; dz \right] dx \, dy$$

Example 5.2 Let E be the wedge in the first octant that is cut from the cylindrical solid $y^2 + z^2 \leq 1$ by the planes $y = x$ and $x = 0$. Evaluate

$$\int \int \int_E z \; dV$$

Solution The solid E and its projection D_{xy} on the xy-plane can be easily sketched (but not shown here). The upper surface of the solid is formed by the cylinder and the lower surface by the xy-plane. Since the portion of the cylinder $y^2 + z^2 = 1$ that lies above the xy-plane has the equation $z = \sqrt{1 - y^2}$, and the xy-plane has the equation $z = 0$, it follows from (3) that

$$\int \int \int_E z \; dV = \int \int_{D_{xy}} \left[\int_0^{\sqrt{1-y^2}} z \; dz \right] dA \tag{4}$$

For the double integral over D_{xy}, the x- and y-integrations can be performed in either order, since D_{xy} is both an X-simple and a Y-simple planar region. We will integrate with respect to x first. With this choice, (4) yields

$$\int \int \int_E z \; dV = \int_0^1 \int_0^y \int_0^{\sqrt{1-y^2}} z \; dz \, dx \, dy = \int_0^1 \int_0^y \frac{1}{2} z^2 \Big]_{z=0}^{\sqrt{1-y^2}} dx \, dy$$

$$= \int_0^1 \int_0^y \frac{1}{2} (1 - y^2) \, dx \, dy = \frac{1}{2} \int_0^1 (1 - y^2) \, x \Big]_{x=0}^y dy$$

$$= \frac{1}{2} \int_0^1 (y - y^3) \, dy = \frac{1}{2} \left[\frac{1}{2} y^2 - \frac{1}{4} y^4 \right]_0^1 = \frac{1}{8}$$

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VOLUME CALCULATED AS A TRIPLE INTEGRAL

Triple integrals have many physical interpretations, some of which we will consider in the following sections. However, in the special case where $f(x, y, z) = 1$, Formula (1) yields

$$\int \int \int_E dV = \lim_{N \to \infty} \sum_{k=1}^{N} \Delta V_k$$

which is the volume of E or $V(E)$; that is,

$$\text{volume of E} = V(E) = \int \int \int_E dV \qquad (5)$$

Example 5.3   Use a triple integral to find the volume of the solid within the cylinder $x^2 + y^2 = 9$ and between the planes $z = 1$ and $x + z = 5$.

Solution   The solid E and its projection D_{xy} on the xy-plane are not shown here. The lower surface of the solid is the plane $z = 1$ and the upper surface is the plane $x + z = 5$ or, equivalently, $z = 5 - x$. Thus, from (3) and (5)

$$\text{volume of E} = V(E) = \int \int \int_E dV = \int_{D_{xy}} \left[\int_1^{5-x} dz \right] dA \qquad (6)$$

For the double integral over D_{xy}, we will integrate with respect to y first. Thus, (6) yields

$$\text{volume of E} = \int_{-3}^{3} \int_{-sqrt9-x^2}^{\sqrt{9-x^2}} \int_1^{5-x} dz\, dy\, dx = \int_{-3}^{3} \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} z \Big]_{z=1}^{5-x} dy\, dx$$

$$= \int_{-3}^{3} \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} (4 - x)\, dy\, dx = \int_{-3}^{3} (8 - 2x)\sqrt{9 - x^2}\, dx$$

$$= 8 \int_{-3}^{3} \sqrt{9 - x^2}\, dx - \int_{-3}^{3} 2x \sqrt{9 - x^2}\, dx$$

$$= 8 \left(\frac{9}{2} \pi \right) - \int_{-3}^{3} 2x \sqrt{9 - x^2}\, dx$$

$$= 8 \left(\frac{9}{2} \pi \right) - 0 = 36\pi \quad \text{(volume units)}$$

Example 5.4   Find the volume of the solid enclosed between the paraboloids

$$z = 5\,x^2 + 5\,y^2 \ \text{ and } \ z = 6 - 7\,x^2 - y^2$$

Solution   The solid E and its projection D_{xy} on the xy-plane is not shown here. The projection D_{xy} is obtained by solving the given equations simutaneously to determine where the paraboloids intersect. We obtain

$$5x^2 + 5y^2 = 6 - 7x^2 - y^2$$

or

$$2x^2 + y^2 = 1 \tag{7}$$

which tells us that the paraboloids intersect in a curve on the elliptic cylinder given by (7). The projection of this intersection on the xy-plane is an ellipse with this same equation. Therefore,

$$
\begin{aligned}
\text{volume of E} &= \int\!\!\int\!\!\int_E dV = \int\!\!\int_{D_{xy}} \left[\int_{5x^2+5y^2}^{6-7x^2-y^2} dz \right] dA \\
&= \int_{-1/\sqrt{2}}^{1/\sqrt{2}} \int_{-\sqrt{1-2y^2}}^{\sqrt{1-2y^2}} \int_{5x^2+5y^2}^{6-7x^2-y^2} dz\, dy\, dx \\
&= \int_{-1/\sqrt{2}}^{1/\sqrt{2}} \int_{-\sqrt{1-2x^2}}^{\sqrt{1-2x^2}} (6 - 12x^2 - 6y^2)\, dy\, dx \\
&= \int_{-1/\sqrt{2}}^{1/\sqrt{2}} \left[6\,(1 - 2x^2)\,y - 2y^3 \right]_{y=-sqrt1-2x^2}^{\sqrt{1-2x^2}} dx \\
&= 8 \int_{-1/\sqrt{2}}^{1/\sqrt{2}} (1 - 2x^2)^{3/2}\, dx = \frac{8}{\sqrt{2}} \int_{-\pi/2}^{\pi/2} \cos^4 \theta\, d\theta = \frac{3\pi}{\sqrt{2}}
\end{aligned}
$$

TRIPLE INTEGRALS IN OTHER ORDERS

In Formula (3) for integrating over a type I solid region E, the z-integration was performed first. However, there are situations in which it is preferable to integrate in a different order. For example, we may prefer to integrate with respect to y first or integrate with respect to x first. These leads us to the following definitions

TRIPLE INTEGRALS OVER **Type II solid regions** IN SPACE

Definition 5.2   [TYPE II SOLID REGION]   We shall say that a solid region E of space is a **Type II Solid** region if it lies between two continuous functions of two variables x and z, that is,

$$E = \{(x,y,z)|\ (x,z) \in D_{xz},\ u_1(x,z) \le y \le u_2(x,z)\}$$

where the plane region D_{xz} is the projection of E in the xz-plane.

Theorem 5.2   Let E be a Type II solid region in space, then we can express the triple integral $\int \int \int_E f(x, y, z) \, dV$ as

$$\int \int \int_E f(x, y, z) \, dV = \int \int_{D_{xz}} \left[\int_{u_1(x,z)}^{u_2(x,z)} f(x, y, z) \, dy \right] dA \tag{8}$$

where D_{xz} is the projection of E onto the xz-plane.

In particular, if D_{xz} is an X-**Simple** planar region, then we describe E as

$$E = \{(x, y, z) | \; \alpha \leq x \leq \beta, \; g_1(x) \leq z \leq g_2(x), \; u_1(x, z) \leq y \leq u_2(x, z)\}$$

and the above integral in (8) becomes

$$\int \int \int_E f(x, y, z) \, dV = \int_\alpha^\beta \int_{g_1(x)}^{g_2(x)} \left[\int_{u_1(x,y)}^{u_2(x,y)} f(x, y, z) \, dy \right] dz \, dx$$

On the other hand, if D_{xz} is a **Z-Simple** planar region, then we describe E as

$$E = \{(x, y, z) | \; h_1(z) \leq x \leq h_2(z), \; \sigma \leq z \leq \tau, \; u_1(x, z) \leq y \leq u_2(x, z)\}$$

and the above integral in (8) becomes

$$\int \int \int_E f(x, y, z) \, dV = \int_\sigma^\tau \int_{h_1(z)}^{h_2(z)} \left[\int_{u_1(x,z)}^{u_2(x,z)} f(x, y, z) \, dy \right] dx \, dz$$

TRIPLE INTEGRALS OVER **Type III solid regions** IN SPACE

Definition 5.3   [TYPE III SOLID REGION]   We shall say that a solid region E of space is a **Type III Solid** region if it lies between two continuous functions of two variables y and z, that is,

$$E = \{(x, y, z) | \; (y, z) \in D_{yz}, \; u_1(y, z) \leq x \leq u_2(y, z)\}$$

where the plane region D_{yz} is the projection of E in the yz-plane.

Theorem 5.3   Let E be a Type III solid region in space, then we can express the triple integral $\iiint\limits_E f(x,y,z)\ dV$ as

$$\iiint\limits_E f(x,y,z)\ dV = \iint\limits_{D_{yz}} \left[\int_{u_1(y,z)}^{u_2(y,z)} f(x,y,z)\ dx \right] dA \tag{9}$$

where D_{yz} is the projection of E onto the yz-plane.

In particular, if D_{yz} is an Y-**Simple** planar region, then we describe E as

$$E = \{(x,y,z)|\ \gamma \le y \le \delta,\ g_1(y) \le z \le g_2(y),\ u_1(y,z) \le x \le u_2(y,z)\}$$

and the above integral in (9) becomes

$$\iiint\limits_E f(x,y,z)\ dV = \int_\gamma^\delta \int_{g_1(y)}^{g_2(y)} \left[\int_{u_1(y,z)}^{u_2(y,z)} f(x,y,z)\ dx \right] dz\, dy$$

On the other hand, if D_{yz} is a **Z-Simple** planar region, then we describe E as

$$E = \{(x,y,z)|\ h_1(z) \le y \le h_2(z),\ \sigma \le z \le \tau,\ u_1(y,z) \le x \le u_2(y,z)\}$$

and the above integral in (9) becomes

$$\iiint\limits_E f(x,y,z)\ dV = \int_\sigma^\tau \int_{h_1(z)}^{h_2(z)} \left[\int_{u_1(y,z)}^{u_2(y,z)} f(x,y,z)\ dx \right] dy\, dz$$

Example 5.5   In Example 2, we evaluated the triple integral

$$\iiint\limits_E z\ dV$$

over the solid cylindrical wedge by integrating first with respect to z. Evaluate this triple integral by integrating first with respect to x.

Solution The solid is bounded in the back by the plane $x = 0$ and in the front by the plane $x = y$, so

$$\iiint\limits_E z\ dV = \int_0^1 \int_0^{\sqrt{1-y^2}} \int_0^y z\ dx\, dz\, dy = \int_0^1 \int_0^{\sqrt{1-y^2}} zx \Big]_{x=0}^y dz\, dy$$

$$= \int_0^1 \int_0^{\sqrt{1-y^2}} zy\, dz\, dy = \int_0^1 \frac{1}{2}z^2 y \Big]_{z=0}^{\sqrt{1-y^2}} dy = \int_0^1 \frac{1}{2}(1-y^2)\, y\, dy = \frac{1}{8}$$

which agrees with the result in Example 2.

Center of gravity, Centroid, Theorem of Pappus, Moment of Inertia

Suppose a rigid physical body is acted on by a gravitational field. Because the body is composed of many particles, each of which is affected by gravity, the action of a constant gravitational field on the body consists of a large number of forces distributed over the entire body. However, these individual forces can be replaced by a single force acting at a point called the **center of gravity** *or the* **center of mass** *of the body. In this section we will show how double and triple integrals can be used to locate centers of gravity.*

DENSITY OF A LAMINA

Let consider an idealized flat object that is thin enough to be viewed as a two-dimensional plane region. Such an object is called a **lamina**. A lamina is called **homogeneous** if its composition is uniform throughout and **inhomogeneous** otherwise. The **density** of a *homogeneous* lamina is defined to be its **mass** per unit area. Thus, the density δ of a homogeneous lamina of mass M and area A is given by $\delta = M/A$.

For an inhomogeneous lamina the composition may vary from point to point, and hence an appropriate definition of "density" must reflect this. To motivate such a definition, suppose that the lamina is placed in an xy-plane. The density at a point (x, y) can be specified by a function $\delta(x, y)$, called the **density function**, which can be interpreted as follows:

Construct a small rectangle centered at (x, y) and let ΔM and ΔA be the mass and area of the portion of the lamina enclosed by this rectangle. If the ratio $\Delta M/\Delta A$ approaches a limiting value as the dimensions (and hence the area) of the rectangle approach zero, then this limit is considered to be the density of the lamina at (x, y). Symbolically,

$$\delta(x, y) = \lim_{\Delta A \to 0} \frac{\Delta M}{\Delta A} \tag{1}$$

From this relationship we obtain the approximation

$$\Delta M \approx \delta(x, y) \, \Delta A \tag{2}$$

which relates the mass and area of a small rectangular portion of the lamina centered at (x, y). It is assumed that as the dimensions of the rectangle tends to zero.

MASS OF A LAMINA

The following result shows how to find the mass of a lamina from its density function.

Definition 6.1   If a lamina with a continuous density function $\delta(x, y)$ occupies a region R in the xy-plane, then its total mass M is given by

$$M \ = \ \text{mass} \ = \ \int \int_R \delta(x, y) \ dA \tag{3}$$

Formula (3) can be motivated by a familiar limiting process that can be outlined as follows:

Imagine the lamina to be subdivided into rectangular pieces using lines parallel to the coordinate axes and excluding from consideration any nonrectangular parts at the boundary.

Assume that there are N such rectangular pieces, and suppose that the kth piece has area ΔA_k. If we let (x_k^*, y_k^*) denote the center of the kth piece, then from Formula (2), the mass ΔM_k of this piece can be approximated by

$$\Delta M_k \ \approx \ \delta(x_k^*, y_k^*) \, \Delta A_k \tag{4}$$

and hence the mass M of the entire lamina can be approximated by

$$M \ \approx \ \sum_{k=1}^{N} \delta(x_k^*, y_k^*) \, \Delta A_k$$

If we now increase N in such a way that the dimensions of the rectangles tend to zero, then it is plausible that the errors in our approximations will approach zero, so

$$M \ = \ \lim_{N \to \infty} \sum_{k=1}^{N} \delta(x_k^*, y_k^*) \, \Delta A_k \ = \ \int \int_R \delta(x, y) \ dA$$

Example 6.1   A triangular lamina with vertices $(0, 0)$, $(0, 1)$, and $(1, 0)$ has density function $\delta(x, y) \ = \ xy$. Find its total mass.

Solution Referring to (3), the mass M of the lamina is

$$M \ = \ \int \int_R \delta(x, y) \ dA \ = \ \int \int_R xy \ dA \ = \ \int_0^1 \int_0^{-x+1} xy \ dy \, dx$$

$$= \ \int_0^1 \left[\frac{1}{2} xy^2 \right]_{y=0}^{-x+1} dx \ = \ \int_0^1 \left[\frac{1}{2} x^3 - x^2 + \frac{1}{2} x \right] dx \ = \ \frac{1}{24} \ (\text{unit of mass})$$

CENTER OF GRAVITY OF A THIN PLATE OR LAMINA

Assume that the acceleration due to the force of gravity is constant and acts downward, and suppose that a lamina occupies a region R in a horizontal xy-plane. It can be shown that there exists a unique point $(\bar{x}, \bar{y})$ (which may or may not belong to R) such that the effect of gravity on the lamina is "equivalent" to that of a single force acting at the point $(\bar{x}, \bar{y})$. This point is called the **center of gravity** of the lamina, and if it is in R, then the lamina will balance horizontally on the point of a support placed at $(\bar{x}, \bar{y})$. For example, the center of gravity of a disk of uniform density is at the center of the disk, and the center of gravity of a rectangular region of uniform density is at the center of the rectangle. For an irregularly shaped lamina or for a lamina in which the density varies from point to point, locating the center of gravity requires calculus.

CENTER OF MASS PROBLEM: Suppose that a lamina with a continuous density function $\delta(x, y)$ occupies a region R in a horizontal xy-plane. Find the coordinates $(\bar{x}, \bar{y})$ of the center of gravity or center of mass of the lamina.

To motivated the solution of this problem, let's consider what happens if we try to balance the lamina on a knife-edge parallel to the x-axis. Suppose the lamina is place on a knife-edge along a line $y = c$ that does not pass through the center of gravity. Because the lamina behaves as if its entire mass is concentrated at the center of gravity $(\bar{x}, \bar{y})$, the lamina will be rotationally unstable and the fore of gravity will cause a rotation about $y = c$. Similarly, the lamina will undergo a rotation if placed on a knife-edge along $y = d$. However, if the knife-edge runs along the line $y = \bar{y}$ through the center of gravity, the lamina will be in perfect balance. Similarly, the lamina will be in perfect balance on a knife-edge along the line $x = \bar{x}$ through the center of gravity. This suggests that the center of gravity of a lamina can be determined as the intersection of two lines of balance, one parallel to the x-axis and the other parallel to the y-axis. In order to find these lines of balance, we will need some preliminary results about rotations.

Children on a see-saw learn by experience that a lighter child can balance a heavier one by sitting farther from the fulcrum or pivot point. This is because the tendency for an object to produce rotation is proportional not only to its mass but also to the distance between the object and the fulcrum. To make this more precise, consider an x-axis, which we view as a weightless beam. If a point-mass m is located on the axis at x, then the tendency for that mass to produce a rotation of the beam about a point a on the axis is measured by the following quantity, called the **moment of m about $\mathbf{x} = \mathbf{a}$**:

$$\begin{bmatrix} \text{moment of } m \\ \text{about } a \end{bmatrix} = m\,(x - a)$$

The number $x - a$ is called the **lever arm**. Depending on whether the mass is to the right or left of a, the lever arm is either the distance between x and a or the negative of this distance. Positive lever arms result in positive moments and clockwise rotations, and negative lever arms result in negative moments and counterclockwise rotations.

Suppose that masses m_1, m_2, ..., m_N are located at positions x_1, x_2, ..., x_N on a coordinate axis and a fulcrum is positioned at the point a. Depending on whether the sum of the moments about a,

$$\sum_{k=1}^{N} m_k\,(x_k\,-\,a)\,=\,m_1\,(x_1\,-\,a)\,+\,m_2\,(x_2\,-\,a)\,+\,\cdots\,+\,m_N\,(x_N\,-\,a)$$

is positive, negative, or zero, a weightless beam along the axis will rotate clockwise about a, rotate counterclockwise about a, or balance perfectly (no rotation about a). In the last cse, the system of masses is said to be in **equilibrium**.

The preceding ideas can be extended to masses distributed in two-dimensional space. If we imagine the xy-plane to be a weightless sheet suporting a point-mass m located at a point (x, y), then the tendency for the mass to produce a rotation of the sheet about the line $x\,=\,a$ is $m\,(x\,-\,a)$, called the **moment of m about x $=$ a**, and the tendency for the mass to produce rotation about the line $y\,=\,c$ is $m\,(y\,-\,c)$, called the **moment of m about y $=$ c**. In summary,

$$\begin{bmatrix} \text{moment of } m \\ \text{about the} \\ \text{line } x\,=\,a \end{bmatrix} = m\,(x\,-\,a) \quad \text{and} \quad \begin{bmatrix} \text{moment of } m \\ \text{about the} \\ \text{line } y\,=\,c \end{bmatrix} = m\,(y\,-\,c) \tag{5-6}$$

If a number of masses are distributed throughout the xy-plane, then the plane (viewed as a weightless sheet) will balance on a knife-edge along the line $x\,=\,a$ if the sum of the moments about the line is zero. Similarly for the line $y\,=\,c$.

We are now ready to solve the CENTER OF GRAVITY PROBLEM above. We imagine the lamina to be subdivided into rectangular pieces using lines parallel to the coordinate axes and excluding from consideration any nonrectangular pieces at the boundary. We assume that there are N such rectangular pieces and the the kth piece has area ΔA_k and mass ΔM_k. We will let (x_k^*, y_k^*) be the center of the kth piece, and we will assume that the entire mass of the kth piece is concentrated at its center. From (4), the mass of the kth piece can be approximated by

$$\Delta M_k \approx \delta(x_k^*, y_k^*)\,\Delta A_k$$

Since the lamina balances on the lines $x\,=\,\bar{x}$ and $y\,=\,\bar{y}$, the sum of the moments of the rectangular pieces about those lnes should be close to zero; that is,

$$\sum_{k=1}^{N} (x_k^*\,-\,\bar{x})\,\Delta M_k\,=\,\sum_{k=1}^{N} (x_k^*\,-\,\bar{x})\,\delta(x_k^*, y_k^*)\,\Delta A_k\,\approx\,0$$

$$\sum_{k=1}^{N} (y_k^*\,-\,\bar{y})\,\Delta M_k\,=\,\sum_{k=1}^{N} (y_k^*\,-\,\bar{y})\,\delta(x_k^*, y_k^*)\,\Delta A_k\,\approx\,0$$

If we now increase N in such a way that the dimensions of the rectangles tend to zero, then it is plausible that the errors in our approximations will aproach zero, so that

$$\lim_{N\to\infty} \sum_{k=1}^{N} (x_k^*\,-\,\bar{x})\,\delta(x_k^*, y_k^*)\,\Delta A_k\,=\,0$$

$$\lim_{N\to\infty} \sum_{k=1}^{N} (y_k^*\,-\,\bar{y})\,\delta(x_k^*, y_k^*)\,\Delta A_k\,=\,0$$

from which we obtain

$$\int\int_R (x - \bar{x})\,\delta(x,y)\,dA = 0$$

$$\int\int_R (y - \bar{y})\,\delta(x,y)\,dA = 0$$

Since $\bar{x}$ and $\bar{y}$ are constant, these equations can be rewritten as

$$\int\int_R x\,\delta(x,y)\,dA = \bar{x}\int\int_R \delta(x,y)\,dA$$

$$\int\int_R y\,\delta(x,y)\,dA = \bar{y}\int\int_R \delta(x,y)\,dA$$

from which we obtain the following formulas for the center of gravity of the lamina:

CENTER OF GRAVITY $(\bar{x}, \bar{y})$ OF A LAMINA

$$\bar{x} = \frac{\int\int_R x\,\delta(x,y)\,dA}{\int\int_R \delta(x,y)\,dA}, \qquad \bar{y} = \frac{\int\int_R y\,\delta(x,y)\,dA}{\int\int_R \delta(x,y)\,dA} \tag{7-8}$$

Observe that in both formulas the denominator is the mass M of the lamina. The numerator in the formula for $\bar{x}$ is denoted by M_y and is called the **first moment of the lamina about the y-axis**; the numerator of the formula for $\bar{y}$ is denoted by M_x and is called the **first moment of the lamina about the x-axis**. Thus, Formulas (7) and (8) can be expressed as

$$\bar{x} = \frac{M_y}{M} = \frac{1}{\text{mass of the lamina}}\int\int_R x\,\delta(x,y)\,dA \tag{9}$$

$$\bar{y} = \frac{M_x}{M} = \frac{1}{\text{mass of the lamina}}\int\int_R y\,\delta(x,y)\,dA \tag{10}$$

Example 6.2 A thin plate covers a triangular region D of the xy-plane bounded by the x-axis and the lines $x = 1$ and $y = 2x$ in the first quadrant. The plate's density at the point $(x, y) \in D$ is $\delta(x, y) = 6x + 6y + 6$. Find the plate's Mass, First Moments and Center-of-Mass about the coordinate axes.

Solution

1. Make a sketch of the lamina $D = \{(x, y) \mid 0 \le x \le 1,\ 0 \le y \le 2x\}$ (Not shown here!).

Calculating

$$M = \int\int_D \delta(x, y)\, dA$$

by integrating first with respect to y and then with respect to x.

2. Finding the y-limits of integration: Any vertical line L passing through D in the direction of increasing y must enter D at $y = 0$ and leaves D at $y = 2x$ and so

$$A(x) = \int_0^{2x} (6x + 6y + 6)dy = \left[6xy + 3y^2 + 6y \right]_{y=0}^{y=2x} = 24x^2 + 12x$$

3. Finding the x-limits of integration: We include all vertical lines cutting through D starting with $x = 0$ and ending with $x = 1$. Thus

$$\begin{aligned}
M = \int\int_D \delta(x, y)\, dA &= \int_0^1 \left[\int_0^{2x} (6x + 6y + 6)\, dy \right] dx \\
&= \int_0^1 (24x^2 + 12x)\, dx \\
&= \left[8x^3 + 6x^2 \right]_{x=0}^{x=1} = 14 \quad \text{(unit of mass)}
\end{aligned}$$

First Moment of the lamina about the x-axis

$$\begin{aligned}
M_x = \int\int_D y\, \delta(x, y)\, dA &= \int_0^1 \left[\int_0^{2x} (6xy + 6y^2 + 6y)\, dy \right] dx \\
&= \int_0^1 \left[3xy^2 + 2y^3 + 3y^2 \right]_{y=0}^{y=2x} dx \\
&= \int_0^1 (28x^3 + 12x^2)\, dx \\
&= \left[7x^4 + 4x^3 \right]_{x=0}^{x=1} = 11
\end{aligned}$$

First Moment of the lamina about the y-axis

$$\begin{aligned}
M_y &= \int\int_D x\,\delta(x,y)\,dA = \int_0^1\left[\int_0^{2x}(6x^2 + 6xy + 6x)\,dy\right]dx \\
&= \int_0^1\left[6x^2y + 3xy^2 + 6xy\right]_{y=0}^{y=2x}dx \\
&= \int_0^1(24x^3 + 12x^2)\,dx \\
&= \left[6x^4 + 4x^3\right]_{x=0}^{x=1} = 10
\end{aligned}$$

Finally, the coordinates of the center of gravity or center of mass are:

$$\bar{x} = \frac{M_y}{M} = \frac{10}{14}\ \text{ and }\ \bar{y} = \frac{M_x}{M} = \frac{11}{14}.$$

In other words, the triangular plate balances at the point (5/7, 11/14) in D.

CENTROIDS

In the special case of a *homogeneous* lamina, the center of gravity is called the **centroid of the lamina** or sometimes the **centroid of the region** R. Because the density function δ is constant for a homogeneous lamina, the factor δ may be moved through the integral signs in (7) and (8) and canceled. Thus, the centroid $(\bar{x},\bar{y})$ is a geometric property of the region R and is given by the following formulas:

CENTROID OF A REGION R

$$\bar{x} = \frac{\int\int_R x\,dA}{\int\int_R dA} = \frac{1}{\text{area of R}}\int\int_R x\,dA \tag{11}$$

$$\bar{y} = \frac{\int\int_R y\,dA}{\int\int_R dA} = \frac{1}{\text{area of R}}\int\int_R y\,dA \tag{12}$$

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Example 6.3   Find the centroid of the semicircular region in the xy-plane bounded by the x-axis and the curve $y = \sqrt{a^2 - x^2}$.

Solution   By symmetry, $\bar{x} = 0$ since the y-axis is obviously a line of balance. From Formula (12),

$$
\begin{aligned}
\bar{y} &= \frac{1}{\text{area of R}} \iint\limits_{R} y \, dA = \frac{1}{\frac{1}{2}\pi a^2} \iint\limits_{R} y \, dA \\
&= \frac{1}{\frac{1}{2}\pi a^2} \int_0^\pi \int_0^a (r \sin\theta) \, r \, dr \, d\theta \quad \text{(Evaluating in Polar coordinates)} \\
&= \frac{1}{\frac{1}{2}\pi a^2} \int_0^\pi \left[\frac{1}{3} r^3 \sin\theta \right]_{r=0}^a d\theta \\
&= \frac{1}{\frac{1}{2}\pi a^2} \left(\frac{1}{3} a^3 \right) \int_0^\pi \sin\theta \, d\theta = \frac{1}{\frac{1}{\pi a^2}} \left(\frac{2}{3} a^3 \right) = \frac{4a}{3\pi}
\end{aligned}
$$

So the centroid is $\left(0, \frac{4a}{3\pi} \right)$.

CENTER OF GRAVITY AND CENTROID OF A SOLID

For a three-dimensional solid E, the formulas for moments, center of gravity, and centroid are similar to those for laminas. If E is *homogeneous*, then its **density** is defined to be its mass per unit volume. Thus, if E is a homogeneous solid of mass M and volume V, then its density δ is given by $\delta = m/V$. If E is inhomogeneous and is in an xyz-coordinate system, then its density at a general point (x, y, z) is specified by a **density function** $\delta(x, y, z)$ whose value at a point can be viewed as a limit:

$$
\delta(x, y, z) = \lim_{\Delta v \to 0} \frac{\Delta M}{\Delta V}
$$

where ΔM and ΔV represent the mass and volume of a rectangular parallelepiped, centered at (x, y, z), whose dimensions tend to zero.

Using the siscussion of laminas as a mode, ou should be able to show that the mass M of a solid with a continuous density function $\delta(x, y, z)$ is

$$
M = \text{mass of } E = \iiint\limits_{E} \delta(x, y, z) \, dV \tag{13}
$$

The formulas for center of gravity and centroid are as follows:

<div align="center">

Center of Gravity $(\bar{x}, \bar{y}, \bar{z})$ **of a Solid** E **Centroid** $(\bar{x}, \bar{y}, \bar{z})$ **of a Solid** E

</div>

$$
\begin{array}{ll}
\bar{x} = \frac{1}{M} M_{yz} = \frac{1}{M} \iiint\limits_{E} x \, \delta(x, y, z) \, dV & \qquad \bar{x} = \frac{1}{V} \iiint\limits_{E} x \, dV \\[2mm]
\bar{y} = \frac{1}{M} M_{xz} = \frac{1}{M} \iiint\limits_{E} y \, \delta(x, y, z) \, dV & \qquad \bar{y} = \frac{1}{V} \iiint\limits_{E} y \, dV \\[2mm]
\bar{z} = \frac{1}{M} M_{xy} = \frac{1}{M} \iiint\limits_{E} z \, \delta(x, y, z) \, dV & \qquad \bar{z} = \frac{1}{V} \iiint\limits_{E} z \, dV
\end{array} \tag{14 - 15}
$$

Observe that in the three formulas for center of gravity the denominator is the mass M of the solid. The numerator in the formula for $\bar{x}$ is denoted by M_{yz} and is called the **first moment of the solid about the yz-plane**; the numerator of the formula for $\bar{y}$ is denoted by M_{xz} and is called the **first moment of the solid about the xz-plane**; and the numerator of the formula for $\bar{z}$ is denoted by M_{xy} is called the **first moment of the solid about the xy-plane.**.

Example 6.4   Find the mass and the center of gravity (or center of mass) of a cylindrical solid of height h and radius a, assuming that the density at each point is proportional to the distance between the point and the base of the solid.

Solution Since the density is proportional to the distance z from the base, the density function has the form $\delta(x, y, z) = \kappa z$, where κ is some (unknown) positive constant of proportionality. From (13) the mass of the solid is

$$M = \int\int\int_E \delta(x, y, z)\ dV = \int_{-a}^{a}\int_{-\sqrt{a^2 - x^2}}^{\sqrt{a^2 - x^2}}\int_{0}^{h} \kappa z\ dz\ dy\ dx$$

$$= \kappa \int_{-a}^{a}\int_{-\sqrt{a^2 - x^2}}^{\sqrt{a^2 - x^2}} \frac{1}{2} h^2\ dy\ dx = \kappa h^2 \int_{-a}^{a}\sqrt{a^2 - x^2}\ dx$$

$$= \frac{1}{2}\kappa h^2 \pi a^2 \quad \text{(Interpret the integral as the area of a semicircle)}$$

Without additional information, the constant κ cannot be determined. However, as we will now see, the value of κ does not affect the center of gravity. From (14),

$$\bar{z} = \frac{1}{M}\int\int\int_E z\, \delta(x, y, z)\ dV = \frac{1}{\frac{1}{2}\kappa h^2 \pi a^2}\int\int\int_E z\, \delta(x, y, z)\ dV$$

$$= \frac{1}{\frac{1}{2}\kappa h^2 \pi a^2}\int_{-a}^{a}\int_{-\sqrt{a^2 - x^2}}^{\sqrt{a^2 - x^2}}\int_{0}^{h} z(\kappa z)\ dz\ dy\ dx$$

$$= \frac{\kappa}{\frac{1}{2}\kappa h^2 \pi a^2}\int_{-a}^{a}\int_{-\sqrt{a^2 - x^2}}^{\sqrt{a^2 - x^2}} \frac{1}{3} h^3\ dy\ dx$$

$$= \frac{\frac{1}{3}\kappa h^3}{\frac{1}{2}\kappa h^2 \pi a^2}\int_{-a}^{a} 2\sqrt{a^2 - x^2}\ dx$$

$$= \frac{\frac{1}{3}\kappa h^3 \pi a^2}{\frac{1}{2}\kappa h^2 \pi a^2} = \frac{2}{3} h$$

Similar calculations using (14) will yield $\bar{x} = \bar{y} = 0$. However, this is evident by inspection, since it follows from the symmetry of the solid and the form of its density function that the center of gravity is on the z-axis. Thus, the center of gravity (or the center of mass) is $\left(0, 0, \frac{2}{3}h\right)$.

Theorem Of Pappus

The following theorem, due to the Greek mathematician Pappus, gives an improtant relationship between the centroid of a plane region R and the volume of the solid generated when the region is revolved about a line.

Theorem 6.1 [Theorem of Pappus] If R is a bounded region and L is a line that lies in the plane of R is entirely on one side of L, then the volume of the solid formed by revolving R about L is given by

$$\text{Volume} = (\text{area of } R) \cdot \left(\begin{array}{c} \text{distance traveled} \\ \text{by the centroid} \end{array} \right)$$

Proof Introduce an xy-coordinate system so that L is along the y-axis and the region R is in the first quadrant. Let R be partitioned into subregions in the usual way and let R_k be a typical rectangle interior to R. If (x_k^*, y_k^*) is the center of R_k and if the area of R_k is $\Delta A_k = \Delta x_k \, \Delta y_k$, then the volume generated by R_k as it revolves about L is

$$2 \pi \, x_k^* \, \Delta x_k \, \Delta y_k = 2\pi \, x_k^* \Delta A_k$$

Therefore, the total volume of the solid is approximately

$$V \approx \sum_{k=1}^{N} 2\pi x_k^* \Delta A_k$$

from which it follows that the exact volume is

$$V = \int \int_R 2 \, \pi \, x \, dA = 2 \, \pi \int \int_R x \, dA$$

Thus, it follows from (11) that

$$V = 2\pi \cdot \bar{x} \cdot [\text{area of } R]$$

This completes the proof since $2\pi\bar{x}$ is the distance traveled by the centroid when R is revolved about the y-axis.

Example 6.5 Use Pappus' Theorem to find the volume V of the torus generated by revolving a circular region of radius β about a line at a distance α (greater than β) from the center of the circle.

Solution By symmetry, the centroid of the circular region is its center. Thus, the distance traveled by the centroid is $2\pi \, \alpha$. Since the area of a circle of radius β is $\pi \, \beta^2$, it follows from Pappus' Theorem that the volume of the torus is

$$V = (2\pi \, \alpha)(\pi \, \beta^2) = 2\pi^2 \alpha \beta^2.$$

MOMENTS OF INERTIA

An object's first moments tell us about balance and about the torque the object experiences about different axes in a gravitational field. If the object is a rotating shaft, however, we are more likely to be interested in how much energy is stored in the shaft or about how much energy is generated by a shaft rotating at a particular angular velocity. This is where the second moment or moment of inertia comes in.

Think of partitioning the shaft into N small blocks where the mass of the kth block is denoted by Δm_k; let r_k denote the distance from the kth block's center of mass to the axis of rotation. If the shaft rotates at a constant angular velocity of $\omega = d\theta/dt$ radians per second, the block's center of gravity will trace its orbit at a linear speed of

$$v_k = \frac{d}{dt}(r_k\,\theta) = r_k\,\frac{d\theta}{dt} = r_k\,\omega$$

The block's kinetic energy will be approximately

$$\frac{1}{2}\,\Delta m_k\,v_k^2 = \frac{1}{2}\,\Delta m_k\,(r_k\,\omega)^2 = \frac{1}{2}\,\omega^2\,r_k^2\,\Delta m_k.$$

The kinetic energy of the shaft will be approximately

$$\sum_{k=1}^{N}\frac{1}{2}\,\omega^2\,r_k^2\,\Delta m_k$$

The integral approached by these Riemann sums as the shaft is partitioned into smaller and smaller blocks gives the shaft's kinetic energy:

$$\text{K.E}_{\text{shaft}} = \int \frac{1}{2}\,\omega^2\,r^2\,dm = \frac{1}{2}\,\omega^2 \int r^2\,dm \tag{16}$$

The factor

$$I = \int r^2\,dm$$

is the **moment of inertia** of the shaft aout its axis of rotation, and we see from (16) that the shaft's kinetic energy is

$$\text{K.E}_{\text{shaft}} = \frac{1}{2}\,I\,\omega^2.$$

The moment of inertia of a shaft resembles in some way the inertial mass of a locomotive. To start a locomotive with mass m moving at a linear velocity v, we need to provide a kinetic energy of $\text{K.E} = (1/2)\,m\,v^2$. To stop the locomotive we have to remove this amount of energy. To start a shaft with moment of inertia I rotating at an angular velocity ω, we need to provide a kinetic energy of $\text{K.E} = (1/2)\,I\,\omega^2$. To stop the shaft we have to take this amount of energy back out. The shaft's moment of inertia is analogous to the locomotive's mass. What makes the locomotive hard to start or stop is its mass. What makes the shaft hard to start or stop is its moment of inertia. The moment of inertia depends not only on the mass of the shaft but also on its distribution. Mass that is farther away from the axis of rotation contributes more to the moment of inertia.

We now derive a formula for the moment of inertia for a solid E in space. If $r(x, y, z)$ is the distance from the point (x, y, z) in E to a line L, then the moment of inertia of the mass $\Delta m_k = \delta(x_k, y_k, z_k)\,\Delta V_k$ about the line L is approximately $\Delta I_k = r^2(x_k, y_k, z_k)\,\Delta m_k$.

The moment of inertia about L of the entire solid E is

$$I_L = \lim_{N\to\infty} \sum_{k=1}^{N} \Delta I_k = \lim_{N\to\infty} \sum_{k=1}^{N} r^2(x_k, y_k, z_k)\,\delta(x_k, y_k, z_k)\,\Delta V_k = \int\int\int_E r^2\,\delta(x, y, z)\,dV$$

If L is the x-axis, then $r^2 = y^2 + z^2$ and

$$I_x = \int\int\int_E (y^2 + z^2)\,\delta(x, y, z)\,dV.$$

Similarly, if L is the y-axis or z-axis we have

$$I_y = \int\int\int_E (x^2 + z^2)\,\delta(x, y, z)\,dV \quad \text{and} \quad I_z = \int\int\int_E (x^2 + y^2)\,\delta(x, y, z)\,dV$$

Example 6.6   Find I_x, I_y, and I_z for a rectangular solid E of constant density δ.

Solution   The formula of I_x gives

$$I_x = \int_{-c/2}^{c/2} \int_{-b/2}^{b/2} \int_{-a/2}^{a/2} (y^2 + z^2)\,\delta\,dx\,dy\,dz.$$

We can avoid some of the wori of integration by observing that $(y^2 + z^2)\,\delta$ is an even function of x, y, and z since δ is constant. The rectangular solid consists of eight symmetric pieces, one in each octant. We can evaluate the integral on one of these pieces and then multiply by 8 to get the total value.

$$I_x = 8 \int_0^{c/2} \int_0^{b/2} \int_0^{a/2} (y^2 + z^2)\,\delta\,dx\,dy\,dz = 4\,a\,\delta \int_0^{c/2} \int_0^{b/2} (y^2 + z^2)\,dy\,dz$$

$$= 4\,a\,\delta \int_0^{c/2} \left[\frac{y^3}{3} + z^2 y\right]_{y=0}^{y=b/2} dz$$

$$= 4\,a\,\delta \int_0^{c/2} \left(\frac{b^3}{24} + \frac{z^2 b}{2}\right) dz$$

$$= 4\,a\,\delta \left(\frac{b^3 c}{48} + \frac{c^3 b}{48}\right) = \frac{abc\delta}{12}(b^2 + c^2) = \frac{M}{12}(b^2 + c^2). \quad (M = V\delta = abc\delta)$$

Similarly,

$$I_y = \frac{M}{12}(a^2 + c^2) \quad \text{and} \quad I_z = \frac{M}{12}(a^2 + b^2).$$

The moment of inertia for thin plates or laminas is given by

About the x-axis: $\qquad\qquad\qquad\qquad I_x \;=\; \int\int_D y^2\,\delta\,dA \qquad\qquad\qquad\qquad \delta \;=\; \delta(x,y)$

About the y-axis: $\qquad\qquad\qquad\qquad I_y \;=\; \int\int_D x^2\,\delta\,dA$

About a line L: $\qquad\qquad\qquad\qquad I_L \;=\; \int\int_D r^2(x,y)\,\delta\,dA \qquad r(x,y) \;=\; \text{distance from } (x,y) \text{ to L}$

About the origin (polar moment): $\quad I_0 \;=\; \int\int_D (x^2 + y^2)\,\delta\,dA \;=\; I_x + I_y$

Example 6.7   A thin plate covers the triangular region bounded by the x-axis and the lines $x = 1$ and $y = 2x$ in the first quadrant. The plate's density at the point (x,y) is $\delta(x,y) = 6x + 6y + 6$. Find the plate's moments of inertia about the coordinates axes and the origin.

Solution   A sketch of the plate is not shown here. The moments of inertia about the x-axis is

$$
\begin{aligned}
I_x &= \int_0^1 \int_0^{2x} y^2\,\delta(x,y)\,dy\,dx \;=\; \int_0^1 \int_0^{2x} (6xy^2 + 6y^3 + 6y^2)\,dy\,dx \\
&= \int_0^1 \left[2xy^3 + \frac{3}{2}y^4 + 2y^3 \right]_{y=0}^{y=2x} dx \;=\; \int_0^1 (40x^4 + 16x^3)\,dx \\
&= [8x^5 + 4x^4]_0^1 \;=\; 12
\end{aligned}
$$

Similarly, the moment of inertia about the y-axis is

$$
I_y \;=\; \int_0^1 \int_0^{2x} x^2\,\delta(x,y)\,dy\,dx \;=\; \frac{39}{5}
$$

Notice that we integrate y^2 times density in calculating I_x and x^2 times density to find I_y.

Since we know I_x and I_y, we do not need to evaluate an integral to find I_0; we can use the equation $I_0 = I_x + I_y$ instead:

$$
I_0 \;=\; 12 + \frac{39}{5} \;=\; \frac{99}{5}
$$

The moment of inertia also plays a role in determining how much a horizontal metal beam will bend under a load. The stiffness of the beam is a constant times I, the moment of inertia of a typical cross-section of the beam about the beam's longitudinal axis. The greater the value of I, the stiffer the beam and the less it will bend under a given load. that is why we use I-beams instead of beams whose cross-sections are square. The flanges at the top and bottom of the beam hold most of the beam's mass away from the longitudinal axis to increase the value of I.

TRIPLE INTEGRALS IN CYLINDRICAL AND SPHERICAL COORDINATES

When a calculation in physics, engineering, or geometry involves a cylinder, cone, or sphere, we can often simplify our work by using cylindrical or spherical coordinates, which are introduced in this discussion. The procedure for transforming to these coordinates and evaluating the resulting triple integrals is similar to the transformation to polar coordinates in the plane studied earlier above.

TRIPLE INTEGRATION IN CYLINDRICAL COORDINATES

We obtain cylindrical coordinates for space by combining polar coordinates in the xy-plane with the usual z-axis. This assigns to every point in space one or more coordinate triples of the form $(r,\ \theta,\ z)$.

Definition 6.2 [CYLINDRICAL COORDINATES] Cylindrical coordinates represent a point P in space by ordered triples $(r,\ \theta,\ z)$ in which

    1. r and θ are polar coordinates for the vertical projection of P on the xy-plane

    2. z is the rectangular vertical coordinate.

The values of x, y, r, and θ in rectangular and cylindrical coordinates are related by the usual equations.

Equations Relating Rectangular (x, y, z) and Cylindrical $(r,\ \theta,\ z)$ Coordinates

$$x = r\,\cos\theta, \qquad y = r\,\sin\theta, \qquad z = z, \qquad r^2 = x^2 + y^2, \qquad \tan\theta = \frac{y}{x}.$$

In cylindrical coordinates, the equation $r = a$ describes not just a circle in the xy-plane but an entire cylinder about the z-axis. The z-axis is given by $r = 0$. The equation $\theta = \theta_0$ describes the plane that contains the z-axis and makes an angle θ_0 with the positive x-axis. And, just as in rectangular coordinates, the equation $z = z_0$ describes a plane perpendicular to the z-axis.

Cylindrical coordinates are good for describing cylinders whose axes run along the z-axis and planes that either contain the z-axis or lie perpendicular to the z-axis. Surfaces like these have equations of constant coordinate value:

$$r = 4. \qquad \text{Cylinder, radius 4, axis the z-axis}$$
$$\theta = \frac{\pi}{3}. \qquad \text{Plane containing the z-axis}$$
$$z = 2. \qquad \text{Plane perpendicular to the z-axis}$$

When computing triple integrals over a region E in cylindrical coordinates, we partition the region into n **small cylindrical wedges**, rather than into rectangular boxes as before. In the kth cylindrical wedge, r, θ and z change

by Δr_k, $\Delta \theta_k$ and Δz_k, and the largest of these numbers among all the cylindrical wedges is called the **norm of the partition**. We define the triple integral as a limit of Riemann sums using these wedges. The volume of such a cylindrical wedge is ΔV_k obtained by taking the area ΔA_k of its base in the $r\theta$-plane and multiplying by the height Δz.

For a point (r_k, θ_k, z_k) in the center of the k-th wedge, we calculated in polar coordinates that $\Delta A_k = r_k \, \Delta r_k \, \Delta \theta_k$. So $\Delta V_k = \Delta z_k \, r_k \, \Delta r_k \, \Delta \theta_k$ and a Riemann sum for f over E has the form

$$S_n = \sum_{k=1}^{n} f(r_k, \theta_k, z_k) \, \Delta z_k \, r_k \, \Delta r_k \, \Delta \theta_k$$

The triple integral of a function f over E is obtained by taking a limit of such Riemann sums with partitions whose norms approach zero

$$\lim_{n \to \infty} S_n = \int \int \int_E f(r, \theta, z) \, dV = \int \int \int_E f(r, \theta, z) \, dz \, r \, dr \, d\theta.$$

Triple integrals in cylindrical coordinates are then evaluated as iterated integrals, as in the following example.

Example 6.8 [FINDING LIMITS OF INTEGRATION IN CYLINDRICAL COORDINATES] FIND THE LIMITS OF INTE-GRATION IN CYLINDRICAL COORDINATES FOR INTEGRATING A FUNCTION $f(r, \theta, z)$ OVER THE REGION E BOUNDED BELOW BY THE PLANE $z = 0$, LATERALLY BY THE CIRCULAR CYLINDER $x^2 + (y - 1)^2 = 1$, AND ABOVE BY THE PARABOLOID $z = x^2 + y^2$.

Solution The base of E is also the region's projection D_{xy} on the xy-plane. The boundary of D_{xy} is the circle $x^2 + (y - 1)^2 = 1$. Its polar coordinate equation is

$$x^2 + (y - 1) = 1$$
$$x^2 + y^2 - 2y + 1 = 1$$
$$r^2 - 2r \sin \theta = 0$$
$$r = 2 \sin \theta.$$

We find the limits of integration, starting with the z-limits. A line M through a typical point $(r, \theta$ in D_{xy} parallel to the z-axis enters E at $z = 0$ and leaves at $z = x^2 + y^2 = r^2$.

Next we find the r-limits of integration. A ray L through (r, θ) from the origin enters D_{xy} at $r = 0$ and leaves at $r = 2 \sin \theta$.

Finally we find the θ-limits of integration. As the ray L sweeps across D_{xy}, the angle θ it makes with the positive x-axis runs from $\theta = 0$ to $\theta = \pi$. The integral is

$$\int \int \int_E f(r, \theta, z) \, dV = \int_0^\pi \int_0^{2 \sin \theta} \int_0^{r^2} f(r, \theta, z) \, dz \, r \, dr \, d\theta.$$

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Example above illustrates a good procedure for finding limits of integration in cylindrical coordinates. The procedure is summarized below as follows:

A PROCEDURE FOR FINDING THE LIMITS OF INTEGRATION FOR TRIPLE INTEGRALS IN CYLINDRICAL COORDINATES

We now give a procedure for finding limits of integration that applies for many regions in space. Regions that are more complicated, and for which this procedure fails, can often be split up into pieces on which the procedure works.

When faced with evaluating a triple integral of the form

$$\int \int \int_E f(r, \theta, z) \, dV$$

over a region E in space in cylindrical coordinates, integrating first with respect to z, then with respect to r, and finally with respect to θ take the following steps:

STEP 1 SKETCH: *Sketch the region E along with its projection D_{xy} on the xy-plane. Label the surfaces and curves that bound E and D_{xy}.*

STEP 2. FIND THE z-LIMITS OF INTEGRATION: *Draw a line M through a typical point (r, θ) of D_{xy} parallel to the z-axis. As z increases, M enters E at $z = g_1(r, \theta)$ and leaves at $z = g_2(r, \theta)$. These are the z-limits of integration.*

STEP 3. FIND THE r-LIMITS OF INTEGRATION: *Draw a ray L through (r, θ) from the origin. The ray enters D_{xy} at $r = h_1(\theta)$ and leaves at $r = h_2(\theta)$. These are the r-limits of integration.*

STEP 4. FIND THE θ-LIMITS OF INTEGRATION: *As the ray L sweeps across D_{xy}, the angle θ it makes with the positive x-axis runs from $\theta = \alpha$ to $\theta = \beta$. These are the θ-limits of integration.*The integral is

$$\int \int \int_E f(r, \theta, z) \, dV = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} \int_{g_1(r,\theta)}^{g_2(r,\theta)} f(r, \theta, z) \, dz \, r \, dr \, d\theta.$$

In this example I evaluate a Triple Integral in cylindrical coordinates using the procedures above. You too, should use these procedures when evaluating your integrals.

Example 6.9 Consider the problem of finding the **centroid** (i.e., the center of mass when the solid has constant density, δ in this case we consider $\delta = 1$) of the solid E in space enclosed by the cylinder $x^2 + y^2 = 4$, bounded above by the paraboloid $z = x^2 + y^2$, and bounded below by the plane $z = 0$.

Solution

STEP 1 SKETCH: *Sketch the region E along with its projection R on the xy-plane. Label the surfaces and curves that bound E and D_{xy}.*

No Sketch provided here. See if you can make a sketch of the solid.

STEP 2. FIND THE z-LIMITS OF INTEGRATION: *Draw a line M through a typical point (r, θ) of D_{xy} parallel to the z-axis. As z increases, M enters E at $z = g_1(r, \theta)$ and leaves at $z = g_2(r, \theta)$. These are the z-limits of integration.*

D_{xy} *is the vertical projection of the solid E on the xy-plane, and it is a closed disk of radius 2 centered at the origin. Any line M passing through D_{xy} at a typical point (r, θ) enters the solid at $z = 0$ and leaves at $z = x^2 + y^2 = r^2$. So our z-limits of integration goes from $z = 0$ to $z = r^2$.*

STEP 3. FIND THE r-LIMITS OF INTEGRATION: *Draw a ray L through (r, θ) from the origin. The ray enters D_{xy} at $r = h_1(\theta)$ and leaves at $r = h_2(\theta)$. These are the r-limits of integration.*

Any ray L through D_{xy} through (r, θ) from the origin, must enter D_{xy} at $r = 0$ and leaves at $r = 2$. So our r-limits of integration goes from $r = 0$ to $r = 2$.

STEP 4. FIND THE θ-LIMITS OF INTEGRATION: *As the ray L sweeps across D_{xy}, the angle θ it makes with the positive x-axis runs from $\theta = \alpha$ to $\theta = \beta$. These are the θ-limits of integration.*

The ray L sweeps across D_{xy}, the angle θ it makes with the positive x-axis runs from $\theta = 0$ to $\theta = 2\pi$. So our θ-limits of integration goes from $\theta = 0$ to $\theta = 2\pi$.

Now we compute M_{xy} (First moment about the xy-plane) which is given by

$$M_{xy} = \int\int\int_E z\, \delta(x, y, z)\, dV = \int\int\int_E z\, dV = \int_0^{2\pi}\int_0^2\int_0^{r^2} z\, dz\, r\, dr\, d\theta = \int_0^{2\pi}\int_0^2 \frac{r^5}{2}\, dr\, d\theta = \frac{32\pi}{3}.$$

Next, we must compute the mass M which is given by

$$M = \int\int\int_E \delta(x, y, z)\, dV = \int\int\int_E dV = \int_0^{2\pi}\int_0^2\int_0^{r^2} dz\, r\, dr\, d\theta = \int_0^{2\pi}\int_0^2 r^3\, dr\, d\theta = 8\pi.$$

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Finally, we have the centroid of the solid E is

$$\bar{x} = \bar{y} = 0 \;\; \text{and} \;\; \bar{z} = \frac{M_{xy}}{M} = \frac{\frac{32\pi}{3}}{8\pi} = \frac{4}{3}.$$

Thus the centroid of the solid has coordinates $(0, 0, 4/3)$. It is important to understand why $\bar{x}$ and $\bar{y}$ are 0. This is because the z-axis the axis of symmetry of the solid and the centroid lies on this axis. This explains why the $\bar{x}$ and $\bar{y}$ are 0 without having to calculating it directly. Notice also that the centroid lies outside of the solid.

TRIPLE INTEGRATION IN SPHERICAL COORDINATES

Spherical coordinates locate points P in space with two angles and one distance. The first coordinate, ρ is the distance from the origin O to the point P (i.e., $\rho = \| OP \|$ see section 13 of textbook). Unlike r (in cylindrical coordinates), the variable ρ **is never negative**. The second coordinate, ϕ, is the angle the position vector $\vec{OP}$ makes with the positive z-axis. It is required to satisfy the inequality $0 \leq \phi \leq \pi$. The third coordinate is the angle θ as measured in cylindrical coordinates.

Definition 6.3 [SPHERICAL COORDINATES] Spherical coordinates represent a point P in space by ordered triples (ρ, ϕ, θ) in which

1. ρ is the distance from the point P to the origin.

2. ϕ is the angle the position vector $\vec{OP}$ makes with the positive z-axis $(0 \leq \phi \leq \pi)$

3. θ is the angle from cylindrical coordinates.

On maps of the Earth, θ is related to the meridian of a point on the Earth and ϕ to its latitude, while ρ is related to elevation above the Earth's surface.

The equation $\rho = a$ describes the sphere of radius a centered at the origin. The equation $\phi = \phi_0$ describes a single cone whose vertex lies at the origin and whose axis lies along the z-axis. (We broaden our interpretation to include the xy-plane as the cone $\phi = \frac{\pi}{2}$.) If ϕ_0 is greater than $\pi/2$, the cone $\phi = \phi_0$ opens downward. The equation $\theta = \theta_0$ describes the half-plane that contains the z-axis and makes an angle θ_0 with the positive x-axis.

EQUATIONS RELATING SPHERICAL COORDINATES TO CARTESIAN AND CYLINDRICAL COORDINATES

$$\begin{aligned}
r &= \rho \sin \phi, & x &= r \cos \theta = \rho \sin \phi \cos \theta, \\
z &= \rho \cos \phi, & y &= r \sin \theta = \rho \sin \phi \sin \theta \\
\rho &= \sqrt{x^2 + y^2 + z^2} = \sqrt{r^2 + z^2}.
\end{aligned} \tag{1}$$

Example 6.10 [CONVERTING RECTANGULAR TO SPHERICAL] Find a spherical coordinate equation for the sphere

$$x^2 + y^2 + (z - 1)^2 = 1.$$

Solution We use Equations (1) to substitute for x, y, and z as follows:

$$x^2 + y^2 + (z-1)^2 = 1$$
$$(\rho \sin \phi \cos \theta)^2 + (\rho \sin \phi \sin \theta)^2 + (\rho \cos \phi - 1)^2 = 1$$
$$\rho^2 \sin^2 \phi \cos^2 \theta + \rho^2 \sin^2 \phi \sin^2 \theta + \rho^2 \cos^2 \phi - 2\rho \cos \phi + 1 = 1$$
$$\rho^2 \sin^2 \phi \underbrace{(\cos^2 \theta + \sin^2 \theta)}_{=1} + \rho^2 \cos^2 \phi = 2\rho \cos \phi$$
$$\rho^2 \underbrace{(\sin^2 \phi + \cos^2 \phi)}_{=1} = 2\rho \cos \phi$$
$$\rho^2 = 2\rho \cos \phi$$
$$\rho = 2 \cos \phi \quad \{\text{finally!}\}$$

Spherical coordinates are good for describing spheres centered at the origin, half-planes hinged along the z-axis, and cones whose vertices lie at the origin and whose axes lie along the z-axis. Surfaces like these have equations of constant coordinate value:

$\rho = 4$.   Sphere, radius 4, centered at the origin

$\phi = \dfrac{\pi}{3}$.   Cone opening up from the origin, making an angle of $\pi/3$ radians with the positive z-axis

$\theta = \dfrac{\pi}{3}$.   Half-plane, hinged along the z-axis, making an angle of $\pi/3$ radians with the positive x-axis

When computing triple integrals over a region E in spherical coordinates, we partition the region into n **spherical wedges**. The size of the k-th spherical wedge, which contains a point $(\rho_k, \phi_k, \theta_k)$ is given by changes by $\Delta \rho_k$, $\Delta \theta_k$, and $\Delta \phi_k$ in ρ, θ and ϕ. Such a spherical wedge has one edge a circular arc of length $\rho_k \Delta \phi_k$, another edge a circular arc of length $\rho_k \sin \phi_k \Delta \theta_k$, and thickness $\Delta \rho_k$. The spherical wedge closely approximates a cube of these dimensions when $\Delta \rho_k$, $\Delta \theta_k$ and $\Delta \phi_k$ are all small. It can be shown that the volume of this spherical wedge ΔV_k is $\Delta V_k = \rho_k^2 \sin \phi_k \Delta \phi_k \Delta \theta_k$ for a point $(\rho_k, \phi_k, \theta_k)$ chosen inside the wedge. The corresponding Riemann sum for a function $F(\rho, \phi, \theta)$ is

$$S_n = \sum_{k=1}^{n} F(\rho_k, \phi_k, \theta_k) \, \rho_k^2 \sin \phi_k \, \Delta \rho_k \, \Delta \phi_k \Delta \theta_k.$$

As the norm of the partition approaches zero, and the spherical wedges get smaller, the Riemann sums have a limit when F is continuous:

$$\lim_{n \to \infty} S_n = \iiint_E F(\rho, \phi, \theta) \, dV = \iiint_E F(\rho, \phi, \theta) \, \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta.$$

In spherical coordinates we have

$$dV = \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta.$$

To evaluate integrals in spherical coordinates, we usually integrate first with respect to ρ. The procedure for finding the limits of integration is shown below. We restrict our attention to integrating over domains that are solids of revolution about the z-axis (or portions thereof) and for which the limits for θ and ϕ are constant.

A PROCEDURE FOR FINDING THE LIMITS OF INTEGRATION FOR TRIPLE INTEGRALS IN SPHERICAL COORDINATES

We now give a procedure for finding limits of integration that applies for many regions in space. Regions that are more complicated, and for which this procedure fails, can often be split up into pieces on which the procedure works.

When faced with evaluating

$$\int \int \int_E F(\rho, \phi, \theta) \, dV$$

over a region E in space in spherical coordinates, integrating first with respect to ρ, then with respect to ϕ, and finally with respect to θ take the following steps:

STEP 1 SKETCH: *Sketch the region E along with its projection D_{xy} on the xy-plane. Label the surfaces and curves that bound E and D_{xy}.*

STEP 2. FIND THE ρ-LIMITS OF INTEGRATION: *Draw a ray M from the origin through E making an angle ϕ with the positive z-axis. Also draw the projection M on the xy-plane (call such a projection L). The ray L makes an angle θ with the positive x-axis. As ρ increases, M enter E at $\rho = g_1(\phi, \theta)$ and leaves at $\rho = g_2(\phi, \theta)$. These are the ρ-limits of integration.*

STEP 3. FIND THE ϕ-LIMITS OF INTEGRATION: *For any given θ, the angle ϕ that M makes with the positive z-axis runs from $\phi = \phi_{min}$ to $\phi = \phi_{max}$. These are the ϕ-limits of integration.*

STEP 4. FIND THE θ-LIMITS OF INTEGRATION: *As the ray L sweeps across D_{xy}, the angle θ it makes with the positive x-axis runs from $\theta = \alpha$ to $\theta = \beta$. These are the θ-limits of integration. The integral is*

$$\int \int \int_E F(\rho, \phi, \theta) \, dV = \int_{\theta=\alpha}^{\theta=\beta} \int_{\phi=\phi_{min}}^{\phi=\phi_{max}} \int_{\rho=g_1(\phi,\theta)}^{\rho=g_2(\phi,\theta)} F(\rho, \phi, \theta) \, \rho^2 \sin \phi \, d\rho \, dr\phi \, d\theta.$$

▷ **Exercise 6.1** *Use the procedure just above to calculate the volume of the "ice-cream cone" E cut from the solid sphere $\rho \leq 1$ by the cone $\phi = \pi/3$.*

YOUR EXAMINATION PAPER BEGINS HERE

INSTRUCTIONS FOR SOLVING THE PROBLEMS:

0. It is important to Read these Notes First before solving any of the problems.

1. Draw a labeled picture of the region of integration for each Double Integral that you do.

2. Give a set description of the solid region of integration for each triple integral that you do. Also state whether the region of integration is x-simple, y-simple, r-simple or θ-simple.

3. Give a set description of the region of integration for each double integral and triple that you do. Also state the type of region of integration.

4. Each Integral (Double or Triple) require a 3 step or 4 step procedure described above. Show how you are using these steps. Draw a sketch (of surfaces and regions of integration) where possible.

5. All problems are to be done in a neat and readable way!

6. This is an Exam. The Rules Apply (Don't forget that). Do all the problems!

Problem #1

A) *Setup a double integral in (Rectangular Coordinates) that would yield the area of plane circular disk $(x^2 + y^2 \leq R^2)$ of radius $R > 0$ centered at the origin $(0,0)$ of the xy-plane.*

B) *Evaluate the double integral in part A using polar coordinates.*

Problem #2

Use double integrals to derive the given formula for the surface area of a right circular cone of radius R and height H. The surface area of a cone is given by the formula

$$\pi R^2 + \pi R\sqrt{R^2 + H^2}.$$

HINT: Consider the right circular cone surface with equation $z = f(x, y) = H - \frac{H}{R}\sqrt{x^2 + y^2}$

Problem #3

Use double integrals to derive the given formula for the surface area of a cap of a sphere of radius R and height H where $0 < H < R$. (The cap of a sphere is the portion of the sphere bounded below by the plane $z = R - H$ and bounded above by the plane $z = R$). The surface area of a cap of a sphere with radius R is given by the formula

$$2\pi RH.$$

QUESTION: Consider what happens to the above formula as H approaches R (Explain). Is the result of this consideration what you expected?

Problem #4

Use a double integral of the form

$$\int\int_D f(x, y)\, dA$$

derive the given formula for the volume of the sphere $x^2 + y^2 + z^2 = R^2$ described in Exercise 0. The volume of such a sphere is given by the formula

$$\frac{4}{3}\pi R^3$$

Problem #5

Use double integrals to derive the given formula for the volume of a right circular cone of radius R and height H. The volume of a cone is given by the formula

$$\frac{1}{3}\pi R^2 H.$$

Hint: Consider the lower cone and you may want to shift this lower cone so that the base of the solid under this lower cone is in the xy-plane. $f(x,y) = H - \frac{H}{R}\sqrt{x^2 + y^2}$

Use a double integral of the form $\int\int_D f(x,y)\ dA$ (Pay close attention to how you write $f(x,y)$) to derive the given formula for the volume a of a cap of a sphere of radius R $(x^2\ +\ y^2\ +\ z^2\ =\ R^2)$ and height H where $0\ <\ H\ <\ R$. (The cap of a sphere is the portion of the sphere bounded below by the plane $z\ =\ R\ -\ H$ and bounded above by the plane $z\ =\ R$). The volume of a cap of a sphere with radius R is given by the formula

$$\frac{1}{3}\pi H^2(3R\ -\ H)$$

and $f(x,y)\ =\ \sqrt{R^2\ -\ (x^2\ +\ y^2)}\ -\ (R\ -\ H)$

QUESTION: Consider what happens to the above formula as H approaches R (Explain?). Is the result of this consideration what you expected?

Problem #7

Write out the other 3 orders of integration for the triple integral given above.

Problem #8

Suppose E is a **Type II solid region in space**. In particular, if the plane region D_{yz}, the projection of the solid region E on the yz-plane is a y-**Simple** plane region or an z-**Simple** plane region, then write out two formulas for

$$\iiint\limits_{E} F(x,y,z)\, dV = \iint\limits_{D_{yz}} \left[\int_{u_1(y,z)}^{u_2(y,z)} F(x,y,z)\ dx \right] dA.$$

Problem #9

Define a **Type III** solid region E.

Suppose E is a **Type III solid region in space** and if your plane region, D_{xz}, the projection of the solid region E on the xz-plane is an x-**Simple** plane region or a z-**Simple** plane region, then write out two formulas again for

$$\int\int\int_E F(x,y,z)\,dV \;=\; \int\int_{D_{xz}} \left[\int_{u_1(x,z)}^{u_2(x,z)} F(x,y,z)\,dy\right] dA.$$

SECTION 17
Problem # 11

Use triple integrals to find the volume of a **right-circular cone** of radius R and height H. Hint: $z = f(x, y) = \frac{H}{R}\sqrt{x^2 + y^2}$ is a cone surface.

Use triple integrals to find the volume of a **right-circular cylinder** of radius R and height H. Hint: $x^2 + y^2 = R^2$ is the surface of a cylinder.

▷ **Exercise 18.1** FIRST STUDY CAREFULLY THIS WORKED OUT EXERCISE: Use a triple integral to find the volume of the solid enclosed by the cylinder $x^2 + y^2 = 9$ and the planes $y + z = 5$ and $z = 1$.

Solution Let E denote this Type I solid enclosed by the cylinder $x^2 + y^2 = 9$ and the planes $y + z = 5$ and $z = 1$. Then $w = F(x, y, z) = 1$ everywhere on E so that

$$V(E) = \int \int \int_E dV$$

where $z = u_1(x, y) = 1$ (the plane $z = 1$) and $z = u_2(x, y) = 5 - y$ and $D_{xy} = \{(x, y) | \ x^2 + y^2 \le 9\}$. Now

$$V(E) = \int \int_{D_{xy}} \left[u_2(x, y) - u_1(x, y) \right] dA$$

$$= \int \int_{D_{xy}} (4 - y) \, dA$$

$$= \int_{-3}^{3} \int_{-\sqrt{9 - x^2}}^{\sqrt{9 - x^2}} (4 - y) \, dy \, dx \quad \{\text{I believe that is integral can be done using polar coordinates}\}$$

$$= \int_{-3}^{3} \left[4y - \frac{y^2}{2} \right]_{y = -\sqrt{9 - x^2}}^{y = \sqrt{9 - x^2}} dx$$

$$= \int_{-3}^{3} \left[4\sqrt{9 - x^2} - \frac{9 - x^2}{2} + 4\sqrt{9 - x^2} + \frac{9 - x^2}{2} \right] dx$$

$$= 8 \int_{-3}^{3} \sqrt{9 - x^2} \, dx \quad \{\text{USE THE METHOD OF TRIGONOMETRIC SUBSTITUTION (PROBLEM \#13)}\}$$

$$= 8 \left[\frac{x}{2} \sqrt{9 - x^2} + \frac{9}{2} \arcsin\left(\frac{x}{3}\right) \right]_{x = -3}^{x = 3}$$

$$= 36 \left(\arcsin(1) - \arcsin(-1) \right) = 36 \cdot \left(\frac{\pi}{2} - \left(-\frac{\pi}{2} \right) \right)$$

$$= 36 \, \pi \quad \text{cubic units.}$$

▷ **Exercise 18.2** TRY THIS EXERCISE IF YOU DARE: Consider a sphere $x^2 + y^2 + z^2 = R^2$ of radius $R > 0$ centered at the origin. Let E be the solid region in space bounded by the xy-plane, the xz-plane, the plane $y = x$ and the above sphere. Now let $F(x, y, z) = 5z$ be a continuous function over E. Can you write (Do not evaluate)

$$\int \int \int_E F(x, y, z) \, dV. \quad \{\text{This is not a volume integral!}\}$$

using our 3 main coordinate systems (Rectangular, cylindrical and spherical coordinates)?

Problem #13

Use polar coordinates to evaluate the integral in the above Exercise 18.1.

Setup and evaluate a Double Integral that calculates the area for the quadrilateral region in the xy-plane bounded by the lines $y = H$, $y = 0$, $x = 0$ and also the line containing the points $(a, 0)$ and (b, H) where $a, b, H > 0$ and $b < a$.

Setup and evaluate a Double Integral to calculate the area for the sector in the polar plane bounded by the rays $\theta = 0$ and $\theta = \theta_0$ and the circle $x^2 + y^2 = R^2$ where θ_0 is any number satisfying the inequality $0 < \theta_0 < 2\pi$.

Problem #16

Setup only (but do not evaluate) a triple integral using Rectangular, cylindrical and spherical coordinates for the volumes of the four 3-dimensional solids mentioned in PROBLEMS 1 through 6

Setup and evaluate a double integral

$$\int\int_D f(x,y)\, dA$$

to show that the volume of a solid rectangular box B with length l, width w, and height h is

$$lwh$$

.

A) Give an appropriate set description of D_{xy}

B) Give the proper definition of $f(x,y)$.

C) Setup and evaluate your double integral $\int\int_{D_{xy}} f(x,y)\, dA$.

Setup and evaluate a double integral

$$\int\int_D f(x,y)\,dA$$

to show that the area of a rectangular region D_{xy} of the xy-plane is the length l of the region times it width w.

A) Give an appropriate set description of D_{xy}

B) Give the proper definition of $f(x,y)$.

C) Setup and evaluate your double integral $\int_{D_{xy}}\int f(x,y)\,dA$.

Problem #19

Consider a right circular cylinder of radius R and height H (which includes the top, bottom and the sides) whose surface area $A(S)$ is given as

$$2\pi R^2 + 2\pi RH.$$

where (S_1 {the top}, S_2 {the bottom}, and S_3 {the side}) denotes various pieces of your cylinder $S = S_1 \cup S_2 \cup S_3$

Use double integrals

$$A(S) = 2A(S_1) + A(S_3) = 2 \iint\limits_{D_1} dA + 4 \iint\limits_{D_2} \sqrt{f_x(x,y)^2 + f_y(x,y)^2 + 1}\ dA$$

to derive the above formula. You must give a description and a sketch of the plane regions D_1 and D_2). D_2 is the projection of the graph of $f(x,y)$ (which you must supply!) in the xy-plane.

Hint: Sketch the right circular cylinder S so the either the x or y-axis acts as a central axis of rotation (or symmetry). Consider only the portion of the surface S_3 in the first octant. This should give you some idea of how to define the surface S_3 as a function f of two variables. Furthermore, you should carefully read through Exercise 0 above before taking on this problem!.

Problem #20

A "torus" is a (donut looking) surface obtained by rotating a circle of radius $R > 0$ around a line l. The center of the circle is $H > 0$ units away from the line and so $H > R > 0$. Such a line does not intersect the circle at any point. Setup double integrals that would give the volume and surface area of a torus. Hint: Think of a bagel sliced horizontally and placing the top half of the bagel in the xy-plane with its center at the origin $(0, 0, 0)$.