COMPACTIFICATION OF REDUCTIVE GROUP SCHEMES

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Abstract. Let G be an isotrivial reductive group over a scheme S. We construct a smooth projective S-scheme containing G as a fiberwise-dense open subscheme equipped with left and right actions of G which extend the translation actions of G on itself. This verifies a conjecture of Česnavičius [Čes22]. When G is adjoint, we recover fiberwise the wonderful compactification. Finally, we give an example of a non-isotrivial torus admitting no equivariant compactification.

1. Introduction

A reductive group scheme G over a base scheme S is called *locally isotrivial* if each point $s \in S$ has a Zariski open neighborhood $U_s \hookrightarrow S$ admitting a finite étale cover $U'_s \to U_s$ such that $\mathrm{G} \times_S U'_s$ is split. This is equivalent to local isotriviality of the central torus [SGA3, Exposé XXIV, Théorème 4.1.5]. A reductive group is called *isotrivial* if U_s can be chosen to be S.

Česnavičius conjectured that every isotrivial reductive group scheme G has a compactification equipped with a left action of G extending that of G on itself:

Conjecture 1.1 ([Čes22, Conjecture 6.2.3])**.** *For an isotrivial reductive group* G *over a Noetherian scheme* S*, there are a projective, finitely presented* S*-scheme* $\overline{\mathrm{G}}$ *equipped with a left* G*-action and a* G*-equivariant* S*-fiberwise dense open immersion*

$$\mathrm{G} \hookrightarrow \overline{\mathrm{G}}.$$

The case where G is an isotrivial torus was established in [Čes22, §6.3] (see also [Kun23, Proposition 7.2.1]). For adjoint G, Li [Li25] confirmed the conjecture using a variant of the Artin–Weil method of birational group laws. In this paper, we construct a smooth projective $\mathrm{G} \times_S \mathrm{G}$-equivariant compactification for any isotrivial reductive group G over an arbitrary base S, thus verifying Conjecture 1.1:

Theorem 1.2. *Let* G *be an isotrivial reductive group scheme over a scheme* S*. Then there exists a smooth projective* S*-scheme* $\overline{\mathrm{G}}$ *containing* G *as a fiberwise-dense open subscheme equipped with a left and right action of* G *extending that on* G *given by left and right multiplication.*

Before proceeding, let us briefly place our results in the context of the existing theory of wonderful compactifications. Over algebraically closed fields, the wonderful compactification for adjoint reductive groups was introduced by de Concini and Procesi [DP83], and extended to arbitrary characteristic by Strickland [Str87]. In an unpublished article, Gabber constructed it for Chevalley group schemes over $\operatorname{Spec} \mathbf{Z}$.

Most recently, Li [Li25] constructed the wonderful compactification for adjoint G over arbitrary bases. Li's method relies on the existence of a 'big cell' in the wonderful compactification $\overline{\mathrm{G}}$. Assuming G is split, this cell is an open dense subscheme $\overline{\Omega} \subset \overline{\mathrm{G}}$ isomorphic to the affine space $\mathrm{U}^- \times_S \mathbf{A}^\ell_S \times_S \mathrm{U}^+$, where ℓ is the rank of G and $\mathrm{U}^\pm$ are the unipotent radicals of opposite Borel subgroups. Since the $\mathrm{G} \times_S \mathrm{G}$-translates of $\overline{\Omega}$ cover $\overline{\mathrm{G}}$, one can expect to recover $\overline{\mathrm{G}}$ as a suitable quotient of $\mathrm{G} \times_S \overline{\Omega} \times_S \mathrm{G}$.

However, in the general non-adjoint case, the correct analogue of $\overline{\Omega}$ is unclear. To overcome this difficulty, we employ a completely different method using the theory of Vinberg monoids and its variants, an approach successfully applied by Martens–Thaddeus [MT16] over algebraically closed fields of characteristic zero.

Date: January 17th, 2026.

When G is semisimple, we can do more:

Theorem 1.3. *Let G be a semisimple reductive group scheme over S. Then there exist projective S-schemes $\overline{\mathrm{G}}$ and $\overline{\mathrm{G}}_{\mathrm{ad}}$ containing G and G_{ad}, respectively, as fiberwise-dense open subschemes, equipped with left and right actions of G and G_{ad} extending those on G and G_{ad} by themselves such that*

- *there is an equivariant morphism $\overline{\mathrm{G}} \to \overline{\mathrm{G}}_{\mathrm{ad}}$ extending the central isogeny $\mathrm{G} \to \mathrm{G}_{\mathrm{ad}}$ which is a normalization over each geometric point of S,*
- *$\overline{\mathrm{G}}_{\mathrm{ad}}$ is S-smooth and agrees with the de Concini–Procesi wonderful compactification over geometric points of S, and*
- *its boundary $\overline{\mathrm{G}}_{\mathrm{ad}} \setminus \mathrm{G}_{\mathrm{ad}}$ is the union of S-smooth relative effective Cartier divisors with relative normal crossings[1].*

All our constructions commute with base change on S. We also verify that our construction agrees with that of [Li25] for adjoint G (Proposition 4.5).

Finally, we show that the isotriviality assumption in Conjecture 1.1 is essential. By analyzing the standard example of a non-isotrivial torus over a nodal rational curve given in [SGA3, Exposé X, §1.6], we obtain the following negative result:

Theorem 1.4. *Let S be the nodal cubic curve over an algebraically closed field, and let T be a non-isotrivial torus over S (constructed in §5). Then there exists no projective S-scheme $\overline{\mathrm{T}}$ containing T as a fiberwise-dense open subscheme such that the left action of T on itself extends to $\overline{\mathrm{T}}$.*

The main difficulty in the proof of Theorem 1.4 is that a potential compactification $\overline{\mathrm{T}} \to S$ need not have normal fibers, preventing the direct application of the classification theory of toric varieties. We overcome this by using Brion's results on the linearization of line bundles [Bri15] to reduce the problem to the setting of toric schemes over discrete valuation rings, where we derive a contradiction.

Remark 1.5. Although we work with base schemes, it is easily seen that proofs of Theorems 1.2 and 1.3 go through even when S is an algebraic space.

Acknowledgments. The author is grateful to Bjorn Poonen and Kęstutis Česnavičius for helpful conversations. The author also thanks Shang Li and Arnab Kundu for their comments on earlier drafts. This work was supported in part by Simons Foundation grant #402472 to Bjorn Poonen.

2. Affine monoids

2.1. **Vinberg monoids.** Let $\mathbf{G}$ be a split reductive group scheme over a connected base scheme S. Let $\mathbf{T}$ be the abstract Cartan. Define $\mathbf{G}_+ := \mathbf{T} \times_S^{\mathrm{Z}_{\mathbf{G}}} \mathbf{G}$ and $\mathbf{T}_{\mathrm{ad}} := \mathbf{T}/\mathrm{Z}_{\mathbf{G}}$, where $\mathrm{Z}_{\mathbf{G}}$ denotes the center of $\mathbf{G}$. Choose a Borel $\mathbf{B} \subset \mathbf{G}$. Evaluating at the system of simple roots gives a canonical toric embedding $\mathbf{T}_{\mathrm{ad}} \hookrightarrow \mathbf{T}_{\mathrm{ad}}^+$ where $\mathbf{T}_{\mathrm{ad}}^+$ is an affine space over S of relative dimension equal to that of $\mathbf{T}_{\mathrm{ad}}$. Here, $\mathbf{T}_{\mathrm{ad}}^+$ is viewed as an S-monoid with unit group $\mathbf{T}_{\mathrm{ad}}$. The *Vinberg monoid* is a certain reductive monoid scheme $\mathrm{V}_{\mathbf{G}}$ over S equipped with an abelianization homomorphism $\mathfrak{a}\colon \mathrm{V}_{\mathbf{G}} \to \mathbf{T}_{\mathrm{ad}}^+$. The properties relevant to us are recorded in Theorem 2.1.1.

Let us briefly recall the construction of the Vinberg monoid given in [XZ19, §3.2] over an algebraically closed field. The same works over $\mathbf{Z}$ and hence for any split reductive group over an arbitrary base. Assume $S = \operatorname{Spec} \mathbf{Z}$ without any loss of generality. Let $\mathbf{X}^\bullet(\mathbf{T}_{\mathrm{ad}})_{\mathrm{pos}}$ be the submonoid of $\mathbf{X}^\bullet(\mathbf{T})$ generated by the simple roots. Let $\mathbf{X}^\bullet(\mathbf{T})^+$ be the submonoid of dominant characters, and let $\mathbf{X}^\bullet(\mathbf{T})^+_{\mathrm{pos}}$ be the submonoid

[1]Following [SGA1, Exposé XIII, §2.1], for an S-scheme X, we say that a relative Cartier divisor $D \subset X$ is *strictly with relative normal crossings* if there exists a finite family $(f_i \in \Gamma(X, \mathcal{O}_X))_{i \in I}$ such that (1) $D = \bigcup_{i \in I} \mathrm{V}_X(f_i)$, and (2) for every $x \in \operatorname{Supp}(D)$, X is smooth at x over S, and the closed subscheme $\mathrm{V}((f_i)_{i \in I(x)}) \subset X$ is smooth over S of codimension $|I(x)|$, where $I(x) = \{i \in I \mid f_i(x) = 0\}$. The divisor D has *relative normal crossings* if étale locally on X it is strictly with relative normal crossings.

generated by $\mathbf{X}^\bullet(\mathbf{T})^+$ and $\mathbf{X}^\bullet(\mathbf{T}_{\mathrm{ad}})_{\mathrm{pos}}$. We equip $\mathbf{X}^\bullet(\mathbf{T})$ with the partial order $\preceq$ defined by $\lambda \preceq \mu$ if $\mu - \lambda \in \mathbf{X}^\bullet(\mathbf{T}_{\mathrm{ad}})_{\mathrm{pos}}$. We write $\lambda \prec \mu$ if $\lambda \preceq \mu$ and $\lambda \neq \mu$.

The coordinate ring $\mathcal{O}(\mathbf{G})$ admits a canonical multi-filtration indexed by $\mathbf{X}^\bullet(\mathbf{T})^+_{\mathrm{pos}}$, induced by the $\mathbf{G} \times \mathbf{G}$ action on $\mathcal{O}(\mathbf{G})$ via left and right translation, given by setting $\mathrm{fil}_\nu \mathcal{O}(\mathbf{G})$, $\nu \in \mathbf{X}^\bullet(\mathbf{T})^+_{\mathrm{pos}}$, as the maximal $\mathbf{G} \times \mathbf{G}$-submodule of $\mathcal{O}(\mathbf{G})$ such that all its weights $(\lambda, \lambda') \in \mathbf{X}^\bullet(\mathbf{T}) \times \mathbf{X}^\bullet(\mathbf{T})$ satisfy $\lambda \preceq -w_0(\nu)$ and $\lambda' \preceq \nu$. Here, w_0 is the longest element of the Weyl group of $\mathbf{G}$. Each piece $\mathrm{fil}_\nu \mathcal{O}(\mathbf{G})$ is finite free over $\mathbf{Z}$ and the associated graded is given by

$$\mathrm{gr}\, \mathcal{O}(\mathbf{G}) := \bigoplus_{\nu \in \mathbf{X}^\bullet(\mathbf{T})^+_{\mathrm{pos}}} \frac{\mathrm{fil}_\nu \mathcal{O}(\mathbf{G})}{\sum_{\lambda \prec \nu} \mathrm{fil}_\lambda \mathcal{O}(\mathbf{G})} = \bigoplus_{\nu \in \mathbf{X}^\bullet(\mathbf{T})^+} \mathrm{S}_{-w_0(\nu)} \otimes \mathrm{S}_\nu,$$

where S_ν denotes the Schur module of highest weight ν, i.e., the induced $\mathbf{G}$-module from the character $-\nu$ of $\mathbf{B}$ [XZ19, Lemma 3.2.1 (4)].

The Vinberg monoid $\mathrm{V}_{\mathbf{G}}$ is defined as the spectrum of the Rees algebra associated to this filtration:

$$\mathrm{V}_{\mathbf{G}} := \mathrm{Spec}\left(\bigoplus_{\nu \in \mathbf{X}^\bullet(\mathbf{T})^+_{\mathrm{pos}}} \mathrm{fil}_\nu \mathcal{O}(\mathbf{G}) \right),$$

endowed with the natural (co)multiplication map. It is an affine monoid $\mathbf{Z}$-scheme of finite type which admits a monoid homomorphism $\mathfrak{a} : \mathrm{V}_{\mathbf{G}} \to \mathbf{T}^+_{\mathrm{ad}} := \mathrm{Spec}\, \mathbf{Z}[\mathbf{X}^\bullet(\mathbf{T}_{\mathrm{ad}})_{\mathrm{pos}}]$ induced by the inclusion of the ring $\mathbf{Z}[\mathbf{X}^\bullet(\mathbf{T}_{\mathrm{ad}})_{\mathrm{pos}}]$ into the Rees algebra. The formation of $(\mathrm{V}_{\mathbf{G}}, \mathfrak{a})$ commutes with base change on S. Indeed, for any ring R, the injection of filtered R-algebras $R \otimes \sum_\nu \mathrm{fil}_\nu \mathcal{O}(\mathbf{G}) \hookrightarrow \sum_\nu \mathrm{fil}_\nu \mathcal{O}(\mathbf{G}_R)$ induces an isomorphism on associated graded algebras since the formation of Schur modules commutes with base change. Thus, $R \otimes \sum_\nu \mathrm{fil}_\nu \mathcal{O}(\mathbf{G}) \hookrightarrow \sum_\nu \mathrm{fil}_\nu \mathcal{O}(\mathbf{G}_R)$ is actually an isomorphism of filtered rings.

Theorem 2.1.1. *The S-monoid $\mathrm{V}_{\mathbf{G}}$ fits into a Cartesian diagram*

$$\begin{tikzcd} \mathbf{G}_+ \arrow[r, hook] \arrow[d] & \mathrm{V}_{\mathbf{G}} \arrow[d] \\ \mathbf{T}_{\mathrm{ad}} \arrow[r, hook] & \mathbf{T}^+_{\mathrm{ad}} \end{tikzcd}$$

and

- $\mathbf{G}_+$ *is the unit group of* $\mathrm{V}_{\mathbf{G}}$*,*
- *the* $\mathbf{T} \times_S \mathbf{G} \times_S \mathbf{G}$ *action on* $\mathbf{G}_+$ *given by* $\mathbf{T}$ *acting on the first component and* $\mathbf{G}$ *acting on itself by left and right multiplication extends to an action on* $\mathrm{V}_{\mathbf{G}}$*,*
- $\mathfrak{a}$ *is faithfully flat.*

Proof. See [XZ19, Proposition 3.2.2]. □

2.2. **Nondegenerate locus.** Choose a Cartan *sub*group $c\colon \mathbf{T} \hookrightarrow \mathbf{B}$. Let $\mathbf{U}_+$ be the unipotent radical of $\mathbf{B}$ and $\mathbf{U}_-$ be that of the opposite Borel.

Proposition 2.2.1. *There is a* canonical section $\bar{\mathfrak{s}}\colon \mathbf{T}^+_{\mathrm{ad}} \to \mathrm{V}_{\mathbf{G}}$ *extending* $\mathfrak{s}\colon t \mapsto (t, c(t)) \pmod{\mathrm{Z}_{\mathbf{G}}}$ *on respective unit groups.*

Proof. We identify $\mathbf{T}$ with the image of c and assume $S = \mathrm{Spec}\, \mathbf{Z}$. The restriction ring map $r : \mathcal{O}(\mathbf{G}) \to \mathcal{O}(\mathbf{T})$ sends the filtered piece $\mathrm{fil}_\nu \mathcal{O}(\mathbf{G})$ into the span of characters e^λ satisfying $\lambda \preceq \nu$. Here, e^λ denotes the character corresponding to λ.

We define a retraction $\phi : \mathcal{O}(\mathrm{V}_{\mathbf{G}}) \to \mathcal{O}(\mathbf{T}^+_{\mathrm{ad}})$ by mapping a homogeneous element $f_\nu \in \mathrm{fil}_\nu \mathcal{O}(\mathbf{G})$ to the coefficient of its highest weight term. Explicitly, we define:

$$\phi(f_\nu) = (\text{coefficient of } e^\nu \text{ in } r(f_\nu)) \cdot e^\nu.$$

This map is multiplicative. Consider homogeneous elements $f \in \mathrm{fil}_\nu \mathcal{O}(\mathbf{G})$ and $g \in \mathrm{fil}_\mu \mathcal{O}(\mathbf{G})$. Their product fg lies in $\mathrm{fil}_{\nu+\mu} \mathcal{O}(\mathbf{G})$. The restriction map is a ring homomorphism, so $r(fg) = r(f)r(g)$. Since $r(f)$ involves only weights $\lambda \preceq \nu$ and $r(g)$ involves only weights $\lambda' \preceq \mu$, the weight $\nu + \mu$ in the product $r(f)r(g)$ can only arise from the product of the term e^ν in $r(f)$ and e^μ in $r(g)$. Indeed, if $\lambda + \lambda' = \nu + \mu$ with $\lambda \preceq \nu$ and $\lambda' \preceq \mu$, then $(\nu - \lambda) + (\mu - \lambda') = 0$. Since $\nu - \lambda$ and $\mu - \lambda'$ are non-negative integer linear combinations of simple roots, this implies $\lambda = \nu$ and $\lambda' = \mu$.

Thus, $\phi(fg) = \phi(f)\phi(g)$. Since ϕ maps the element $1 \cdot e^\nu \in \mathcal{O}(\mathrm{V}_\mathbf{G})$ to $e^\nu \in \mathcal{O}(\mathbf{T}^+_{\mathrm{ad}})$, it forms the required section $\bar{\mathfrak{s}}$ of the abelianization. □

Proposition 2.2.2. *The multiplication map* $m\colon \mathbf{U}_- \times_S \mathbf{T} \times_S \mathbf{T}^+_{\mathrm{ad}} \times_S \mathbf{U}_+ \to \mathrm{V}_\mathbf{G}$ *given by* $(u_-, t, a, u_+) \mapsto u_- t \bar{\mathfrak{s}}(a) u_+$ *is an open embedding.*

Proof. We first prove this in the case when S is the spectrum of an algebraically closed field k. By [AB04, §7.5], $\mathrm{V}_\mathbf{G}$ is a normal irreducible k-variety. Therefore, m is a birational morphism of normal irreducible k-varieties. By Zariski's main theorem, it suffices to verify that m is injective on k-points. Checking this is equivalent to showing that

$$u_- t \bar{\mathfrak{s}}(a_1) u_+ = \bar{\mathfrak{s}}(a_2) \implies u_- = u_+ = t = 1, a_1 = a_2$$

for $u_- \in \mathbf{U}^-(k), u_+ \in \mathbf{U}^+(k), t \in \mathbf{T}(k), a_1, a_2 \in \mathbf{T}^+_{\mathrm{ad}}(k)$. We may assume that $a_1 = e_I$ where e_I is an idempotent corresponding to a subset I of the set of simple roots without any loss of generality. There is a map from the source of m to $\mathbf{T}^+_{\mathrm{ad}}$ given by $(u_-, t, a, u_+) \mapsto ta$. Then m is a $\mathbf{T}^+_{\mathrm{ad}}$-morphism where the target is viewed as a $\mathbf{T}^+_{\mathrm{ad}}$-scheme via $\mathfrak{a}$. Therefore, $te_I = a_2$ as elements of $\mathbf{T}^+_{\mathrm{ad}}$. Hence,

$$u_- t \bar{\mathfrak{s}}(e_I) u_+ = (t, t)\bar{\mathfrak{s}}(e_I) \implies u_- \bar{\mathfrak{s}}(e_I) u_+ = (1, t)\bar{\mathfrak{s}}(e_I) \implies (t^{-1} u_-)\bar{\mathfrak{s}}(e_I) u_+ = \bar{\mathfrak{s}}(e_I).$$

In [DG16, Appendix C], it is shown that the $\mathbf{P} \times_k \mathbf{P}^-$-stabilizer of $\bar{\mathfrak{s}}(e_I)$, where $(\mathbf{P}, \mathbf{P}^-)$ is a certain pair of opposite parabolics of $\mathbf{G}$ depending on I, is $\mathbf{P} \times_\mathbf{M} \mathbf{P}^-$, where $\mathbf{M}$ is the Levi factor. It is also shown that

$$\mathbf{P} = \{g \in \mathbf{G} \mid g \cdot \bar{\mathfrak{s}}(e_I) = \bar{\mathfrak{s}}(e_I) \cdot g \cdot \bar{\mathfrak{s}}(e_I)\} \quad \text{and} \quad \mathbf{P}^- = \{g \in \mathbf{G} \mid \bar{\mathfrak{s}}(e_I) \cdot g = \bar{\mathfrak{s}}(e_I) \cdot g \cdot \bar{\mathfrak{s}}(e_I)\}.$$

From the above descriptions, one deduces that the $\mathbf{G} \times_k \mathbf{G}$-stabilizer of $\bar{\mathfrak{s}}(e_I)$ is actually contained in the $\mathbf{P} \times_k \mathbf{P}^-$-stabilizer and hence equals $\mathbf{P} \times_\mathbf{M} \mathbf{P}^-$. We conclude that $t = u_- = u_+ = 1$ and hence m is injective on k-points.

Let us now consider the case of general S. Since the formation of m commutes with base change, it follows by fibral criteria that m is étale [Stacks, Tag 039E]. Thus, m is an étale monomorphism and we conclude by [Stacks, Tag 025G]. □

Lemma 2.2.3. *Let H be a group scheme locally of finite presentation and flat over a scheme S acting on an S-scheme X. Let $U \subseteq X$ be an open subscheme. Then the saturation $H \cdot U$ of U is an H-stable open subscheme of X. If in addition*

- *U is S-flat then so is $H \cdot U$,*
- *U is locally of finite presentation over S then so is $H \cdot U$,*
- *U is S-smooth then so is $H \cdot U$.*

Proof. The H-saturation $H \cdot U$ is defined as the image of the composition

$$H \times_S U \hookrightarrow H \times_S X \xrightarrow{(h,x) \mapsto (h, h \cdot x)} H \times_S X \xrightarrow{(h,x) \mapsto x} X.$$

The middle map is an isomorphism and the last map is a projection. Since flat morphisms of locally finite presentation are universally open [Stacks, Tag 01UA], it follows that $H \cdot U$ is open. The induced map $H \times_S U \to H \cdot U$ is fppf. Since flatness and being locally of finite presentation are fppf local on source [Stacks, Tag 06ET, Tag 06EV], S-flatness of $H \cdot U$ is equivalent to that of $H \times_S U$ and the same is true for being locally of finite presentation. When U is smooth, we reduce to the case of S an algebraically closed point by [Stacks, Tag 01V8], in which case smoothness follows because translations of U cover $H \cdot U$. □

The $\mathbf{G} \times_S \mathbf{G}$-saturation of the image of m in Proposition 2.2.2 is called the *nondegenerate locus*, denoted $\mathrm{V}^\circ_{\mathbf{G}}$. It is independent of the choice of Cartan subgroup, $\mathbf{T} \times_S \mathbf{G} \times_S \mathbf{G}$-stable, and contains $\mathbf{G}_+$ by construction.

Corollary 2.2.4. *$\mathfrak{a}$ restricted to $\mathrm{V}^\circ_{\mathbf{G}}$ is smooth.*

3. Proof of Theorem 1.2

3.1. **Preliminary reductions.** Let $\mathbf{G}$ be the split form of G. Fix a pinning of $\mathbf{G}$. This canonically determines a presentation of the automorphism group scheme $\mathrm{Aut}_{\mathbf{G}/S}$ as a semidirect product of $\mathbf{G}_{\mathrm{ad}}$ and $\mathrm{Out}_{\mathbf{G}/S}$ and hence a unique quasi-split inner form $\mathbf{G}_{\text{q-ép}}$ of G endowed with a quasi-pinning[1] [SGA3, Exposé XXIV, Corollaire 3.12].

Since G is assumed to be isotrivial, the étale local $\mathbf{G}_{\mathrm{ad}}$-torsor corresponding to $\mathbf{G}_{\text{q-ép}}$ is isotrivial too. As finite étale morphisms satisfy effective descent for projective schemes, it suffices to prove the theorem for $\mathbf{G}_{\text{q-ép}}$. Indeed, this essentially boils down to the fact that finite group quotients of projective schemes are representable by projective schemes.

Choose a finite étale Galois cover $S' \to S$ splitting $\mathbf{G}_{\text{q-ép}}$ with Galois group Γ. The data of $\mathbf{G}_{\text{q-ép}}$ is then equivalent to the data of a Γ-action on the $\mathbf{G}_{\text{q-ép}} \times_S S'$. Thus, it suffices to find an equivariant compactification equipped an action of Γ extending that on $\mathbf{G}_{\text{q-ép}} \times_S S'$. The quasi-pinning on $\mathbf{G}_{\text{q-ép}}$ induces a pinning on $\mathbf{G}_{\text{q-ép}} \times_S S'$. Note that Γ additionally acts on the pinning of $\mathbf{G}_{\text{q-ép}} \times_S S'$.

From now, we replace our setup with a pinned reductive group $(\mathbf{G}, \mathbf{B}, \mathbf{T}, \{u_\alpha\}_{\alpha\in\Delta})$ over S equipped with an action of a finite group Γ.

3.2. **Cox-Vinberg hybrid.** We perform the Cox-Vinberg construction introduced in [MT16, §6] in a Γ-equivariant fashion. As usual, let $\mathbf{X}_\bullet(\mathbf{T})$ be the coweight lattice, $\mathbf{X}_\bullet(\mathbf{T})^+_{\mathbf{Q}}$ the dominant chamber, and W the Weyl group. Of course, the dominant chamber is Γ-stable.

Lemma 3.2.1. *There exists a Γ-stable fan Σ which is a subdivision of the rational polyhedral set $\mathbf{X}_\bullet(\mathbf{T})^+_{\mathbf{Q}}$ such that $W\Sigma$, the W-saturation of Σ, is smooth and projective.*

Proof. Subdivide the Weyl chambers in $\mathbf{X}_\bullet(\mathbf{T})$ to obtain a projective fan Σ'. Now apply [CHS05, Théorème 1] to Σ' with $W \times \Gamma$ as the finite group acting on $\mathbf{X}_\bullet(\mathbf{T})$. This yields a new smooth projective $W \times \Gamma$-stable fan Σ'' which is a subdivision of Σ'. Then take $\Sigma = \Sigma'' \cap \mathbf{X}_\bullet(\mathbf{T})^+_{\mathbf{Q}}$. □

Remark 3.2.2. When $\mathbf{G}$ is semisimple, we have the canonical choice of taking Σ to be the fan consisting of the single cone $\mathbf{X}_\bullet(\mathbf{T})^+_{\mathbf{Q}}$. Note that this fan does not depend on the choice of Galois cover $S' \to S$.

Choose primitive lattice generators $\beta = \{\beta_i\}_{i\in I}$ of all the rays in Σ. The finite group Γ stabilizes β and hence lifts to an action on the finite set I. The β_i's induce monoid homomorphisms $\overline{\beta}_i \colon \mathbf{A}^1_S \to \mathbf{T}^+_{\mathrm{ad}}$. Multiplying these, we get a monoid homomorphism $\mathbf{A}^I_S \to \mathbf{T}^+_{\mathrm{ad}}$. Define $\mathrm{V}_{\mathbf{G},\beta}$ so that the following square is Cartesian:

$$\begin{tikzcd} \mathrm{V}_{\mathbf{G},\beta} \arrow[r] \arrow[d] & \mathrm{V}_{\mathbf{G}} \arrow[d, "\mathfrak{a}"] \\ \mathbf{A}^I_S \arrow[r] & \mathbf{T}^+_{\mathrm{ad}} \end{tikzcd}$$

[1] A quasi-pinning is the data of a Killing pair $(\mathbf{B}, \mathbf{T})$ a section $s \in \mathrm{H}^0(\mathrm{Dyn}\,\mathbf{G}_{\text{q-ép}}, \mathfrak{g}^{\mathfrak{D}})^\times$ where $\mathrm{Dyn}\,\mathbf{G}_{\text{q-ép}}$ is the scheme of Dynkin diagrams of $\mathbf{G}_{\text{q-ép}}$ and $\mathfrak{g}^{\mathfrak{D}}$ is a certain line bundle on $\mathrm{Dyn}\,\mathbf{G}_{\text{q-ép}}$. When $\mathbf{G}_{\text{q-ép}}$ is split and Δ a system of simple roots, $\mathrm{Dyn}\,\mathbf{G}_{\text{q-ép}} \simeq S \times \Delta$ and $\mathfrak{g}^{\mathfrak{D}}$ is the line bundle which restricts to the eigenspace $\mathfrak{g}_\alpha$ over $S \times \{\alpha\}$ for each $\alpha \in \Delta$. That is, the notion of quasi-pinning coincides with that of pinning for split reductive groups. See [SGA3, Exposé XXIV, §3.9] for more details.

Then $\mathrm{V}_{\mathbf{G},\beta}$ is a Γ-equivariant reductive monoid scheme over S such that all diagrams in the above square are Γ-equivariant. Also, it has $\mathbf{G}_{\mathrm{m},S}^I \times_S \mathbf{G}$ as its group of units. For any $\sigma \subseteq I$, let $U_\sigma := \{x \in \mathbf{A}_S^I \colon x_i \neq 0 \text{ if } i \notin \sigma\}$. Then let $\mathbf{A}_{S,\beta}^\circ$ be the union of all U_α such that $\langle \beta_i \colon i \in \sigma \rangle$ is a cone in Σ. Define $\mathrm{V}_{\mathbf{G},\beta}^\circ := \mathbf{A}_{S,\beta}^\circ \times_{\mathbf{T}_{\mathrm{ad}}^+} \mathrm{V}_{\mathbf{G}}^\circ$. This is called the *nondegenerate locus* in [MT16].

3.3. **Compactification as a GIT quotient.** We apply geometric invariant theory developed over general bases in [Ses77]. As in [MT16, §8], we consider quotients $\mathrm{V}_{\mathbf{G},\beta} /\!\!/_\rho \mathbf{G}_{\mathrm{m},S}^I$ with respect to a suitable linearization ρ on the trivial line bundle. Note that the formation of such GIT quotients is compatible with arbitrary base change and in particular, passing to geometric fibers, by virtue of *linear* reductivity of tori. Semistable and stable loci are defined as open subschemes and their formation commutes with arbitrary base change essentially by construction [Ses77, §II]. In particular, one can use Hilbert-Mumford criterion along geometric fibers to identify stable and semistable geometric points. Therefore, the same argument as in the proof of [MT16, Theorem 8.1] works to show that there is a linearization ρ such that $\mathrm{V}_{\mathbf{G},\beta}^\circ$ is the semistable (and stable) subscheme. As a result, the GIT quotient $\overline{\mathbf{G}} := \mathrm{V}_{\mathbf{G},\beta} /\!\!/_\rho \mathbf{G}_{\mathrm{m},S}^I$ contains $\mathbf{G}$ as a fiberwise-dense open subscheme by virtue of compatibility with formation of this quotient and restricting to geometric points. It also acquires an action of Γ, being a geometric quotient of $\mathrm{V}_{\mathbf{G},\beta}^\circ$ by $\mathbf{G}_{\mathrm{m},S}^I$.

Proposition 3.3.1. *$\overline{\mathbf{G}}$ is separated and finitely presented over S.*

Proof. Let $\mathbf{G}_{\mathrm{Ch}}$ be the Chevalley group scheme over $\operatorname{Spec} \mathbf{Z}$ for $\mathbf{G}$. The same fan Σ as in Lemma 3.2.1 can be used to produce a compactification $\mathbf{G}_{\mathrm{Ch}} \hookrightarrow \overline{\mathbf{G}_{\mathrm{Ch}}}$ so that $\overline{\mathbf{G}} = \overline{\mathbf{G}_{\mathrm{Ch}}} \times_{\mathbf{Z}} S$. Therefore, it suffices to check that $\overline{\mathbf{G}_{\mathrm{Ch}}} \to \operatorname{Spec} \mathbf{Z}$ is separated and finitely presented, which is readily true by standard properties of GIT as $\mathbf{Z}$ is universally Japanese (c.f. [Ses77, §4]). ☐

Proposition 3.3.2. *$\overline{\mathbf{G}}$ is S-smooth.*

Proof. Proposition 2.2.2 gives an open cell $m_\beta \colon \mathbf{U}_- \times_S \mathbf{T} \times_S \mathbf{A}_S^I \times_S \mathbf{U}_+ \hookrightarrow \mathrm{V}_{\mathbf{G},\beta}$ by base changing m. The $\mathbf{G} \times_S \mathbf{G}$ translates of m_β cover $\mathrm{V}_{\mathbf{G},\beta}^\circ$ since the same is true for m and $\mathrm{V}_{\mathbf{G}}^\circ$ by definition. This open embedding is equivariant for the obvious action of $\mathbf{G}_{\mathrm{m},S}^I$ on $\mathbf{T}$ and $\mathbf{A}_S^I$. Passing to GIT quotients with the same linearization as before, we thus get an open embedding

$$\mathbf{U}_- \times_S \overline{\mathbf{T}} \times_S \mathbf{U}_+ \hookrightarrow \overline{\mathbf{G}},$$

where $\overline{\mathbf{T}}$ is the toric scheme for $\mathbf{T}$ corresponding to the fan Σ. This toric scheme can be constructed as $\mathbf{T} \times_S \mathbf{A}_S^I /\!\!/_\rho \mathbf{G}_{\mathrm{m},S}^I$ with semistable (and stable) subscheme equal to $\mathbf{T} \times_S \mathbf{A}_{S,\beta}^\circ$ (see, for e.g., [CLS11, §5.1]). Since $W\Sigma$ is smooth by construction (Lemma 3.2.1), so is Σ. Therefore, $\overline{\mathbf{T}}$ is smooth. The desired result follows by Lemma 2.2.3 since the $\mathbf{G} \times_S \mathbf{G}$-saturation of $\mathbf{U}_- \times_S \overline{\mathbf{T}} \times_S \mathbf{U}_+$ is $\overline{\mathbf{G}}$. ☐

The proof of Proposition 3.3.2 also shows

Proposition 3.3.3. *There is an open cell*

$$\Omega \colon \mathbf{U}_- \times_S \overline{\mathbf{T}} \times_S \mathbf{U}_+ \hookrightarrow \overline{\mathbf{G}}$$

whose $\mathbf{G} \times_S \mathbf{G}$-saturation is $\overline{\mathbf{G}}$.

What remains is to check projectivity of $\overline{\mathbf{G}}$. In [MT16], properness is shown by realizing $\overline{\mathbf{G}}$ as the coarse space associated to a certain proper moduli stack of bundles. We resort to an alternative approach as we do not have a modular interpretation at hand.

Proposition 3.3.4. *$\overline{\mathbf{G}}$ is S-projective.*

Proof. As in the proof of Proposition 3.3.1, we may assume $S = \operatorname{Spec} \mathbf{Z}$. Because $\overline{\mathbf{G}}$ is constructed as a GIT quotient, it suffices to check properness. We use valuative criterion for properness. For this, we may assume that S is the spectrum of a discrete valuation ring R with fraction field K. By [Rom13, Lemma 4.1.1], it is enough to check that any K-point $x_K \in \mathbf{G}(K)$ extends to an R-point of $\overline{\mathbf{G}}$. Using the Cartan decomposition of $\mathbf{G}(K)$, we can assume $x_K = \lambda(\pi) \in \mathbf{T}(K)$ for some dominant coweight λ and uniformizer π. Such a point induces an R-point of $\mathbf{T}^+_{\mathrm{ad}}$ which in turn induces an R-point x_R of $\mathrm{V}^\circ_{\mathbf{G}}$ by applying the canonical section $\bar{\mathfrak{s}}$ (c.f. §2.2). Base changing, we get a map $\mathbf{A}^I_S \times_{\mathbf{T}^+_{\mathrm{ad}}} x_R \to \mathrm{V}_{\mathbf{G},\beta}$ and choosing an arbitrary S-morphism of affine spaces $\mathbf{A}^I_S \to \mathbf{T}^+_{\mathrm{ad}}$, not necessarily a section, we get a map $x_R \to \mathrm{V}_{\mathbf{G},\beta}$. By construction, the generic point of x_R lies inside the unit group $\mathbf{G}^I_{\mathrm{m},S} \times_S \mathbf{G}$ where its second component is x_K. Then the image of x_R along the GIT quotient morphism gives the desired point. □

Remark 3.3.5. It is likely possible to obtain a complete combinatorial classification of all equivariant compactifications of an isotrivial reductive group scheme by following the methods of [MT16], but we do not pursue this here.

4. The semisimple case

We prove Theorem 1.3 in this section. Let G be a semisimple reductive group scheme over S. By [SGA3, Exposé XXIV, Théorème 4.1.5], G is locally isotrivial. This allows us to carry out the construction of §3 Zariski locally on S where we always choose Σ according to Remark 3.2.2. Since this choice is independent of the finite étale Galois covers, they glue to give $\overline{\mathrm{G}} \to S$. By [Vin95, Theorem 8] and [Bou15, Proposition 1.3] and compatibility of such GIT quotients with base change, geometric fibers of $\overline{\mathrm{G}}_{\mathrm{ad}}$ are indeed classical wonderful compactifications of de Concini and Procesi (also see [MT16, Theorem 5.4]). We have shown that $\overline{\mathrm{G}}_{\mathrm{ad}}$ is smooth projective over S. There is a morphism $\overline{\mathrm{G}} \to \overline{\mathrm{G}}_{\mathrm{ad}}$ coming from a natural map of Cox-Vinberg monoids, extending the central isogeny $\mathrm{G} \to \mathrm{G}_{\mathrm{ad}}$. By [MT16, Lemma 9.2], $(\overline{\mathrm{G}})_s$ is the normalization of G_s in $(\overline{\mathrm{G}}_{\mathrm{ad}})_s$ for each geometric point $s \to S$. What remains is to prove the statement about the boundary.

Assumption 4.1. From now on, we assume G is adjoint.

Proposition 4.2. $\overline{\mathrm{G}} \setminus \mathrm{G}$ *is the union of S-smooth relative effective Cartier divisor with normal crossings.*

Proof. By construction, the formation of $\overline{\mathrm{G}} \setminus \mathrm{G} \hookrightarrow \overline{\mathrm{G}}$ commutes with base change on S. Since smoothness, being a relative divisor, and having normal crossings are étale local on the base, we may assume that G is split and $S = \operatorname{Spec} \mathbf{Z}$. By Proposition 3.3.3, there is an open cell

$$\Omega \colon \mathrm{U}_- \times_S \overline{\mathrm{T}} \times_S \mathrm{U}_+ \hookrightarrow \overline{\mathrm{G}}$$

where the toric embedding $\mathrm{T} \hookrightarrow \overline{\mathrm{T}}$ looks like $\prod_{\alpha \in \Delta} \mathbf{G}_{\mathrm{m},S} \hookrightarrow \prod_{\alpha \in \Delta} \mathbf{A}^1_S$. The complement $\overline{\mathrm{T}} \setminus \mathrm{T}$ is clearly an S-flat divisor, therefore by Lemma 2.2.3, so is the whole boundary $\overline{\mathrm{G}} \setminus \mathrm{G}$. By [Stacks, Tag 062Y] and the fact that translates of Ω cover $\overline{\mathrm{G}}$ when S is an algebraically closed point, it follows that $D := \overline{\mathrm{G}} \setminus \mathrm{G}$ is a relative effective Cartier divisor. Since G is connected, irreducible components, say D_i, of this divisor are $\mathrm{G} \times_S \mathrm{G}$-stable. Since Ω is dense in $\overline{\mathrm{G}}$, it intersects each D_i in a dense open subscheme, which implies that D_i is the $\mathrm{G} \times_S \mathrm{G}$-saturation of the dense open $\Omega \cap D_i$ inside D_i. Hence, we obtain that D_i is S-smooth by Lemma 2.2.3. What remains is to check that $D = \sum_i D_i$ has relative normal crossings. We know that D_s has normal crossings for each geometric point $s \to S$ since translates of Ω_s cover $\overline{\mathrm{G}}_s$. The desired result now follows from Lemma 4.4. □

Remark 4.3. In fact, we end up proving a bit more when G is split: irreducible components of the boundary divisor are indexed by a set of system of simple roots Δ and $\mathrm{G} \times_S \mathrm{G}$-stable subschemes of $\overline{\mathrm{G}} \setminus \mathrm{G}$ correspond to subsets of Δ.

Lemma 4.4. *Let X be a smooth S-scheme equipped with a relative effective Cartier divisor D with S-smooth irreducible components. Assume that for each geometric point $s \in S$, D_s has normal crossings in X_s. Then D has relative normal crossings over S.*

Proof. Let $x \in X$ be a point with image $s \in S$, and $D_1, \ldots, D_n$ the irreducible components of D passing through x. Choose an affine open neighborhood U of x such that $D_i \cap U$ is cut out by $f_i = 0$ for $f_i \in$

$\mathrm{H}^0(U, \mathcal{O}_X)$. The images of f_i in $\Omega_{X/S} \otimes \kappa(x)$ are linearly independent because $\Omega_{X/S} \otimes \kappa(x) \simeq \Omega_{X_s/s} \otimes \kappa(x)$ and the same is true in $\Omega_{X_{\overline{s}}/\overline{s}} \otimes \overline{\kappa(x)}$ by assumption, where $\overline{s}$ is an arbitrary geometric point lying over s. Let N be the rank of $\Omega_{X/S}$ at x. By Nakamaya and possibly shrinking U, extend these to a set of functions $\{f_1, \ldots, f_n, f_{n+1}, \ldots, f_N\}$ each of which vanish at x such that their differentials form a basis for the trivial vector bundle $\Omega_{U/S}$. We thus obtain an *étale* S-morphism

$$f \colon U \xrightarrow{(f_1,\ldots,f_N)} \mathbf{A}_S^N$$

such that $D \cap U$ is the preimage of the relative normal crossing divisor $x_1 x_2 \cdots x_n = 0$ in $\mathbf{A}_S^N$. □

Proposition 4.5. $\overline{\mathrm{G}}$ *agrees with the equivariant compactification* $\mathcal{X}$ *of [Li25, Theorem 1].*

Proof. Firstly, it suffices to check this for quasi-split G because our $\overline{\mathrm{G}}$ and the compactification $\mathcal{X}$ of [Li25, Theorem 1] is obtained by first making the same constructions for split G and then performing the obvious inner twist. The construction for quasi-split G is basically the same as the construction for split G where everything is equipped with the action of a finite abstract group preserving a pinning. Thus, we reduce to the split case. By Proposition 3.3.3, we have an open cell $\Omega \colon \mathrm{U}_- \times_S \overline{\mathrm{T}} \times_S \mathrm{U}_+ \hookrightarrow \overline{\mathrm{G}}$ whose $\mathrm{G} \times_S \mathrm{G}$-saturation is $\overline{\mathrm{G}}$. The method of [Li25] starts by defining a *rational* $\mathrm{G} \times_S \mathrm{G}$-action π on Ω [Li25, Theorem 3.4] and then defining $\mathcal{X}$ as an appropriate fppf sheaf quotient of $\mathrm{G} \times_S \Omega \times_S \mathrm{G}$. Due to the uniqueness assertion of [Li25, Theorem 3.4], we reduce to checking that the rational S-morphism $\mathrm{G} \times_S \Omega \times_S \mathrm{G} \dashrightarrow \Omega$ induced by the action map $\mathrm{G} \times_S \Omega \times_S \mathrm{G} \to \overline{\mathrm{G}}$ given by $(g_1, \omega, g_2) \mapsto g_1 \omega g_2$ satisfies the conditions of *loc. cit.*, but this is clear. □

5. A torus admitting no equivariant compactification

We prove Theorem 1.4 in this section. Let k be an algebraically closed field. We recall the construction of [SGA3, Exposé X, §1.6]. Let S_1 be the Néron 1-gon, obtained by glueing sections 0 and ∞ of $\mathbf{P}^1_k$. It can be realized as the nodal cubic curve S_1 equipped with the normalization $\pi_1 \colon \mathbf{P}^1_k \to S_1$. Let S_∞ be the Néron ∞-gon which comes equipped with a finite morphism $\pi_\infty \colon \mathbf{P}^1_k \times \mathbf{Z} \to S_\infty$ which glues the ∞-section of $\mathbf{P}^1_k \times \{i\}$ with the 0-section of $\mathbf{P}^1_k \times \{i+1\}$. There is an infinite connected étale Galois cover $S_\infty \to S_1$ with Galois group $\mathbf{Z}$ which is covered by the trivial $\mathbf{Z}$-torsor $\mathbf{P}^1_k \times \mathbf{Z} \to \mathbf{P}^1_k$ where $\mathbf{Z}$ acts on the source via $j \cdot (x, i) = (x, i + j)$.

Define an action of $\mathbf{Z}$ on the constant torus $\mathbf{G}^2_{\mathrm{m},k} \times S_\infty$ by $1 \cdot (t, x) = (Mt, 1 \cdot x)$ where M is the *infinite* order automorphism of $\mathbf{G}^2_{\mathrm{m},k}$ given by $(t_1, t_2) \mapsto (t_1, t_1 t_2)$. By Galois descent, we thus obtain a rank 2 torus $\mathrm{T}_{S_1} \to S_1$. Since T_{S_1} has infinite monodromy by construction, T_{S_1} is a quasi-isotrivial[1] torus that is not isotrivial. Alternatively, T_{S_1} can be constructed by taking the constant torus $\mathbf{G}^2_{\mathrm{m},k} \times \mathbf{P}^1_k \to \mathbf{P}^1_k$ and glueing the fibers over 0 and ∞ via the automorphism M.

Proposition 5.1. *There is no projective* S_1*-scheme* $\overline{\mathrm{T}}_{S_1}$ *containing* T_{S_1} *as a fiberwise dense open subscheme such that the translation action of* T_{S_1} *on itself extends to* $\overline{\mathrm{T}}_{S_1}$.

Assume the contrary that such a $\overline{\mathrm{T}}_{S_1}$ exists. First, replace $\overline{\mathrm{T}}_{S_1}$ by its reduction so that it's a genuine k-variety– we know that it is irreducible because it contains a torus as an open dense subvariety. Let X be the base change of $\overline{\mathrm{T}}_{S_1}$ along the normalization $\pi_1 \colon \mathbf{P}^1_k \to S_1$. Let $\mathrm{T} := \mathbf{G}^2_{\mathrm{m},k} \times_k \mathbf{P}^1_k$ be the open dense torus inside X.

Since X is irreducible, it must be flat over $\mathbf{P}^1_k$. Set $\mathrm{T} := \mathbf{G}^2_{\mathrm{m},\mathbf{P}^1_k}$. We use the following result of Brion:

Theorem 5.2 ([Bri15, Theorem 4.8])**.** *Let* G *be a split torus over a field* k. *Then every quasiprojective* k*-variety* X *equipped with an action of* G *admits a finite étale* G*-equivariant cover* $f \colon Y \to X$, *where* Y *is the union of open affine* G*-stable subvarieties.*

[1] This means it is split by an étale cover.

We may view X as a projective variety with $\mathbf{G}^2_{\mathrm{m},k}$-action by identifying $\mathrm{T} \times_{\mathbf{P}^1_k} X \simeq \mathbf{G}^2_{\mathrm{m},k} \times_k X$ in the source of the action map. By applying Theorem 5.2, we get a $\mathbf{G}^2_{\mathrm{m},k}$-equivariant finite étale cover $\pi\colon Y \to X$ such that Y is the union of open affine $\mathbf{G}^2_{\mathrm{m},k}$-stable subvarieties.

Lemma 5.3. $\pi\colon \pi^{-1}(\mathrm{T}) \to \mathrm{T}$ *is a trivial cover.*

Proof. Any finite étale cover of T comes from a finite étale cover $U \to \mathbf{G}^2_{\mathrm{m},k}$. Indeed, the projection $\mathrm{T} \to \mathbf{G}^2_{\mathrm{m},k}$ is a $\mathbf{P}^1_k$-bundle and hence induces an isomorphism on étale fundamental groups by the homotopy exact sequence. A connected finite étale cover $U \to \mathbf{G}^2_{\mathrm{m},k}$ must be an étale self-isogeny induced by a linear endomorphism of the character lattice. Equivariance forces such an isogeny to be an isomorphism. □

Label the components of $\pi^{-1}(\mathrm{T})$ as $\mathrm{T}_1, \mathrm{T}_2, \ldots, \mathrm{T}_n$. Each of these are abstractly isomorphic to T and map down to $\mathrm{T} \subset X$ via identity. The $\mathbf{G}^2_{\mathrm{m},k}$-action on $\overline{\mathrm{T}}_i$ upgrades to a T_i-action extending the one on T_i by itself. The union $\bigcup_i \mathrm{T}_i \subset Y$ is a $\mathbf{P}^1_k$-fiberwise open-dense subscheme since the same is true for $\mathrm{T} \subset X$. Denote by t a closed point of $\mathbf{P}^1_k$. The fiber Y_t has irreducible components given by the (scheme-theoretic) closures of $(\mathrm{T}_1)_t, (\mathrm{T}_2)_t, \ldots, (\mathrm{T}_n)_t$. In particular, each fiber of Y has n irreducible components. Every closure $\overline{\mathrm{T}}_i$, $1 \leqslant i \leqslant n$, is faithfully flat over $\mathbf{P}^1_k$ and hence has fibers of pure dimension 2. Therefore, each $(\overline{\mathrm{T}}_i)_t$ is the union of closures of a nonempty subset of $\{(\mathrm{T}_1)_t, (\mathrm{T}_2)_t, \ldots, (\mathrm{T}_n)_t\}$. For $i \neq j$, $(\overline{\mathrm{T}}_i)_t$ and $(\overline{\mathrm{T}}_j)_t$ cannot have an irreducible component in common, say the closure of $(\mathrm{T}_h)_t$, because Y_t is generically reduced. Alternatively, if ξ is the generic point of $(\mathrm{T}_h)_t$, the local ring $\mathcal{O}_{Y,\xi} \simeq \mathcal{O}_{\mathrm{T}_h,\xi}$ is a DVR and hence cannot contain more than one minimal prime. We thus obtain:

Proposition 5.4. *Every* $\overline{\mathrm{T}}_i$ *is a flat projective variety over* $\mathbf{P}^1_k$ *containing the split torus* T_i *as a fiberwise dense open subvariety such that the action of* T_i *on itself extends to* $\overline{\mathrm{T}}_i$ *in a way that* $\overline{\mathrm{T}}_i$ *can be covered by* $\mathbf{G}^2_{\mathrm{m},k}$*-stable open affine subvarieties. Furthermore, there is an isomorphism between the normalizations of the* 0*-fiber and* ∞*-fiber of* $\overline{\mathrm{T}}_i$ *restricting to* $(x, y) \mapsto (x, xy)$ *on* $\mathbf{G}^2_{\mathrm{m},k}$*.*

Proof. The first part is clear from the previous discussion. For the second part, note that there is a finite birational morphism $(\overline{\mathrm{T}}_i)_t \to X_t$ because it restricts to identity on the toral part and hence induces an isomorphism on normalizations by Zariski's main theorem. □

Set $W := \overline{\mathrm{T}}_1$ and rename T_1 as T for ease of notation. Denote $W \to \mathbf{P}^1_k$ by f. Let $U \subset W$ be a nonempty $\mathbf{G}^2_{\mathrm{m},k}$-stable open affine subvariety. Then $\mathrm{V} := f(U)$ is a nonempty open subvariety of $\mathbf{P}^1_k$. Of course, U is $\mathbf{P}^1_k$-fiberwise open and hence intersects $\mathrm{T}_{\mathrm{V}} = \mathbf{G}^2_{m,\mathrm{V}}$ fiberwise. Due to $\mathbf{G}^2_{\mathrm{m},k}$-invariance, U must contain whole of T_{V}. The $\mathbf{G}^2_{\mathrm{m},k}$-action on U can be upgraded to an action of T_{V}. Therefore,

Proposition 5.5. *Let* $\underline{x}\colon \operatorname{Spec} \mathcal{O}_{\mathbf{P}^1_k, x} \to \mathbf{P}^1_k$ *be the local scheme at a closed point* $x \in \mathbf{P}^1_k$*. Then* $W_{\underline{x}}$ *can be covered by* $\mathrm{T}_{\underline{x}}$*-stable affine open neighborhoods of* $\mathrm{T}_{\underline{x}} \subset W_{\underline{x}}$*.*

Lemma 5.6. *Let* X *be an integral affine scheme over a discrete valuation ring* R *which contains a split* R*-torus* T *as a* fiberwise dense *open subscheme such that the action of* T *on itself extends to* X*. Then the fibers of the normalization* $\widetilde{X}$ *as an* R*-scheme are irreducible and normal.*

Proof. Let M be the character lattice of T. Such an affine toric scheme X is given by a graded sub-R-algebra A of the M-graded group ring $R[M]$, which in turn is equivalent to the data of a finitely generated submonoid Q of M which generates M as a group. The last bit ensures that X contains $\mathbf{G}^n_{\mathrm{m},R}$ as a open dense subscheme. The normalization of A then corresponds to the saturation monoid $\widetilde{Q}$ of Q defined as $\{a \in M\colon na \in Q \text{ for some } n \in \mathbf{Z}\}$. The special fiber of $\widetilde{X}$ is then the spectrum of the monoid ring $k[\widetilde{Q}]$, where k is the residue field of R. Since $k[\widetilde{Q}]$ is a subring of the integral domain $k[M]$, it follows that the special fiber of $\widetilde{X}$ is irreducible. Since $\widetilde{Q}$ is saturated, $k[\widetilde{Q}]$ is also integrally closed. □

Let $\widetilde{W_{\underline{x}}} \to W_{\underline{x}}$ be the normalization. By Proposition 5.5 and Lemma 5.6, it follows that $\widetilde{W_\ell}$ is a flat projective normal $\mathrm{T}_{\underline{x}}$-toric scheme over $R := k[t]_{(t)}$ with irreducible normal fibers. Consider $\widetilde{W_{\underline{0}}}$ and $\widetilde{W_{\underline{\infty}}}$. The generic fibers of these are identified and there is an isomorphism between their special fibers which restricts to $(x, y) \mapsto (x, xy)$ on respective toral parts. By the theory of normal toric schemes over DVRs [Kem+73, item e) at p. 192], these are classified by two *complete* rational polyhedral fans Σ_1 and Σ_2 in $\mathbf{R}^2 \times \mathbf{R}_{\geqslant 0}$, respectively. Let $\pi\colon \mathbf{R}^2 \times \mathbf{R}_{\geqslant 0} \to \mathbf{R}^2$ be the natural projection where we often view the target as sitting inside $\mathbf{R}^2 \times \mathbf{R}_{\geqslant 0}$ as $\mathbf{R}^2 \times \{0\}$. We recall the following two facts:

- By [Kem+73, item e') at p. 192], the recession fan of Σ_i, defined as the image of $\Sigma_i \cap (\mathbf{R}^2 \times \{0\})$ along π, classify the respective generic fibers, and are therefore equal, to say Σ_∂.
- There is an embedding of fans $\{0\} \times \mathbf{R}_{\geqslant 0} \hookrightarrow \Sigma_i$ corresponding to the open embedding of toric schemes $\mathbf{G}^2_{\mathrm{m},R} \hookrightarrow \widetilde{W_{\underline{0}}}$ and $\mathbf{G}^2_{\mathrm{m},R} \hookrightarrow \widetilde{W_{\underline{\infty}}}$. Indeed, the toric scheme $\mathbf{G}^2_{\mathrm{m},R} \to \operatorname{Spec} R$ is classified by the cone $\{0\} \times \mathbf{R}_{\geqslant 0}$. Therefore, the components of the special fiber containing $\mathbf{G}^2_{\mathrm{m},k}$ are classified by the (complete) fan $\Delta_i := \{\pi(\sigma)\colon \sigma \in \Sigma_i,\ \{0\} \times \mathbf{R}_{\geqslant 0} \subseteq \sigma\}$ as a toric variety. Furthermore, there is a one-to-one correspondence between irreducible components of the special fiber and vertices of the polyhedral complex $\Sigma_i \cap (\mathbf{R}^2 \times \{1\})$ (c.f. [Wal13, Proposition 7.15] or [BPS18, Remark 3.5.9]).

We thus conclude that every ray in Σ_i not equal to $\{0\} \times \mathbf{R}_{\geqslant 0}$ must be contained in the boundary $\mathbf{R}^2 \times \{0\}$. That is, $\widetilde{W_{\underline{x}}}$ is a constant family– the base change of a normal toric variety over k. Now, the fact that there is an isomorphism between the special fibers of $\widetilde{W_{\underline{0}}}$ and $\widetilde{W_{\underline{\infty}}}$ extending $(x, y) \mapsto (x, xy)$ on the toral part corresponds to the fact that $A(\Delta_1) = \Delta_2$ where $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. However, $\Delta_1 = \Sigma_\partial = \Delta_2$. Thus, Σ_∂ is a complete fan in $\mathbf{R}^2$ which is stable under the automorphism A. There must exist a ray $\ell \in \Sigma_\partial$ which is not contained in the x-axis for otherwise it would not be complete. Then $\{A^n \ell\colon n \in \mathbf{Z}\}$ is an infinite set. This contradicts the finiteness of Σ_∂. The proof is complete.

The extension of these results to the general setting of toric schemes will be explored in a future paper.

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