

COMPACTIFICATION OF REDUCTIVE GROUP SCHEMES

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ABSTRACT. Let G be an isotrivial reductive group over S . We construct a smooth projective S -scheme containing G as a fiberwise-dense open subscheme equipped with left and right actions of G which extend the translation actions of G on itself. This verifies a conjecture of Česnavičius [Čes22]. When G is adjoint, we recover fiberwise the wonderful compactification. Finally, we give an example of a non-isotrivial torus admitting no equivariant compactification.

1. INTRODUCTION

A reductive group scheme G over a base scheme S is called *locally isotrivial* if each point $s \in S$ has a Zariski open neighborhood $U_s \hookrightarrow S$ admitting a finite étale cover $U'_s \rightarrow U_s$ such that $G \times_S U'_s$ is split. This is equivalent to local isotriviality of the central torus [SGA3, Exposé XXIV, Théorème 4.1.5]. A reductive group is called *isotrivial* if U_s can be chosen to be S .

Česnavičius conjectured that every isotrivial reductive group scheme G has a compactification equipped with a left action of G extending that of G on itself:

Conjecture 1.1 ([Čes22, Conjecture 6.2.3]). *For an isotrivial reductive group G over a Noetherian scheme S , there are a projective, finitely presented S -scheme \overline{G} equipped with a left G -action and a G -equivariant S -fiberwise dense open immersion*

$$G \hookrightarrow \overline{G}.$$

When G is an isotrivial torus, this is shown in [Čes22, §6.3]. Using a variant of the Artin–Weil method of birational group laws, [Li25] considers the case of adjoint G and obtains an equivariant compactification whose geometric fibers agree with classical wonderful compactifications. In this paper, we construct a smooth projective $G \times_S G$ -equivariant compactification for any isotrivial reductive group over an arbitrary base, thus verifying Conjecture 1.1:

Theorem 1.2. *Let G be an isotrivial reductive group scheme over a scheme S . Then there exists a smooth projective S -scheme \overline{G} containing G as a fiberwise-dense open subscheme equipped with a left and right action of G extending that on G given by left and right multiplication.*

When G is semisimple, we can do more:

Theorem 1.3. *Let G be a semisimple reductive group scheme over S . Then there exist projective S -schemes \overline{G} and \overline{G}_{ad} containing G and G_{ad} , respectively, as fiberwise-dense open subschemes, equipped with left and right actions of G and G_{ad} extending those on G and G_{ad} by themselves such that*

- *there is an equivariant morphism $\overline{G} \rightarrow \overline{G}_{\text{ad}}$ extending the central isogeny $G \rightarrow G_{\text{ad}}$ which is a normalization over each geometric point of S ,*
- *\overline{G}_{ad} is smooth and agrees with the de Concini–Procesi wonderful compactification over geometric points of S , and*

- its boundary $\overline{G}_{\text{ad}} \setminus G_{\text{ad}}$ is the union of S -smooth relative effective Cartier divisors with relative normal crossings¹.

All our constructions commute with base change on S . We also verify that our construction agrees with that of [Li25] for adjoint G (Proposition 4.5). In §5, we prove that Conjecture 1.1 is false without the isotriviality assumption by showing that the standard example of a non-isotrivial torus over the nodal rational curve [SGA3, Exposé X, §1.6] does not admit an equivariant compactification.

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2. AFFINE MONOIDS

2.1. Vinberg monoids. Let G be a split reductive group scheme over a connected base S . Let T be an abstract Cartan. Define $G_+ := T \times_S^{\mathbf{Z}_G} G$ and $T_{\text{ad}} := T/\mathbf{Z}_G$. Choose a Borel $B \subset G$. Evaluating at the system of simple roots gives a canonical toric embedding $T_{\text{ad}} \hookrightarrow T_{\text{ad}}^+$ where T_{ad}^+ is an affine space over S of relative dimension equal to that of T_{ad} . Here, T_{ad}^+ is viewed as an S -monoid with unit group T_{ad} . Then there is a reductive monoid scheme V_G over S , called the *Vinberg monoid*, equipped with an abelianization homomorphism $\alpha: V_G \rightarrow T_{\text{ad}}^+$.

Let us briefly recall the construction of the Vinberg monoid given in [XZ19, §3.2]. The same works over \mathbf{Z} and hence for any split reductive group over an arbitrary base. Let us assume $S = \text{Spec } \mathbf{Z}$. Let $\mathbf{X}^\bullet(T_{\text{ad}})_{\text{pos}}$ be the submonoid of $\mathbf{X}^\bullet(T)$ generated by the simple roots. We equip $\mathbf{X}^\bullet(T)$ with the partial order \preceq defined by $\lambda \preceq \mu$ if $\mu - \lambda \in \mathbf{X}^\bullet(T_{\text{ad}})_{\text{pos}}$.

The coordinate ring $\mathbf{Z}[G]$ admits a canonical multi-filtration indexed by $\mathbf{X}^\bullet(T_{\text{ad}})_{\text{pos}}$ induced by the $G \times_S G$ action on $\mathbf{Z}[G]$ via left and right translation. For $\nu \in \mathbf{X}^\bullet(T_{\text{ad}})_{\text{pos}}$, let $\text{fil}_\nu \mathbf{Z}[G]$ be the \mathbf{Z} -submodule of matrix coefficients of representations V_λ with highest weight $\lambda \preceq \nu$. Explicitly, this is the saturated submodule such that for any weight space $V(\mu_1, \mu_2)$ under the $(G \times_S G)$ -action, non-vanishing implies $\mu_1 \preceq -w_0(\nu)$ and $\mu_2 \preceq \nu$.

The Vinberg monoid V_G is defined as the spectrum of the Rees algebra associated to this filtration:

$$V_G := \text{Spec} \left(\bigoplus_{\nu \in \mathbf{X}^\bullet(T_{\text{ad}})_{\text{pos}}} \text{fil}_\nu \mathbf{Z}[G] \right).$$

The abelianization homomorphism $\pi: V_G \rightarrow T_{\text{ad}}^+ := \text{Spec } \mathbf{Z}[\mathbf{X}^\bullet(T_{\text{ad}})_{\text{pos}}]$ is induced by the inclusion of the base ring $\mathbf{Z}[\mathbf{X}^\bullet(T_{\text{ad}})_{\text{pos}}]$ into the Rees algebra.

Theorem 2.1.1. *The monoid V_G fits into a Cartesian diagram*

$$\begin{array}{ccc} G_+ & \hookrightarrow & V_G \\ \downarrow & & \downarrow \\ T_{\text{ad}} & \hookrightarrow & T_{\text{ad}}^+ \end{array}$$

and

- G_+ is the unit group of V_G ,

¹Following [SGA1, Exposé XIII, §2.1], for an S -scheme X , we say that a relative Cartier divisor $D \subset X$ is *strictly with relative normal crossings* if there exists a finite family $(f_i \in \Gamma(X, \mathcal{O}_X))_{i \in I}$ such that (1) $D = \bigcup_{i \in I} V_X(f_i)$, and (2) for every $x \in \text{Supp}(D)$, X is smooth at x over S , and the closed subscheme $V((f_i)_{i \in I(x)}) \subset X$ is smooth over S of codimension $|I(x)|$, where $I(x) = \{i \in I \mid f_i(x) = 0\}$. The divisor D has *relative normal crossings* if étale locally on X it is strictly with relative normal crossings.

- the $\mathbf{T} \times_S \mathbf{G} \times_S \mathbf{G}$ action on \mathbf{G}_+ given by \mathbf{T} acting on the first component and \mathbf{G} acting on itself by left and right multiplication extends to an action on $V_{\mathbf{G}}$,
- \mathfrak{a} is faithfully flat.

Proof. See [XZ19, Proposition 3.2.2]. \square

2.2. Nondegenerate locus. Choose a Cartan subgroup $c: \mathbf{T} \hookrightarrow \mathbf{B}$. Let \mathbf{U}_+ be the unipotent radical of \mathbf{B} and \mathbf{U}_- be that of the opposite Borel.

Proposition 2.2.1. *There is a canonical section $\bar{s}: \mathbf{T}_{\text{ad}}^+ \rightarrow V_{\mathbf{G}}$ extending $s: t \mapsto (t, c(t)) \pmod{Z_{\mathbf{G}}}$ on respective unit groups.*

Proof. We identify \mathbf{T} with the image of c and assume $S = \text{Spec } \mathbf{Z}$. The restriction ring map $r: \mathbf{Z}[\mathbf{G}] \rightarrow \mathbf{Z}[\mathbf{T}]$ sends the filtered piece $\text{fil}_{\nu} \mathbf{Z}[\mathbf{G}]$ into the span of characters e^{λ} satisfying $\lambda \preceq \nu$. Here, e^{λ} denotes the character corresponding to λ .

We define a retraction $\phi: \mathbf{Z}[V_{\mathbf{G}}] \rightarrow \mathbf{Z}[\mathbf{T}_{\text{ad}}^+]$ by composing the restriction with the projection onto the trivial weight space. Explicitly, for a homogeneous element $f_{\nu} \in \text{fil}_{\nu} \mathbf{Z}[\mathbf{G}]$, we define

$$\phi(f_{\nu}) = (\text{coefficient of } e^0 \text{ in } r(f_{\nu})) \cdot e^{\nu}.$$

This map is well-defined: if the trivial character e^0 appears in the restriction of f_{ν} , then we must have $0 \preceq \nu$. This condition holds for all ν in the grading monoid $\mathbf{X}^*(\mathbf{T}_{\text{ad}})_{\text{pos}}$ since it is generated by simple roots. Since ϕ maps the element $1 \cdot e^{\nu} \in \mathbf{Z}[V_{\mathbf{G}}]$ to $e^{\nu} \in \mathbf{Z}[\mathbf{T}_{\text{ad}}^+]$, it forms a section of the abelianization. Restricted to the unit group, the condition $\lambda = 0$ corresponds to the identity element of the fiber, matching the definition of \bar{s} . \square

Proposition 2.2.2. *The multiplication map $m: \mathbf{U}_- \times_S \mathbf{T} \times_S \mathbf{T}_{\text{ad}}^+ \times_S \mathbf{U}_+ \rightarrow V_{\mathbf{G}}$ given by $(u_-, t, a, u_+) \mapsto u_- t \bar{s}(a) u_+$ is an open embedding.*

Proof. We assume $S = \text{Spec } \mathbf{Z}$. By [Vin95, Proposition 14], m is an open embedding over geometric points of S , hence in particular, fiberwise unramified. By fibral criteria for flatness, m is flat. Thus, m is an étale monomorphism and we conclude by [Stacks, Tag 025G]. \square

Lemma 2.2.3. *Let H be a flat group scheme locally of finite presentation over a scheme S acting on an S -scheme X . Let $U \subseteq X$ be an open subscheme. Then the saturation $H \cdot U$ of U is an H -stable open subscheme of X . If in addition*

- U is S -flat then so is $H \cdot U$,
- U is locally of finite presentation over S then so is $H \cdot U$,
- U is S -smooth then so is $H \cdot U$.

Proof. The H -saturation $H \cdot U$ is defined as the image of the composition

$$H \times_S U \hookrightarrow H \times_S X \xrightarrow{(h,x) \mapsto (h, h \cdot x)} H \times_S X \xrightarrow{(h,x) \mapsto x} X.$$

The middle map is an isomorphism and the last map is a projection. Since flat morphisms of locally finite presentation are universally open [Stacks, Tag 01UA], it follows that $H \cdot U$ is open. The induced map $H \times_S U \rightarrow H \cdot U$ is fppf. Since flatness and being locally of finite presentation are fppf local on source [Stacks, Tag 06ET, Tag 06EV], S -flatness of $H \cdot U$ is equivalent to that of $H \times_S U$ and the same is true for being locally of finite presentation. When U is smooth, we reduce to the case of S an algebraically closed point by [Stacks, Tag 01V8], in which case smoothness follows because translations of U cover $H \cdot U$. \square

The $\mathbf{G} \times_S \mathbf{G}$ -saturation of the image of m in Proposition 2.2.2 is called the *nondegenerate locus*, denoted $V_{\mathbf{G}}^{\circ}$. It is independent of the choice of Cartan subgroup, $\mathbf{T} \times_S \mathbf{G} \times_S \mathbf{G}$ -stable, and contains \mathbf{G}_+ by construction.

Corollary 2.2.4. $V_{\mathbf{G}}^{\circ}$ is smooth over S .

3. PROOF OF THEOREM 1.2

3.1. Preliminary reductions. Let \mathbf{G} be the split form of G . Fix a pinning of \mathbf{G} . This canonically determines a presentation of the automorphism group scheme $\mathrm{Aut}_{\mathbf{G}/S}$ as a semidirect product of \mathbf{G}_{ad} and $\mathrm{Out}_{\mathbf{G}/S}$ and hence a unique quasi-split inner form $\mathbf{G}_{\mathrm{q}\text{-}\acute{\mathrm{e}}\mathrm{p}}$ of G endowed with a quasi-pinning¹ [SGA3, Exposé XXIV, Corollaire 3.12].

Since G is assumed to be isotrivial, the étale local \mathbf{G}_{ad} -torsor corresponding to $\mathbf{G}_{\mathrm{q}\text{-}\acute{\mathrm{e}}\mathrm{p}}$ is isotrivial too. As finite étale morphisms satisfy effective descent for projective schemes, it suffices to prove the theorem for $\mathbf{G}_{\mathrm{q}\text{-}\acute{\mathrm{e}}\mathrm{p}}$. Indeed, this essentially boils down to the fact that finite group quotients of projective schemes are representable by projective schemes.

Choose a finite étale Galois cover $S' \rightarrow S$ splitting $\mathbf{G}_{\mathrm{q}\text{-}\acute{\mathrm{e}}\mathrm{p}}$ with Galois group Γ . The data of $\mathbf{G}_{\mathrm{q}\text{-}\acute{\mathrm{e}}\mathrm{p}}$ is then equivalent to the data of a Γ -action on the $\mathbf{G}_{\mathrm{q}\text{-}\acute{\mathrm{e}}\mathrm{p}} \times_S S'$. Thus, it suffices to find an equivariant compactification equipped an action of Γ extending that on $\mathbf{G}_{\mathrm{q}\text{-}\acute{\mathrm{e}}\mathrm{p}} \times_S S'$. The quasi-pinning on $\mathbf{G}_{\mathrm{q}\text{-}\acute{\mathrm{e}}\mathrm{p}}$ induces a pinning on $\mathbf{G}_{\mathrm{q}\text{-}\acute{\mathrm{e}}\mathrm{p}} \times_S S'$. Note that Γ additionally acts on the pinning of $\mathbf{G}_{\mathrm{q}\text{-}\acute{\mathrm{e}}\mathrm{p}} \times_S S'$.

From now, we replace our setup with a pinned reductive group $(\mathbf{G}, \mathbf{B}, \mathbf{T}, \{u_\alpha\}_{\alpha \in \Delta})$ over S equipped with an action of a finite group Γ .

3.2. Cox-Vinberg hybrid. We perform the Cox-Vinberg construction introduced in [MT16, §6] in a Γ -equivariant fashion. As usual, let $\mathbf{X}_\bullet(\mathbf{T})$ be the coweight lattice, $\mathbf{X}_\bullet(\mathbf{T})_{\mathbf{Q}}^+$ the dominant chamber, and W the Weyl group. Of course, the dominant chamber is Γ -stable.

Lemma 3.2.1. *There exists a Γ -stable fan Σ which is a subdivision of the rational polyhedral set $\mathbf{X}_\bullet(\mathbf{T})_{\mathbf{Q}}^+$ such that $W\Sigma$, the W -saturation of Σ , is smooth and projective.*

Proof. Subdivide the Weyl chambers in $\mathbf{X}_\bullet(\mathbf{T})$ to obtain a projective fan Σ' . Now apply [CHS05, Théorème 1] to Σ' with $W \times \Gamma$ as the finite group acting on $\mathbf{X}_\bullet(\mathbf{T})$. This yields a new smooth projective $W \times \Gamma$ -stable fan Σ'' which is a subdivision of Σ' . Then take $\Sigma = \Sigma'' \cap \mathbf{X}_\bullet(\mathbf{T})_{\mathbf{Q}}^+$. \square

Remark 3.2.2. When \mathbf{G} is semisimple, we have the canonical choice of taking Σ to be the fan consisting of the single cone $\mathbf{X}_\bullet(\mathbf{T})_{\mathbf{Q}}^+$. Note that this fan does not depend on the choice of Galois cover $S' \rightarrow S$.

Choose primitive lattice generators $\beta = \{\beta_i\}_{i \in I}$ of all the rays in Σ . The finite group Γ stabilizes β and hence lifts to an action on the finite set I . The β_i 's induce monoid homomorphisms $\bar{\beta}_i: \mathbf{A}_S^1 \rightarrow \mathbf{T}_{\mathrm{ad}}^+$. Multiplying these, we get a monoid homomorphism $\mathbf{A}_S^I \rightarrow \mathbf{T}_{\mathrm{ad}}^+$. Define $V_{\mathbf{G}, \beta}$ so that the following square is Cartesian:

$$\begin{array}{ccc} V_{\mathbf{G}, \beta} & \longrightarrow & V_{\mathbf{G}} \\ \downarrow & & \downarrow \mathfrak{a} \\ \mathbf{A}_S^I & \longrightarrow & \mathbf{T}_{\mathrm{ad}}^+ \end{array}$$

Then $V_{\mathbf{G}, \beta}$ is a Γ -equivariant reductive monoid scheme over S such that all diagrams in the above square are Γ -equivariant. Also, it has $\mathbf{G}_{\mathrm{m}, S}^I \times_S \mathbf{G}$ as its group of units. For any $\sigma \subseteq I$, let $U_\sigma := \{x \in \mathbf{A}_S^I : x_i \neq 0 \text{ if } i \notin \sigma\}$. Then let $\mathbf{A}_{S, \beta}^\circ$ be the union of all U_α such that $\langle \beta_i : i \in \sigma \rangle$ is a cone in Σ . Define $V_{\mathbf{G}, \beta}^\circ := \mathbf{A}_{S, \beta}^\circ \times_{\mathbf{T}_{\mathrm{ad}}^+} V_{\mathbf{G}}^\circ$. This is called the *nondegenerate locus* in [MT16].

¹A quasi-pinning is the data of a Killing pair (\mathbf{B}, \mathbf{T}) a section $s \in H^0(\mathrm{Dyn} \mathbf{G}_{\mathrm{q}\text{-}\acute{\mathrm{e}}\mathrm{p}}, \mathfrak{g}^\mathfrak{D})^\times$ where $\mathrm{Dyn} \mathbf{G}_{\mathrm{q}\text{-}\acute{\mathrm{e}}\mathrm{p}}$ is the scheme of Dynkin diagrams of $\mathbf{G}_{\mathrm{q}\text{-}\acute{\mathrm{e}}\mathrm{p}}$ and $\mathfrak{g}^\mathfrak{D}$ is a certain line bundle on $\mathrm{Dyn} \mathbf{G}_{\mathrm{q}\text{-}\acute{\mathrm{e}}\mathrm{p}}$. When $\mathbf{G}_{\mathrm{q}\text{-}\acute{\mathrm{e}}\mathrm{p}}$ is split and Δ a system of simple roots, $\mathrm{Dyn} \mathbf{G}_{\mathrm{q}\text{-}\acute{\mathrm{e}}\mathrm{p}} \simeq S \times \Delta$ and $\mathfrak{g}^\mathfrak{D}$ is the line bundle which restricts to the eigenspace \mathfrak{g}_α over $S \times \{\alpha\}$ for each $\alpha \in \Delta$. That is, the notion of quasi-pinning coincides with that of pinning for split reductive groups. See [SGA3, Exposé XXIV, §3.9] for more details.

3.3. Compactification as a GIT quotient. We apply geometric invariant theory developed over general bases in [Ses77]. As in [MT16, §8], we consider quotients $V_{\mathbf{G}, \beta} //_{\rho} \mathbf{G}_{m,S}^I$ with respect to a suitable linearization ρ on the trivial line bundle. Note that the formation of such GIT quotients is compatible with arbitrary base change and in particular, passing to geometric fibers, by virtue of *linear* reductivity of tori. Semistable and stable loci are defined as open subschemes and their formation commutes with arbitrary base change essentially by construction [Ses77, §II]. In particular, one can use Hilbert-Mumford criterion along geometric fibers to identify stable and semistable geometric points. Therefore, the same argument as in the proof of [MT16, Theorem 8.1] works to show that there is a linearization ρ such that $V_{\mathbf{G}, \beta}^o$ is the semistable (and stable) subscheme. As a result, the GIT quotient $\overline{\mathbf{G}} := V_{\mathbf{G}, \beta} //_{\rho} \mathbf{G}_{m,S}^I$ contains \mathbf{G} as a fiberwise-dense open subscheme by virtue of compatibility with formation of this quotient and restricting to geometric points. It also acquires an action of Γ , being a geometric quotient of $V_{\mathbf{G}, \beta}^o$ by $\mathbf{G}_{m,S}^I$.

Proposition 3.3.1. $\overline{\mathbf{G}}$ is separated and finitely presented over S .

Proof. Let \mathbf{G}_{Ch} be the Chevalley group scheme over $\text{Spec } \mathbf{Z}$ for \mathbf{G} . The same fan Σ as in Lemma 3.2.1 can be used to produce a compactification $\mathbf{G}_{\text{Ch}} \hookrightarrow \overline{\mathbf{G}_{\text{Ch}}}$ so that $\overline{\mathbf{G}} = \mathbf{G}_{\text{Ch}} \times_{\mathbf{Z}} S$. Therefore, it suffices to check that $\overline{\mathbf{G}_{\text{Ch}}} \rightarrow \text{Spec } \mathbf{Z}$ is separated and finitely presented, which is readily true by standard properties of GIT as \mathbf{Z} is universally Japanese (c.f. [Ses77, §4]). \square

Proposition 3.3.2. $\overline{\mathbf{G}}$ is S -smooth.

Proof. Proposition 2.2.2 gives an open cell $m_{\beta}: \mathbf{U}_- \times_S \mathbf{T} \times_S \mathbf{A}_S^I \times_S \mathbf{U}_+ \hookrightarrow V_{\mathbf{G}, \beta}$ by base changing m . The $\mathbf{G} \times_S \mathbf{G}$ translates of m_{β} cover $V_{\mathbf{G}, \beta}^o$ since the same is true for m and $V_{\mathbf{G}}^o$ by definition. This open embedding is equivariant for the obvious action of $\mathbf{G}_{m,S}^I$ on \mathbf{T} and \mathbf{A}_S^I . Passing to GIT quotients with the same linearization as before, we thus get an open embedding

$$\mathbf{U}_- \times_S \overline{\mathbf{T}} \times_S \mathbf{U}_+ \hookrightarrow \overline{\mathbf{G}},$$

where $\overline{\mathbf{T}}$ is the toric scheme for \mathbf{T} corresponding to the fan Σ . This toric scheme can be constructed as $\mathbf{T} \times_S \mathbf{A}_S^I //_{\rho} \mathbf{G}_{m,S}^I$ with semistable (and stable) subscheme equal to $\mathbf{T} \times_S \mathbf{A}_{S,\beta}^o$ (see, for e.g., [CLS11, §5.1]). Since $W\Sigma$ is smooth by construction (Lemma 3.2.1), so is Σ . Therefore, $\overline{\mathbf{T}}$ is smooth. The desired result follows by Lemma 2.2.3 since the $\mathbf{G} \times_S \mathbf{G}$ -saturation of $\mathbf{U}_- \times_S \overline{\mathbf{T}} \times_S \mathbf{U}_+$ is $\overline{\mathbf{G}}$. \square

The proof of Proposition 3.3.2 also shows

Proposition 3.3.3. There is an open cell

$$\Omega: \mathbf{U}_- \times_S \overline{\mathbf{T}} \times_S \mathbf{U}_+ \hookrightarrow \overline{\mathbf{G}}$$

whose $\mathbf{G} \times_S \mathbf{G}$ -saturation is $\overline{\mathbf{G}}$.

What remains is to check projectivity of $\overline{\mathbf{G}}$. In [MT16], properness is shown by realizing $\overline{\mathbf{G}}$ as the coarse space associated to a certain proper moduli stack of bundles. We resort to an alternative approach as we don't have a modular interpretation at hand.

Proposition 3.3.4. $\overline{\mathbf{G}}$ is S -projective.

Proof. As in the proof of Proposition 3.3.1, we may assume $S = \text{Spec } \mathbf{Z}$. Because $\overline{\mathbf{G}}$ is constructed as a GIT quotient, it suffices to check properness. We use valuative criterion for properness. For this, we may assume that S is the spectrum of a discrete valuation ring R with fraction field K . By [Rom13, Lemma 4.1.1], it is enough to check that any K -point $x_K \in \mathbf{G}(K)$ extends to an R -point of $\overline{\mathbf{G}}$. Using the Cartan decomposition of $\mathbf{G}(K)$, we can assume $x_K = \lambda(\pi) \in \mathbf{T}(K)$ for some dominant coweight λ and uniformizer π . Such a point induces an R -point of \mathbf{T}_{ad}^+ which in turn induces an R -point x_R of $V_{\mathbf{G}}^o$ by applying the canonical section \bar{s} (c.f. §2.2). Base changing, we get a map $\mathbf{A}_S^I \times_{\mathbf{T}_{\text{ad}}^+} x_R \rightarrow V_{\mathbf{G}, \beta}$ and choosing an arbitrary S -morphism of affine spaces $\mathbf{A}_S^I \rightarrow \mathbf{T}_{\text{ad}}^+$, not necessarily a section, we get a map

$x_R \rightarrow V_{\mathbf{G}, \beta}$. By construction, the generic point of x_R lies inside the unit group $\mathbf{G}_{m,S}^I \times_S \mathbf{G}$ where its second component is x_K . Then the image of x_R along the GIT quotient morphism gives the desired point. \square

Remark 3.3.5. It is likely possible to obtain a complete combinatorial classification of all equivariant compactifications of an isotrivial reductive group scheme by following the methods of [MT16], but we do not pursue this here.

4. THE SEMISIMPLE CASE

We prove Theorem 1.3 in this section. Let G be a semisimple reductive group scheme over S . By [SGA3, Exposé XXIV, Théorème 4.1.5], G is locally isotrivial. This allows us to carry out the construction of §3 Zariski locally on S where we always choose Σ according to Remark 3.2.2. Since this choice is independent of the finite étale Galois covers, they glue to give $\overline{G} \rightarrow S$. By [Vin95, Theorem 8] and compatibility of such GIT quotients with base change, geometric fibers of \overline{G}_{ad} are indeed classical wonderful compactifications of de Concini and Procesi. We have shown that \overline{G}_{ad} is smooth projective over S . There is a morphism $\overline{G} \rightarrow \overline{G}_{ad}$ coming from a natural map of Cox-Vinberg monoids, extending the central isogeny $G \rightarrow G_{ad}$. By [MT16, Lemma 9.2], $(\overline{G})_s$ is the normalization of G_s in $(\overline{G}_{ad})_s$ for each geometric point $s \rightarrow S$. What remains is to prove the statement about the boundary.

Assumption 4.1. From now on, we assume G is adjoint.

Proposition 4.2. $\overline{G} \setminus G$ is the union of S -smooth relative effective Cartier divisor with normal crossings.

Proof. By construction, the formation of $\overline{G} \setminus G \hookrightarrow \overline{G}$ commutes with base change on S . Since smoothness, being a relative divisor, and having normal crossings are étale local on the base, we may assume that G is split and $S = \text{Spec } \mathbf{Z}$. By Proposition 3.3.3, there is an open cell

$$\Omega: U_- \times_S \overline{T} \times_S U_+ \hookrightarrow \overline{G}$$

where the toric embedding $T \hookrightarrow \overline{T}$ looks like $\prod_{\alpha \in \Delta} \mathbf{G}_{m,S} \hookrightarrow \prod_{\alpha \in \Delta} \mathbf{A}_S^1$. The complement $\overline{T} \setminus T$ is clearly an S -flat divisor, therefore by Lemma 2.2.3, so is the whole boundary $\overline{G} \setminus G$. By [Stacks, Tag 062Y] and the fact that translates of Ω cover \overline{G} when S is an algebraically closed point, it follows that $D := \overline{G} \setminus G$ is a relative effective Cartier divisor. Since G is connected, irreducible components, say D_i , of this divisor are $G \times_S G$ -stable. Since Ω is dense in \overline{G} , it intersects each D_i in a dense open subscheme, which implies that D_i is the $G \times_S G$ -saturation of the dense open $\Omega \cap D_i$ inside D_i . Hence, we obtain that D_i is S -smooth by Lemma 2.2.3. What remains is to check that $D = \sum_i D_i$ has relative normal crossings. We know that D_s has normal crossings for each geometric point $s \rightarrow S$ since translates of Ω_s cover \overline{G}_s . The desired result now follows from Lemma 4.4. \square

Remark 4.3. In fact, we end up proving a bit more when G is split: irreducible components of the boundary divisor are indexed by a set of system of simple roots Δ and $G \times_S G$ -stable subschemes of $\overline{G} \setminus G$ correspond to subsets of Δ .

Lemma 4.4. Let X be a smooth S -scheme equipped with a relative effective Cartier divisor D with S -smooth irreducible components. Assume that for each geometric point $s \in S$, D_s has normal crossings in X_s . Then D has relative normal crossings over S .

Proof. Let $x \in X$ be a point with image $s \in S$, and D_1, \dots, D_n the irreducible components of D passing through x . Choose an affine open neighborhood U of x such that $D_i \cap U$ is cut out by $f_i = 0$ for $f_i \in H^0(U, \mathcal{O}_X)$. The images of f_i in $\Omega_{X/S} \otimes \kappa(x)$ are linearly independent because $\Omega_{X/S} \otimes \kappa(x) \simeq \Omega_{X_s/s} \otimes \kappa(x)$ and the same is true in $\Omega_{X_{\bar{s}}/\bar{s}} \otimes \kappa(\bar{x})$ by assumption, where \bar{s} is an arbitrary geometric point lying over s . Let N be the rank of $\Omega_{X/S}$ at x . By Nakamaya and possibly shrinking U , extend these to a set of functions $\{f_1, \dots, f_n, f_{n+1}, \dots, f_N\}$ each of which vanish at x such that their differentials form a basis for the trivial

vector bundle $\Omega_{U/S}$. We thus obtain an étale S -morphism

$$f: U \xrightarrow{(f_1, \dots, f_N)} \mathbf{A}_S^N$$

such that $D \cap U$ is the preimage of the relative normal crossing divisor $x_1 x_2 \cdots x_n = 0$ in \mathbf{A}_S^N . \square

Proposition 4.5. \overline{G} agrees with the equivariant compactification \mathcal{X} of [Li25, Theorem 1].

Proof. Firstly, it suffices to check this for quasi-split G because our \overline{G} and the compactification \mathcal{X} of [Li25, Theorem 1] is obtained by first making the same constructions for split G and then performing the obvious inner twist. The construction for quasi-split G is basically the same as the construction for split G where everything is equipped with the action of a finite abstract group preserving a pinning. Thus, we reduce to the split case. By Proposition 3.3.3, we have an open cell $\Omega: U_- \times_S \overline{T} \times_S U_+ \hookrightarrow \overline{G}$ whose $G \times_S G$ -saturation is \overline{G} . The method of [Li25] starts by defining a rational $G \times_S G$ -action π on Ω [Li25, Theorem 3.4] and then defining \mathcal{X} as an appropriate fppf sheaf quotient of $G \times_S \Omega \times_S G$. Due to the uniqueness assertion of [Li25, Theorem 3.4], we reduce to checking that the rational S -morphism $G \times_S \Omega \times_S G \dashrightarrow \Omega$ induced by the action map $G \times_S \Omega \times_S G \rightarrow \overline{G}$ given by $(g_1, \omega, g_2) \mapsto g_1 \omega g_2$ satisfies the conditions of *loc. cit.*, but this is clear. \square

5. A TORUS ADMITTING NO EQUIVARIANT COMPACTIFICATION

Let k be an algebraically closed field. We recall the construction of [SGA3, Exposé X, §1.6]. Let S_1 be the Néron 1-gon, obtained by glueing sections 0 and ∞ of \mathbf{P}_k^1 . It can be realized as the nodal cubic curve $\text{Proj } k[x, y, z]/(y^2z - x^3 + x^2z)$ equipped with the normalization $\pi_1: \mathbf{P}_k^1 \rightarrow S_1$. Let S_∞ be the Néron ∞ -gon which comes equipped with a finite morphism $\pi_\infty: \mathbf{P}_k^1 \times \mathbf{Z} \rightarrow S_\infty$ which glues the ∞ -section of $\mathbf{P}_k^1 \times \{i\}$ with the 0-section of $\mathbf{P}_k^1 \times \{i+1\}$. There is an infinite connected étale Galois cover $S_\infty \rightarrow S_1$ with Galois group \mathbf{Z} which is covered by the trivial \mathbf{Z} -torsor $\mathbf{P}_k^1 \times \mathbf{Z} \rightarrow \mathbf{P}_k^1$ where k acts on the source via $j \cdot (x, i) = (x, i+j)$.

Define an action of \mathbf{Z} on the constant torus $\mathbf{G}_{m,k}^2 \times S_\infty$ by $1 \cdot (t, x) = (Mt, 1 \cdot x)$ where M is the *infinite* order automorphism of $\mathbf{G}_{m,k}^2$ given by $(t_1, t_2) \mapsto (t_1, t_1 t_2)$. By Galois descent, we thus obtain a rank 2 torus $T_{S_1} \rightarrow S_1$. Since T_{S_1} has infinite monodromy by construction, T_{S_1} is a quasi-isotrivial¹ torus that is not isotrivial. Alternatively, T_{S_1} can be constructed by taking the constant torus $\mathbf{G}_{m,k}^2 \times \mathbf{P}_k^1 \rightarrow \mathbf{P}_k^1$ and glueing the fibers over 0 and ∞ via the automorphism M .

Proposition 5.1. *There is no projective S_1 -scheme \overline{T}_{S_1} containing T_{S_1} as a fiberwise dense open subscheme such that the translation action of T_{S_1} on itself extends to \overline{T}_{S_1} .*

Assume the contrary that such a \overline{T}_{S_1} exists. First, replace \overline{T}_{S_1} by its reduction so that it's a genuine k -variety— we know that it is irreducible because it contains a torus as an open dense subvariety. Let X be the base change of \overline{T}_{S_1} along the normalization $\pi_1: \mathbf{P}_k^1 \rightarrow S_1$. Let $T := \mathbf{G}_{m,k}^2 \times_k \mathbf{P}_k^1$ be the open dense torus inside X .

Since X is irreducible, it must be flat over \mathbf{P}_k^1 . Set $T := \mathbf{G}_{m,\mathbf{P}_k^1}^2$. We use the following result of Brion:

Theorem 5.2 ([Bri15, Theorem 4.8]). *Let G be a split torus over a field k . Then every quasiprojective k -variety X equipped with an action of G admits a finite étale G -equivariant cover $f: Y \rightarrow X$, where Y is the union of open affine G -stable subvarieties.*

We may view X as a projective variety with $\mathbf{G}_{m,k}^2$ -action by identifying $T \times_{\mathbf{P}_k^1} X \simeq \mathbf{G}_{m,k}^2 \times_k X$ in the source of the action map. By applying Theorem 5.2, we get a $\mathbf{G}_{m,k}^2$ -equivariant finite étale cover $\pi: Y \rightarrow X$ such that Y is the union of open affine $\mathbf{G}_{m,k}^2$ -stable subvarieties.

¹This means it is split by an étale cover.

Lemma 5.3. $\pi: \pi^{-1}(T) \rightarrow T$ is a trivial cover.

Proof. Any finite étale cover of T comes from a finite étale cover $U \rightarrow \mathbf{G}_{m,k}^2$. Indeed, the projection $T \rightarrow \mathbf{G}_{m,k}^2$ is a \mathbf{P}_k^1 -bundle and hence induces an isomorphism on étale fundamental groups by the homotopy exact sequence. A connected finite étale cover $U \rightarrow \mathbf{G}_{m,k}^2$ must be an étale self-isogeny induced by a linear endomorphism of the character lattice. Equivariance forces such an isogeny to be an isomorphism. \square

Label the components of $\pi^{-1}(T)$ as T_1, T_2, \dots, T_n . Each of these are abstractly isomorphic to T and map down to $T \subset X$ via identity. The $\mathbf{G}_{m,k}^2$ -action on \overline{T}_i upgrades to a T_i -action extending the one on T_i by itself. The union $\bigcup_i T_i \subset Y$ is a \mathbf{P}_k^1 -fiberwise open-dense subscheme since the same is true for $T \subset X$. Denote by t a closed point of \mathbf{P}_k^1 . The fiber Y_t has irreducible components given by the (scheme-theoretic) closures of $(T_1)_t, (T_2)_t, \dots, (T_n)_t$. In particular, each fiber of Y has n irreducible components. Every closure \overline{T}_i , $1 \leq i \leq n$, is faithfully flat over \mathbf{P}_k^1 and hence has fibers of pure dimension 2. Therefore, each $(\overline{T}_i)_t$ is the union of closures of a nonempty subset of $\{(T_1)_t, (T_2)_t, \dots, (T_n)_t\}$. For $i \neq j$, $(\overline{T}_i)_t$ and $(\overline{T}_j)_t$ cannot have an irreducible component in common, say the closure of $(T_h)_t$, because Y_t is generically reduced. Alternatively, if ξ is the generic point of $(T_h)_t$, the local ring $\mathcal{O}_{Y,\xi} \simeq \mathcal{O}_{T_h,\xi}$ is a DVR and hence cannot contain more than one minimal prime. We thus obtain:

Proposition 5.4. Every \overline{T}_i is a flat projective variety over \mathbf{P}_k^1 containing the split torus T_i as a fiberwise dense open subvariety such that the action of T_i on itself extends to \overline{T}_i in a way that \overline{T}_i can be covered by $\mathbf{G}_{m,k}^2$ -stable open affine subvarieties. Furthermore, there is an isomorphism between the normalizations of the 0-fiber and ∞ -fiber of \overline{T}_i restricting to $(x, y) \mapsto (x, xy)$ on $\mathbf{G}_{m,k}^2$.

Proof. The first part is clear from the previous discussion. For the second part, note that there is a finite birational morphism $(\overline{T}_i)_t \rightarrow X_t$ because it restricts to identity on the toral part and hence induces an isomorphism on normalizations by Zariski's main theorem. \square

Set $W := \overline{T}_1$ and rename T_1 as T for ease of notation. Denote $W \rightarrow \mathbf{P}_k^1$ by f . Let $U \subset W$ be a nonempty $\mathbf{G}_{m,k}^2$ -stable open affine subvariety. Then $V := f(U)$ is a nonempty open subvariety of \mathbf{P}_k^1 . Of course, U is \mathbf{P}_k^1 -fiberwise open and hence intersects $T_V = \mathbf{G}_{m,V}^2$ fiberwise. Due to $\mathbf{G}_{m,k}^2$ -invariance, U must contain whole of T_V . The $\mathbf{G}_{m,k}^2$ -action on U can be upgraded to an action of T_V . Therefore,

Proposition 5.5. Let $\underline{x}: \text{Spec } \mathcal{O}_{\mathbf{P}_k^1,x} \rightarrow \mathbf{P}_k^1$ be the local scheme at a closed point $x \in \mathbf{P}_k^1$. Then $W_{\underline{x}}$ can be covered by $T_{\underline{x}}$ -stable affine open neighborhoods of $T_{\underline{x}} \subset W_{\underline{x}}$.

Lemma 5.6. Let X be an integral affine scheme over a discrete valuation ring R which contains a split R -torus T as a fiberwise dense open subscheme such that the action of T on itself extends to X . Then the fibers of the normalization \tilde{X} as an R -scheme are irreducible and normal.

Proof. Let M be the character lattice of T . Such an affine toric scheme X is given by a graded sub- R -algebra A of the M -graded group ring $R[M]$, which in turn is equivalent to the data of a finitely generated submonoid Q of M which generates M as a group. The last bit ensures that X contains $\mathbf{G}_{m,R}^n$ as a open dense subscheme. The normalization of A then corresponds to the saturation monoid \tilde{Q} of Q defined as $\{a \in M : na \in Q \text{ for some } n \in \mathbf{Z}\}$. The special fiber of \tilde{X} is then the spectrum of the monoid ring $k[\tilde{Q}]$, where k is the residue field of R . Since $k[\tilde{Q}]$ is a subring of the integral domain $k[M]$, it follows that the special fiber of \tilde{X} is irreducible. Since \tilde{Q} is saturated, $k[\tilde{Q}]$ is also integrally closed. \square

Let $\widetilde{W}_{\underline{x}} \rightarrow W_{\underline{x}}$ be the normalization. By Proposition 5.5 and Lemma 5.6, it follows that $\widetilde{W}_{\underline{x}}$ is a flat projective normal $T_{\underline{x}}$ -toric scheme over $R := k[t]_{(t)}$ with irreducible normal fibers. Consider \widetilde{W}_0 and \widetilde{W}_{∞} . The generic fibers of these are identified and there is an isomorphism between their special fibers which restricts to $(x, y) \mapsto (x, xy)$ on respective toral parts. By the theory of normal toric schemes over DVRs [Kem+73, item e] at p. 192], these are classified by two complete rational polyhedral fans Σ_1 and Σ_2 in

$\mathbf{R}^2 \times \mathbf{R}_{\geq 0}$, respectively. Let $\pi: \mathbf{R}^2 \times \mathbf{R}_{\geq 0} \rightarrow \mathbf{R}^2$ be the natural projection where we often view the target as sitting inside $\mathbf{R}^2 \times \mathbf{R}_{\geq 0}$ as $\mathbf{R}^2 \times \{0\}$. We recall the following two facts:

- By [Kem+73, item e') at p. 192], the recession fan of Σ_i , defined as the image of $\Sigma_i \cap (\mathbf{R}^2 \times \{0\})$ along π , classify the respective generic fibers, and are therefore equal, to say Σ_∂ .
- There is an embedding of fans $\{0\} \times \mathbf{R}_{\geq 0} \hookrightarrow \Sigma_i$ corresponding to the open embedding of toric schemes $\mathbf{G}_{m,R}^2 \hookrightarrow \widetilde{W}_0$ and $\mathbf{G}_{m,R}^2 \hookrightarrow \widetilde{W}_\infty$. Indeed, the toric scheme $\mathbf{G}_{m,R}^2 \rightarrow \text{Spec } R$ is classified by the cone $\{0\} \times \mathbf{R}_{\geq 0}$. Therefore, the components of the special fiber containing $\mathbf{G}_{m,k}^2$ are classified by the (complete) fan $\Delta_i := \{\pi(\sigma): \sigma \in \Sigma_i, \{0\} \times \mathbf{R}_{\geq 0} \subseteq \sigma\}$ as a toric variety. Furthermore, there is a one-to-one correspondence between irreducible components of the special fiber and vertices of the polyhedral complex $\Sigma_i \cap (\mathbf{R}^2 \times \{1\})$ (c.f. [Wal13, Proposition 7.15] or [BPS18, Remark 3.5.9]).

We thus conclude that every ray in Σ_i not equal to $\{0\} \times \mathbf{R}_{\geq 0}$ must be contained in the boundary $\mathbf{R}^2 \times \{0\}$. That is, \widetilde{W}_x is a constant family— the base change of a normal toric variety over k . Now, the fact that there is an isomorphism between the special fibers of \widetilde{W}_0 and \widetilde{W}_∞ extending $(x,y) \mapsto (x,xy)$ on the toral part corresponds to the fact that $A(\Delta_1) = \Delta_2$ where $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. However, $\Delta_1 = \Sigma_\partial = \Delta_2$. Thus, Σ_∂ is a complete fan in \mathbf{R}^2 which is stable under the automorphism A . There must exist a ray $\ell \in \Sigma_\partial$ which is not contained in the x -axis for otherwise it wouldn't be complete. Then $\{A^n\ell: n \in \mathbf{Z}\}$ is an infinite set. This contradicts the finiteness of Σ_∂ . The proof is complete.

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