A NOTE ON COMPACTIFICATION OF REDUCTIVE GROUP SCHEMES

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Abstract. Let G be an isotrivial reductive group over S. We construct a smooth projective S-scheme containing G as a fiberwise-dense open subscheme equipped with left and right actions of G which extend the translation actions of G on itself. This verifies a conjecture of Česnavičius [Čes22]. When G is semisimple, we recover fiberwise the wonderful compactification. Finally, we give an example of a non-isotrivial torus with no equivariant compactification.

1. Introduction

A reductive group scheme G over a base scheme S is called *locally isotrivial* if each point $s \in S$ has a Zariski open neighborhood $U_s \hookrightarrow S$ admitting a finite étale cover $U'_s \to U_s$ such that $\mathrm{G} \times_S U'_s$ is split. This is equivalent to local isotriviality of the central torus [SGA3, Exposé XXIV, Théorème 4.1.5]. A reductive group is called *isotrivial* if U_s can be chosen to be S.

Česnavičius conjectured the existence of a compactification for isotrivial reductive group schemes equipped with a left action of G extending the left translation of G on itself:

Conjecture 1.1 ([Čes22, Conjecture 6.2.3])**.** *For an isotrivial reductive group* G *over a Noetherian scheme* S*, there are a projective, finitely presented* S*-scheme* $\overline{\mathrm{G}}$ *equipped with a left* G*-action and a* G*-equivariant* S*-fiberwise dense open immersion*

$$\mathrm{G} \hookrightarrow \overline{\mathrm{G}}.$$

When G is an isotrivial torus, this is shown in [Čes22, §6.3]. Using a variant of Artin-Weil method of birational group laws, [Li25] considers the case of adjoint G and obtains an equivariant compactification whose geometric fibers agree with classical wonderful compactifications. In this paper, we construct a smooth $\mathrm{G} \times_S \mathrm{G}$-equivariant compactification for any isotrivial reductive group over an arbitrary base, thus verifying Conjecture 1.1:

Theorem 1.2. *Let* G *be an isotrivial reductive group scheme over a scheme* S*. Then there exists a smooth projective* S*-scheme* $\overline{\mathrm{G}}$ *containing* G *as a fiberwise-dense open subscheme equipped with a left and right action of* G *extending that on* G *given by left and right multiplication.*

When G is semisimple, we can do more:

Theorem 1.3. *Let* G *be a semisimple reductive group scheme over* S*. Then there exists a projective* S*-scheme* $\overline{\mathrm{G}}$ *containing* G *as a fiberwise-dense open subscheme equipped with left and right actions of* G *extending that on* G *by itself. There is a canonical morphism* $\overline{\mathrm{G}} \to \overline{\mathrm{G}}_{\mathrm{ad}}$ *extending the central isogeny* $\mathrm{G} \to \mathrm{G}_{\mathrm{ad}}$ *and which is a normalization over each geometric point of* S*. When* G *is adjoint,*

- $\overline{\mathrm{G}}$ *is smooth and it agrees with the de Concini-Procesi wonderful compactification over geometric points of* S*,*
- *and its boundary* $\overline{\mathrm{G}} \setminus \mathrm{G}$ *is the union of* S*-smooth relative effective Cartier divisors with relative normal crossings*[1].

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[1]For an S-scheme X, we say that a relative Cartier divisor $D \subset X$ is *strictly with relative normal crossings*, if there exists a finite family $(f_i \in \Gamma(X, \mathcal{O}_X))_{i \in I}$ such that (1) $D = \bigcup_{i \in I} \mathrm{V}_X(f_i)$, (2) for every $x \in \mathrm{Supp}(D)$, X is smooth at x over S, and the closed subscheme $\mathrm{V}((f_i)_{i \in I(x)}) \subset X$ is smooth over S of codimension $|I(x)|$, where $I(x) = \{i \in I \mid f_i(x) = 0\}$. The divisor D has *relative normal crossings*, if étale locally on X it is strictly with relative normal crossings.

Of course, all our constructions commute with base change on S. We also verify that our construction agrees with that of [Li25] for adjoint G (Proposition 4.5). In §5, we prove that Conjecture 1.1 is false without the isotriviality assumption by showing that the standard example of a non-isotrivial torus over the nodal rational curve [SGA3, Exposé X, §1.6] doesn't admit an equivariant compactification.

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2. Affine monoids

2.1. **Vinberg monoids.** Let $\mathbf{G}$ be a split reductive group scheme over a connected base S. Let $\mathbf{T}$ be an abstract Cartan. Define $\mathbf{G}_+ := \mathbf{T} \times_S^{\mathrm{Z}_\mathbf{G}} \mathbf{G}$ and $\mathbf{T}_{\mathrm{ad}} := \mathbf{T}/\mathrm{Z}_\mathbf{G}$. Choose a Borel $\mathbf{B} \subset \mathbf{G}$. Evaluating at the system of positive roots gives a canonical toric embedding $\mathbf{T}_{\mathrm{ad}} \hookrightarrow \mathbf{T}_{\mathrm{ad}}^+$ where $\mathbf{T}_{\mathrm{ad}}^+$ is an affine space over S of relative dimension equal to that of $\mathbf{T}_{\mathrm{ad}}$. Here, $\mathbf{T}_{\mathrm{ad}}^+$ is viewed as an S-monoid with unit group $\mathbf{T}_{\mathrm{ad}}$. Then there is a reductive monoid scheme $\mathrm{V}_\mathbf{G}$ over S, called the *Vinberg monoid*, equipped with an abelianization homomorphism $\mathfrak{a}\colon \mathrm{V}_\mathbf{G} \to \mathbf{T}_{\mathrm{ad}}^+$.

Let us briefly recall the construction of the Vinberg monoid given in [XZ19, §3.2]. The same works over $\mathbf{Z}$ and hence for any split reductive group over an arbitrary base. Let us assume $S = \operatorname{Spec} \mathbf{Z}$. Let $\mathbf{X}^\bullet(\mathbf{T}_{\mathrm{ad}})_{\mathrm{pos}}$ be the submonoid of $\mathbf{X}^\bullet(\mathbf{T})$ generated by the simple roots. We equip $\mathbf{X}^\bullet(\mathbf{T})$ with the partial order $\preceq$ defined by $\lambda \preceq \mu$ if $\mu - \lambda \in \mathbf{X}^\bullet(\mathbf{T}_{\mathrm{ad}})_{\mathrm{pos}}$.

The coordinate ring $\mathbf{Z}[\mathbf{G}]$ admits a canonical multi-filtration indexed by $\mathbf{X}^\bullet(\mathbf{T}_{\mathrm{ad}})_{\mathrm{pos}}$ induced by the $\mathbf{G} \times_S \mathbf{G}$ action on $\mathbf{Z}[\mathbf{G}]$ via left and right translation. For $\nu \in \mathbf{X}^\bullet(\mathbf{T}_{\mathrm{ad}})_{\mathrm{pos}}$, let $\mathrm{fil}_\nu \mathbf{Z}[\mathbf{G}]$ be the $\mathbf{Z}$-submodule of matrix coefficients of representations V_λ with highest weight $\lambda \preceq \nu$. Explicitly, this is the saturated submodule such that for any weight space $\mathrm{V}(\mu_1, \mu_2)$ under the $(\mathbf{G} \times_S \mathbf{G})$-action, non-vanishing implies $\mu_1 \preceq -w_0(\nu)$ and $\mu_2 \preceq \nu$.

The Vinberg monoid $\mathrm{V}_\mathbf{G}$ is defined as the spectrum of the Rees algebra associated to this filtration:

$$\mathrm{V}_\mathbf{G} := \operatorname{Spec}\left(\bigoplus_{\nu \in \mathbf{X}^\bullet(\mathbf{T}_{\mathrm{ad}})_{\mathrm{pos}}} \mathrm{fil}_\nu \mathbf{Z}[\mathbf{G}] \right).$$

The abelianization homomorphism $\pi : \mathrm{V}_\mathbf{G} \to \mathbf{T}_{\mathrm{ad}}^+ := \operatorname{Spec} \mathbf{Z}[\mathbf{X}^\bullet(\mathbf{T}_{\mathrm{ad}})_{\mathrm{pos}}]$ is induced by the inclusion of the base ring $\mathbf{Z}[\mathbf{X}^\bullet(\mathbf{T}_{\mathrm{ad}})_{\mathrm{pos}}]$ into the Rees algebra.

Theorem 2.1.1. *The monoid $\mathrm{V}_\mathbf{G}$ fits into a Cartesian diagram*

$$\begin{tikzcd} \mathbf{G}_+ \arrow[r, hook] \arrow[d] & \mathrm{V}_\mathbf{G} \arrow[d] \\ \mathbf{T}_{\mathrm{ad}} \arrow[r, hook] & \mathbf{T}_{\mathrm{ad}}^+ \end{tikzcd}$$

and

- $\mathbf{G}_+$ *is the unit group of* $\mathrm{V}_\mathbf{G}$*,*
- *the* $\mathbf{T} \times_S \mathbf{G} \times_S \mathbf{G}$ *action on* $\mathbf{G}_+$ *given by* $\mathbf{T}$ *acting on the first component and* $\mathbf{G}$ *acting on itself by left and right multiplication extends to an action on* $\mathrm{V}_\mathbf{G}$*,*
- $\mathfrak{a}$ *is faithfully flat.*

Proof. See [XZ19, Proposition 3.2.2]. ∎

2.2. **Nondegenerate locus.** Choose a Cartan *sub*group $c\colon \mathbf{T} \hookrightarrow \mathbf{B}$. Let $\mathbf{U}_+$ be the unipotent radical of $\mathbf{B}$ and $\mathbf{U}_-$ be that of the opposite Borel.

Proposition 2.2.1. *There is a* canonical section $\bar{\mathfrak{s}}\colon \mathbf{T}_{\mathrm{ad}}^+ \to \mathrm{V}_{\mathbf{G}}$ *extending* $\mathfrak{s}\colon t \mapsto (t, c(t)) \pmod{\mathrm{Z}_{\mathbf{G}}}$ *on respective unit groups.*

Proof. We identify $\mathbf{T}$ with the image of c and assume $S = \operatorname{Spec} \mathbf{Z}$. The restriction ring map $r\colon \mathbf{Z}[\mathbf{G}] \to \mathbf{Z}[\mathbf{T}]$ sends the filtered piece $\mathrm{fil}_\nu \mathbf{Z}[\mathbf{G}]$ into the span of characters e^λ satisfying $\lambda \preceq \nu$. Here, e^λ denotes the character corresponding to λ.

We define a retraction $\phi : \mathbf{Z}[\mathrm{V}_{\mathbf{G}}] \to \mathbf{Z}[\mathbf{T}_{\mathrm{ad}}^+]$ by composing the restriction with the projection onto the trivial weight space. Explicitly, for a homogeneous element $f_\nu \in \mathrm{fil}_\nu \mathbf{Z}[\mathbf{G}]$, we define

$$\phi(f_\nu) = (\text{coefficient of } e^0 \text{ in } r(f_\nu)) \cdot e^\nu.$$

This map is well-defined: if the trivial character e^0 appears in the restriction of f_ν, then we must have $0 \preceq \nu$. This condition holds for all ν in the grading monoid $\mathbf{X}^\bullet(\mathbf{T}_{\mathrm{ad}})_{\mathrm{pos}}$ since it is generated by simple roots. Since ϕ maps the element $1 \cdot e^\nu \in \mathbf{Z}[\mathrm{V}_{\mathbf{G}}]$ to $e^\nu \in \mathbf{Z}[\mathbf{T}_{\mathrm{ad}}^+]$, it forms a section of the abelianization. Restricted to the unit group, the condition $\lambda = 0$ corresponds to the identity element of the fiber, matching the definition of $\mathfrak{s}$. ☐

Proposition 2.2.2. *The multiplication map* $m\colon \mathbf{U}_- \times_S \mathbf{T} \times_S \mathbf{T}_{\mathrm{ad}}^+ \times_S \mathbf{U}_+ \to \mathrm{V}_{\mathbf{G}}$ *given by* $(u_-, t, a, u_+) \mapsto u_- t \bar{\mathfrak{s}}(a) u_+$ *is an open embedding.*

Proof. We assume $S = \operatorname{Spec} \mathbf{Z}$. By [Vin95, Proposition 14], m is an open embedding over geometric points of S, hence in particular, fiberwise unramified. By fibral criteria for flatness, m is flat. Thus, m is an étale monomorphism and we conclude by [Stacks, Tag 025G]. ☐

Lemma 2.2.3. *Let H be a flat group scheme locally of finite presentation over a scheme S acting on an S-scheme X. Let $U \subseteq X$ be an open subscheme. Then the saturation $H \cdot U$ of U is an H-stable open subscheme of X. If in addition*

- *U is S-flat then so is $H \cdot U$,*
- *U is locally of finite presentation over S then so is $H \cdot U$,*
- *U is S-smooth then so is $H \cdot U$.*

Proof. The H-saturation $H \cdot U$ is defined as the image of the composition

$$H \times_S U \hookrightarrow H \times_S X \xrightarrow{(h,x) \mapsto (h, h \cdot x)} H \times_S X \xrightarrow{(h,x) \mapsto x} X.$$

The middle map is an isomorphism and the last map is a projection. Since flat morphisms of locally finite presentation are universally open [Stacks, Tag 01UA], it follows that $H \cdot U$ is open. The induced map $H \times_S U \to H \cdot U$ is fppf. Since flatness and being locally of finite presentation are fppf local on source [Stacks, Tag 06ET, Tag 06EV], S-flatness of $H \cdot U$ is equivalent to that of $H \times_S U$ and the same is true for being locally of finite presentation. When U is smooth, we reduce to the case of S an algebraically closed point by [Stacks, Tag 01V8], in which case smoothness follows because translations of U cover $H \cdot U$. ☐

The $\mathbf{G} \times_S \mathbf{G}$-saturation of the image of m in Proposition 2.2.2 is called the *nondegenerate locus*, denoted $\mathrm{V}_{\mathbf{G}}^\circ$. It is independent of the choice of Cartan subgroup, $\mathbf{T} \times_S \mathbf{G} \times_S \mathbf{G}$-stable, and contains $\mathbf{G}_+$ by construction.

Corollary 2.2.4. $\mathrm{V}_{\mathbf{G}}^\circ$ *is smooth over S.*

3. Proof of Theorem 1.2

3.1. **Preliminary reductions.** Let $\mathbf{G}$ be the split form of G. Fix a pinning of $\mathbf{G}$. This canonically determines a presentation of the automorphism group scheme $\mathrm{Aut}_{\mathbf{G}/S}$ as a semidirect product of $\mathbf{G}_{\mathrm{ad}}$ and $\mathrm{Out}_{\mathbf{G}/S}$ and hence a unique quasi-split inner form $\mathbf{G}_{\text{q-ép}}$ of G endowed with a quasi-pinning[1] [SGA3, Exposé XXIV, Corollaire 3.12].

Since G is assumed to be isotrivial, the étale local $\mathbf{G}_{\mathrm{ad}}$-torsor corresponding to $\mathbf{G}_{\text{q-ép}}$ is isotrivial too. As finite étale morphisms satisfy effective descent for projective schemes, it suffices to prove the theorem for $\mathbf{G}_{\text{q-ép}}$. Indeed, this essentially boils down to the fact that finite group quotients of projective schemes are representable by projective schemes.

Choose a finite étale Galois cover $S' \to S$ splitting $\mathbf{G}_{\text{q-ép}}$ with Galois group Γ. The data of $\mathbf{G}_{\text{q-ép}}$ is then equivalent to the data of a Γ-action on the $\mathbf{G}_{\text{q-ép}} \times_S S'$. Thus, it suffices to find an equivariant compactification equipped an action of Γ extending that on $\mathbf{G}_{\text{q-ép}} \times_S S'$. The quasi-pinning on $\mathbf{G}_{\text{q-ép}}$ induces a pinning on $\mathbf{G}_{\text{q-ép}} \times_S S'$. Note that Γ additionally acts on the pinning of $\mathbf{G}_{\text{q-ép}} \times_S S'$.

From now, we replace our setup with a pinned reductive group $(\mathbf{G}, \mathbf{B}, \mathbf{T}, \{u_\alpha\}_{\alpha\in\Delta})$ over S equipped with an action of a finite group Γ.

3.2. **Cox-Vinberg hybrid.** We perform the Cox-Vinberg construction introduced in [MT16, §6] in a Γ-equivariant fashion. As usual, let $\mathbf{X}_\bullet(\mathbf{T})$ be the coweight lattice, $\mathbf{X}_\bullet(\mathbf{T})^+_{\mathbf{Q}}$ the dominant chamber, and W the Weyl group. Of course, the dominant chamber is Γ-stable.

Lemma 3.2.1. *There exists a Γ-stable fan Σ which is a subdivision of the rational polyhedral set $\mathbf{X}_\bullet(\mathbf{T})^+_{\mathbf{Q}}$ such that $W\Sigma$, the W-saturation of Σ, is smooth and projective.*

Proof. Subdivide the Weyl chambers in $\mathbf{X}_\bullet(\mathbf{T})$ to obtain a projective fan Σ'. Now apply [CHS05, Théorème 1] to Σ' with $W \times \Gamma$ as the finite group acting on $\mathbf{X}_\bullet(\mathbf{T})$. This yields a new smooth projective $W \times \Gamma$-stable fan Σ'' which is a subdivision of Σ'. Then take $\Sigma = \Sigma'' \cap \mathbf{X}_\bullet(\mathbf{T})^+_{\mathbf{Q}}$. □

Remark 3.2.2. When $\mathbf{G}$ is semisimple, we have the canonical choice of taking Σ to be the fan consisting of the single cone $\mathbf{X}_\bullet(\mathbf{T})^+_{\mathbf{Q}}$. Note that this fan doesn't depend on the choice of Galois cover $S' \to S$.

Choose primitive lattice generators $\beta = \{\beta_i\}_{i\in I}$ of all the rays in Σ. The finite group Γ stabilizes β and hence lifts to an action on the finite set I. The β_i's induce monoid homomorphisms $\overline{\beta}_i \colon \mathbf{A}^1_S \to \mathbf{T}^+_{\mathrm{ad}}$. Multiplying these, we get a monoid homomorphism $\mathbf{A}^I_S \to \mathbf{T}^+_{\mathrm{ad}}$. Define $\mathrm{V}_{\mathbf{G},\beta}$ so that the following square is Cartesian:

$$\begin{tikzcd} \mathrm{V}_{\mathbf{G},\beta} \arrow[r] \arrow[d] & \mathrm{V}_{\mathbf{G}} \arrow[d, "\mathfrak{a}"] \\ \mathbf{A}^I_S \arrow[r] & \mathbf{T}^+_{\mathrm{ad}} \end{tikzcd}$$

Then $\mathrm{V}_{\mathbf{G},\beta}$ is a Γ-equivariant reductive monoid scheme over S such that all diagrams in the above square are Γ-equivariant. Also, it has $\mathbf{G}^I_{\mathrm{m},S} \times_S \mathbf{G}$ as its group of units. For any $\sigma \subseteq I$, let $U_\sigma := \{x \in \mathbf{A}^I_S \colon x_i \neq 0 \text{ if } i \notin \sigma\}$. Then let $\mathbf{A}^\circ_{S,\beta}$ be the union of all U_α such that $\langle \beta_i \colon i \in \sigma\rangle$ is a cone in Σ. Define $\mathrm{V}^\circ_{\mathbf{G},\beta} := \mathbf{A}^\circ_{S,\beta} \times_{\mathbf{T}^+_{\mathrm{ad}}} \mathrm{V}^\circ_{\mathbf{G}}$. This is called the *nondegenerate locus* in [MT16].

[1] A quasi-pinning is the data of a Killing pair $(\mathbf{B}, \mathbf{T})$ a section $s \in \mathrm{H}^0(\mathrm{Dyn}\,\mathbf{G}_{\text{q-ép}}, \mathfrak{g}^{\mathfrak{D}})^\times$ where $\mathrm{Dyn}\,\mathbf{G}_{\text{q-ép}}$ is the scheme of Dynkin diagrams of $\mathbf{G}_{\text{q-ép}}$ and $\mathfrak{g}^{\mathfrak{D}}$ is a certain line bundle on $\mathrm{Dyn}\,\mathbf{G}_{\text{q-ép}}$. When $\mathbf{G}_{\text{q-ép}}$ is split and Δ a system of simple roots, $\mathrm{Dyn}\,\mathbf{G}_{\text{q-ép}} \simeq S \times \Delta$ and $\mathfrak{g}^{\mathfrak{D}}$ is the line bundle which restricts to the eigenspace $\mathfrak{g}_\alpha$ over $S \times \{\alpha\}$ for each $\alpha \in \Delta$. That is, the notion of quasi-pinning coincides with that of pinning for split reductive groups. See [SGA3, Exposé XXIV, §3.9] for more details.

3.3. **Compactification as a GIT quotient.** We apply geometric invariant theory developed over general bases in [Ses77]. As in [MT16, §8], we consider quotients $\mathrm{V}_{\mathbf{G},\beta} /\!\!/_{\rho} \mathbf{G}_{\mathrm{m},S}^I$ with respect to a suitable linearization ρ on the trivial line bundle. Note that the formation of such GIT quotients is compatible with arbitrary base change and in particular, passing to geometric fibers, by virtue of *linear* reductivity of tori. Semistable and stable loci are defined as open subschemes and their formation commutes with arbitrary base change essentially by construction [Ses77, §II]. In particular, one can use Hilbert-Mumford criterion along geometric fibers to identify stable and semistable geometric points. Therefore, the same argument as in the proof of [MT16, Theorem 8.1] works to show that there is a linearization ρ such that $\mathrm{V}^{\circ}_{\mathbf{G},\beta}$ is the semistable (and stable) subscheme. As a result, the GIT quotient $\overline{\mathbf{G}} := \mathrm{V}_{\mathbf{G},\beta} /\!\!/_{\rho} \mathbf{G}_{\mathrm{m},S}^I$ contains $\mathbf{G}$ as a fiberwise-dense open subscheme by virtue of compatibility with formation of this quotient and restricting to geometric points. It also acquires an action of Γ, being a geometric quotient of $\mathrm{V}^{\circ}_{\mathbf{G},\beta}$ by $\mathbf{G}_{\mathrm{m},S}^I$.

Proposition 3.3.1. *$\overline{\mathbf{G}}$ is separated and finitely presented over S.*

Proof. Let $\mathbf{G}_{\mathrm{Ch}}$ be the Chevalley group scheme over $\operatorname{Spec} \mathbf{Z}$ for $\mathbf{G}$. The same fan Σ as in Lemma 3.2.1 can be used to produce a compactification $\mathbf{G}_{\mathrm{Ch}} \hookrightarrow \overline{\mathbf{G}_{\mathrm{Ch}}}$ so that $\overline{\mathbf{G}} = \overline{\mathbf{G}_{\mathrm{Ch}}} \times_{\mathbf{Z}} S$. Therefore, it suffices to check that $\overline{\mathbf{G}_{\mathrm{Ch}}} \to \operatorname{Spec} \mathbf{Z}$ is separated and finitely presented, which is readily true by standard properties of GIT as $\mathbf{Z}$ is universally Japanese (c.f. [Ses77, §4]). $\square$

Proposition 3.3.2. *$\overline{\mathbf{G}}$ is S-smooth.*

Proof. Proposition 2.2.2 gives an open cell $m_\beta \colon \mathbf{U}_- \times_S \mathbf{T} \times_S \mathbf{A}^I_S \times_S \mathbf{U}_+ \hookrightarrow \mathrm{V}_{\mathbf{G},\beta}$ by base changing m. The $\mathbf{G} \times_S \mathbf{G}$ translates of m_β cover $\mathrm{V}^{\circ}_{\mathbf{G},\beta}$ since the same is true for m and $\mathrm{V}^{\circ}_{\mathbf{G}}$ by definition. This open embedding is equivariant for the obvious action of $\mathbf{G}^I_{\mathrm{m},S}$ on $\mathbf{T}$ and $\mathbf{A}^I_S$. Passing to GIT quotients with the same linearization as before, we thus get an open embedding

$$\mathbf{U}_- \times_S \overline{\mathbf{T}} \times_S \mathbf{U}_+ \hookrightarrow \overline{\mathbf{G}},$$

where $\overline{\mathbf{T}}$ is the toric scheme for $\mathbf{T}$ corresponding to the fan Σ. This toric scheme can be constructed as $\mathbf{T} \times_S \mathbf{A}^I_S /\!\!/_{\rho} \mathbf{G}^I_{\mathrm{m},S}$ with semistable (and stable) subscheme equal to $\mathbf{T} \times_S \mathbf{A}^{\circ}_{S,\beta}$ (see, for e.g., [CLS11, §5.1]). Since $W\Sigma$ is smooth by construction (Lemma 3.2.1), so is Σ. Therefore, $\overline{\mathbf{T}}$ is smooth. The desired result follows by 2.2.3 since the $\mathbf{G} \times_S \mathbf{G}$-saturation of $\mathbf{U}_- \times_S \overline{\mathbf{T}} \times_S \mathbf{U}_+$ is $\overline{\mathbf{G}}$. $\square$

The proof of Proposition 3.3.2 also shows

Proposition 3.3.3. *There is an open cell*

$$\Omega \colon \mathbf{U}_- \times_S \overline{\mathbf{T}} \times_S \mathbf{U}_+ \hookrightarrow \overline{\mathbf{G}}$$

whose $\mathbf{G} \times_S \mathbf{G}$-saturation is $\overline{\mathbf{G}}$.

What remains is to check projectivity of $\overline{\mathbf{G}}$. In [MT16], properness is shown by realizing $\overline{\mathbf{G}}$ as the coarse space associated to a certain proper moduli stack of bundles. We resort to an alternative approach as we don't have a modular interpretation at hand.

Proposition 3.3.4. *$\overline{\mathbf{G}}$ is S-projective.*

Proof. As in the proof of Proposition 3.3.1, we may assume $S = \operatorname{Spec} \mathbf{Z}$. Because $\overline{\mathbf{G}}$ is constructed as a GIT quotient, it suffices to check properness. We use valuative criterion for properness. For this, we may assume that S is the spectrum of a discrete valuation ring R with fraction field K. By [Rom13, Lemma 4.1.1], it is enough to check that any K-point $x_K \in \mathbf{G}(K)$ extends to an R-point of $\overline{\mathbf{G}}$. Using the Cartan decomposition of $\mathbf{G}(K)$, we can assume $x_K = \lambda(\pi) \in \mathbf{T}(K)$ for some dominant coweight λ and uniformizer π. Such a point induces an R-point of $\mathbf{T}^+_{\mathrm{ad}}$ which in turn induces an R-point x_R of $\mathrm{V}^{\circ}_{\mathbf{G}}$ by applying the canonical section $\mathfrak{s}$ (c.f. §2.2). Base changing, we get a map $\mathbf{A}^I_S \times_{\mathbf{T}^+_{\mathrm{ad}}} x_R \to \mathrm{V}_{\mathbf{G},\beta}$ and choosing an arbitrary S-morphism of affine spaces $\mathbf{A}^I_S \to \mathbf{T}^+_{\mathrm{ad}}$, not necessarily a section, we get a map

$x_R \to \mathrm{V}_{\mathbf{G},\beta}$. By construction, the generic point of x_R lies inside the unit group $\mathbf{G}_{\mathrm{m},S}^I \times_S \mathbf{G}$ where its second component is x_K. Then the image of x_R along the GIT quotient morphism gives the desired point. □

Remark 3.3.5. It is likely possible to obtain a complete combinatorial classification of all equivariant compactifications of an isotrivial reductive group scheme by following the methods of [MT16], but we do not pursue this here.

4. The semisimple case

We prove Theorem 1.3 in this section. Let G be a semisimple reductive group scheme over S. By [SGA3, Exposé XXIV, Théorème 4.1.5], G is locally isotrivial. This allows us to carry out the construction of §3 Zariski locally on S where we always choose Σ according to Remark 3.2.2. Since this choice is independent of the finite étale Galois covers, they glue to give $\overline{\mathrm{G}} \to S$. By [Vin95, Theorem 8] and compatibility of such GIT quotients with base change, geometric fibers of $\overline{\mathrm{G}}_{\mathrm{ad}}$ are indeed classical wonderful compactifications of de Concini and Procesi. We have shown that $\overline{\mathrm{G}}_{\mathrm{ad}}$ is smooth projective over S. There is a morphism $\overline{\mathrm{G}} \to \overline{\mathrm{G}}_{\mathrm{ad}}$ coming from a natural map of Cox-Vinberg monoids, extending the central isogeny $\mathrm{G} \to \mathrm{G}_{\mathrm{ad}}$. By [MT16, Lemma 9.2], $(\overline{\mathrm{G}})_s$ is the normalization of G_s in $(\overline{\mathrm{G}}_{\mathrm{ad}})_s$ for each geometric point $s \to S$. What remains is to prove the statement about the boundary.

Assumption 4.1. From now on, we assume G is adjoint.

Proposition 4.2. $\overline{\mathrm{G}} \setminus \mathrm{G}$ *is the union of S-smooth relative effective Cartier divisor with normal crossings.*

Proof. By construction, the formation of $\overline{\mathrm{G}} \setminus \mathrm{G} \hookrightarrow \overline{\mathrm{G}}$ commutes with base change on S. Since smoothness, being a relative divisor, and having normal crossings are étale local on the base, we may assume that G is split and $S = \operatorname{Spec} \mathbf{Z}$. By Proposition 3.3.3, there is an open cell

$$\Omega \colon \mathrm{U}_- \times_S \overline{\mathrm{T}} \times_S \mathrm{U}_+ \hookrightarrow \overline{\mathrm{G}}$$

where the toric embedding $\mathrm{T} \hookrightarrow \overline{\mathrm{T}}$ looks like $\prod_{\alpha\in\Delta} \mathbf{G}_{\mathrm{m},S} \hookrightarrow \prod_{\alpha\in\Delta} \mathbf{A}^1_S$. The complement $\overline{\mathrm{T}} \setminus \mathrm{T}$ is clearly an S-flat divisor, therefore by Lemma 2.2.3, so is the whole boundary $\overline{\mathrm{G}} \setminus \mathrm{G}$. By [Stacks, Tag 062Y] and the fact that translates of Ω cover $\overline{\mathrm{G}}$ when S is an algebraically closed point, it follows that $D := \overline{\mathrm{G}} \setminus \mathrm{G}$ is a relative effective Cartier divisor. Since G is connected, irreducible components, say D_i, of this divisor are $\mathrm{G} \times_S \mathrm{G}$-stable. Since Ω is dense in $\overline{\mathrm{G}}$, it intersects each D_i in a dense open subscheme, which implies that D_i is the $\mathrm{G} \times_S \mathrm{G}$-saturation of the dense open $\Omega \cap D_i$ inside D_i. Hence, we obtain that D_i is S-smooth by Lemma 2.2.3. What remains is to check that $D = \sum_i D_i$ has relative normal crossings. We know that D_s has normal crossings for each geometric point $s \to S$ since translates of Ω_s cover $\overline{\mathrm{G}}_s$. The desired result now follows from Lemma 4.4. □

Remark 4.3. In fact, we end up proving a bit more when G is split: irreducible components of the boundary divisor are indexed by a set of system of simple roots Δ and $\mathrm{G} \times_S \mathrm{G}$-stable subschemes of $\overline{\mathrm{G}} \setminus \mathrm{G}$ correspond to subsets of Δ.

Lemma 4.4. *Let X be a smooth S-scheme equipped with a relative effective Cartier divisor D with S-smooth irreducible components. Assume that for each geometric point $s \in S$, D_s has normal crossings in X_s. Then D has relative normal crossings over S.*

Proof. Let $x \in X$ be a point with image $s \in S$, and $D_1, \ldots, D_n$ the irreducible components of D passing through x. Choose an affine open neighborhood U of x such that $D_i \cap U$ is cut out by $f_i = 0$ for $f_i \in \mathrm{H}^0(U, \mathcal{O}_X)$. The images of f_i in $\Omega_{X/S} \otimes \kappa(x)$ are linearly independent because $\Omega_{X/S} \otimes \kappa(x) \simeq \Omega_{X_s/s} \otimes \kappa(x)$ and the same is true in $\Omega_{X_{\overline{s}}/\overline{s}} \otimes \overline{\kappa(x)}$ by assumption, where $\overline{s}$ is an arbitrary geometric point lying over s. Let N be the rank of $\Omega_{X/S}$ at x. By Nakamaya and possibly shrinking U, extend these to a set of functions $\{f_1, \ldots, f_n, f_{n+1}, \ldots, f_N\}$ each of which vanish at x such that their differentials form a basis for the trivial vector bundle $\Omega_{U/S}$. We thus obtain an *étale* S-morphism

$$f\colon U \xrightarrow{(f_1,\dots,f_N)} \mathbf{A}_S^N$$

such that $D \cap U$ is the preimage of the relative normal crossing divisor $x_1 x_2 \cdots x_n = 0$ in $\mathbf{A}_S^N$. □

Proposition 4.5. $\overline{\mathrm{G}}$ *agrees with the equivariant compactification* $\mathcal{X}$ *of [Li25, Theorem 1].*

Proof. Firstly, it suffices to check this for quasi-split G because our $\overline{\mathrm{G}}$ and the compactification $\mathcal{X}$ of [Li25, Theorem 1] is obtained by first making the same constructions for split G and then performing the obvious inner twist. The construction for quasi-split G is basically the same as the construction for split G where everything is equipped with the action of a finite abstract group preserving a pinning. Thus, we reduce to the split case. By Proposition 3.3.3, we have an open cell $\Omega\colon \mathrm{U}_- \times_S \overline{\mathrm{T}} \times_S \mathrm{U}_+ \hookrightarrow \overline{\mathrm{G}}$ whose $\mathrm{G} \times_S \mathrm{G}$-saturation is $\overline{\mathrm{G}}$. The method of [Li25] starts by defining a *rational* $\mathrm{G} \times_S \mathrm{G}$-action π on Ω [Li25, Theorem 3.4] and then defining $\mathcal{X}$ as an appropriate fppf sheaf quotient of $\mathrm{G} \times_S \Omega \times_S \mathrm{G}$. Due to the uniqueness assertion of [Li25, Theorem 3.4], we reduce to checking that the rational S-morphism $\mathrm{G} \times_S \Omega \times_S \mathrm{G} \dashrightarrow \Omega$ induced by the action map $\mathrm{G} \times_S \Omega \times_S \mathrm{G} \to \overline{\mathrm{G}}$ given by $(g_1, \omega, g_2) \mapsto g_1 \omega g_2$ satisfies the conditions of *loc. cit.*, but this is clear. □

5. A torus admitting no equivariant compactification

Let k be an algebraically closed field. We recall the construction of [SGA3, Exposé X, §1.6]. Let S_1 be the Néron 1-gon, obtained by glueing sections 0 and ∞ of $\mathbf{P}_k^1$. It can be realized as the nodal cubic curve $\operatorname{Proj} k[x,y,z]/(y^2 z - x^3 + x^2 z)$ equipped with the normalization $\pi_1\colon \mathbf{P}_k^1 \to S_1$. Let S_∞ be the Néron ∞-gon which comes equipped with a finite morphism $\pi_\infty\colon \mathbf{P}_k^1 \times \mathbf{Z} \to S_\infty$ which glues the ∞-section of $\mathbf{P}_k^1 \times \{i\}$ with the 0-section of $\mathbf{P}_k^1 \times \{i+1\}$. There is an infinite connected étale Galois cover $S_\infty \to S_1$ with Galois group $\mathbf{Z}$ which is covered by the trivial $\mathbf{Z}$-torsor $\mathbf{P}_k^1 \times \mathbf{Z} \to \mathbf{P}_k^1$ where k acts on the source via $j \cdot (x, i) = (x, i+j)$.

Define an action of $\mathbf{Z}$ on the constant torus $\mathbf{G}_{\mathrm{m},k}^2 \times S_\infty$ by $1 \cdot (t, x) = (Mt, 1 \cdot x)$ where M is the *infinite* order automorphism of $\mathbf{G}_{\mathrm{m},k}^2$ given by $(t_1, t_2) \mapsto (t_1, t_1 t_2)$. By Galois descent, we thus obtain a rank 2 torus $\mathrm{T}_{S_1} \to S_1$. Since T_{S_1} has infinite monodromy by construction, T_{S_1} is a quasi-isotrivial[1] torus that is not isotrivial. Alternatively, T_{S_1} can be constructed by taking the constant torus $\mathbf{G}_{\mathrm{m},k}^2 \times \mathbf{P}_k^1 \to \mathbf{P}_k^1$ and glueing the fibers over 0 and ∞ via the automorphism M.

Proposition 5.1. *There is no projective S_1-scheme $\overline{\mathrm{T}}_{S_1}$ containing T_{S_1} as a fiberwise dense open subscheme such that the translation action of T_{S_1} on itself extends to $\overline{\mathrm{T}}_{S_1}$.*

Assume the contrary that such a $\overline{\mathrm{T}}_{S_1}$ exists. First, replace $\overline{\mathrm{T}}_{S_1}$ by its reduction so that it's a genuine k-variety– we know that it is irreducible because it contains a torus as an open dense subvariety. Let X be the base change of $\overline{\mathrm{T}}_{S_1}$ along the normalization $\pi_1\colon \mathbf{P}_k^1 \to S_1$. Let $\mathrm{T} := \mathbf{G}_{\mathrm{m},k}^2 \times_k \mathbf{P}_k^1$ be the open dense torus inside X.

Since X is irreducible, it must be flat over $\mathbf{P}_k^1$. Set $\mathrm{T} := \mathbf{G}_{\mathrm{m},\mathbf{P}_k^1}^2$. We use the following result of Brion:

Theorem 5.2 ([Bri15, Theorem 4.8])**.** *Let* G *be a split torus over a field k. Then every quasiprojective k-variety X equipped with an action of* G *admits a finite étale* G*-equivariant cover $f\colon Y \to X$, where Y is the union of open affine* G*-stable subvarieties.*

We may view X as a projective variety with $\mathbf{G}_{\mathrm{m},k}^2$-action by identifying $\mathrm{T} \times_{\mathbf{P}_k^1} X \simeq \mathbf{G}_{\mathrm{m},k}^2 \times_k X$ in the source of the action map. By applying Theorem 5.2, we get a $\mathbf{G}_{\mathrm{m},k}^2$-equivariant finite étale cover $\pi\colon Y \to X$ such that Y is the union of open affine $\mathbf{G}_{\mathrm{m},k}^2$-stable subvarieties.

[1]This means it is split by an étale cover.

Lemma 5.3. *$\pi\colon \pi^{-1}(\mathrm{T}) \to \mathrm{T}$ is a trivial cover.*

Proof. Any finite étale cover of T comes from a finite étale cover $U \to \mathbf{G}_{\mathrm{m},k}^2$. Indeed, the projection $\mathrm{T} \to \mathbf{G}_{\mathrm{m},k}^2$ is a $\mathbf{P}_k^1$-bundle and hence induces an isomorphism on étale fundamental groups by the homotopy exact sequence. A connected finite étale cover $U \to \mathbf{G}_{\mathrm{m},k}^2$ must be an étale self-isogeny induced by a linear endomorphism of the character lattice. Equivariance forces such an isogeny to be an isomorphism. □

Label the components of $\pi^{-1}(\mathrm{T})$ as $\mathrm{T}_1, \mathrm{T}_2, \ldots, \mathrm{T}_n$. Each of these are abstractly isomorphic to T and map down to $\mathrm{T} \subset X$ via identity. The $\mathbf{G}_{\mathrm{m},k}^2$-action on $\overline{\mathrm{T}}_i$ upgrades to a T_i-action extending the one on T_i by itself. The union $\bigcup_i \mathrm{T}_i \subset Y$ is a $\mathbf{P}_k^1$-fiberwise open-dense subscheme since the same is true for $\mathrm{T} \subset X$. Denote by t a closed point of $\mathbf{P}_k^1$. The fiber Y_t has irreducible components given by the (scheme-theoretic) closures of $(\mathrm{T}_1)_t, (\mathrm{T}_2)_t, \ldots, (\mathrm{T}_n)_t$. In particular, each fiber of Y has n irreducible components. Every closure $\overline{\mathrm{T}}_i$, $1 \leqslant i \leqslant n$, is faithfully flat over $\mathbf{P}_k^1$ and hence has fibers of pure dimension 2. Therefore, each $(\overline{\mathrm{T}}_i)_t$ is the union of closures of a nonempty subset of $\{(\mathrm{T}_1)_t, (\mathrm{T}_2)_t, \ldots, (\mathrm{T}_n)_t\}$. For $i \neq j$, $(\overline{\mathrm{T}}_i)_t$ and $(\overline{\mathrm{T}}_j)_t$ cannot have an irreducible component in common, say the closure of $(\mathrm{T}_h)_t$, because Y_t is generically reduced. Alternatively, if ξ is the generic point of $(\mathrm{T}_h)_t$, the local ring $\mathcal{O}_{Y,\xi} \simeq \mathcal{O}_{\mathrm{T}_h,\xi}$ is a DVR and hence cannot contain more than one minimal prime. We thus obtain:

Proposition 5.4. *Every $\overline{\mathrm{T}}_i$ is a flat projective variety over $\mathbf{P}_k^1$ containing the split torus T_i as a fiberwise dense open subvariety such that the action of T_i on itself extends to $\overline{\mathrm{T}}_i$ in a way that $\overline{\mathrm{T}}_i$ can be covered by $\mathbf{G}_{\mathrm{m},k}^2$-stable open affine subvarieties. Furthermore, there is an isomorphism between the normalizations of the 0-fiber and ∞-fiber of $\overline{\mathrm{T}}_i$ restricting to $(x, y) \mapsto (x, xy)$ on $\mathbf{G}_{\mathrm{m},k}^2$.*

Proof. The first part is clear from the previous discussion. For the second part, note that there is a finite birational morphism $(\overline{\mathrm{T}}_i)_t \to X_t$ because it restricts to identity on the toral part and hence induces an isomorphism on normalizations by Zariski's main theorem. □

Set $W := \overline{\mathrm{T}}_1$ and rename T_1 as T for ease of notation. Denote $W \to \mathbf{P}_k^1$ by f. Let $U \subset W$ be a nonempty $\mathbf{G}_{\mathrm{m},k}^2$-stable open affine subvariety. Then $\mathrm{V} := f(U)$ is a nonempty open subvariety of $\mathbf{P}_k^1$. Of course, U is $\mathbf{P}_k^1$-fiberwise open and hence intersects $\mathrm{T}_{\mathrm{V}} = \mathbf{G}_{m,\mathrm{V}}^2$ fiberwise. Due to $\mathbf{G}_{\mathrm{m},k}^2$-invariance, U must contain whole of T_{V}. The $\mathbf{G}_{\mathrm{m},k}^2$-action on U can be upgraded to an action of T_{V}. Therefore,

Proposition 5.5. *Let $\underline{x}\colon \operatorname{Spec} \mathcal{O}_{\mathbf{P}_k^1,x} \to \mathbf{P}_k^1$ be the local scheme at a closed point $x \in \mathbf{P}_k^1$. Then $W_{\underline{x}}$ can be covered by $\mathrm{T}_{\underline{x}}$-stable affine open neighborhoods of $\mathrm{T}_{\underline{x}} \subset W_{\underline{x}}$.*

Lemma 5.6. *Let X be an integral affine scheme over a discrete valuation ring R which contains a split R-torus T as a* fiberwise dense *open subscheme such that the action of T on itself extends to X. Then the fibers of the normalization $\widetilde{X}$ as an R-scheme are irreducible and normal.*

Proof. Let M be the character lattice of T. Such an affine toric scheme X is given by a graded sub-R-algebra A of the M-graded group ring $R[M]$, which in turn is equivalent to the data of a finitely generated submonoid Q of M which generates M as a group. The last bit ensures that X contains $\mathbf{G}_{\mathrm{m},R}^n$ as a open dense subscheme. The normalization of A then corresponds to the saturation monoid $\widetilde{Q}$ of Q defined as $\{a \in M \colon na \in Q \text{ for some } n \in \mathbf{Z}\}$. The special fiber of $\widetilde{X}$ is then the spectrum of the monoid ring $k[\widetilde{Q}]$, where k is the residue field of R. Since $k[\widetilde{Q}]$ is a subring of the integral domain $k[M]$, it follows that the special fiber of $\widetilde{X}$ is irreducible. Since $\widetilde{Q}$ is saturated, $k[\widetilde{Q}]$ is also integrally closed. □

Let $\widetilde{W_{\underline{x}}} \to W_{\underline{x}}$ be the normalization. By Proposition 5.5 and Lemma 5.6, it follows that $\widetilde{W_\ell}$ is a flat projective normal $\mathrm{T}_{\underline{x}}$-toric scheme over $R := k[t]_{(t)}$ with irreducible normal fibers. Consider $\widetilde{W_{\underline{0}}}$ and $\widetilde{W_{\underline{\infty}}}$. The generic fibers of these are identified and there is an isomorphism between their special fibers which restricts to $(x, y) \mapsto (x, xy)$ on respective toral parts. By the theory of normal toric schemes over DVRs [Kem+73, item e) at p. 192], these are classified by two *complete* rational polyhedral fans Σ_1 and Σ_2 in

$\mathbf{R}^2 \times \mathbf{R}_{\geqslant 0}$, respectively. Let $\pi\colon \mathbf{R}^2 \times \mathbf{R}_{\geqslant 0} \to \mathbf{R}^2$ be the natural projection where we often view the target as sitting inside $\mathbf{R}^2 \times \mathbf{R}_{\geqslant 0}$ as $\mathbf{R}^2 \times \{0\}$. We recall the following two facts:

- By [Kem+73, item e') at p. 192], the recession fan of Σ_i, defined as the image of $\Sigma_i \cap (\mathbf{R}^2 \times \{0\})$ along π, classify the respective generic fibers, and are therefore equal, to say Σ_∂.
- There is an embedding of fans $\{0\} \times \mathbf{R}_{\geqslant 0} \hookrightarrow \Sigma_i$ corresponding to the open embedding of toric schemes $\mathbf{G}^2_{\mathrm{m},R} \hookrightarrow \widetilde{W}_{\underline{0}}$ and $\mathbf{G}^2_{\mathrm{m},R} \hookrightarrow \widetilde{W}_{\underline{\infty}}$. Indeed, the toric scheme $\mathbf{G}^2_{\mathrm{m},R} \to \operatorname{Spec} R$ is classified by the cone $\{0\} \times \mathbf{R}_{\geqslant 0}$. Therefore, the components of the special fiber containing $\mathbf{G}^2_{\mathrm{m},k}$ are classified by the (complete) fan $\Delta_i := \{\pi(\sigma)\colon \sigma \in \Sigma_i,\ \{0\} \times \mathbf{R}_{\geqslant 0} \subseteq \sigma\}$ as a toric variety. Furthermore, there is a one-to-one correspondence between irreducible components of the special fiber and vertices of the polyhedral complex $\Sigma_i \cap (\mathbf{R}^2 \times \{1\})$ (c.f. [Wal13, Proposition 7.15] or [BPS18, Remark 3.5.9]).

We thus conclude that every ray in Σ_i not equal to $\{0\} \times \mathbf{R}_{\geqslant 0}$ must be contained in the boundary $\mathbf{R}^2 \times \{0\}$. That is, $\widetilde{W}_{\underline{x}}$ is a constant family– the base change of a normal toric variety over k. Now, the fact that there is an isomorphism between the special fibers of $\widetilde{W}_{\underline{0}}$ and $\widetilde{W}_{\underline{\infty}}$ extending $(x, y) \mapsto (x, xy)$ on the toral part corresponds to the fact that $A(\Delta_1) = \Delta_2$ where $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. However, $\Delta_1 = \Sigma_\partial = \Delta_2$. Thus, Σ_∂ is a complete fan in $\mathbf{R}^2$ which is stable under the automorphism A. There must exist a ray $\ell \in \Sigma_\partial$ which is not contained in the x-axis for otherwise it wouldn't be complete. Then $\{A^n \ell\colon n \in \mathbf{Z}\}$ is an infinite set. This contradicts the finiteness of Σ_∂. The proof is complete.

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