# LSU Algebra Question Bank Solution

Ayanava Mandal

April 2025

# Contents

# Chapter 1

# Group Theory

**G1**: Let $H$ be a normal subgroup of a group $G$, and let $K$ be a subgroup of $H$.

(a) Give an example of this situation where $K$ is not a normal subgroup of $G$,

(b) Prove that if the normal subgroup $H$ is cyclic, then $K$ is normal in $G$.

**Solution 1.** *(a) Let $G = S_4$, $H = A_4$, and $K = \{e, (123), (132)\}$.*

*(b) Let $H =< h >$ be cyclic. Let $K =< k >$ where $k = h^a$ for some $a \in \mathbb{N}$.*
*Since $H$ is normal, $ghg^{-1} = h^b \in H$ for some $b$.*
*$gkg^{-1} = gh^a g^{-1} = (ghg^{-1})^a = h^{ba} = k^b \in K$. So, $K$ is normal in $G$.*

$\square$

***G2:*** *Prove that every finite group of order at least three has a nontrivial automorphism.*

**Solution 2.** *We will try this in two cases:*
*Case 1: The group is not abelian. Let $g \notin Z(G)$. Let $\phi_g$ be the nontrivial automorphism $h \mapsto ghg^{-1}$.*
*Case 2: The group is abelian. If there is an element of order not equal to 2, the inverse map is a nontrivial automorphism. If every element is of order 2: $G = (\mathbb{Z}/2\mathbb{Z})^n$, where $n > 1$. Swap 2 elements.* $\square$

***G3:***

*(a) State the structure theorem for finitely generated Abelian group.*

*(b) If $p$ and $q$ are distinct primes, determine the number of nonisomorphic Abelian groups of order $p^3q^4$.*

**Solution 3.** *(a) If $G$ is finitely generated Abelian group, $G$ is isomorphic to $\mathbb{Z}^n \times \mathbb{Z}_{a_1} \times \cdots \times \mathbb{Z}_{a_r}$ where $a_i \mid a_{i+1}$, $\mathbb{Z}_a = \mathbb{Z}/a\mathbb{Z}$ cyclic group of order $a$.*

*(b) Let $P(n)$ be the partition function. The number of nonisomorphic Abelian groups of order $p^3q^4 = P(3)P(4) = 3 \times 5 = 15$.*

$\square$

***G4:*** *Let $G = \mathrm{GL}(2, \mathbb{F}_p)$ be the group of invertible $2 \times 2$ matrices with entries in the finite field $\mathbb{F}_p$, where $p$ is a prime.*

*(a) Show that $G$ has order $\left(p^2-1\right)\left(p^2-p\right)$.*

*(b) Show that for $p = 2$ the group $G$ is isomorphic to the symmetric group $S_3$.*

**Solution 4.** *(a) Choosing a invertible $2 \times 2$ matrix is equivalent to choosing two linearly independent vectors(which will be the columns of the matrix) from the space $\mathbb{F}_p^2$. We can choose a nonzero vector in $\mid \mathbb{F}_p^2 \mid -1 = p^2 - 1$ ways and the second vector can't be a multiple of the first vector(there are $p$ of them). So, we can choose the second vector in $p^2 - p$ ways.*

*(b) The group is of order 6. We just have to show that it is not abelian. Show for the elements $a = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $b = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. $ab = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, ba = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$.*

$\square$

***G5:*** *Let $G$ be the group of units of the ring $\mathbb{Z}/247\mathbb{Z}$.*

*(a) Determine the order of $G$ (note that $247 = 13 \cdot 19$ ).*

*(b) Determine the structure of $G$ (as in the classification theorem for finitely generated abelian groups). Hint: Use the Chinese Remainder Theorem.*

**Solution 5.** *We will discuss a bit about the group of units. Let* $N = 2^k p_1^{k_1} \cdots p_n^{k_n}$ *where* $p_i s$ *are odd primes. By CRT, we have* $(\mathbb{Z}_N) = (\mathbb{Z}_{2^k}) \times (\mathbb{Z}_{p_1^{k_1}}) \times \cdots \times (\mathbb{Z}_{p_n^{k_n}})$. *We have the unit group*

$$(\mathbb{Z}_N)^\times = (\mathbb{Z}_{2^k})^\times \times (\mathbb{Z}_{p_1^{k_1}})^\times \times \cdots \times (\mathbb{Z}_{p_n^{k_n}})^\times$$

*. For odd prime powers, we have that the unit group is cyclic* $(\mathbb{Z}_{p^k})^\times = \mathbb{Z}_{p^k - p^{k-1}}$.
*For* $2^k$ *we have* $(\mathbb{Z}_2)^\times = \mathbb{Z}_1$ *the trivial group,* $(\mathbb{Z}_4)^\times = \mathbb{Z}_2$ *cyclic and* $(\mathbb{Z}_{2^k})^\times = \mathbb{Z}_2 \times \mathbb{Z}_{2^{k-2}}$ *noncyclic groups for* $2^k \geq 8$. *So the only time where the unit group is cyclic is* $N = 1, 2, 4, p^k, 2p^k$ *where* $p$ *is an odd prime.*
*So, for* $N = 247$ *the order of the group is* $12 \times 18 = 216$. *And the structure of* $G$ *is* $\mathbb{Z}_{12} \times \mathbb{Z}_{18} = \mathbb{Z}_3 \times \mathbb{Z}_4 \times \mathbb{Z}_9 \times \mathbb{Z}_2 = \mathbb{Z}_6 \times \mathbb{Z}_{36}$.

□

***G6:*** *Let* $G$ *be the group of invertible* $2 \times 2$ *upper triangular matrices with entries in* $\mathbb{R}$. *Let* $D \subseteq G$ *be the subgroup of invertible diagonal matrices and let* $U \subseteq G$ *be the subgroup of matrices of the form* $\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}$ *where* $x \in \mathbb{R}$ *is arbitrary.*

(a) *Show that* $U$ *is a normal subgroup of* $G$ *and that* $G/U$ *is isomorphic to* $D$.

(b) *True or False (with justification):* $G \cong U \times D$

**Solution 6.** □

***G7:*** *Let* $G$ *be a group and let* $Z$ *denote the center of* $G$.

(a) *Show that* $Z$ *is a normal subgroup of* $G$.

(b) *Show that if* $G/Z$ *is cyclic, then* $G$ *must be abellan.*

(c) *Let* $D_6$ *be the dihedral group of order 6 . Find the center of* $D_6$.

**Solution 7.** □

***G8:*** *List all abelian groups of order 8 up to isomorphism. Identify which group on your list is isomorphic to each of the following groups of order 8 . Justify your answer.*

(a) $(Z/15Z)^* =$ *the group of units of the ring* $Z/15Z$.

(b) *The roots of the equation* $z^8 - 1 = 0 \mathrm{inC}$.

(c) $\mathbb{F}_8^+$ *=the additive group of the field* $\mathbb{F}_8$ *with eight elements.*

**Solution 8.** □

***G9:*** *Let* $S_9$ *denote the symmetric group on 9 elements.*

(a) *Find an element of* $S_9$ *of order 20.*

(b) *Show that there is no element of* $S_9$ *of order 18 .*

**Solution 9.** □

***G10:*** $G = \left\{ \begin{bmatrix} a & b \\ 0 & a^{-1} \end{bmatrix} : a, b \in \mathbb{R}, a > 0 \right\}$ *and* $N = \left\{ \begin{bmatrix} 1 & c \\ 0 & 1 \end{bmatrix} : c \in \mathbb{R} \right\}$ *are groups under matrix multiplication.*

(a) *Show that* $N$ *is a normal subgroup of* $G$ *and that* $G/N$ *is isomorphic to the multiplicative group of positive real numbers* $\mathbb{R}^+$.

# Chapter 2

# Ring Theory

***R1:***

## Chapter 3

# Module Theory

***M1:***

Chapter 4

# Linear Algebra

***L1:***