LSU Algebra Question Bank Solution

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Contents

Chapter 1

Group Theory

G1: Let H be a normal subgroup of a group G, and let K be a subgroup of H.

(a) Give an example of this situation where K is not a normal subgroup of G,

(b) Prove that if the normal subgroup H is cyclic, then K is normal in G.

Solution 1. *(a) Let $G = S_4$, $H = A_4$, and $K = \{e, (123), (132)\}$.*

(b) Let $H =< h >$ be cyclic. Let $K =< k >$ where $k = h^a$ for some $a \in \mathbb{N}$. Since H is normal, $ghg^{-1} = h^b \in H$ for some b.
$gkg^{-1} = gh^a g^{-1} = (ghg^{-1})^a = h^{ba} = k^b \in K$. So, K is normal in G.

□

G2: *Prove that every finite group of order at least three has a nontrivial automorphism.*

Solution 2. *We will try this in two cases:*
Case 1: The group is not abelian. Let $g \notin Z(G)$. Let ϕ_g be the nontrivial automorphism $h \mapsto ghg^{-1}$.
Case 2: The group is abelian. If there is an element of order not equal to 2, the inverse map is a nontrivial automorphism. If every element is of order 2: $G = (\mathbb{Z}/2\mathbb{Z})^n$, where $n > 1$. Swap 2 elements. □

G3:

(a) State the structure theorem for finitely generated Abelian group.

(b) If p and q are distinct primes, determine the number of nonisomorphic Abelian groups of order p^3q^4.

Solution 3. *(a) If G is finitely generated Abelian group, G is isomorphic to $\mathbb{Z}^n \times \mathbb{Z}_{a_1} \times \cdots \times \mathbb{Z}_{a_r}$ where $a_i \mid a_{i+1}$, $\mathbb{Z}_a = \mathbb{Z}/a\mathbb{Z}$ cyclic group of order a.*

(b) Let $P(n)$ be the partition function. The number of nonisomorphic Abelian groups of order $p^3q^4 = P(3)P(4) = 3 \times 5 = 15$.

□

G4:

Chapter 2

Ring Theory