

LSU Algebra Question Bank Solution

Ayanava Mandal

April 2025

Contents

1	Group Theory	1
2	Ring Theory	3
3	Module Theory	5
4	Linear Algebra	7

Chapter 1

Group Theory

G1: Let H be a normal subgroup of a group G , and let K be a subgroup of H .

- (a) Give an example of this situation where K is not a normal subgroup of G ,
- (b) Prove that if the normal subgroup H is cyclic, then K is normal in G .

Solution 1. (a) Let $G = S_4$, $H = A_4$, and $K = \{e, (123), (132)\}$.

- (b) Let $H = \langle h \rangle$ be cyclic. Let $K = \langle k \rangle$ where $k = h^a$ for some $a \in \mathbb{N}$.
Since H is normal, $ghg^{-1} = h^b \in H$ for some b .
 $gkg^{-1} = gh^a g^{-1} = (ghg^{-1})^a = h^{ba} = k^b \in K$. So, K is normal in G .

□

G2: Prove that every finite group of order at least three has a nontrivial automorphism.

Solution 2. We will try this in two cases:

Case 1: The group is not abelian. Let $g \notin Z(G)$. Let ϕ_g be the nontrivial automorphism $h \mapsto ghg^{-1}$.

Case 2: The group is abelian. If there is an element of order not equal to 2, the inverse map is a nontrivial automorphism. If every element is of order 2: $G = (\mathbb{Z}/2\mathbb{Z})^n$, where $n > 1$. Swap 2 elements.

□

G3:

- (a) State the structure theorem for finitely generated Abelian group.
- (b) If p and q are distinct primes, determine the number of nonisomorphic Abelian groups of order $p^3 q^4$.

Solution 3. (a) If G is finitely generated Abelian group, G is isomorphic to $\mathbb{Z}^n \times \mathbb{Z}_{a_1} \times \cdots \times \mathbb{Z}_{a_r}$ where $a_i \mid a_{i+1}$, $\mathbb{Z}_a = \mathbb{Z}/a\mathbb{Z}$ cyclic group of order a .

- (b) Let $P(n)$ be the partition function. The number of nonisomorphic Abelian groups of order $p^3 q^4 = P(3)P(4) = 3 \times 5 = 15$.

□

G4: Let $G = \text{GL}(2, \mathbb{F}_p)$ be the group of invertible 2×2 matrices with entries in the finite field \mathbb{F}_p , where p is a prime.

- (a) Show that G has order $(p^2 - 1)(p^2 - p)$.
- (b) Show that for $p = 2$ the group G is isomorphic to the symmetric group S_3 .

Solution 4. (a) Choosing a invertible 2×2 matrix is equivalent to choosing two linearly independent vectors (which will be the columns of the matrix) from the space \mathbb{F}_p^2 . We can choose a nonzero vector in $|\mathbb{F}_p^2| - 1 = p^2 - 1$ ways and the second vector can't be a multiple of the first vector (there are p of them). So, we can choose the second vector in $p^2 - p$ ways.

- (b) The group is of order 6. We just have to show that it is not abelian. Show for the elements $a = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $b = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. $ab = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$, $ba = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$.

□

G5: Let G be the group of units of the ring $\mathbb{Z}/247\mathbb{Z}$.

- (a) Determine the order of G (note that $247 = 13 \cdot 19$).
- (b) Determine the structure of G (as in the classification theorem for finitely generated abelian groups).
Hint: Use the Chinese Remainder Theorem.

Solution 5.

G6: Let G be the group of invertible 2×2 upper triangular matrices with entries in \mathbb{R} . Let $D \subseteq G$ be the subgroup of invertible diagonal matrices and let $U \subseteq G$ be the subgroup of matrices of the form $\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}$ where $x \in \mathbb{R}$ is arbitrary. □

- (a) Show that U is a normal subgroup of G and that G/U is isomorphic to D .
- (b) True or False (with justification): $G \cong U \times D$

Solution 6.

G7: Let G be a group and let Z denote the center of G . □

- (a) Show that Z is a normal subgroup of G .
- (b) Show that if G/Z is cyclic, then G must be abelian.
- (c) Let D_6 be the dihedral group of order 6. Find the center of D_6 .

Solution 7.

G8: List all abelian groups of order 8 up to isomorphism. Identify which group on your list is isomorphic to each of the following groups of order 8. Justify your answer. □

- (a) $(\mathbb{Z}/15\mathbb{Z})^*$ = the group of units of the ring $\mathbb{Z}/15\mathbb{Z}$.
- (b) The roots of the equation $z^8 - 1 = 0$ in \mathbb{C} .
- (c) \mathbb{F}_8^+ = the additive group of the field \mathbb{F}_8 with eight elements.

Solution 8.

G9: Let S_9 denote the symmetric group on 9 elements. □

- (a) Find an element of S_9 of order 20.
- (b) Show that there is no element of S_9 of order 18.

Solution 9.

G10: $G = \left\{ \begin{bmatrix} a & b \\ 0 & a^{-1} \end{bmatrix} : a, b \in \mathbb{R}, a > 0 \right\}$ and $N = \left\{ \begin{bmatrix} 1 & c \\ 0 & 1 \end{bmatrix} : c \in \mathbb{R} \right\}$ are groups under matrix multiplication. □

- (a) Show that N is a normal subgroup of G and that G/N is isomorphic to the multiplicative group of positive real numbers \mathbb{R}^+ .

Chapter 2

Ring Theory

R1:

Chapter 3

Module Theory

M1:

Chapter 4

Linear Algebra

L1:

