LSU Algebra Question Bank Solution

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Contents

Chapter 1

Group Theory

G1: Let H be a normal subgroup of a group G, and let K be a subgroup of H.

(a) Give an example of this situation where K is not a normal subgroup of G,

(b) Prove that if the normal subgroup H is cyclic, then K is normal in G.

Solution 1. *(a) Let $G = S_4$, $H = A_4$, and $K = \{e, (123), (132)\}$.*

(b) Let $H =< h >$ be cyclic. Let $K =< k >$ where $k = h^a$ for some $a \in \mathbb{N}$.
Since H is normal, $ghg^{-1} = h^b \in H$ for some b.
$gkg^{-1} = gh^a g^{-1} = (ghg^{-1})^a = h^{ba} = k^b \in K$. So, K is normal in G.

□

G2: *Prove that every finite group of order at least three has a nontrivial automorphism.*

Solution 2. *We will try this in two cases:*
Case 1: The group is not abelian. Let $g \notin Z(G)$. Let ϕ_g be the nontrivial automorphism $h \mapsto ghg^{-1}$.
Case 2: The group is abelian. If there is an element of order not equal to 2, the inverse map is a nontrivial automorphism. If every element is of order 2: $G = (\mathbb{Z}/2\mathbb{Z})^n$, where $n > 1$. Swap 2 elements. □

G3:

(a) State the structure theorem for finitely generated Abelian group.

(b) If p and q are distinct primes, determine the number of nonisomorphic Abelian groups of order p^3q^4.

Solution 3. *(a) If G is finitely generated Abelian group, G is isomorphic to $\mathbb{Z}^n \times \mathbb{Z}_{a_1} \times \cdots \times \mathbb{Z}_{a_r}$ where $a_i \mid a_{i+1}$, $\mathbb{Z}_a = \mathbb{Z}/a\mathbb{Z}$ cyclic group of order a.*

(b) Let $P(n)$ be the partition function. The number of nonisomorphic Abelian groups of order $p^3q^4 = P(3)P(4) = 3 \times 5 = 15$.

□

G4:*Let $G = \mathrm{GL}(2, \mathbb{F}_p)$ be the group of invertible 2×2 matrices with entries in the finite field $\mathbb{F}_p$, where p is a prime.*

(a) Show that G has order $\left(p^2 - 1\right)\left(p^2 - p\right)$.

(b) Show that for $p = 2$ the group G is isomorphic to the symmetric group S_3.

Solution 4. *(a) Choosing a invertible 2×2 matrix is equivalent to choosing two linearly independent vectors(which will be the columns of the matrix) from the space $\mathbb{F}_p^2$. We can choose a nonzero vector in $\mid \mathbb{F}_p^2 \mid -1 = p^2 - 1$ ways and the second vector can't be a multiple of the first vector(there are p of them). So, we can choose the second vector in $p^2 - p$ ways.*

(b) The group is of order 6. We just have to show that it is not abelian. Show for the elements $a = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $b = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. $ab = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, ba = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$.

□

G5: *Let G be the group of units of the ring $\mathbb{Z}/247\mathbb{Z}$.*

(a) Determine the order of G (note that $247 = 13 \cdot 19$).

(b) Determine the structure of G (as in the classification theorem for finitely generated abelian groups). Hint: Use the Chinese Remainder Theorem.

Solution 5. $\square$

G6: *Let G be the group of invertible 2×2 upper triangular matrices with entries in $\mathbb{R}$. Let $D \subseteq G$ be the subgroup of invertible diagonal matrices and let $U \subseteq G$ be the subgroup of matrices of the form $\left[\begin{array}{ll}1 & x \\ 0 & 1\end{array}\right]$ where $x \in \mathbb{R}$ is arbitrary.*

(a) *Show that U is a normal subgroup of G and that G / U is isomorphic to D.*

(b) *True or False (with justification): $G \cong U \times D$*

Solution 6. $\square$

G7: *Let G be a group and let Z denote the center of G.*

(a) *Show that Z is a normal subgroup of G.*

(b) *Show that if G / Z is cyclic, then G must be abellan.*

(c) *Let D_6 be the dihedral group of order 6 . Find the center of D_6.*

Solution 7. $\square$

G8: *List all abelian groups of order 8 up to isomorphism. Identify which group on your list is isomorphic to each of the following groups of order 8 . Justify your answer.*

(a) *$(Z / 15 Z)^*=$ the group of units of the ring $Z / 15 Z$.*

(b) *The roots of the equation $z^8-1=0 \mathrm{inC}$.*

(c) *$\mathbb{F}_8^{+}=$the additive group of the field $\mathbb{F}_8$ with eight elements.*

Solution 8. $\square$

G9: *Let S_9 denote the symmetric group on 9 elements.*

(a) *Find an element of S_9 of order 20.*

(b) *Show that there is no element of S_9 of order 18 .*

Solution 9. $\square$

G10: *$G=\left\{\left[\begin{array}{cc}a & b \\ 0 & a^{-1}\end{array}\right]: a, b \in \mathbb{R}, a>0\right\}$ and $N=\left\{\left[\begin{array}{ll}1 & c \\ 0 & 1\end{array}\right]: c \in \mathbb{R}\right\}$ are groups under matrix multiplication.*

(a) *Show that N is a normal subgroup of G and that G / N is isomorphic to the multiplicative group of positive real numbers $\mathbb{R}^{+}$.*

Chapter 2

Ring Theory