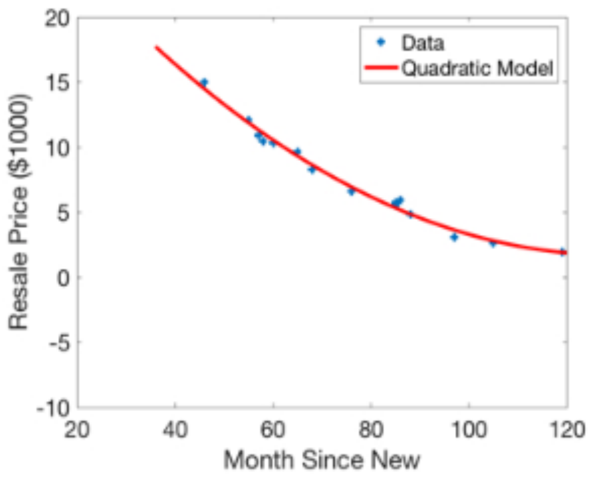
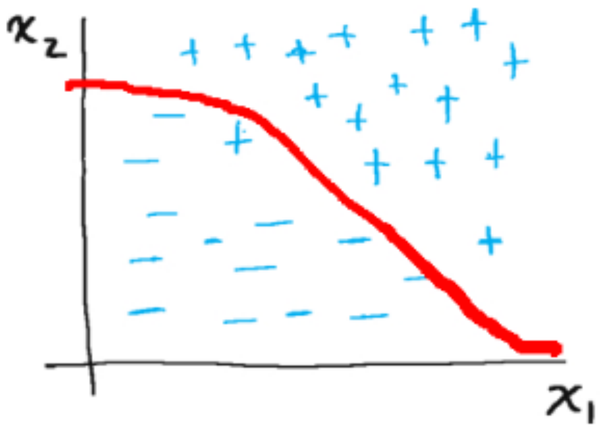
CS/ECE/ME 532
Activity 6

- Solving least squares



$$\begin{bmatrix} 1 & t_1 & t_1^2 \\ 1 & t_2 & t_2^2 \\ & \vdots & \\ 1 & t_n & t_n^2 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_n \end{bmatrix}$$



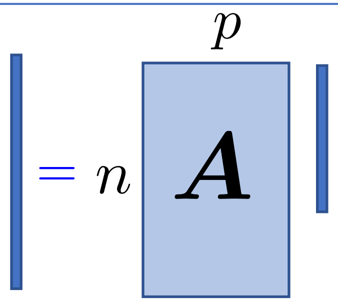
$$\text{sign}(\boldsymbol{x}_1^T \boldsymbol{w}) = -1$$
$$\text{sign}(\boldsymbol{x}_2^T \boldsymbol{w}) = +1$$
$$\vdots$$
$$\downarrow$$
$$\boldsymbol{X}\boldsymbol{w} \approx \boldsymbol{y}$$

$$\boldsymbol{y} = \boldsymbol{A}\boldsymbol{x}$$

x that satisfies this is a called *a solution*

Option 3: *No solution*

- Usually what happens with real data:
- Find approximate solution



$$\boldsymbol{x}^* = \arg\min_{\boldsymbol{x}} ||\boldsymbol{A}\boldsymbol{x} - \boldsymbol{y}||_2^2$$

this is a called *a least squares solution*

$$\boldsymbol{x}^* = (\boldsymbol{A}^T\boldsymbol{A})^{-1}\boldsymbol{A}^T\boldsymbol{y}$$

when does this inverse exist?

Invertible Matrix Theorem

The invertible matrix theorem is a theorem in linear algebra which gives a series of equivalent conditions for an $n \times n$ square matrix $\mathbf{A}$ to have an inverse. In particular, $\mathbf{A}$ is invertible if and only if any (and hence, all) of the following hold:

1. $\mathbf{A}$ is row-equivalent to the $n \times n$ identity matrix $\mathbf{I}_n$.
2. $\mathbf{A}$ has n pivot positions.
3. The equation $\mathbf{A}\mathbf{x} = \mathbf{0}$ has only the trivial solution $\mathbf{x} = \mathbf{0}$.
4. The columns of $\mathbf{A}$ form a linearly independent set.
5. The linear transformation $\mathbf{x} \mapsto \mathbf{A}\mathbf{x}$ is one-to-one.
6. For each column vector $\mathbf{b} \in \mathbb{R}^n$, the equation $\mathbf{A}\mathbf{x} = \mathbf{b}$ has a unique solution.
7. The columns of $\mathbf{A}$ span $\mathbb{R}^n$.
8. The linear transformation $\mathbf{x} \mapsto \mathbf{A}\mathbf{x}$ is a surjection.
9. There is an $n \times n$ matrix $\mathbf{C}$ such that $\mathbf{C}\mathbf{A} = \mathbf{I}_n$.
10. There is an $n \times n$ matrix $\mathbf{D}$ such that $\mathbf{A}\mathbf{D} = \mathbf{I}_n$.
11. The transpose matrix $\mathbf{A}^{\mathrm{T}}$ is invertible.
12. The columns of $\mathbf{A}$ form a basis for $\mathbb{R}^n$.
13. The column space of $\mathbf{A}$ is equal to $\mathbb{R}^n$.
14. The dimension of the column space of $\mathbf{A}$ is n.
15. The rank of $\mathbf{A}$ is n.
16. The null space of $\mathbf{A}$ is $\{\mathbf{0}\}$.
17. The dimension of the null space of $\mathbf{A}$ is 0.
18. 0 fails to be an eigenvalue of $\mathbf{A}$.
19. The determinant of $\mathbf{A}$ is not zero.
20. The orthogonal complement of the column space of $\mathbf{A}$ is $\{\mathbf{0}\}$.
21. The orthogonal complement of the null space of $\mathbf{A}$ is $\mathbb{R}^n$.
22. The row space of $\mathbf{A}$ is $\mathbb{R}^n$.
23. The matrix $\mathbf{A}$ has n non-zero singular values.

Positive definiteness (P.D.)

$$\boldsymbol{Q} \succ 0$$

$$\Updownarrow$$

$$\boldsymbol{x}^T\boldsymbol{Q}\boldsymbol{x} > 0 \text{ for all } \boldsymbol{x} \neq 0$$

$$\Downarrow$$

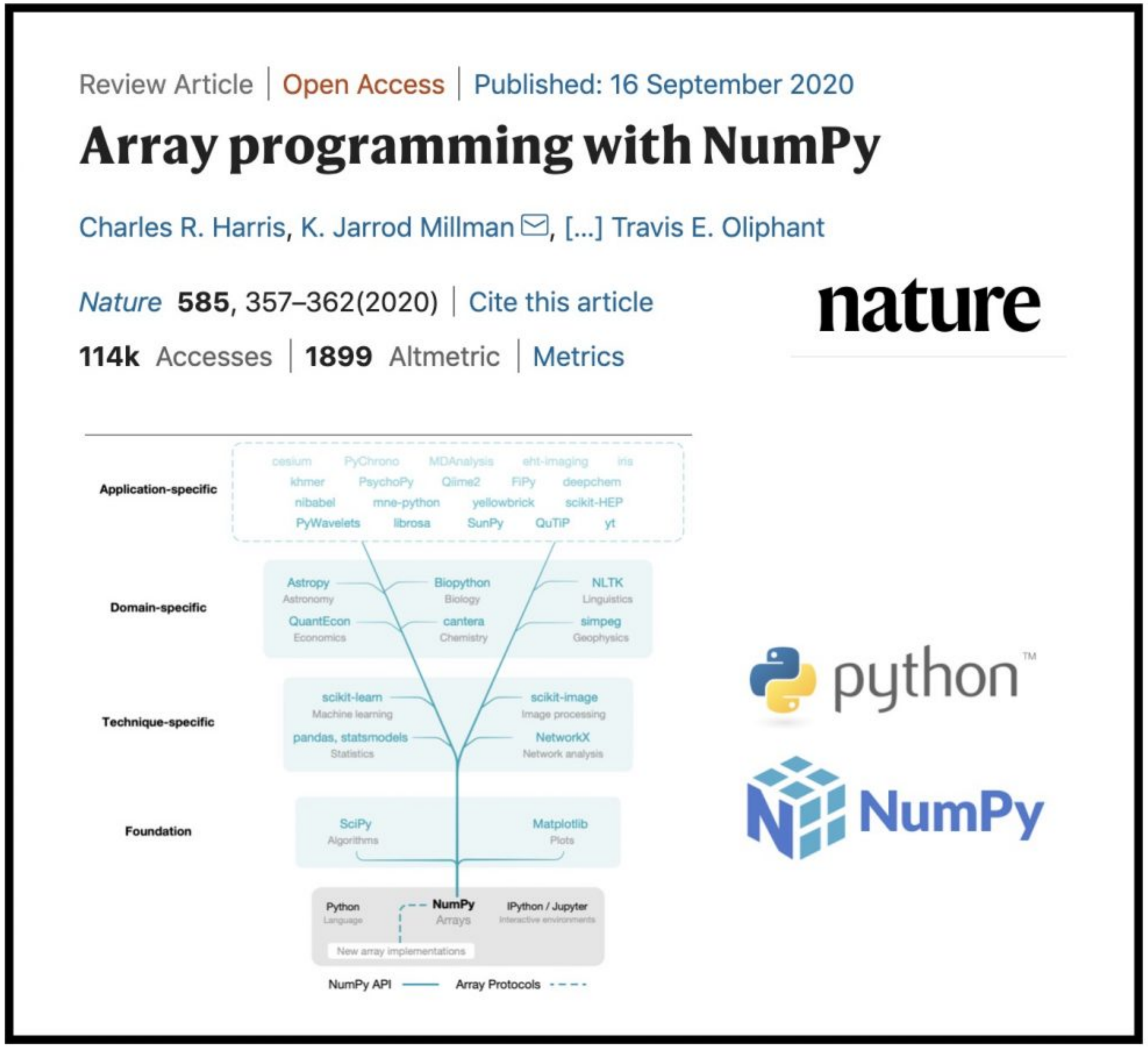
$$\boldsymbol{Q} \text{ is invertible}$$

$$\text{if } \operatorname{rank}(\boldsymbol{A}) = p, \text{ then } \boldsymbol{A}^T\boldsymbol{A} \text{ is invertible}$$

$$\Downarrow$$

$$\text{least squares solution exists and is unique}$$

numpy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Array programming with NumPy

Charles R. Harris, K. Jarrod Millman, [...] Travis E. Oliphant

Nature **585**, 357–362(2020) | Cite this article

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Useful commands

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

- A = np.array([[1, 2], [3, 4]])

$$b = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

- b = np.array([[1], [2], [3]])

$$Ab$$

- A@b