Ayar Deep Hazra Activity 4

1) a) $X = \begin{bmatrix} 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

If we look at the rows,

$\alpha_1 r_1 + \alpha_2 r_2 \neq 0$ for any $\alpha_1, \alpha_2 \neq 0$.

They are linearly independent.

Thus, Rank = 2

b) A set of linearly independent columns would be $\{a_1, a_3\}$

Yes, there is more than one set.

There are 5 sets of linearly independent columns.

$\{a_1, a_3\}, \{a_3, a_4\}, \{a_4, a_5\}, \{a_1, a_4\},$

$\{a_1, a_5\}$

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c) $\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} [w_1 \ w_2] = \begin{bmatrix} a \\ b \\ -1 \end{bmatrix}$

$$w_1 = a$$

$$w_1 + w_2 = b$$

$$w_2 = -1$$

$\therefore$ $a - 1 = b$

We can choose any values that satisfie the above eq^n

Let, $(a, b) = (5, 4)$

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2. $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}$

a) $b = \begin{bmatrix} 8 \\ 6 \\ -2 \end{bmatrix}$ Let $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 8 \\ 6 \\ -2 \end{bmatrix}$$

$x_1 = 8$

$x_1 + x_2 = 6$

$x_2 = -2$

Yes, solution exists for X.

Thus, $x = \begin{bmatrix} 8 \\ -2 \end{bmatrix}$

b) $b = \begin{bmatrix} 4 \\ 6 \\ 1 \end{bmatrix}$

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \\ 1 \end{bmatrix}$$

Thus, x has no solution as system is not consistent.

$x_1 = 4$

$x_1 + x_2 = 6$

$x_2 = 1$

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c) Since, b is the weighted sum of the columns of A, the augmented matrix's $[A \vdots b]$ rank must be equal to the rank of matrix A, for there to be a solution to the equation.

If for some reason, the addition of column b altered/increased the rank of A, then there would be no solution.

Since, A is full rank, we know this is not possible anyhow.

$\Rightarrow$ $\text{rank}\{A\} = \text{rank}\{[A \vdots b]\}$ guarantees that a solution exists.

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3. a) i) $A = \begin{bmatrix} 1 & -2 \\ -1 & 2 \\ -2 & 4 \end{bmatrix}$ $b = \begin{bmatrix} 2 \\ -2 \\ -4 \end{bmatrix}$ $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$$Ax = b$$

Rank $\{[A]\}$ → The two columns are linearly dependent, thus rank by default is 1.
Rank $\{[A]\} = 1$.

Rank $\{[A \vdots b]\}$ → The three columns in this augmented matrix are also linearly dependendent. No combination of the three columns taken two at a time are linearly independent. Thus, Rank $\{[A \vdots b]\} = 1$.

Thus, Rank $\{[A]\}$ = Rank $\{[A \vdots b]\} = 1$.

Thus the system of linear equations has a solution, by definition.

ii) $\dim\{x\} = 2$ (column vector with 2 rows)

Since Rank $\{A\} = 1$ (from above.)

Rank$\{A\} < \dim\{x\}$

iii) Thus the solution is not unique, by definition.

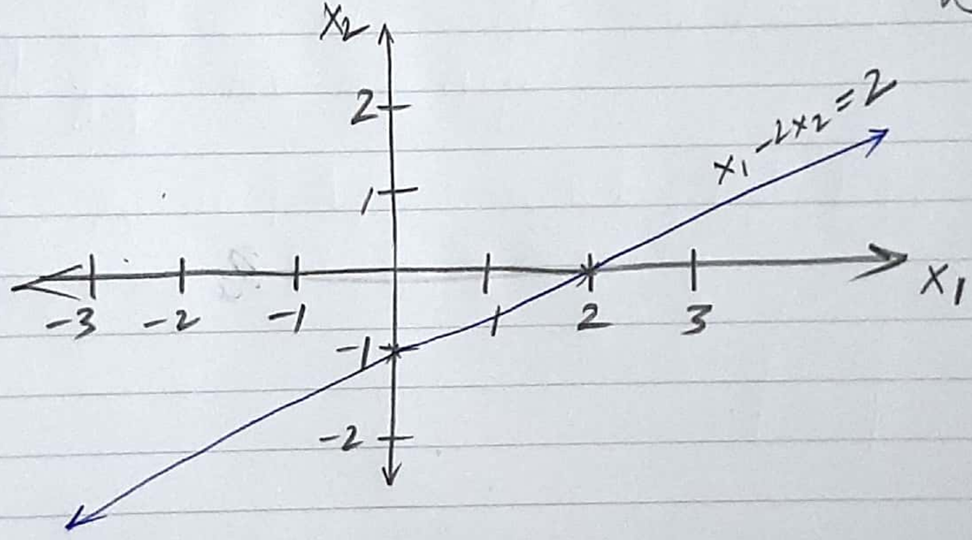
iii) $\begin{bmatrix} 1 & -2 \\ -1 & 2 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ -4 \end{bmatrix}$

$x_1 - 2x_2 = 2$

$-x_1 + 2x_2 = -2 \Rightarrow x_1 - 2x_2 = 2$

$-2x_1 + 4x_2 = -4 \Rightarrow x_1 - 2x_2 = 2$

(It's the same equation, in effect).



b) The rank of a matrix is the number of linearly independent columns it has.

Say $Ax = b$, where x is a column vector.

If the ~~If~~ x has the same number of rows as lin. independent columns in A, then system has an unique solution. If it has more rows than linearly independent colums, then there are infinite solutions

Thus, if rank$\{A\} <$ dim$\{x\}$ then there are more than one solutions in the system.

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