Ayan Deep Hazra Activity 7

1. a) $X = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$

Let $\vec{u_1} = x_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

$$u_1 = \frac{\tilde{u_1}}{\|\tilde{u_1}\|_2} = \frac{1}{\sqrt{1^2+1^2}}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}$$

$$\tilde{u_2} = x_2 - \mathrm{proj}_{u_1} x_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \cdot \frac{1}{\sqrt{2}}\left(\begin{bmatrix} 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right)$$

$$= \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}(1+0+0)$$

$$= \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 1/2 \\ 1/2 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1/2 \\ -1/2 \\ 1 \end{bmatrix}$$

$$u_2 = \frac{\tilde{u_2}}{\|\tilde{u_2}\|_2} = \frac{1}{\sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{-1}{2}\right)^2 + (1)^2}}\begin{bmatrix} 1/2 \\ -1/2 \\ 1 \end{bmatrix}$$

$$= \sqrt{\frac{2}{3}}\begin{bmatrix} 1/2 \\ -1/2 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} \\ \sqrt{\frac{2}{3}} \end{bmatrix}$$

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The object spanned by span $\{\tilde{v}_1, \tilde{v}_2\}$ is a plane. $(v_1, v_2) = \left(\begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}, \begin{bmatrix} 1/\sqrt{6} \\ -1/\sqrt{6} \\ \sqrt{2}/\sqrt{3} \end{bmatrix} \right)$

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b) $\tilde{X} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$

i) yes the columns of x span the same space as the columns of $\tilde{X}$.

ii) $\tilde{u}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = x_1$

$$u_1 = \frac{\tilde{u}_1}{\|\tilde{u}_1\|_2} = \frac{1}{\sqrt{1^2+0+1^2}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$\tilde{u}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - \text{proj}_{u_1} x_2$$

$$= \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \cdot \left(\frac{1}{\sqrt{2}} [1\ 0\ 1] \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right)$$

$$= \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} (1)$$

$$= \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 1/2 \\ 0 \\ 1/2 \end{bmatrix}$$

$$= \begin{bmatrix} 1/2 \\ 1 \\ -1/2 \end{bmatrix}$$

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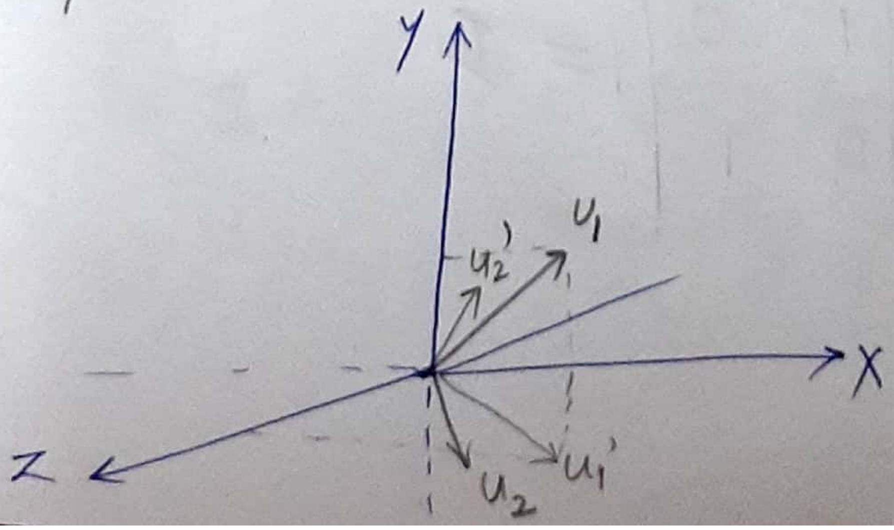
$$u_2' = \frac{\tilde{v}_2}{\|\tilde{v}_2\|_2} = \frac{1}{\sqrt{\left(\frac{1}{2}\right)^2 + (1)^2 + \left(\frac{-1}{2}\right)^2}} \begin{bmatrix} 1/2 \\ 1 \\ -1/2 \end{bmatrix}$$

$$= \sqrt{\frac{2}{3}} \begin{bmatrix} 1/2 \\ 1 \\ -1/2 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{\sqrt{6}} \\ \sqrt{\frac{2}{3}} \\ \frac{-1}{\sqrt{6}} \end{bmatrix}$$

$$(u_1', u_2') = \left(\begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}, \begin{bmatrix} 1/\sqrt{6} \\ \sqrt{2}/\sqrt{3} \\ -1/\sqrt{6} \end{bmatrix} \right)$$

The geometric object is a plane that is rotated along the origin as compared to the one in part a.



iii) The basis spanned by Gram-Schmidt are not unique.
The space spanned depends on the order of the columns.

2. a) $U = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{2} & -1/\sqrt{6} \\ 0 & \sqrt{2}/\sqrt{3} \end{bmatrix}$

b) $\boxed{U^T U}$

$U^T = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{6} & -1/\sqrt{6} & \sqrt{2}/\sqrt{3} \end{bmatrix}$

$U^T U = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{6} & -1/\sqrt{6} & \sqrt{2}/\sqrt{3} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} \\ 0 & \frac{\sqrt{2}}{\sqrt{3}} \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$

c) Define $U \cdot X = U[a_1 \; a_2]$

Since X has dimensions 3×2

& U has dimensions 3×2

$[a_1 \; a_2]$ must have dimensions 2×2

Thus a_1 & a_2 are column vectors with dim $= 2$.

→ Thus dimension of a_1 is 2×1.

→ If $[a_1 \; a_2] = \begin{bmatrix} \alpha_1 & \alpha_2 \\ \beta_1 & \beta_2 \end{bmatrix}$

The rows of $[a_1 \; a_2]$ weight the columns of U

d) $X = UA$

$U^T X = U^T U A$ (multiplying by U^T)

$U^T X = I A$ ($U^T U = I$ by definition)

$U^T X = A$ (Identity property)

$$U^T X = A$$

$$\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{6} & -1/\sqrt{6} & \sqrt{2}/\sqrt{3} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} = [a_1 \quad a_2]$$

$$\begin{bmatrix} \sqrt{2} & 1/\sqrt{2} \\ 0 & \sqrt{3}/\sqrt{2} \end{bmatrix} = [a_1 \quad a_2]$$

Thus $A = [a_1 \quad a_2] = \begin{bmatrix} \sqrt{2} & 1/\sqrt{2} \\ 0 & \sqrt{3}/\sqrt{2} \end{bmatrix}$

$$a_1 = \begin{bmatrix} \sqrt{2} \\ 0 \end{bmatrix} \qquad a_2 = \begin{bmatrix} 1/\sqrt{2} \\ \sqrt{3}/\sqrt{2} \end{bmatrix}$$

If we define A as in (c)

we get $A = \begin{bmatrix} \alpha_1 & \alpha_2 \\ \beta_1 & \beta_2 \end{bmatrix}$

Thus $\alpha_1 = \sqrt{2}$ $\qquad \alpha_2 = 1/\sqrt{2}$

$\beta_1 = 0$ $\qquad \beta_2 = \sqrt{3}/\sqrt{2}$

3. a) → Since X is linearly independent & $n \cdot p$ matrix $(n > p)$. The rank of X is p.

→ U is orthonormal for p-dimensional space, thus $rank\{U\} = p$.

By definition, T's rank is also P.

→ But since, T's dimensions are $p \times p$, it is full rank. Since full rank matrices are invertible, T is invertible.

b) Given $P_X = X(X^T X)^{-1} . X^T$

Given $X = UT$

$$P_{(UT)} = UT\left((UT)^T (UT)\right)^{-1} . (UT)^T$$

$$= UT\left(T^T U^T . U . T\right)^{-1} (UT)^T$$

as U is orthonormal, $U^T . U = I$

$$= UT \, (T^T . T)^{-1} \, T^T . U^T$$

$$= UT \, (T)^{-1} (T^T)^{-1} T^T . U^T$$

$$= U . I . I . U^T = UU^T$$

Given, $P_U = U(U^T U)^{-1} U^T$

$$= U(U)^{-1}(U^T)^{-1} U^T$$

$$= U(U)^{-1}(U^T)^{-1} U^T$$

$$= U.U^T$$

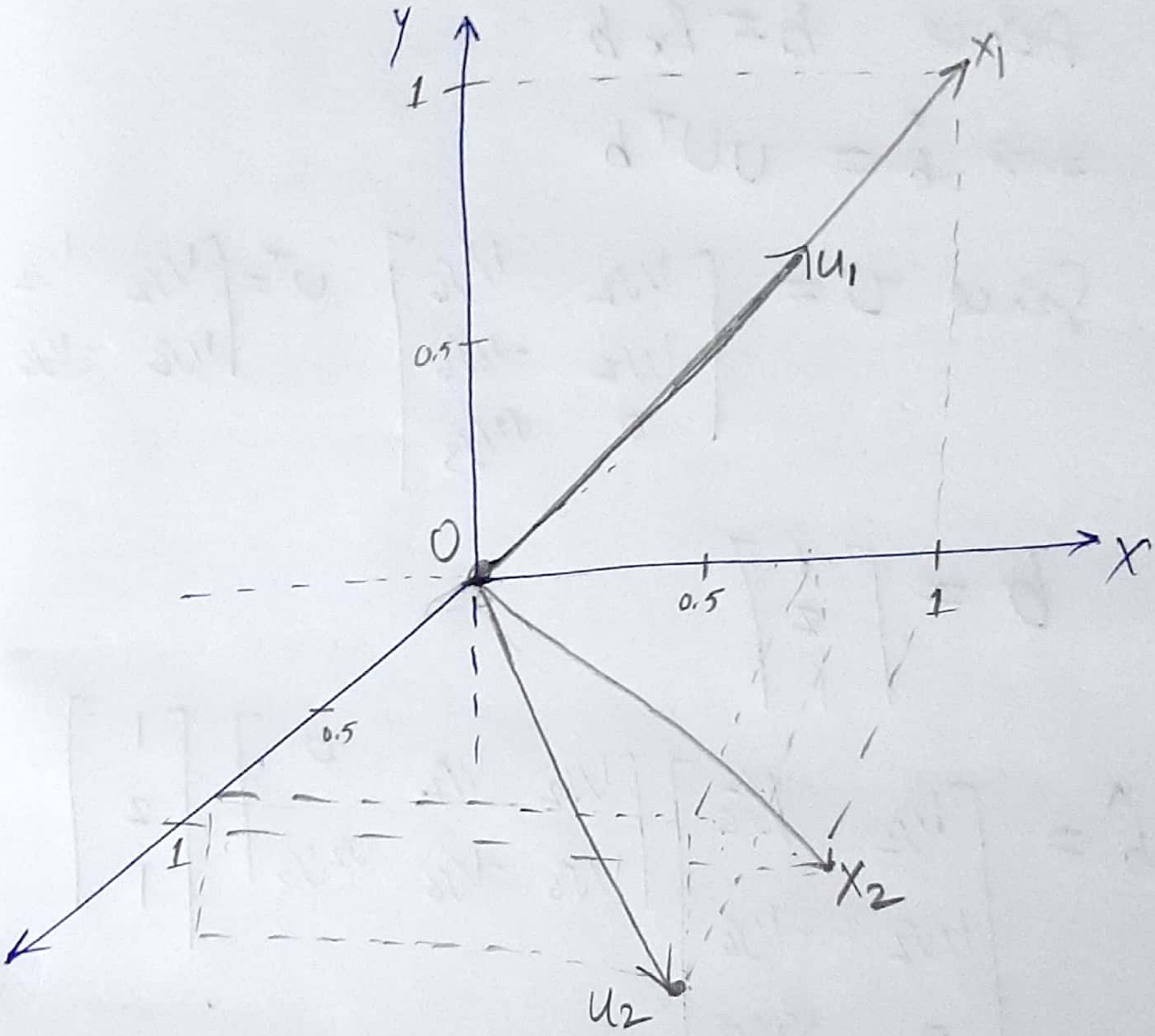
Thus, $P_X = P_U = UU^T$

c) from (b), $P_U = UU^T$

4. a) $X = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$ $b = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

$$U = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{2} & -1/\sqrt{6} \\ 0 & \sqrt{2}/\sqrt{3} \end{bmatrix} \quad X = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

let $X = [x_1 \quad x_2]$ $U = [u_1 \quad u_2]$



b)

$$X = UT, \quad P_X = X(X^T X)^{-1} X$$

$$\therefore P_X = (UT)\left((UT)^T (UT)\right)^{-1} (UT)^T$$

$$= (UT)(T^T U^T UT)^{-1} (T^T U^T)$$

$$= (UT)(T^T . I . T)^{-1} (T^T U^T)$$

$$= (UT)\, T^{-1} (T^T)^{-1} (T^T)(U^T)$$

$$= U . I . I . U^T$$

since $\hat{b} = P_X b$

$\Rightarrow \hat{b} = UU^T b$

since $U = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{2} & -1/\sqrt{6} \\ 0 & \sqrt{2}/\sqrt{3} \end{bmatrix}$ $U^T = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{6} & -1/\sqrt{6} & \sqrt{2}/\sqrt{3} \end{bmatrix}$

$b = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

$$\hat{b} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{2} & -1/\sqrt{6} \\ 0 & \sqrt{2}/\sqrt{3} \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{6} & -1/\sqrt{6} & \sqrt{2}/\sqrt{3} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2/3 & 1/3 & 1/3 \\ 1/3 & 2/3 & -1/3 \\ 1/3 & -1/3 & 2/3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 5/3 \\ 4/3 \\ 1/3 \end{bmatrix}$$