Activity 6, Ayan Deep Hazra

1\. a) $A = \begin{bmatrix} 1 & 0 \\ 1 & -1 \\ 0 & 1 \end{bmatrix}$ $d = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$

$\min_w \| d - Aw \|_2$

$w^* = (A^T A)^{-1} A^T d$ is the least squares solution,

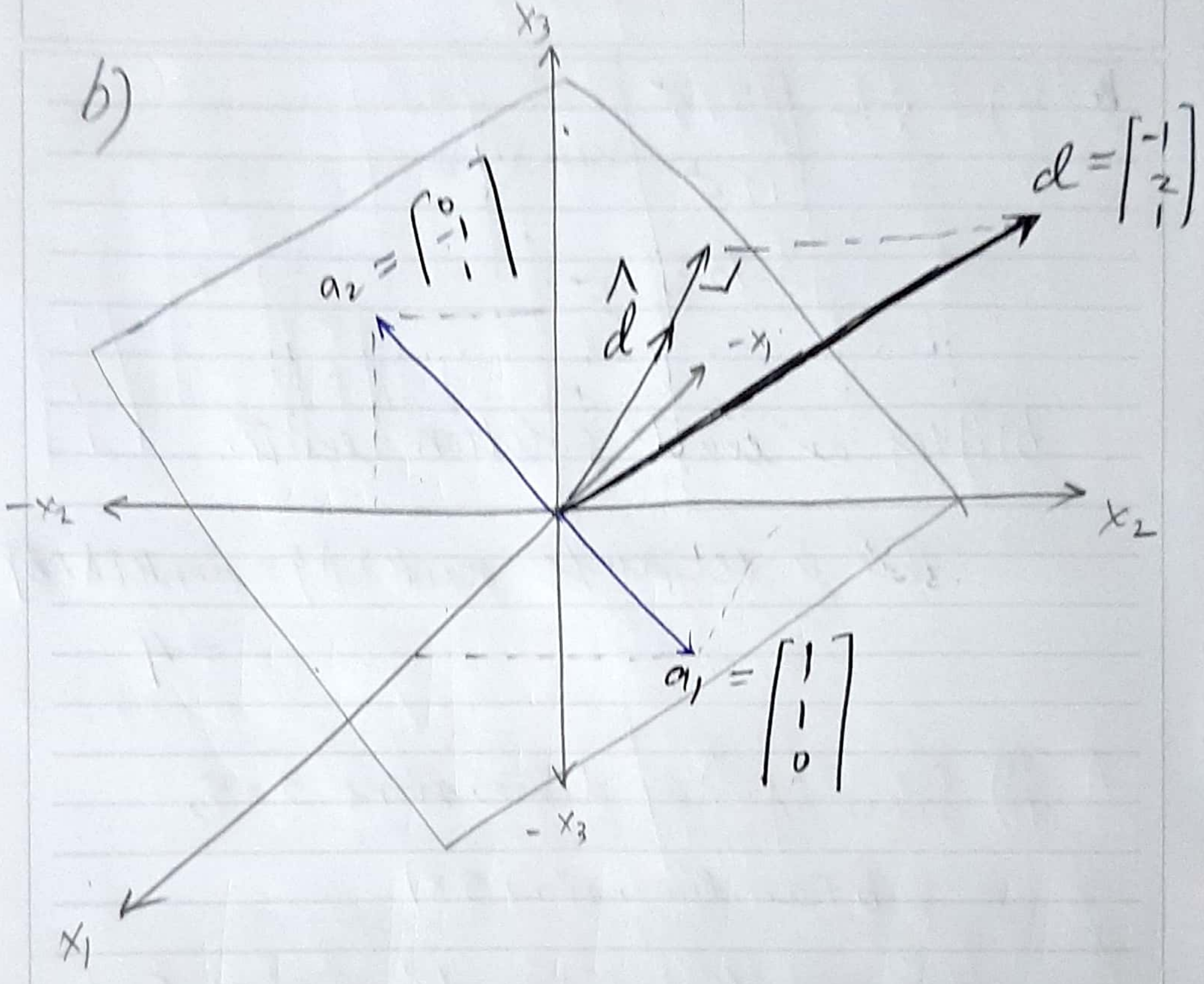
$$= \left(\begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & -1 \\ 0 & 1 \end{bmatrix} \right)^{-1} A^T d$$

$$= \left(\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$$

$$= \frac{1}{3} \begin{bmatrix} 2 & 1 & 1 \\ 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$$

$$= \frac{1}{3} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1/3 \\ -1/3 \end{bmatrix}$$

b)



x_3
$d = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$
$a_2 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$
$\hat{d}$
$-x_1$
$-x_2$
x_2
$a_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$
$-x_3$
x_1

2. a) $x = \begin{bmatrix} 4.25 \\ 17.5 \\ 3.75 \end{bmatrix}$

b) i) Yes an exact solution exists.

This is because $\text{rank}\{A\} = \text{rank}[A:b]$
$= 4$

ii) Since A has dimensions 5×5,

b has dimensions 5×1

we get that x has dimensions 5×1

Thus $\dim\{x\} = 5$

→ Thus, the solution is non unique, by definition.

as $\text{rank}\{A\} = 4$,

$\dim\{x\} = 5$ $\quad$ ($\because \text{rank}\{A\} < \dim\{x\}$)

iii) Given A, b, the solution w to the modified least-squares problem is

$$w = \begin{bmatrix} 4 \\ 4 \\ 9 \\ 4 \end{bmatrix}$$

If A' is matrix A without first column, we have,

$$\text{rank}\{A'\} = \dim\{W\} = 4$$

Thus an unique solution exists & the resulting squared error is 0.

Working page

3.) $T = [t_1 \; t_2] \quad W^T = \begin{bmatrix} w_1^T \\ w_2^T \end{bmatrix}$

$t_1 = \begin{bmatrix} 0.5 \\ 0.5 \\ 0.5 \\ 0.5 \end{bmatrix} \quad t_2 = \begin{bmatrix} 0.5 \\ -0.5 \\ -0.5 \\ 0.5 \end{bmatrix} \quad w_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad w_2 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$

$$A = TW^T$$

$$A = [t_1 \; t_2] \begin{bmatrix} w_1^T \\ w_2^T \end{bmatrix}$$

$$A = t_1 w_1^T + t_2 w_2^T$$

$$A = \begin{bmatrix} 1 & -0.5 & 1 \\ 0 & 1.5 & 0 \\ 0 & 1.5 & 0 \\ 1 & -0.5 & 1 \end{bmatrix}$$

a) Rank $\{A\} = 2$

b) t_1 & t_2 are linearly independent.

Thus the $\dim\{\text{span}\{t_1, t_2\}\} = 2$.

c) No.

$$A^T A = \begin{bmatrix} 2 & -1 & 2 \\ -1 & 5 & -1 \\ 2 & -1 & 2 \end{bmatrix}$$ is not full rank (2 lin-dependent rows)

Thus $(A^T A)$ is not positive definite.

by definition (& proof sheet)

d) We know the solution for the L-S problem must satisfy $A^T A X = A^T b$.

Let $A' = A^T A$ $\quad b' = A^T b$

$\Rightarrow A' X = b'$

$$A' = \begin{bmatrix} 1 & 0 & 0 & 1 \\ -0.5 & 1.5 & 1.5 & -0.5 \\ 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -0.5 & 1 \\ 0 & 1.5 & 0 \\ 0 & 1.5 & 0 \\ 1 & -0.5 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 2 \\ -1 & 5 & -1 \\ 2 & -1 & 2 \end{bmatrix}$$

$$b' = A^{T} b$$

$$= \begin{bmatrix} 1 & 0 & 0 & 1 \\ -0.5 & 1.5 & 1.5 & -0.5 \\ 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 4 \end{bmatrix}$$

$A'^{\,3\times 3}$ $\quad b'^{\,3\times 1}$

Thus x has dimensions $3\times 1 \rightarrow \dim\{x\} = 3$

A' has rank of 2 as 2 columns are linearly dependent

Thus $\text{rank}\{A'\} = 2$

Since $\text{rank}\{A'\} < \dim\{x\}$,

we can determine that system has non unique solution.

e) If we substitute $x = W\tilde{x}$ to $\min_x \|b - Ax\|$,

we will have $\min_{\tilde{x}} \|b - AW\tilde{x}\| =$

$\min_{\tilde{x}} \|b - (AW)\tilde{x}\| = \min_{\tilde{x}} = \|b - C\tilde{x}\|$

where $C = AW$

Solution to the LS problem is,

$C^T C \tilde{x} = C^T b$ or $D\tilde{x} = E$ where $D = C^T C$, $E = C^T b$

$$A = \begin{vmatrix} 1 & -0.5 & 1 \\ 0 & 1.5 & 0 \\ 0 & 1.5 & 0 \\ 1 & -0.5 & 1 \end{vmatrix} \quad W^T = \begin{vmatrix} 1 & 1 & 1 \\ 1 & -2 & 1 \end{vmatrix} \quad W = \begin{vmatrix} 1 & 1 \\ 1 & -2 \\ 1 & 1 \end{vmatrix}$$

$$C = \begin{vmatrix} 1 & -0.5 & 1 \\ 0 & 1.5 & 0 \\ 0 & 1.5 & 0 \\ 1 & -0.5 & 1 \end{vmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1.5 & 3 \\ 1.5 & -3 \\ 1.5 & -3 \\ 1.5 & 3 \end{bmatrix} \quad b = \begin{bmatrix} 2 \\ 1 \\ 1 \\ 2 \end{bmatrix}$$

$$C^T = \begin{bmatrix} 1.5 & 1.5 & 1.5 & 1.5 \\ 3 & -3 & -3 & 3 \end{bmatrix} \quad C^T b = E = \begin{bmatrix} 1.5 & 1.5 & 1.5 & 1.5 \\ 3 & -3 & -3 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 9 \\ 6 \end{bmatrix}$$

$$D = C^T C = \begin{bmatrix} 1.5 & 1.5 & 1.5 & 1.5 \\ 3 & -3 & -3 & 3 \end{bmatrix} \begin{bmatrix} 1.5 & 3 \\ 1.5 & -3 \\ 1.5 & -3 \\ 1.5 & 3 \end{bmatrix} = \begin{bmatrix} 9 & 0 \\ 0 & 36 \end{bmatrix}$$

$$\begin{bmatrix} 9 & 0 \\ 0 & 36 \end{bmatrix} \tilde{x} = \begin{bmatrix} 9 \\ 6 \end{bmatrix}$$

or $D\tilde{x} = E$

we know this system has solutions depending on relation of rank{D}, dim{E}

rank{D} = 2 as matrix is composed of 2 lin- independent columns.

dim{$\tilde{x}$} = 2 as observed from matrix multiplication.

since rank{D} = dim{$\tilde{x}$}, we get that the system has an unique solution