CS/ECE/ME532 Period 7 Activity

Estimated Time: 15 min for each problem

1. Let $\boldsymbol{X} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$.

a) Use the Gram-Schmidt orthogonalization procedure and hand calculation to find an orthonormal basis for the space spanned by the columns of $\boldsymbol{X}$. What geometric object is described by the span of these bases?

b) Now interchange the columns of $\boldsymbol{X}$, that is, define $\tilde{\boldsymbol{X}} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$

i. Do the columns of $\boldsymbol{X}$ span the same space as the columns of $\tilde{\boldsymbol{X}}$?

ii. Use the Gram-Schmidt orthogonalization procedure to find an orthonormal basis for the space spanned by the columns of $\tilde{\boldsymbol{X}}$. How does the geometric object described by the span of this set of orthonormal bases compare to the one in Part a?

iii. Are the bases obtained by the Gram-Schmidt procedure unique? Does the space spanned depend on the order of the columns?

2. Let $\boldsymbol{X} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$ as in the previous problem.

a) Place the orthonormal bases you found as columns of a matrix $\boldsymbol{U}$.

b) Find $\boldsymbol{U}^T\boldsymbol{U}$.

c) Since $\boldsymbol{U}$ contains a basis for space spanned by the columns of $\boldsymbol{X}$ you decide to write each column of $\boldsymbol{X}$ as a linear combination of the columns of $\boldsymbol{U}$: $\boldsymbol{X} = \boldsymbol{U}\begin{bmatrix} \boldsymbol{a}_1 & \boldsymbol{a}_2 \end{bmatrix}$. What is the dimension of $\boldsymbol{a}_1$? Briefly describe the meaning of $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$.

d) Let $\boldsymbol{A} = \begin{bmatrix} \boldsymbol{a}_1 & \boldsymbol{a}_2 \end{bmatrix}$ so that $\boldsymbol{X} = \boldsymbol{U}\boldsymbol{A}$. Multiply both sides of this equation by $\boldsymbol{U}^T$ and solve for $\boldsymbol{A}$.

3. Let the columns of an n-by-p $(n > p)$ matrix $\boldsymbol{X}$ be linearly independent and $\boldsymbol{U}$ be an orthonormal basis for the p-dimensional space spanned by the columns of $\boldsymbol{X}$.

a) It can be shown that $\boldsymbol{X} = \boldsymbol{U}\boldsymbol{T}$ where $\boldsymbol{T}$ is a p-by-p invertible matrix. Briefly explain why $\boldsymbol{T}$ should be invertible without resorting to a mathematical proof. That is, explain why this result is intuitively reasonable.

b) Use the result in the previous item to show that the projection onto the space spanned by $\boldsymbol{X}$ is identical to that onto the space spanned by $\boldsymbol{U}$. That is, show $\boldsymbol{P}_x = \boldsymbol{X}(\boldsymbol{X}^T\boldsymbol{X})^{-1}\boldsymbol{X}^T = \boldsymbol{P}_U = \boldsymbol{U}(\boldsymbol{U}^T\boldsymbol{U})^{-1}\boldsymbol{U}^T$. *Hint:* Recall that $(\boldsymbol{A}\boldsymbol{B})^{-1} = \boldsymbol{B}^{-1}\boldsymbol{A}^{-1}$.

c) Express $\boldsymbol{P}_U$ without a matrix inverse.

4. Consider the matrix and vector

$$\boldsymbol{X} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \boldsymbol{b} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}.$$

Note that $\boldsymbol{X}$ is defined identically in the preceding problems.

a) Make a sketch of the orthonormal bases $\boldsymbol{U}$ and the columns of $\boldsymbol{X}$ in three dimensions.

b) Use $\boldsymbol{U}$ and the result of the previous problem to compute the LS estimate $\hat{\boldsymbol{b}} = \boldsymbol{X}(\boldsymbol{X}^T\boldsymbol{X})^{-1}\boldsymbol{X}^T\boldsymbol{b}$.

5. Let $\boldsymbol{z} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and define $\boldsymbol{Q} = \boldsymbol{z}\boldsymbol{z}^T$.

a) Sketch the surface $y = \boldsymbol{x}^T\boldsymbol{Q}\boldsymbol{x}$ where $\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. If you find 3-D sketching too difficult, you may draw a contour map with labeled contours.

b) Let $\boldsymbol{w} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. Sketch the subspace spanned by $\boldsymbol{z}$ and the subspace spanned by $\boldsymbol{w}$ on your sketch of the surface $y = \boldsymbol{x}^T\boldsymbol{Q}\boldsymbol{x}$.

c) Does the problem $\min_{\boldsymbol{x}} \boldsymbol{x}^T\boldsymbol{Q}\boldsymbol{x}$ have a unique solution?

d) Is $\boldsymbol{Q} \succ 0$? Is $\boldsymbol{Q} \succeq 0$?