Eigendecomposition, SVD, and Power Iterations

Objectives

- Define eigenvectors and eigenvalues
- Relate the eigendecomposition to SVD
- Power iterations for computing eigenvector with largest eigenvalue

Eigendecomposition applies to square matrices

Eigenvector $\underline{e}_i$, eigenvalue λ_i, $\underline{B}$ $(k \times k)$

$$\underline{B}\,\underline{e}_i = \lambda_i\,\underline{e}_i$$

matrix mult $\Longleftrightarrow$ scalar mult

$$\underline{e}_i \longrightarrow \boxed{\underline{B}} \longrightarrow \lambda_i\,\underline{e}_i \qquad i = 1, 2, \cdots, k$$

- K eigenvalues, possibly complex valued
- Distinct $\lambda_i \Rightarrow$ linearly independent $\underline{e}_i$
- Symmetric $\underline{B} \Rightarrow K$ orthonormal $\underline{e}_i$ $\quad \underline{E}\,\underline{E}^T = \underline{E}^T\underline{E} = \underline{I}$

$$\underline{B}\,\underline{e}_i = \lambda_i\,\underline{e}_i \Rightarrow \underline{B}\left[\underline{e}_1\ \underline{e}_2 \cdots \underline{e}_k\right] = \left[\underline{e}_1\ \underline{e}_2 \cdots \underline{e}_k\right]\begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ & & \ddots & \\ 0 & & & \lambda_k \end{bmatrix}$$

$$\underline{B}\,\underline{E} = \underline{E}\,\underline{\Lambda} \Rightarrow \underline{B} = \underline{E}\,\underline{\Lambda}\,\underline{E}^T = \sum_{i=1}^{K} \lambda_i\,\underline{e}_i\,\underline{e}_i^T$$

Symmetric PSD matrices and SVD

$$\underline{A} = [\underline{a}_1\ \underline{a}_2 \cdots \underline{a}_M] = \begin{bmatrix} \underline{x}_1^T \\ \underline{x}_2^T \\ \vdots \\ \underline{x}_N^T \end{bmatrix} = \underline{U}\,\underline{\Sigma}\,\underline{V}^T \quad (N \times M,\ N > M)$$

full SVD

1) $\underline{B} = \underline{A}\,\underline{A}^T = \sum_{i=1}^{M} \underline{a}_i \underline{a}_i^T$

$$\underline{B} = \underline{U}\,\underline{\Sigma}\,\underline{V}^T \underline{V}\,\underline{\Sigma}^T \underline{U}^T = \underline{U}\,\underline{\Sigma}\,\underline{\Sigma}^T \underline{U}^T = \underline{U} \begin{bmatrix} \sigma_1^2 & & & \\ & \ddots & \sigma_M^2 & \underline{0} \\ \underline{0} & & 0 & \\ & & & \ddots\ 0 \end{bmatrix} \underline{U}^T$$

left SV of $\underline{A} \Longleftrightarrow$ eigenvectors $\underline{B}$

$$\lambda_i = \begin{cases} \sigma_i^2 & i = 1, 2, \cdots, M \\ 0 & i = M+1, \cdots, N \end{cases}$$

2) $\underline{B} = \underline{A}^T \underline{A} = \sum_{i=1}^{N} \underline{x}_i \underline{x}_i^T$

$$= \underline{V}\,\underline{\Sigma}^T \underline{U}^T \underline{U}\,\underline{\Sigma}\,\underline{V}^T = \underline{V}\,\underline{\Sigma}^T \underline{\Sigma}\,\underline{V}^T = \underline{V} \begin{bmatrix} \sigma_1^2 & & \underline{0} \\ & \ddots & \\ \underline{0} & & \sigma_M^2 \end{bmatrix} \underline{V}^T$$

right SV of $\underline{A} \Longleftrightarrow$ eigenvectors $\underline{B}$, $\lambda_i = \sigma_i^2$, $i = 1, 2, \cdots M$

Power iteration for computing 1^{st} principal component

$\underline{A}: N \times M$, $N \gg M$ want $\underline{v}_1$ 1^{st} principal component

right SV of $\underline{A}$, eigenvector of $\underline{B} = \underline{A}^T\underline{A}$ $(M \times M)$

Power Iteration

pick $\underline{c}_0$ (random)

for $k = 1, 2, \ldots$ to converge

$$\underline{c}_k = \underline{B}\,\underline{c}_{k-1} / \|\underline{B}\,\underline{c}_{k-1}\|_2$$

end

$$\underline{v}_1 = \underline{c}_{end}$$

$$\underline{B}\,\underline{c}_{k-1} = \underbrace{\underline{B}\cdot\underline{B}\cdots\underline{B}}_{k \text{ times}}\,\underline{c}_0 = \underline{B}^k\underline{c}_0$$

$$\underline{B}^k = \underbrace{\underline{V}\underline{\Lambda}\underline{V}^T\underline{V}\underline{\Lambda}\underline{V}^T\cdots\underline{V}\underline{\Lambda}\underline{V}^T}_{k \text{ times}}$$

$$= \underline{V}\,\underline{\Lambda}^k\underline{V}^T$$

Let $\underline{c}_0 = \underline{V}\underline{g} = \underline{V}\begin{bmatrix} g_1 \\ g_2 \\ \vdots \\ g_M \end{bmatrix}$

$$\underline{B}^k\underline{c}_0 = \underline{V}\,\underline{\Lambda}^k\underline{V}^T\underline{V}\,\underline{g}$$

$$= \underline{V}\,\underline{\Lambda}^k\underline{g}$$

Power iteration...

$$\underline{c}_k = \underline{B}\,\underline{c}_{k-1} / \|\underline{B}\,\underline{c}_{k-1}\|_2 = \underline{V}\,\underline{\Lambda}^k \underline{g} \,/\, \|\underline{V}\,\underline{\Lambda}^k \underline{g}\|_2$$

$$\underline{V}\,\underline{\Lambda}^k \underline{g} = \begin{bmatrix} \underline{v}_1 & \underline{v}_2 & \cdots & \underline{v}_m \end{bmatrix} \begin{bmatrix} \lambda_1^k & & & \underline{0} \\ & \lambda_2^k & & \\ & & \ddots & \\ \underline{0} & & & \lambda_m^k \end{bmatrix} \begin{bmatrix} g_1 \\ g_2 \\ \vdots \\ g_m \end{bmatrix}$$

but $\frac{\lambda_i}{\lambda_1} < 1$

so

$$= \lambda_1^k g_1 \underline{V} \begin{bmatrix} 1 & & & \underline{0} \\ & \left(\frac{\lambda_2}{\lambda_1}\right)^k & & \\ & & \ddots & \\ \underline{0} & & & \left(\frac{\lambda_m}{\lambda_1}\right)^k \end{bmatrix} \begin{bmatrix} 1 \\ g_2/g_1 \\ \vdots \\ g_m/g_1 \end{bmatrix}$$

$$\left(\frac{\lambda_i}{\lambda_1}\right)^k \rightarrow 0$$

$$\underline{V}\,\underline{\Lambda}^k \underline{g} \rightarrow \lambda_1^k g_1 \underline{V} \begin{bmatrix} 1 & & 0 \\ & 0 \ddots & \\ 0 & & 0 \end{bmatrix} \begin{bmatrix} 1 \\ \vdots \\ \vdots \end{bmatrix} = \lambda_1^k g_1 \underline{v}_1$$

$$\underline{c}_k \rightarrow \frac{\lambda_1^k g_1 \underline{v}_1}{\|\lambda_1^k g_1 \underline{v}_1\|_2} = \frac{\underline{v}_1}{\|\underline{v}_1\|_2} = \underline{v}_1$$

Copyright 2019
Barry Van Veen