ECE 532 Assignment 2

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1. a) $A = \begin{bmatrix} +0.92 & +0.92 \\ -0.92 & +0.92 \\ +0.92 & -0.92 \\ -0.92 & -0.92 \end{bmatrix}$

If we write the columns of A as $\alpha_1 c_1 + \alpha_2 c_2 = 0$, we see that the only way to satisfy the equation is if $\alpha_1 = \alpha_2 = 0$. Thus the columns of A are linearly independent.

b) $A = \begin{bmatrix} +1 & +1 & +1 \\ -1 & +1 & -1 \\ +1 & -1 & -1 \end{bmatrix}$

If we take the columns of A two at a time, we see that for any two columns c_1, c_2 in A, $\alpha_1 c_1 + \alpha_2 c_2 = 0$ gives us $\alpha_1 = \alpha_2 = 0$, so, the columns of A taken two at a time are linearly independent.

Thus, the columns of A are linearly independent.

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c) $A = \begin{bmatrix} 1 & 2 & 2 \\ 3 & 4 & 5 \\ 5 & 6 & 8 \end{bmatrix}$

If we write the columns of A as

$\alpha_1 C_1 + \alpha_2 C_2 + \alpha_3 C_3 = 0$

we see that, for $\alpha_1 = -1$, $\alpha_2 = \frac{-1}{2}$, $\alpha_3 = 1$

$$-1 \cdot \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} + 1 \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix}$$

$$= \begin{bmatrix} -1 \\ -3 \\ -5 \end{bmatrix} + \begin{bmatrix} -1 \\ -2 \\ -3 \end{bmatrix} + \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix}$$

$$= \begin{bmatrix} (2-1-1) \\ (5-3-2) \\ (8-3-5) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Since $\alpha_1, \alpha_2, \alpha_3 \neq 0$, we get that the columns of A are linearly dependent for some $\{\alpha_1, \alpha_2, \alpha_3\}$

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d) $A = \begin{bmatrix} +5 & +2 \\ -5 & +2 \\ +5 & -2 \end{bmatrix}$

we notice that columns of A are linearly independent as there is no non-zero α_1 & α_2 that satisfies $\alpha_1 c_1 + \alpha_2 c_2 = 0$.

Thus, the matrix is full rank with $\text{rank}\{A\} = 2$.

e) For given A, $A = \begin{bmatrix} +5 & +2 \\ -5 & +2 \\ +5 & -2 \end{bmatrix}$

we have $A^T A = \begin{bmatrix} +5 & -5 & +5 \\ +2 & +2 & -2 \end{bmatrix} \begin{bmatrix} +5 & +2 \\ -5 & +2 \\ +5 & -2 \end{bmatrix} = \begin{bmatrix} 75 & -10 \\ -10 & 12 \end{bmatrix}$

let $A^T A = B$, given $Bw = d$

we see that $\text{rank}(B) = 2$ as the columns of B are linearly independent.

If we take $d = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$, then $[B|d]$ would be

$\begin{bmatrix} 75 & -10 & d_1 \\ -10 & 12 & d_2 \end{bmatrix}$ which has a rank 2 still as rank $\leq$ dim $[B|d]$

Thus, since $\text{rank}(B) = \text{rank}[B|d] = 2$

a solution to $Bw = d$ exists.

→ Given $B_{2\times2}$ & $B|d_{2\times3}$, we get that

w has dimentions of 2×3

Thus, $\dim(w) = 2$.

Since, an unique solution exists

if $\text{rank}(B) = \dim(w)$.

Since, $\text{rank}(B) = \dim(w) = 2$,

we know an unique solution exists.

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2a) A vector norm $\|\cdot\|$ satisfies

① $\|x\| > 0$ for all x

② $\|x\| = 0$ iff $x = 0$

③ $\|bx\| = |b| \, \|x\|$ for all $b \in \mathbb{R}$, $x \in \mathbb{R}^n$

④ $\|x+y\| \leq \|x\| + \|y\|$

Let $f(x) = \|x\| = \|x\|_a + \|x\|_b$,

① Since, by definition $\|x\|_a \geq 0$ and $\|x\|_b \geq 0$, we can add those two to get $\|x\|_a + \|x\|_b \geq 0$
$\Rightarrow \|x\| \geq 0$.

② If $\|x\| = 0$, then $\|x\|_a + \|x\|_b = 0$ which would mean $\|x\|_a = 0$ & $\|x\|_b = 0$ as norms are greater than/equal to 0.

③ $f(\lambda x) = \|\lambda x\| = \|\lambda x\|_a + \|\lambda x\|_b =$

$|\lambda| \cdot \|x\|_a + |\lambda| \cdot \|x\|_b = |\lambda| \left(\|x\|_a + \|x\|_b \right)$

$= |\lambda| \, \|x\|$

④ $f(x+y) = \|x+y\|$

$= \|x+y\|_a + \|x+y\|_b \leq (\|x\|_a + \|y\|_a) + (\|x\|_b + \|y\|_b)$

$\Rightarrow \|x+y\| \leq (\|x\|_a + \|x\|_b) + (\|y\|_a + \|y\|_b)$

$\Rightarrow \|x+y\| \leq (\|x\|) + (\|y\|)$

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Since $f(x) = \|x\|$ satisfies all 4 properties, we can say that it is a norm.

b) We want to draw the norm ball

$$f(x) = \|x\|_1 + \|x\|_\infty = 1$$

$$\Rightarrow \quad |x_1| + |x_2| + \max\{|x_1|, |x_2|\} = 1$$

In the first quadrant $x_1, x_2 > 0$

and $|x_1| = x_1$ & $|x_2| = x_2$

Thus equation can be,

$$x_1 + x_2 + \max\{x_1, x_2\} = 1$$

* At $x_1 = x_2$, we have $3x_1 = 1$ or $x_1 = x_2 = 1/3$

* At $x_1 > x_2$, eq^n is $x_1 + x_2 + x_1 = 1$

$$\boxed{x_2 = -2x_1 + 1}$$

* At $x_2 > x_1$, eq^n is $x_1 + x_2 + x_2 = 1$

$$2x_2 = -x_1 + 1$$

$$\boxed{x_2 = -\frac{1}{2}x_1 + \frac{1}{2}}$$

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Similarly in the second quadrant, $x_1 < 0,\ x_2 > 0$

thus $|x_1| = -x_1$, $|x_2| = x_2$

Thus eqn is $-x_1 + x_2 + \max\{-x_1, x_2\} = 1$

* At $-x_1 = x_2$, we have $-x_1 + x_2 - x_1 = 1$

$-3x_1 = 1 \Rightarrow x_1 = \frac{-1}{3}, x_2 = \frac{1}{3}$

* At $-x_1 < x_2$, we have $-x_1 + x_2 + x_2 = 1$

$2x_2 = x_1 + 1$

$\boxed{x_2 = \frac{x_1}{2} + \frac{1}{2}}$

* At $-x_1 > x_2$, we have $-x_1 + x_2 - x_1 = 1$

$\boxed{x_2 = 2x_1 + 1}$

Similarly in the third quadrant, $x_1 < 0, x_2 < 0$

thus $|x_1| = -x_1$, $|x_2| = -x_2$

thus eqn is $-x_1 - x_2 + \max\{-x_1, -x_2\} = 1$

* At $-x_1 = -x_2$, we have $-x_1 - x_1 - x_1 = 1$

$-3x_1 = 1 \Rightarrow x_1 = x_2 = \frac{-1}{3}$

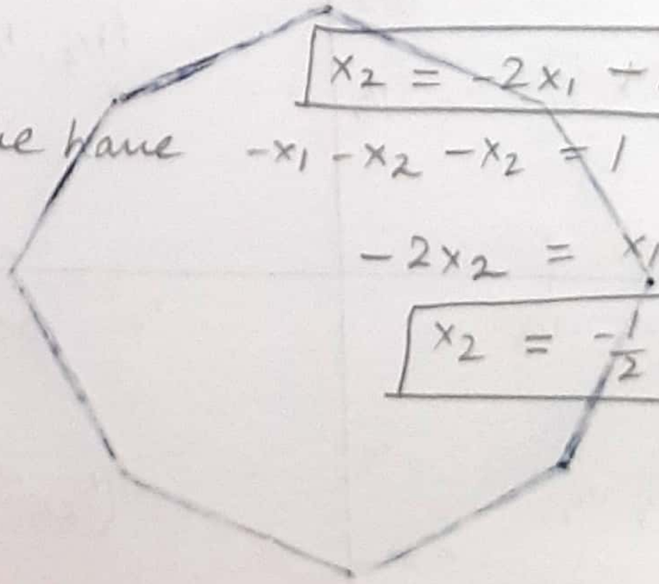
* At $-x_1 > -x_2$, we have $-x_1 - x_2 - x_1 = 1$

$\boxed{x_2 = -2x_1 - 1}$

* At $-x_1 < -x_2$, we have $-x_1 - x_2 - x_2 = 1$

$-2x_2 = x_1 + 1$

$\boxed{x_2 = \frac{-1}{2}x_1 \frac{-1}{2}}$



Similarly in the fourth quadrant, $x_1 > 0, x_2 < 0$

thus $|x_1| = x_1 \quad |x_2| = -x_2$

thus eqⁿ is $-x_1 - x_2 + \max\{x_1, -x_2\} = 1$

* At $x_1 = -x_2$, we have $x_1 + x_1 + x_1 = 1$

$$x_1 = \frac{1}{3}, \quad x_2 = \frac{-1}{3}$$

* At $x_1 > -x_2$, we have $x_1 - x_2 + x_1 = 1$

$$\boxed{x_2 = 2x_1 - 1}$$

* At $x_1 < -x_2$, we have $x_1 - x_2 - x_2 = 1$

$$2x_2 = x_1 - 1$$

$$\boxed{x_2 = \frac{x_1}{2} - \frac{1}{2}}$$

Using these 8 equations and 4 points, we get.