Orthonormality, Subspaces and Projections

Objectives

- Define and understand orthonormal basis
- Define and understand projections
- List methods for finding orthonormal basis

Definition of orthonormal basis

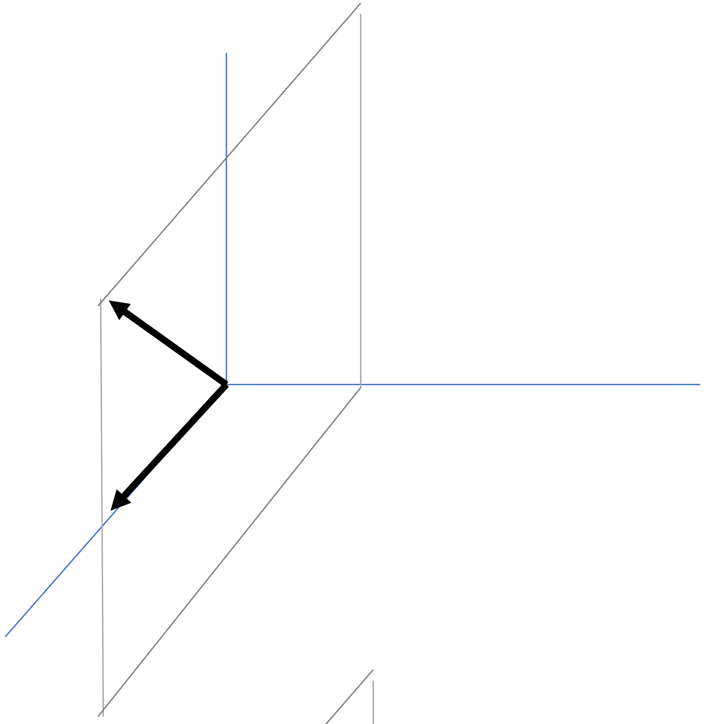
An orthonormal basis for a set of vectors $\boldsymbol{x}_1, \boldsymbol{x}_2 \ldots$ is another set of vectors $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots$ such that:

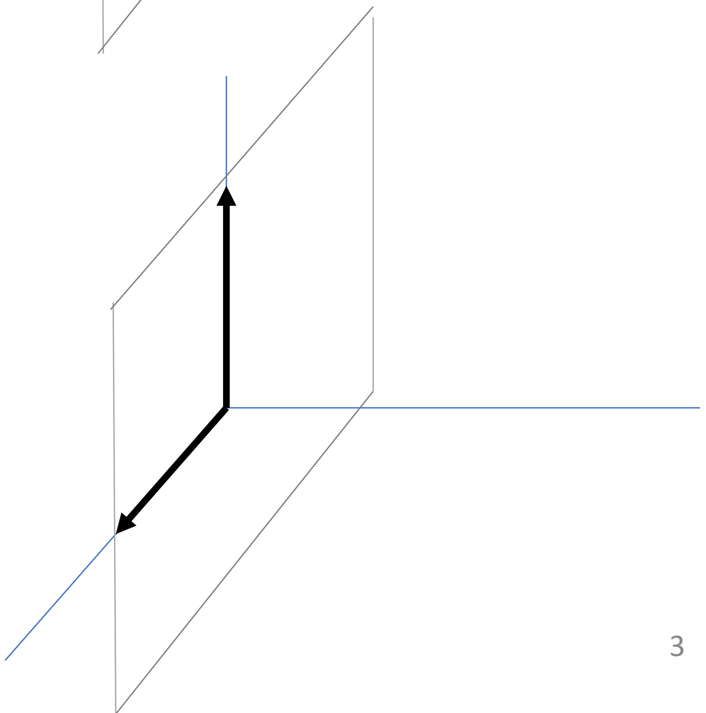
1. $\boldsymbol{u}_i^T \boldsymbol{u}_j = 0$ for all $i \neq j$
2. $\boldsymbol{u}_i^T \boldsymbol{u}_i = 1$ for all i
3. $\text{span}\{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n\} = \text{span}\{\boldsymbol{u}_1, \ldots, \boldsymbol{u}_m\}$

subspace

$$\text{span}\{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n\} = \left\{ \boldsymbol{x} : \boldsymbol{x} = \sum_{i=1}^{n} w_i \boldsymbol{x}_i,\ w_i \in \mathbb{R}, i = 1, \ldots, n \right\}$$

$$\boldsymbol{X} = \begin{bmatrix} | & & | \\ \boldsymbol{x}_1 & \ldots & \boldsymbol{x}_n \\ | & & | \end{bmatrix} \qquad \boldsymbol{U} = \begin{bmatrix} | & & | \\ \boldsymbol{u}_1 & \ldots & \boldsymbol{u}_m \\ | & & | \end{bmatrix}$$





Properties of orthonormal basis

m is the dimension of the subspace

$$\boldsymbol{X} = \begin{bmatrix} | & & | \\ \boldsymbol{x}_1 & \dots & \boldsymbol{x}_n \\ | & & | \end{bmatrix} \qquad \boldsymbol{U} = \begin{bmatrix} | & & | \\ \boldsymbol{u}_1 & \dots & \boldsymbol{u}_m \\ | & & | \end{bmatrix}$$

Properties of orthonormal basis:

1. $m \leq n$
2. $m \leq \dim(\boldsymbol{x}_i)$
3. $\boldsymbol{U}^T\boldsymbol{U} = \boldsymbol{I}_{m\times m}$
4. if $\boldsymbol{U}$ is square, $\boldsymbol{U}^{-1} = \boldsymbol{U}^T$
5. if $\boldsymbol{U}$ is square, $\boldsymbol{U}\boldsymbol{U}^T = \boldsymbol{I}_{m\times m}$

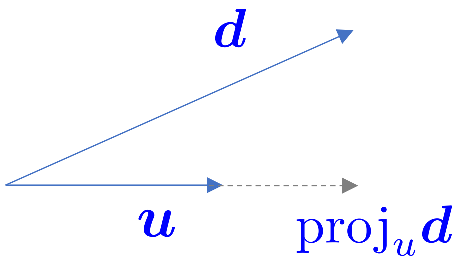
Examples of bases for $\mathbb{R}^n$:

1. Euclidean basis
2. Haar Wavelets
3. Rotation matrices
4. DFT coefficients (or Fourier basis)

Projections

$$\boldsymbol{X} = \begin{bmatrix} | & & | \\ \boldsymbol{x}_1 & \dots & \boldsymbol{x}_n \\ | & & | \end{bmatrix} \quad \boldsymbol{U} = \begin{bmatrix} | & & | \\ \boldsymbol{u}_1 & \dots & \boldsymbol{u}_m \\ | & & | \end{bmatrix}$$

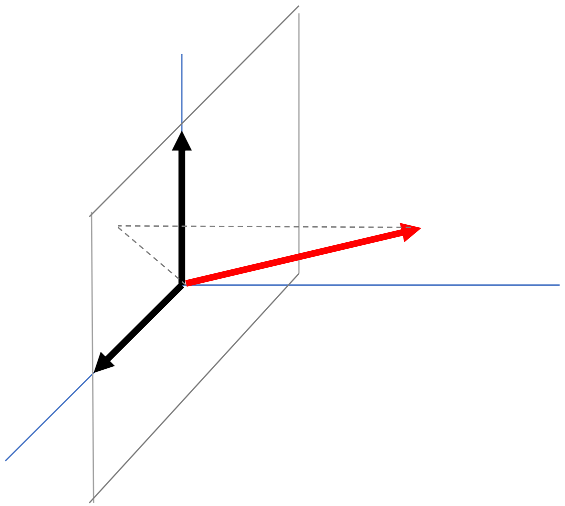
$n \leq \dim(\boldsymbol{x}_i)$



$$\text{proj}_u \boldsymbol{d} = \boldsymbol{u}(\boldsymbol{u}^T \boldsymbol{d})$$

projection of $\boldsymbol{d}$ onto $\boldsymbol{u}$

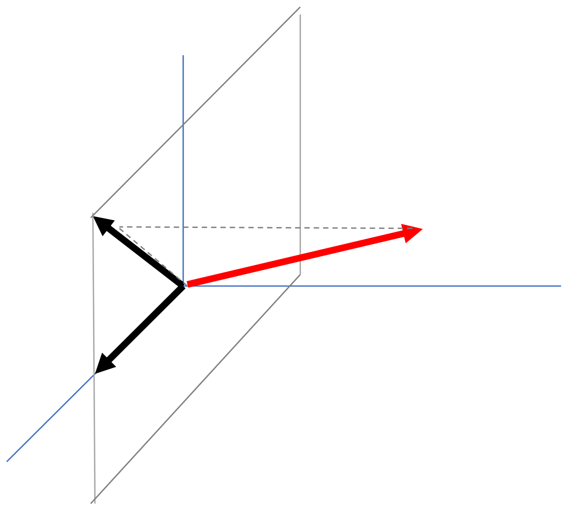
amount of $\boldsymbol{d}$ in the direction of $\boldsymbol{u}$



$$\text{proj}_{\boldsymbol{X}} \boldsymbol{d} = \sum_{i=1}^{m} \boldsymbol{u}_i (\boldsymbol{u}_i^T \boldsymbol{d})$$

$$= \begin{bmatrix} | & & | \\ \boldsymbol{u}_1 & \dots & \boldsymbol{u}_m \\ | & & | \end{bmatrix} \begin{bmatrix} \boldsymbol{u}_1^T \boldsymbol{d} \\ \vdots \\ \boldsymbol{u}_m^T \boldsymbol{d} \end{bmatrix}$$

$$= \boldsymbol{U}\boldsymbol{U}^T \boldsymbol{d}$$



$$\text{proj}_{\boldsymbol{X}} \boldsymbol{X} = \boldsymbol{U}\boldsymbol{U}^T \boldsymbol{X} = \boldsymbol{X}$$

$$\boldsymbol{U}\boldsymbol{U}^T = \boldsymbol{X}(\boldsymbol{X}^T \boldsymbol{X})^{-1} \boldsymbol{X}^T$$

$$\text{proj}_{\boldsymbol{X}} \boldsymbol{d} = \boldsymbol{X}(\boldsymbol{X}^T \boldsymbol{X})^{-1} \boldsymbol{X}^T \boldsymbol{d}$$

- Finding Orthonormal Basis

$$\boldsymbol{X} = \begin{bmatrix} | & & | \\ \boldsymbol{x}_1 & \dots & \boldsymbol{x}_n \\ | & & | \end{bmatrix} \xrightarrow{\quad ? \quad} \boldsymbol{U} = \begin{bmatrix} | & & | \\ \boldsymbol{u}_1 & \dots & \boldsymbol{u}_m \\ | & & | \end{bmatrix}$$

  - Gram-Schmidt orthogonalization

  - scipy.linalg.orth(X), orth(X)

  - The SVD (Singular Value Decomposition)