

HW5 Ayan Deep Kagra ECE 532

1. a) Given $X = USV^T$

$$\text{where } U = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & -1 \\ 1 & -1 \end{bmatrix} \quad S = \begin{bmatrix} 1 & 0 \\ 0 & \delta \end{bmatrix} \quad V = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$\text{let } y = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\text{If } \delta = 0.1, \text{ we have condition number} = \frac{\sigma_1}{\sigma_2} = \frac{1}{\delta} = \frac{1}{0.1} = \underline{\underline{10}}$$

$$\text{If } \delta = 10^{-8}, \text{ we have condition number} = \frac{\sigma_1}{\sigma_2} = \frac{1}{\delta} = \frac{1}{10^{-8}} = \underline{\underline{10^8}}$$

$$\underline{\underline{Xw = y}}$$

We can use the least squares solution

$$\min_w \|Xw - y\|_2^2, \quad w = (X^T X)^{-1} X^T y$$

$$X = USV^T$$

$$\text{Thus, } w = ((USV^T)^T USV^T)^{-1} (USV^T)^T y$$

$$\omega = V S^{-1} U^T y$$

$$\text{Also, } S^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{\delta} \end{bmatrix}$$

for $\delta = 0.1$

$$\omega = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{\delta} \end{bmatrix} \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\omega = \frac{1}{2\sqrt{2}} \begin{bmatrix} 1 & \frac{1}{\delta} \\ 1 & -\frac{1}{\delta} \end{bmatrix} \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

$$\omega = \frac{1}{2\sqrt{2}} \begin{bmatrix} 2 + \frac{2}{\delta} \\ 2 - \frac{2}{\delta} \end{bmatrix}$$

* Thus for $\delta = 0.1$,

$$\omega = \frac{1}{2\sqrt{2}} \begin{bmatrix} 2 + \frac{2}{0.1} \\ 2 - \frac{2}{0.1} \end{bmatrix}$$

$$\omega = \frac{1}{2\sqrt{2}} \begin{bmatrix} 22 \\ -18 \end{bmatrix}$$

$$\omega = \frac{1}{\sqrt{2}} \begin{bmatrix} 11 \\ -9 \end{bmatrix}$$

thus

$$\left. \| \omega \|_2^2 \right|_{\gamma=0.1} = \frac{1}{\sqrt{2}} \left(\sqrt{11^2 + (-9)^2} \right)^2$$

$$= \frac{1}{\sqrt{2}} (121 + 81)$$

$$= \frac{202}{\sqrt{2}}$$

for $\gamma = 10^{-8}$

$$\omega = \frac{1}{2\sqrt{2}} \begin{bmatrix} 2 + \frac{2}{10^{-8}} \\ 2 - \frac{2}{10^{-8}} \end{bmatrix}$$

$$\omega = \frac{1}{2\sqrt{2}} \begin{bmatrix} \sim 2 \times 10^8 \\ \sim -2 \times 10^8 \end{bmatrix}$$

$$\omega = \frac{1}{\sqrt{2}} \begin{bmatrix} 10^8 \\ -10^8 \end{bmatrix}$$

$$\left. \| \omega \|_2^2 \right|_{\gamma=10^{-8}} = \frac{1}{\sqrt{2}} \left(\sqrt{(10^8)^2 + (-10^8)^2} \right)^2$$

$$= \frac{1}{\sqrt{2}} 2 \cdot 10^8 = \underline{\underline{\sqrt{2} \times 10^8}}$$

b) let $y = y_0 + y_\epsilon$

where $y = \begin{bmatrix} 1 + \epsilon \\ 0 \\ 0 \\ 1 \end{bmatrix}$ & $y_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ & $y_\epsilon = \begin{bmatrix} \epsilon \\ 0 \\ 0 \\ 0 \end{bmatrix}$

Thus, $w = VS^{-1}U^T y$

$\Rightarrow w = w_0 + w_\epsilon$ given

$w = \underbrace{VS^{-1}U^T y_0}_{w_0} + \underbrace{VS^{-1}U^T y_\epsilon}_{w_\epsilon}$

We have $VS^{-1}U^T y_0$ from (a)

$w_0 = VS^{-1}U^T y_0 = \frac{1}{2\sqrt{2}} \begin{bmatrix} 2 + \frac{2}{\delta} \\ 2 - \frac{2}{\delta} \end{bmatrix}$

Similarly,

$w_\epsilon = VS^{-1}U^T y_\epsilon = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & \frac{1}{\delta} \\ 1 & -\frac{1}{\delta} \end{bmatrix} \cdot \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} \epsilon \\ 0 \\ 0 \\ 0 \end{bmatrix}$

$w_\epsilon = \frac{1}{2\sqrt{2}} \begin{bmatrix} 1 & \frac{1}{\delta} \\ 1 & -\frac{1}{\delta} \end{bmatrix} \cdot \begin{bmatrix} \epsilon \\ \epsilon \end{bmatrix}$

$$w_E = \frac{1}{2/2} \begin{bmatrix} E + \frac{E}{\delta} \\ E - \frac{E}{\delta} \end{bmatrix}$$

for $E = 0.01$ & $\delta = 0.1$

$$w_E = \frac{1}{2/2} \begin{bmatrix} 0.01 + \frac{0.01}{0.1} \\ 0.01 - \frac{0.01}{0.1} \end{bmatrix} = \frac{1}{2/2} \begin{bmatrix} 0.11 \\ -0.09 \end{bmatrix}$$

$$\|w_E\|_2^2 = \left(\sqrt{\left(\frac{0.11}{2/2} \right)^2 + \left(\frac{-0.09}{2/2} \right)^2} \right)^2$$

$$= 0.002525$$

for $E = 0.01$ & $\delta = 10^{-8}$

$$w_E = \frac{1}{2/2} \begin{bmatrix} 0.01 + \frac{0.01}{10^{-8}} \\ 0.01 - \frac{0.01}{10^{-8}} \end{bmatrix} = \frac{1}{2/2} \begin{bmatrix} 10^6 \\ -10^6 \end{bmatrix}$$

$$\|w_E\|_2^2 = 250,000,000,000$$

That the norm of the perturbation
depends on δ as $\frac{1}{\delta^2}$.

we see this from

$$w_\epsilon = \frac{1}{2^{1/2}} \begin{bmatrix} \epsilon + \frac{\epsilon}{\delta} \\ \epsilon - \frac{\epsilon}{\delta} \end{bmatrix}$$

$$\|w_\epsilon\|_2^2 = \sqrt{\left[\frac{1}{2^{1/2}} \left(\epsilon + \frac{\epsilon}{\delta} \right) \right]^2 + \left[\frac{1}{2^{1/2}} \left(\epsilon - \frac{\epsilon}{\delta} \right) \right]^2}$$

$$\|w_\epsilon\|_2^2 = \frac{1}{8} \left(\epsilon^2 + \frac{\epsilon^2}{\delta^2} + \frac{2\epsilon^2}{\delta} \right) + \frac{1}{8} \left(\epsilon^2 + \frac{\epsilon^2}{\delta^2} - \frac{2\epsilon^2}{\delta} \right)$$

$$\|w_\epsilon\|_2^2 = \frac{1}{4} \epsilon^2 + \frac{\epsilon^2}{\delta^2}$$

$$\text{Thus } \|w_\epsilon\|_2^2 \propto \frac{1}{\delta^2}$$

$$\Rightarrow \|w_\epsilon\|_2^2 \propto (\text{condition number})^2$$

$$\text{as condition number} = \frac{1}{\delta}$$

$$c) (X^T X)^{-1} X^T \approx \sum_{i=1}^n \frac{1}{\sigma_i^2} V_i^0 U_i^{0T}$$

for $n=1$, we have,

$$\begin{aligned} (X^T X)^{-1} X^T &= \frac{1}{\sigma_1} V_1 U_1^T \\ &= \frac{1}{1} \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1/2 & 1/2 & 1/2 & 1/2 \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} \\ \frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} \end{bmatrix} \end{aligned}$$

$$\begin{aligned} w &= (X^T X)^{-1} X^T y \\ &= (X^T X)^{-1} X^T (y_0 + y_\epsilon) \\ &= \underbrace{(X^T X)^{-1} X^T y_0}_{w_0} + \underbrace{(X^T X)^{-1} X^T y_\epsilon}_{w_\epsilon} \end{aligned}$$

$$w_0 = \begin{bmatrix} \frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} \\ \frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$w_\epsilon = \begin{bmatrix} \frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} \\ \frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} \end{bmatrix} \begin{bmatrix} \epsilon \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{\epsilon}{2\sqrt{2}} \\ \frac{\epsilon}{2\sqrt{2}} \end{bmatrix}$$

we see that w_ϵ does not depend on γ , just ϵ .

Thus for both $\gamma = 0.1$ & $\gamma = 10^{-8}$, for $\epsilon = 0.01$ we have,

$$w_\epsilon = \begin{bmatrix} \frac{0.01}{2\sqrt{2}} \\ \frac{0.01}{2\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 0.00353 \\ 0.00353 \end{bmatrix}$$

$$\begin{aligned} \|w_\epsilon\|_2^2 &= \left(\sqrt{(0.00353)^2 + (0.00353)^2} \right)^2 \\ &= 0.0000125 + 0.0000125 \\ &= 0.000025 \end{aligned}$$

which is much less than the norms in
the two cases for (b).