HW5 Ayan Deep Hazra ECE 532

1\. a) Given $X = USV^T$

where $U = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & -1 \\ 1 & 1 \end{bmatrix}$ $S = \begin{bmatrix} 1 & 0 \\ 0 & \gamma \end{bmatrix}$ $V = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

Let $y = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

If $\gamma = 0.1$, we have condition number $= \frac{\sigma_1}{\sigma_2} = \frac{1}{\gamma}$

$= \frac{1}{0.1} = 10$

If $\gamma = 10^{-8}$, we have condition number $= \frac{\sigma_1}{\sigma_2} = \frac{1}{\gamma}$

$= \frac{1}{10^{-8}} = 10^8$

$Xw = y$

We can use the least squares solution

$\min_w \|Xw - y\|_2^2$, $w = (X^TX)^{-1}X^Ty$

$X = USV^T$

Thus, $w = \left((USV^T)^T USV^T\right)^{-1}(USV^T)^T y$

$$w = V S^{-1} U^T y \qquad \text{Also, } S^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{\gamma} \end{bmatrix}$$

for $\gamma = 0.1$

$$w = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{\gamma} \end{bmatrix} \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$w = \frac{1}{2\sqrt{2}} \begin{bmatrix} 1 & \frac{1}{\gamma} \\ 1 & -\frac{1}{\gamma} \end{bmatrix} \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

$$w = \frac{1}{2\sqrt{2}} \begin{bmatrix} 2 + \frac{2}{\gamma} \\ 2 - \frac{2}{\gamma} \end{bmatrix}$$

* thus for $\gamma = 0.1$,

$$w = \frac{1}{2\sqrt{2}} \begin{bmatrix} 2 + \frac{2}{0.1} \\ 2 - \frac{2}{0.1} \end{bmatrix}$$

$$w = \frac{1}{2\sqrt{2}} \begin{bmatrix} 22 \\ -18 \end{bmatrix}$$

$$w = \frac{1}{\sqrt{2}} \begin{bmatrix} 11 \\ -9 \end{bmatrix}$$

thus

$$\|w\|_2^2\Big|_{\gamma=0.1} = \frac{1}{\sqrt{2}}\left(\sqrt{11^2+(-9)^2}\right)^2$$

$$= \frac{1}{\sqrt{2}}(121+81)$$

$$= \frac{202}{\sqrt{2}}$$

for $\gamma = 10^{-8}$

$$w = \frac{1}{2\sqrt{2}}\begin{bmatrix} 2+\frac{2}{10^{-8}} \\ 2-\frac{2}{10^{-8}} \end{bmatrix}$$

$$w = \frac{1}{2\sqrt{2}}\begin{bmatrix} \sim 2\times 10^{8} \\ \sim -2\times 10^{8} \end{bmatrix}$$

$$w = \frac{1}{\sqrt{2}}\begin{bmatrix} 10^{8} \\ -10^{8} \end{bmatrix}$$

$$\|w\|_2^2\Big|_{\gamma=10^{-8}} = \frac{1}{\sqrt{2}}\left(\sqrt{(10^8)^2+(-10^8)^2}\right)^2$$

$$= \frac{1}{\sqrt{2}}\,2.10^{8} = \sqrt{2}\times 10^{8}$$

b) let $y = y_0 + y_\varepsilon$

where $y = \begin{bmatrix} 1+\varepsilon \\ 0 \\ 0 \\ 1 \end{bmatrix}$ & $y_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ & $y_\varepsilon = \begin{bmatrix} \varepsilon \\ 0 \\ 0 \\ 0 \end{bmatrix}$

Thus, $w = VS^{-1}U^T y$

$\Rightarrow$ $w = w_0 + w_\varepsilon$ given

$$w = \underbrace{VS^{-1}U^T y_0}_{w_0} + \underbrace{VS^{-1}U^T y_\varepsilon}_{w_\varepsilon}$$

we have $VS^{-1}U^T y_0$ from (a)

$$w_0 = VS^{-1}U^T y_0 = \frac{1}{2\sqrt{2}} \begin{bmatrix} 2+\frac{2}{\delta} \\ 2-\frac{2}{\delta} \end{bmatrix}$$

Similarly,

$$w_\varepsilon = VS^{-1}U^T y_\varepsilon = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & \frac{1}{\delta} \\ 1 & \frac{-1}{\delta} \end{bmatrix} \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} \varepsilon \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$w_\epsilon = \frac{1}{2\sqrt{2}} \begin{bmatrix} 1 & \frac{1}{\delta} \\ 1 & \frac{-1}{\delta} \end{bmatrix} \cdot \begin{bmatrix} \epsilon \\ \epsilon \end{bmatrix}$$

$$w_\epsilon = \frac{1}{2\sqrt{2}} \begin{bmatrix} \epsilon + \frac{\epsilon}{\gamma} \\ \epsilon - \frac{\epsilon}{\gamma} \end{bmatrix}$$

For $\epsilon = 0.01$ & $\gamma = 0.1$

$$w_\epsilon = \frac{1}{2\sqrt{2}} \begin{bmatrix} 0.01 + \frac{0.01}{0.1} \\ 0.01 - \frac{0.01}{0.1} \end{bmatrix} = \frac{1}{2\sqrt{2}} \begin{bmatrix} 0.11 \\ -0.09 \end{bmatrix}$$

$$\|w_\epsilon\|_2^2 = \left(\sqrt{\left(\frac{0.11}{2\sqrt{2}}\right)^2 + \left(\frac{-0.09}{2\sqrt{2}}\right)^2}\right)^2$$

$$= 0.002525$$

For $\epsilon = 0.01$ & $\gamma = 10^{-8}$

$$w_\epsilon = \frac{1}{2\sqrt{2}} \begin{bmatrix} 0.01 + \frac{0.01}{10^{-8}} \\ 0.01 - \frac{0.01}{10^{-8}} \end{bmatrix} = \frac{1}{2\sqrt{2}} \begin{bmatrix} 10^6 \\ -10^6 \end{bmatrix}$$

$$\|w_\epsilon\|_2^2 = 250,000,000,000$$

Thus the norm of the perturbation depends on γ as $\frac{1}{\gamma^2}$.

We see this from

$$w_\epsilon = \frac{1}{2\sqrt{2}} \begin{bmatrix} \epsilon + \frac{\epsilon}{\gamma} \\ \epsilon - \frac{\epsilon}{\gamma} \end{bmatrix}$$

$$\|w_\epsilon\|_2^2 = \left(\sqrt{\left[\frac{1}{2\sqrt{2}}\left(\epsilon + \frac{\epsilon}{\gamma}\right)\right]^2 + \left[\frac{1}{2\sqrt{2}}\left(\epsilon - \frac{\epsilon}{\gamma}\right)\right]^2}\right)^2$$

$$\|w_\epsilon\|_2^2 = \frac{1}{8}\left(\epsilon^2 + \frac{\epsilon^2}{\gamma^2} + \cancel{\frac{2\epsilon^2}{\gamma}}\right) + \frac{1}{8}\left(\epsilon^2 + \frac{\epsilon^2}{\gamma^2} - \cancel{\frac{2\epsilon^2}{\gamma}}\right)$$

$$\|w_\epsilon\|_2^2 = \frac{1}{4}\epsilon^2 + \frac{\epsilon^2}{\gamma^2}$$

Thus $\|w_\epsilon\|_2^2 \propto \frac{1}{\gamma^2}$

c) $(X^T X)^{-1} X^T \approx \sum_{i=1}^{r} \frac{1}{\sigma_i} v_i u_i^T$

for $r = 1$, we have,

$$(X^T X)^{-1} X^T = \frac{1}{\sigma_1} v_1 u_1^T$$

$$= \frac{1}{1} \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} \\ \frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} \end{bmatrix}$$

$$\omega = (X^T X)^{-1} X^T y$$

$$= (X^T X)^{-1} X^T (y_0 + y_\varepsilon)$$

$$= \underbrace{(X^T X)^{-1} X^T y_0}_{\omega_0} + \underbrace{(X^T X)^{-1} X^T y_\varepsilon}_{\omega_\varepsilon}$$

$$w_0 = \begin{bmatrix} \frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} \\ \frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$w_\epsilon = \begin{bmatrix} \frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} \\ \frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} \end{bmatrix} \begin{bmatrix} \epsilon \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{\epsilon}{2\sqrt{2}} \\ \frac{\epsilon}{2\sqrt{2}} \end{bmatrix}$$

we see that w_ϵ does not depend on γ, just ϵ.

Thus for both $\gamma = 0.1$ & $\gamma = 10^{-8}$, for $\epsilon = 0.01$ we have,

$$w_\epsilon = \begin{bmatrix} \frac{0.01}{2\sqrt{2}} \\ \frac{0.01}{2\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 0.00353 \\ 0.00353 \end{bmatrix}$$

$$\|w_\epsilon\|_2^2 = \left(\sqrt{(0.00353)^2 + (0.00353)^2}\right)^2$$

$$= 0.0000125 + 0.0000125$$

$$= 0.000025$$

which is much less than the norms in the two cases for (b).