Representing Functions as Inner Products

Objectives:

- introduce notation for vectors
- review inner products
- use inner products to represent functions
- interpret vectors and inner products geometrically

A vector is a collection of values arranged as a row or a column

$$\underline{w} = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} \qquad \underline{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix}$$

vectors: lower case underscore symbols

We will assume vectors are columns

Use transpose to write as row

$$\underline{w}^T = \begin{bmatrix} 2 & 3 & -1 \end{bmatrix}$$

$$\underline{a}^T = \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \end{bmatrix}$$

Inner Product

$$\underline{a} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}, \underline{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} \qquad c = \sum_{i=1}^{n} a_i b_i = \underline{a}^T \underline{b} = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

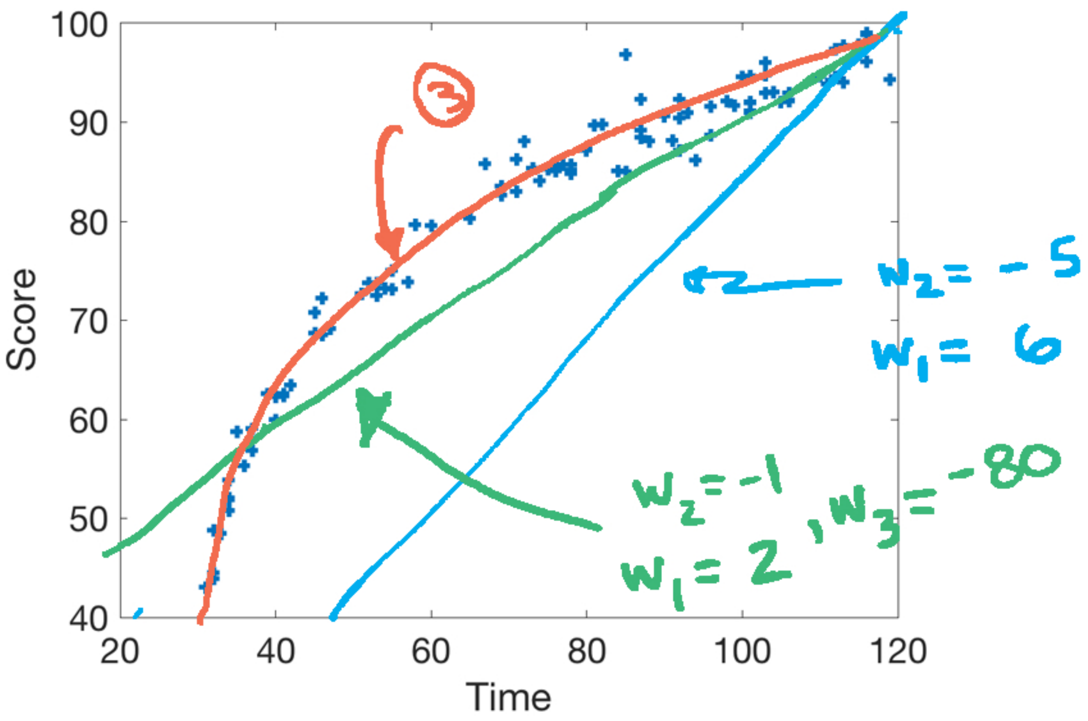
$$= \underline{b}^T \underline{a}$$

$\uparrow$ scalar

Inner products can represent many functions

predict relationship between score and time

$$s = f(t)$$



① $\underline{x} = \begin{bmatrix} s \\ t \end{bmatrix}, \ \underline{w} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$ $\qquad \underline{x}^T \underline{w} = 0$

$$s w_1 + t w_2 = 0$$

$$\Rightarrow s = -\frac{w_2}{w_1} t$$ line, slope $-\frac{w_2}{w_1}$

② $\underline{x} = \begin{bmatrix} s \\ t \\ 1 \end{bmatrix}, \ \underline{w} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$ $\qquad \underline{x}^T \underline{w} = 0$

$$s w_1 + t w_2 + w_3 = 0$$

$$\Rightarrow s = -\frac{w_2}{w_1} t - \frac{w_3}{w_1}$$

slope $-\frac{w_2}{w_1}$

intercept $\frac{-w_3}{w_1}$

③ $\underline{x}^T = [s \ t^3 \ t^2 \ t \ 1]$

$\underline{w}^T = [w_1 \ w_2 \ w_3 \ w_4 \ w_5]$

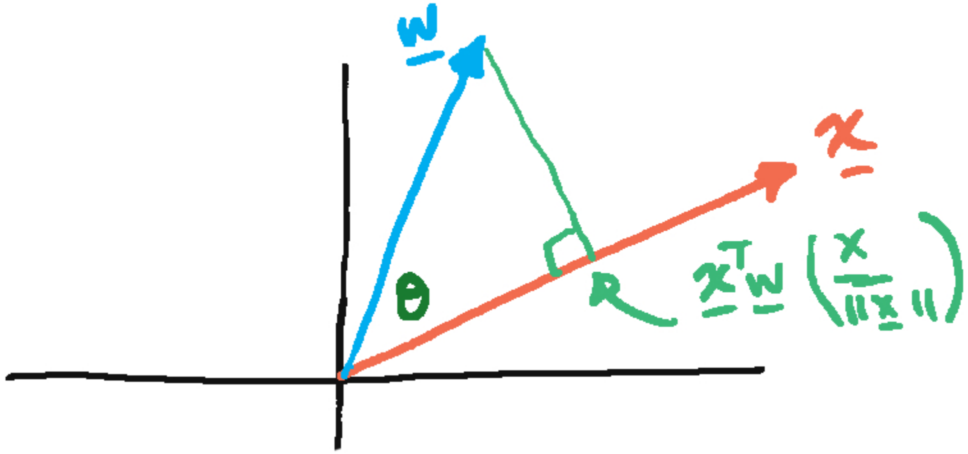
$$\underline{x}^T \underline{w} = 0$$

$$\Rightarrow S = \frac{-w_2}{w_1} t^3 - \frac{w_3}{w_1} t^2 - \frac{w_4}{w_1} t - \frac{w_5}{w_1}$$

Orthogonality

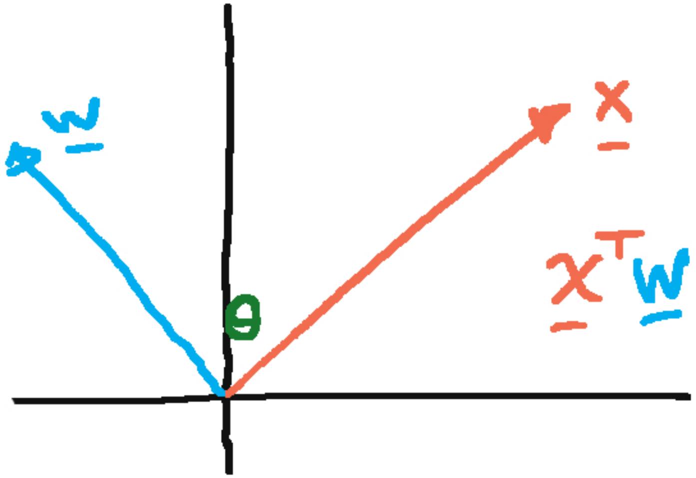
$\underline{x}$ and $\underline{w}$ are orthogonal iff $\underline{x}^T\underline{w} = 0$

Recall $\underline{x}^T\underline{w} = |\underline{x}|\,|\underline{w}|\cos\theta$



$|\underline{x}| = (\underline{x}^T\underline{x})^{1/2}$

$|\underline{w}| = (\underline{w}^T\underline{w})^{1/2}$

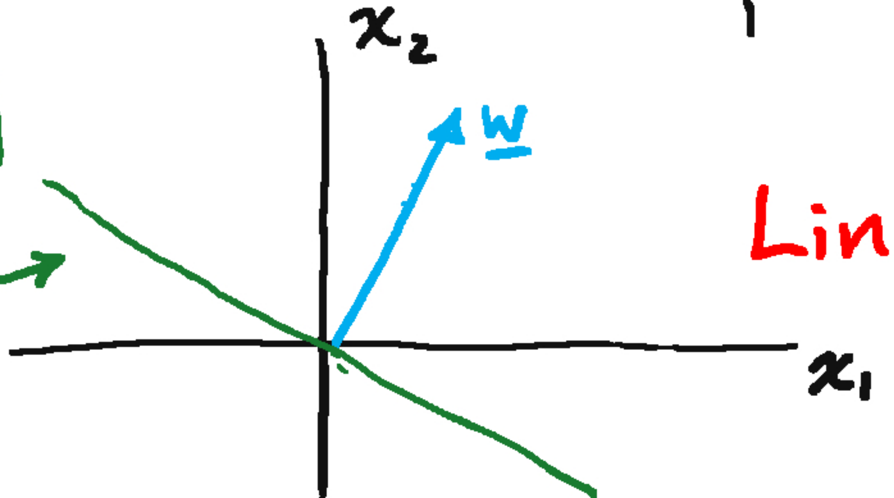


$\underline{x}^T\underline{w} = 0$

Consider $\{\underline{x} : \underline{x}^T\underline{w} = 0\}$

$x_1 w_1 + x_2 w_2 = 0$

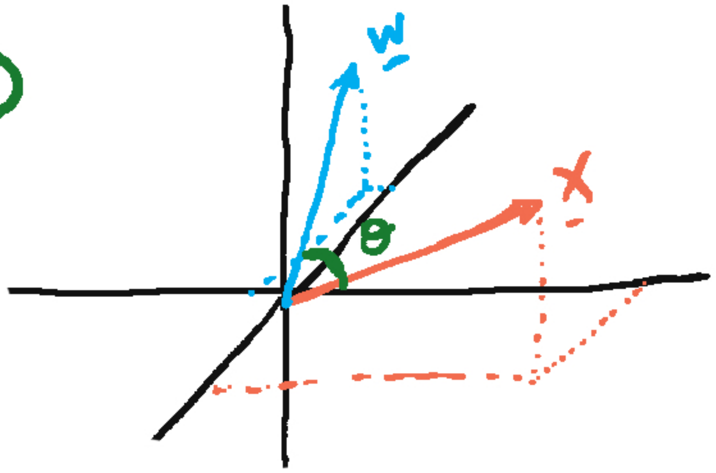
$x_2 = -\frac{w_1}{w_2} x_1$



Line!

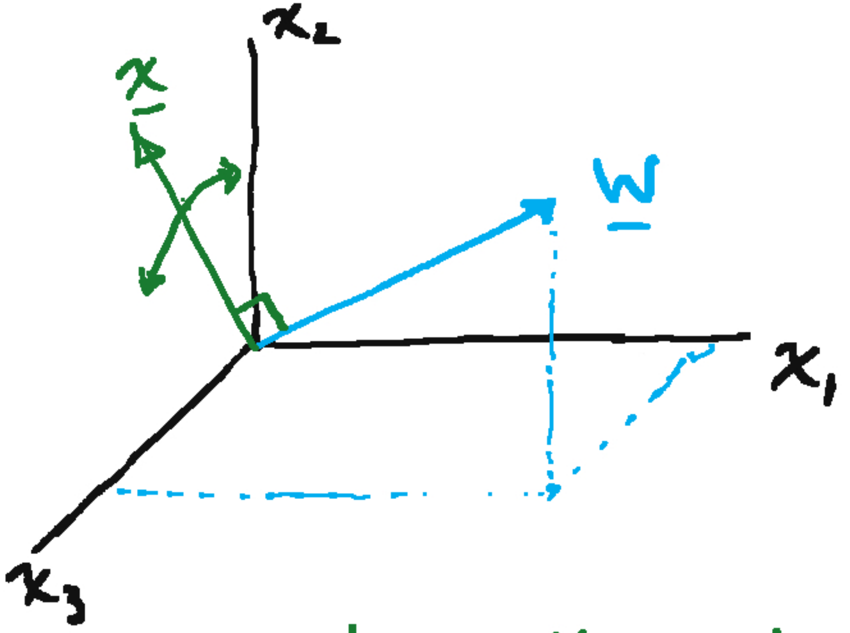
Geometric Concepts Apply to Higher Dimensions

3-D



$\underline{x}^T\underline{w} = |\underline{x}||\underline{w}|\cos\theta$

What about $\{\underline{x} : \underline{x}^T\underline{w} = 0\}$?



n-D space

$$\underline{x}^T\underline{w} = |\underline{x}||\underline{w}|\cos\theta$$

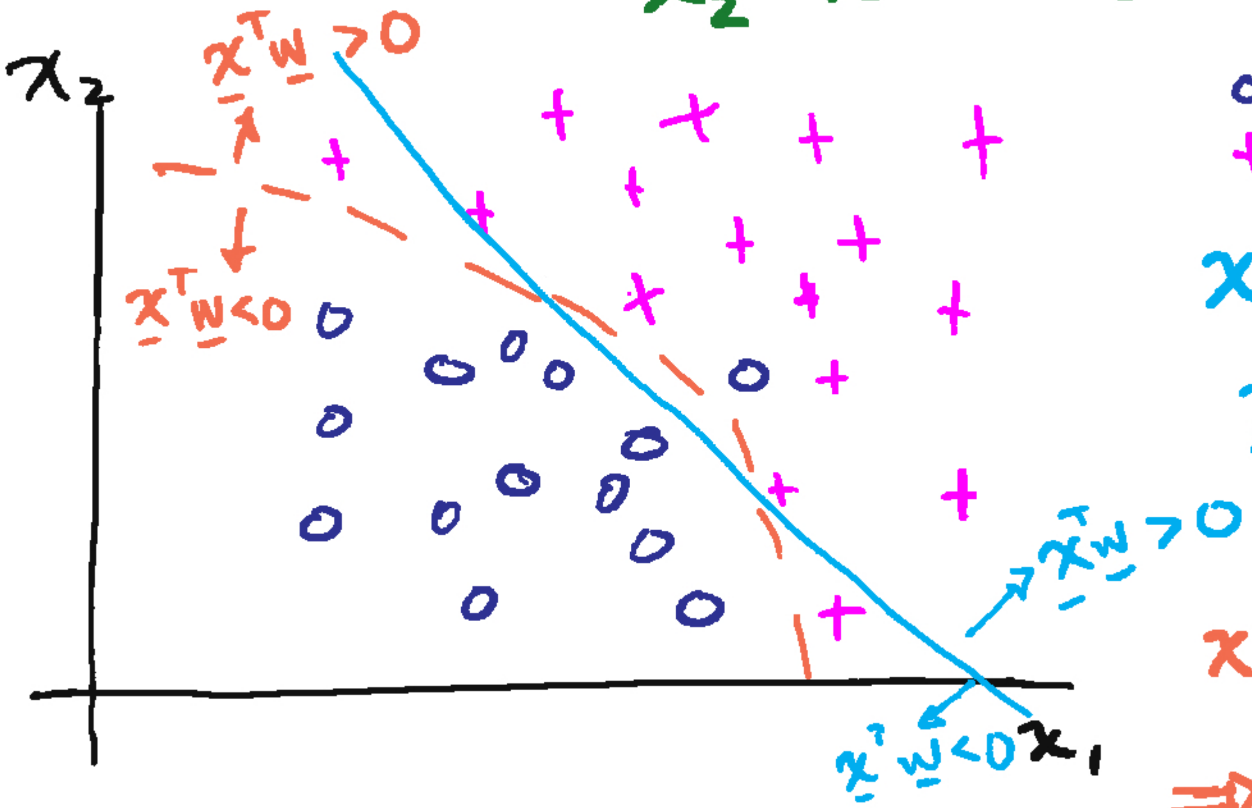
$$|\underline{x}| = (\underline{x}^T\underline{x})^{1/2}, \quad |\underline{w}| = (\underline{w}^T\underline{w})^{1/2}$$

$\underline{x}^T\underline{w} = 0$ is an n-1 dim space $\perp$ to $\underline{w}$

$\underline{x}^T\underline{w} = 0$ is a plane through $\underline{x} = 0$ $\perp$ to $\underline{w}$

Classification Application

Features: x_1 systolic blood pressure
x_2 total cholesterol



o no heart disease
+ heart disease

$x_2 = m x_1 + b \Rightarrow \underline{x}^T \underline{w} = 0$

$\underline{x}^T = [\,x_2 \;\; x_1 \;\; 1\,], \; \underline{w} = \begin{bmatrix} 1 \\ -m \\ -b \end{bmatrix}$

$x_2 = c_1 x_1^3 + c_2 x_1^2 + c_3 x_1 + c_4$

$\Rightarrow \underline{x}^T \underline{w} = 0$

$\underline{x}^T = [\,x_2 \;\; x_1^3 \;\; x_1^2 \;\; x_1 \;\; 1\,]$ $\quad \underline{w}^T = [\,1 \;\; -c_1 \;\; -c_2 \;\; -c_3 \;\; -c_4\,]$