Uniqueness of Solutions to Learning Problems

Objectives

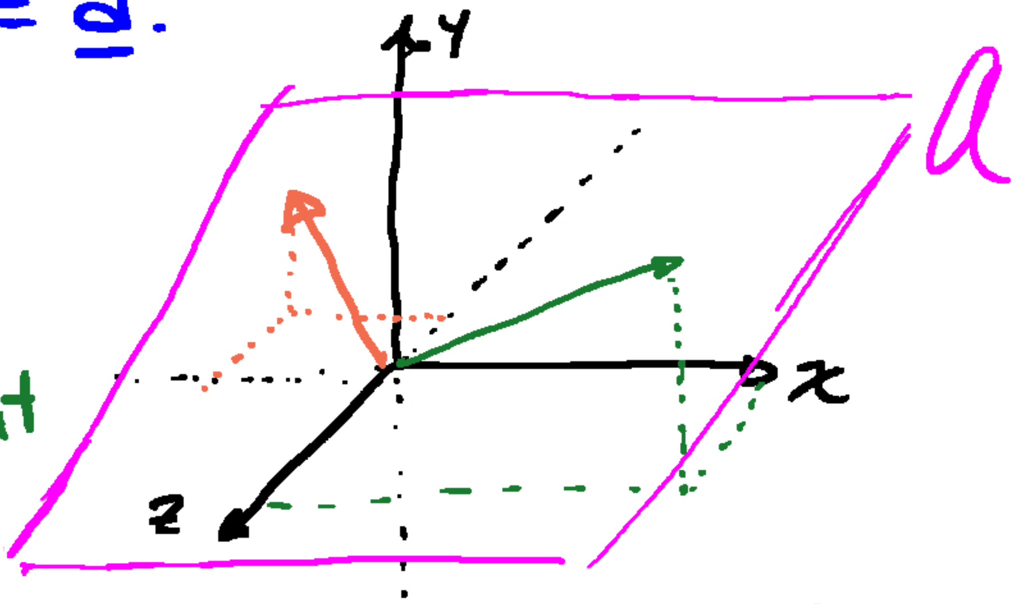
- Revisit conditions for a solution to $\underline{A}\,\underline{w} = \underline{d}$
- Conditions for a unique solution
- Overview approaches to find solutions

Many machine learning problems require solving $\underline{A}\underline{w} = \underline{d}$.

$$\underline{d} = \underline{a}_1 w_1 + \underline{a}_2 w_2$$

$\Rightarrow$ $\underline{d}, \underline{a}_1, \underline{a}_2$ linearly dependent

$$\text{rank}\{\underline{A}\} = \text{rank}\{[\underline{A} \,\vdots\, \underline{d}]\}$$



$\mathcal{a}$ = span{columns of $\underline{A}$}

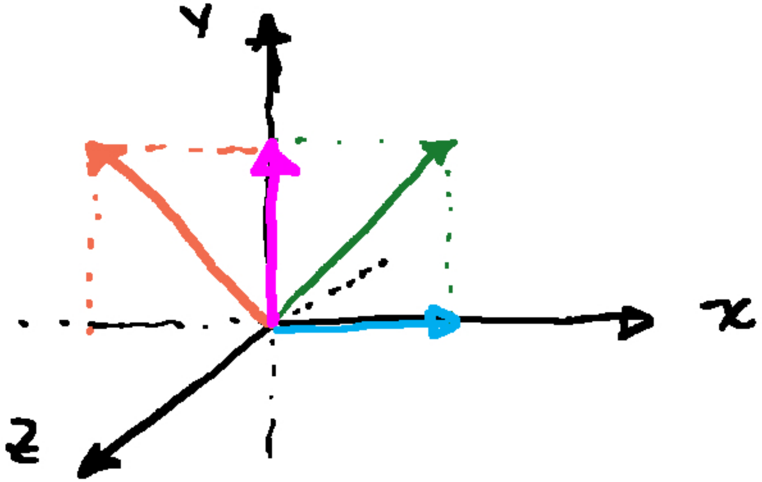
all vectors that can be written $\sum_{i=1}^{M} \underline{a}_i w_i$

$$\underline{A} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \\ 0 & 0 \end{bmatrix}, \quad \underline{d} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \quad \tilde{\underline{d}} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

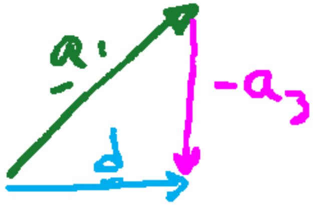
$\underline{d} \in \mathcal{a}$ $\quad$ $\underline{d} = -\underline{a}_1 + \underline{a}_2$

$\tilde{\underline{d}} \notin \mathcal{a}$

Ex: $\underline{A} = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, $\underline{d} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

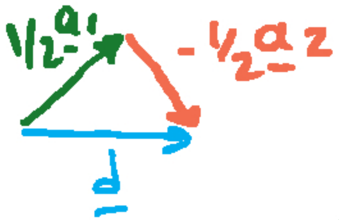


$\underline{d} = \underline{a}_1 - \underline{a}_3$



or

$\underline{d} = \frac{1}{2}(\underline{a}_1 - \underline{a}_2)$



Solution to $\underline{A}\underline{w} = \underline{d}$ may not be unique!

Nonuniqueness: Suppose $\underline{A}\underline{w} = \underline{d}$. Does $\underline{f} \neq \underline{0}$ exist so that $\tilde{\underline{w}} = \underline{w} + \underline{f}$ also satisfies $\underline{A}\tilde{\underline{w}} = \underline{d}$?

$$\underline{A}\tilde{\underline{w}} = \underline{d} \Rightarrow \underline{A}\underline{w} + \underline{A}\underline{f} = \underline{d} \Rightarrow (\underline{A}\underline{w} - \underline{d}) + \underline{A}\underline{f} = 0 \Rightarrow \underline{A}\underline{f} = 0$$

$\sum_{i=1}^{M} \underline{a}_i f_i = 0$ for $\underline{f} \neq 0$

Nonunique iff cols. $\underline{A}$ are lin. dep.

Example (cont):

$$\underline{A} = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad \underline{d} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$\text{rank}\{\underline{A}\} = 2$

$\text{rank}\{[\underline{A} : \underline{d}]\} = 2$

$$\underline{a}_1 - \underline{a}_3 \Rightarrow \underline{w} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \qquad \frac{1}{2}(\underline{a}_1 - \underline{a}_2) \Rightarrow \underline{\tilde{w}} = \begin{bmatrix} 1/2 \\ -1/2 \\ 0 \end{bmatrix} = \underline{w} + \underbrace{\begin{bmatrix} -1/2 \\ -1/2 \\ 1 \end{bmatrix}}_{\underline{f}}$$

$\underline{A}\,\underline{f} = \underline{0}$ ✓

Note: $\underline{A}\,\gamma\underline{f} = 0$ for all γ

$\Rightarrow$ infinite number of solutions

Characterizing Solutions to $\underline{A}\,\underline{w} = \underline{d}$:

1) $\text{rank}\{\underline{A}\} < \text{rank}\{[\underline{A} : \underline{d}]\}$

no solution

2) $\text{rank}\{\underline{A}\} = \text{rank}\{[\underline{A} : \underline{d}]\}$

$\text{rank}\{\underline{A}\} = \dim\{\underline{w}\}$

unique soln

$\text{rank}\{\underline{A}\} < \dim\{\underline{w}\}$

nonunique soln

Solving $\underline{A}\,\underline{w} = \underline{d}$

Finding rank: best with a computer (approximate)

toy example - guess and check

Finding $\underline{w}$:

use computer

mannally - algebraic manipulation

Gaussian elimination