CS/ECE/ME532 Period 11 Activity

Estimated time: 45 *mins for Q1 and* 20 *mins for Q2.*

1. See `period_11.ipynb`.

2. Let a 4-by-2 matrix $\boldsymbol{X}$ have SVD $\boldsymbol{X} = \boldsymbol{U}\boldsymbol{S}\boldsymbol{V}^T$ where $\boldsymbol{U} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & -1 \\ 1 & 1 \end{bmatrix}$, $\boldsymbol{S} = \begin{bmatrix} 1 & 0 \\ 0 & \gamma \end{bmatrix}$, and $\boldsymbol{V} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$.

a) Express the solution to the least-squares problem $\arg\min_{\boldsymbol{w}} ||\boldsymbol{X}\boldsymbol{w} - \boldsymbol{y}||_2^2$ as a function of $\boldsymbol{U}$, $\boldsymbol{S}, \boldsymbol{V}$, and $\boldsymbol{y}$.

b) Let $\boldsymbol{y} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$. Find the weights $\boldsymbol{w}$ that minimize $||\boldsymbol{X}\boldsymbol{w} - \boldsymbol{y}||_2^2$ as a function of γ. Calculate $||\boldsymbol{X}\boldsymbol{w} - \boldsymbol{y}||_2^2$ and $||\boldsymbol{w}||_2^2$ as a function of γ for this value of $\boldsymbol{w}$. What happens to $||\boldsymbol{w}||_2^2$ as $\gamma \to 0$?

c) Now consider a "low-rank" inverse. Instead of writing

$$(\boldsymbol{X}^T\boldsymbol{X})^{-1}\boldsymbol{X}^T = \sum_{i=1}^{p} \frac{1}{\sigma_i}\boldsymbol{v}_i\boldsymbol{u}_i^T$$

where p is the number of columns of $\boldsymbol{X}$ (assumed less than the number of rows), we approximate

$$(\boldsymbol{X}^T\boldsymbol{X})^{-1}\boldsymbol{X}^T \approx \sum_{i=1}^{r} \frac{1}{\sigma_i}\boldsymbol{v}_i\boldsymbol{u}_i^T$$

In this approximation we only invert the largest r singular values, and ignore all of them smaller than σ_r. If $r = 1$, use the low-rank inverse to find $\boldsymbol{w}$, $||\boldsymbol{y} - \boldsymbol{X}\boldsymbol{w}||_2^2$, and $||\boldsymbol{w}||_2^2$ when $\boldsymbol{y} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ as in part b). Compare $||\boldsymbol{y} - \boldsymbol{X}\boldsymbol{w}||_2^2$, and $||\boldsymbol{w}||_2^2$ to the results for part b).