CS/ECE/ME 532
Period 11

- Today: Unit 3 – Low Rank Decompositions

- Low rank decompositions:
  - k-means
  - The singular value decomposition (SVD)
- Properties of the SVD
- Least squares and the SVD

- Next week: Unit 4 – Applications of the SVD

Today – Properties of the SVD and LS+SVD

Low rank decompositions:

basis vectors or patterns

$$\boldsymbol{A} \approx \boldsymbol{U}\boldsymbol{W}^T = \begin{bmatrix} u_{1,1} & u_{1,2} \\ u_{2,1} & u_{2,2} \\ u_{3,1} & u_{3,2} \\ u_{4,1} & u_{4,2} \end{bmatrix} \begin{bmatrix} w_{1,1} & \dots & w_{1,6} \\ w_{2,1} & \dots & w_{2,6} \end{bmatrix}$$

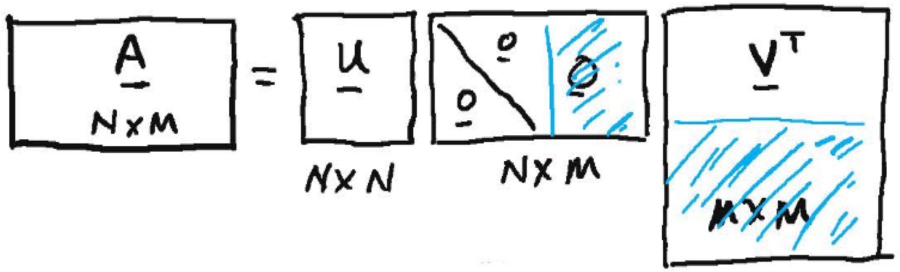
k-means: cluster centers

We'd like decomposition with:

1. columns of ***U*** to be orthonormal: $\boldsymbol{U}^T\boldsymbol{U} = \boldsymbol{I}$
2. unique, meaningful ordering
   - find first basis vector that well aligned with columns of ***A:***

$$\text{large } |\boldsymbol{u}^T\boldsymbol{a}_i| \qquad \boldsymbol{u}_1 = \arg \max_{||\boldsymbol{u}||_2=1} ||\boldsymbol{u}^T\boldsymbol{A}||_2$$

Answer: these two requirements **define** the SVD!

$\boldsymbol{A} =$ 

SVD defines the **'best'** rank r approximation (EY, 1936):

$$\min ||\boldsymbol{A} - \tilde{\boldsymbol{A}}||_F$$

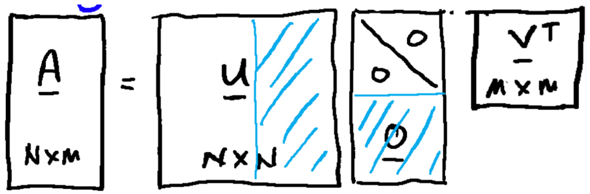
over all matrices $\tilde{\boldsymbol{A}}$ with rank $\leq r$

given by $\tilde{\boldsymbol{A}} = \sum_{i=1}^r \sigma_i \boldsymbol{u}_i \boldsymbol{v}_i^T$

SVD for least squares:

$$\min_{\boldsymbol{w}} ||\boldsymbol{A}\boldsymbol{w} - \boldsymbol{y}||^2$$

$$\boldsymbol{w}^* = (\boldsymbol{A}^T\boldsymbol{A})^{-1}\boldsymbol{A}^T\boldsymbol{y}$$

$\boldsymbol{A} =$ 

$$\boldsymbol{w}^* = \boldsymbol{V}\Sigma^{-1}\boldsymbol{U}^T\boldsymbol{y}$$

economy SVD

Problems when small singular values!
Fix by dropping corresponding singular vectors.