~~ECE~~ 532 Activity 11 Ayan Deep Hazra

2. a) $w = V S^{-1} U^T y$

b) $w = V S^{-1} U^T y$ where

$$V = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \quad S = \begin{bmatrix} 1 & 0 \\ 0 & \gamma \end{bmatrix}$$

$$U = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & -1 \\ 1 & 1 \end{bmatrix} \quad y = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$w = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \cdot \frac{1}{\gamma}\begin{bmatrix} \gamma & 0 \\ 0 & 1 \end{bmatrix} \cdot \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$w = \frac{1}{2\sqrt{2}\gamma}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} \gamma & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$w = \frac{1}{2\sqrt{2}\gamma}\begin{bmatrix} \gamma & 1 \\ \gamma & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

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$$W = \frac{1}{2\sqrt{2}\gamma}\begin{bmatrix} \gamma+1 & \gamma-1 & \gamma-1 & \gamma+1 \\ \gamma-1 & \gamma+1 & \gamma+1 & \gamma-1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$W = \frac{1}{2\sqrt{2}\gamma}\begin{bmatrix} 2\gamma+2 \\ 2\gamma-2 \end{bmatrix}$$

$$X = USV^T$$

$$= \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & \gamma \end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$= \frac{1}{2\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ \gamma & -\gamma \end{bmatrix} = \frac{1}{2\sqrt{2}}\begin{bmatrix} 1+\gamma & 1-\gamma \\ 1-\gamma & 1+\gamma \\ 1-\gamma & 1+\gamma \\ 1+\gamma & 1-\gamma \end{bmatrix}$$

$\Rightarrow Xw - y$

$$= \frac{1}{2\sqrt{2}} \begin{bmatrix} 1+\gamma & 1-\gamma \\ 1-\gamma & 1+\gamma \\ 1-\gamma & 1+\gamma \\ 1+\gamma & 1-\gamma \end{bmatrix} \frac{1}{2\sqrt{2}\gamma} \begin{bmatrix} 2\gamma+2 \\ 2\gamma-2 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$= \frac{1}{8\gamma} \begin{bmatrix} (2(\gamma+1)^2 - 2(\gamma-1)^2) \\ 0 \\ 0 \\ (2(\gamma+1)^2 - 2(\gamma-1)^2) \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$= \frac{1}{8\gamma} \begin{bmatrix} 2(4\gamma) \\ 0 \\ 0 \\ 2(4\gamma) \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Thus, $\|Xw - y\|_2^2 = 0$

& $\|w\|_2^2 = \left\| \frac{1}{2\sqrt{2\gamma}} \begin{bmatrix} 2\gamma+2 \\ 2\gamma-2 \end{bmatrix} \right\|_2^2$

$$\|w\|_2^2 = \frac{1}{\sqrt{2\gamma}} \left\| \begin{pmatrix} \gamma+1 \\ \gamma-1 \end{pmatrix} \right\|_2^2$$

$$\|w\|_2^2 = \frac{1}{\sqrt{2\gamma}} \left(\sqrt{(\gamma+1)^2 + (\gamma-1)^2} \right)^2$$

$$\|w\|_2^2 = \frac{1}{\sqrt{2\gamma}} \left(\gamma^2 + 1 + \cancel{2\gamma} + \gamma^2 + 1 - \cancel{2\gamma} \right)$$

$$\|w\|_2^2 = \frac{1}{\sqrt{2\gamma}} \left(2\gamma^2 + 2 \right)$$

$$\lim_{\gamma \to 0} \|w\|_2^2 = \lim_{\gamma \to 0} \frac{1}{\sqrt{2\gamma}} (2\gamma^2 + 2) \to \infty$$

c) If $n=1$, by definition

$$(X^TX)^{-1}X^T \approx \frac{1}{\sigma_1} v_i u_i^T$$

$$w = (X^TX)^{-1}X^T y \quad , y = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

$$= \frac{1}{\sigma_1} v_1 u_1^T y$$

$$= \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \frac{1}{2} \begin{bmatrix} 1 & 1 & \cdots & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

$$= \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \frac{1}{2} \begin{bmatrix} 2 \end{bmatrix}$$

$$= \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \end{bmatrix}$$

$$= \frac{1}{\sqrt{2}}$$

Thus $\|w\|_2^2 = \left(\frac{1}{\sqrt{2}}\right)^2 = \frac{1}{2}$

$$X_1 = U_1 S_1 V_1^T$$

$$= \frac{1}{2}\begin{bmatrix}1\\1\\1\\1\end{bmatrix}\begin{bmatrix}1 & 0\end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix}1\\1\\1\\1\end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\end{bmatrix}$$

$$= \frac{1}{2\sqrt{2}}\begin{bmatrix}1\\1\\1\\1\end{bmatrix}$$

Thus $\|y - Xw\|_2^2$ gives us

$$\left\| \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{2\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \frac{1}{2} \right\|_2^2$$

$$= \left\| \begin{matrix} 1 - \frac{1}{4\sqrt{2}} \\ \frac{-1}{4\sqrt{2}} \\ \frac{-1}{4\sqrt{2}} \\ 1 - \frac{1}{4\sqrt{2}} \end{matrix} \right\|_2^2 = 2\left(1 - \frac{1}{4\sqrt{2}}\right)^2 + 2\left(\frac{1}{4\sqrt{2}}\right)^2$$

$$= 2\left[1 + \frac{1}{32} - \frac{1}{2\sqrt{2}}\right] + \frac{2}{32}$$

$$= 2 + \frac{1}{16} - \frac{1}{2\sqrt{2}} + \frac{1}{16}$$

$$= 2 + \frac{1}{8} - \frac{1}{2\sqrt{2}}$$

$\|w\|_2^2$ is a constant value in (c) & depends on δ in (b).

Thus, we see $\|y - Xw\|_2^2$ in both cases is different. When we use just $n=1$, we get a +ve value for $\|y - Xw\|_2^2$.