CS/ECE/ME532 Period 3 Activity

Estimated Time: 15 min for P1, 15 min for P2, 15 min for P3, 15 min for P4

1) You collect ratings of three classes from four of your friends.

Class	Bao	Julia	Vivek	Jamal
CS760	36	72	90	54
ECE533	40	80	100	60
Math521	20	40	50	30

a) Express these ratings in a matrix $\boldsymbol{X}$ whose columns represent the ratings of each friend, and rows the ratings of each class.

b) Suppose $\boldsymbol{X}$ is expressed as the outer product of a taste vector $\boldsymbol{t}$ and an affinity weight vector $\boldsymbol{w}$, that is, $\boldsymbol{X} = \boldsymbol{t}\boldsymbol{w}^T$ and assume $\boldsymbol{t} = \begin{bmatrix} 9 \\ 10 \\ 5 \end{bmatrix}$. Find $\boldsymbol{w}$.

c) Your friend Brianna wasn't able to complete your survey, but did rate ECE533 as 30. Assuming Brianna has the same taste profile as the rest of your friends, what would her ratings for CS760 and Math521 be?

SOLUTION:

a) rows are classes, columns are friends' ratings

$$\boldsymbol{X} = \begin{bmatrix} 36 & 72 & 90 & 54 \\ 40 & 80 & 100 & 60 \\ 20 & 40 & 50 & 30 \end{bmatrix}$$

b) Write $\boldsymbol{X}$ as the outer product $\begin{bmatrix} 9 \\ 10 \\ 5 \end{bmatrix} \begin{bmatrix} w_1 & w_2 & w_3 & w_4 \end{bmatrix}$. Thus,

$$\boldsymbol{X} = \begin{bmatrix} 9w_1 & 9w_2 & 9w_3 & 9w_4 \\ 10w_1 & 10w_2 & 10w_3 & 10w_4 \\ 5w_1 & 5w_2 & 5w_3 & 5w_4 \end{bmatrix}$$

This form implies $w_1 = 4, w_2 = 8, w_3 = 10, w_4 = 6$.

c) If Brianna has the same taste profile, then her ratings will be of the form $\begin{bmatrix} 9 \\ 10 \\ 5 \end{bmatrix} w_5$.

We can use his known rating for ECE533 to solve for w_5: $10w_5 = 30$, so $w_5 = 3$. This predicts her rating of CS760 is 27 and Math521 is 15.

2) Suppose a 4×5 rating matrix $\boldsymbol{X}$ reflecting ratings of four movies by five people is decomposed as the product of a rank-2 taste matrix $\boldsymbol{T} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & -1 \\ 1 & 1 \end{bmatrix}$ and rank-2 affinity weight matrix $\boldsymbol{W} = \begin{bmatrix} 8 & 6 & 4 & 5 & 4 \\ 2 & 3 & 3 & -2 & -1 \end{bmatrix}$, that is, $\boldsymbol{X} = \boldsymbol{T}\boldsymbol{W}$.

a) Find $\boldsymbol{X}$.

b) Write $\boldsymbol{X} = \boldsymbol{t}_1\boldsymbol{w}_1^T + \boldsymbol{t}_2\boldsymbol{w}_2^T$ where $\boldsymbol{t}_1, \boldsymbol{t}_2, \boldsymbol{w}_1, \boldsymbol{w}_2$ are column vectors and the first elements of $\boldsymbol{t}_1$ and $\boldsymbol{t}_2$ are given by $[\boldsymbol{t}_1]_1 = [\boldsymbol{t}_2]_1 = 1$. Find one choice for $\boldsymbol{t}_1, \boldsymbol{t}_2, \boldsymbol{w}_1$ and $\boldsymbol{w}_2$.

SOLUTION:

a)

$$\boldsymbol{X} = \boldsymbol{T}\boldsymbol{W} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 8 & 6 & 4 & 5 & 4 \\ 2 & 3 & 3 & -2 & -1 \end{bmatrix}$$

so multiplying gives

$$\boldsymbol{X} = \begin{bmatrix} 10 & 9 & 7 & 3 & 3 \\ 6 & 3 & 1 & 7 & 5 \\ 6 & 3 & 1 & 7 & 5 \\ 10 & 9 & 7 & 3 & 3 \end{bmatrix}$$

b) We can use the outer product form for the matrix multiplication to write

$$\boldsymbol{X} = \boldsymbol{t}_1\boldsymbol{w}_1^T + \boldsymbol{t}_2\boldsymbol{w}_2^T = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 8 & 6 & 4 & 5 & 4 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} \begin{bmatrix} 2 & 3 & 3 & -2 & -1 \end{bmatrix}$$

so $\boldsymbol{t}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \boldsymbol{t}_2 = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}, \boldsymbol{w}_1^T = \begin{bmatrix} 8 & 6 & 4 & 5 & 4 \end{bmatrix}, \boldsymbol{w}_2^T = \begin{bmatrix} 2 & 3 & 3 & -2 & -1 \end{bmatrix}$.

3) Let $\boldsymbol{X} = \begin{bmatrix} \boldsymbol{A} & \boldsymbol{B} \\ \boldsymbol{c}^T & \boldsymbol{d}^T \end{bmatrix}$ where $\boldsymbol{A}$ is 2×3, $\boldsymbol{B}$ is 2×2, $\boldsymbol{c}^T$ is 1×3, and $\boldsymbol{d}^T$ is 1×2.

a) Express the product $\boldsymbol{R} = \boldsymbol{X}\boldsymbol{X}^T$ in terms of $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{c}, \boldsymbol{d}$.

b) What are the dimensions of $\boldsymbol{R}$?

SOLUTION:

a)

$$\boldsymbol{R} = \boldsymbol{X}\boldsymbol{X}^T = \begin{bmatrix} \boldsymbol{A} & \boldsymbol{B} \\ \boldsymbol{c}^T & \boldsymbol{d}^T \end{bmatrix} \begin{bmatrix} \boldsymbol{A} & \boldsymbol{B} \\ \boldsymbol{c}^T & \boldsymbol{d}^T \end{bmatrix}^T$$

which is

$$\begin{aligned} \boldsymbol{R} &= \begin{bmatrix} \boldsymbol{A} & \boldsymbol{B} \\ \boldsymbol{c}^T & \boldsymbol{d}^T \end{bmatrix} \begin{bmatrix} \boldsymbol{A}^T & \boldsymbol{c} \\ \boldsymbol{B}^T & \boldsymbol{d} \end{bmatrix} \\ &= \begin{bmatrix} \boldsymbol{A}\boldsymbol{A}^T + \boldsymbol{B}\boldsymbol{B}^T & \boldsymbol{A}\boldsymbol{c} + \boldsymbol{B}\boldsymbol{d} \\ \boldsymbol{c}^T\boldsymbol{A}^T + \boldsymbol{d}^T\boldsymbol{B}^T & \boldsymbol{c}^T\boldsymbol{c} + \boldsymbol{d}^T\boldsymbol{d} \end{bmatrix} \end{aligned}$$

b) Note that $\boldsymbol{A}\boldsymbol{A}^T + \boldsymbol{B}\boldsymbol{B}^T$ is 2-by-2, $\boldsymbol{A}\boldsymbol{c} + \boldsymbol{B}\boldsymbol{d}$ is 2-by-1, $\boldsymbol{c}^T\boldsymbol{A}^T + \boldsymbol{d}^T\boldsymbol{B}^T$ is 1-by-2, and $\boldsymbol{c}^T\boldsymbol{c} + \boldsymbol{d}^T\boldsymbol{d}$ is 1-by-1, so $\boldsymbol{R}$ is 3-by-3.

4) **a)** Let $\boldsymbol{y} = \boldsymbol{A}\boldsymbol{x}$. Denote the columns of $\boldsymbol{A}$ as $\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_n$. Express $\boldsymbol{y}$ as a weighted sum of the columns of $\boldsymbol{A}$.

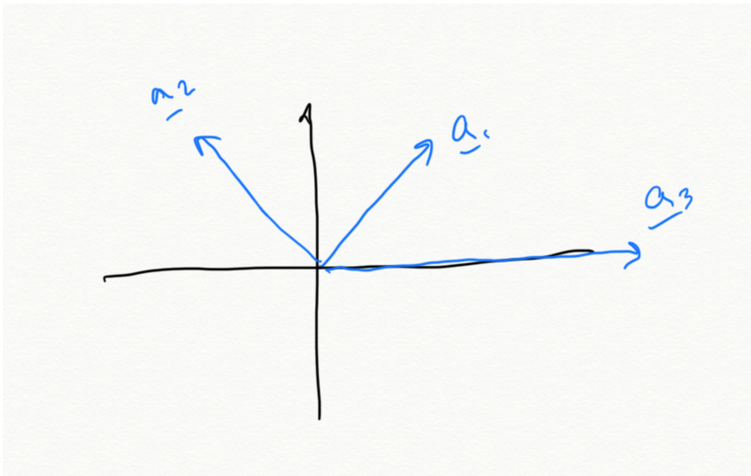
b) Let $\boldsymbol{A} = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 1 & 0 \end{bmatrix}$. Consider the columns of $\boldsymbol{A}$ as vectors in $\mathbb{R}^2$, and plot them with the first element on the horizontal axis, and the second element on the vertical axis.

c) Let $\boldsymbol{y} = \boldsymbol{A}\boldsymbol{x}$ with $\boldsymbol{x} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ and $\boldsymbol{A}$ as in part (b). Draw a picture to find $\boldsymbol{y}$ by expressing it as a weighted sum of vectors you ploted in (b).

SOLUTION:

a) $\boldsymbol{y} = \sum_{i=1}^{n} x_i \boldsymbol{a}_i$

b)



c)