CS/ECE/ME532 Period 4 Activity

Estimated Time: 15 min for P1, 10 min for P2, 15 min for P3

1) *Matrix Rank.* Let $\boldsymbol{X} = \begin{bmatrix} 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

   a) What is the rank of $\boldsymbol{X}$?

   b) Find a set of linearly independent columns in $\boldsymbol{X}$. Is there more than one set? How many sets of linearly independent columns can you find?

   c) A matrix $\boldsymbol{A} = \begin{bmatrix} 1 & 0 & a \\ 1 & 1 & b \\ 0 & 1 & -1 \end{bmatrix}$. Find the relationship between b and a so that rank$\{\boldsymbol{A}\} = 2$. *Hint:* find a, b so that the third column is a weighted sum of the first two columns. Note that there are many choices for a, b that result in rank 2.

SOLUTION:

a) $\boldsymbol{X}$ is rank two. Note that the rank is the number of linearly independent columns (or rows). Let $\boldsymbol{v}_j$ be the j^{th} column of $\boldsymbol{X}$. Vectors $\boldsymbol{v}_j$ are linearly independent if

$$\sum_j \alpha_j \boldsymbol{v}_j = 0 \text{ implies } \alpha_j = 0 \text{ for all } j$$

Note that $\boldsymbol{v}_1$ is linearly independent since $\alpha_1 \boldsymbol{v}_1 = \boldsymbol{0}$ implies $\alpha_1 = 0$. $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ are not linearly independent since $\alpha_1 \boldsymbol{v}_1 + \alpha_2 \boldsymbol{v}_2 = 0$ for any α_2 when $\alpha_1 = 0$. However, $\boldsymbol{v}_1$ and $\boldsymbol{v}_3$ are linearly independent since $\alpha_1 \boldsymbol{v}_1 + \alpha_3 \boldsymbol{v}_3 = 0$ requires $\alpha_1 = \alpha_3 = 0$. Thus the rank is at least two. But $\boldsymbol{v}_4$ is not linearly independent of $\boldsymbol{v}_1$ and $\boldsymbol{v}_3$ since $\alpha_1 \boldsymbol{v}_1 + \alpha_3 \boldsymbol{v}_3 + \alpha_4 \boldsymbol{v}_4 = 0$ for $\alpha_1 = \alpha_4 = 1$ and $\alpha_3 = -1$. Similarly $\boldsymbol{v}_5$ is not linearly independent of $\boldsymbol{v}_3$ since $\boldsymbol{v}_3 + \boldsymbol{v}_5 = 0$. Note that the main point of this problem is to apply the definition of linear independence by inspection - it is not to use a formal procedure for finding rank.

b) From the previous part we know that the following pairs of vectors are linearly independent: $\{\boldsymbol{v}_1, \boldsymbol{v}_3\}$, $\{\boldsymbol{v}_1, \boldsymbol{v}_4\}$, $\{\boldsymbol{v}_1, \boldsymbol{v}_5\}$, $\{\boldsymbol{v}_3, \boldsymbol{v}_4\}$, $\{\boldsymbol{v}_4, \boldsymbol{v}_5\}$.

c) We require

$$\alpha_1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + \alpha_3 \begin{bmatrix} a \\ b \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \text{ for some } \alpha_1, \alpha_2, \alpha_3 \neq 0$$

There are many possibilities. Suppose we set $\alpha_2 = \alpha_3$ so the third row is zero. Then we require

$$\alpha_1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} a \\ b \\ -1 \end{bmatrix} = \begin{bmatrix} \alpha_1 + a \\ \alpha_1 + 1 + b \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

which may be solved by setting $\alpha_1 = -a$ and requiring $b = a - 1$.

2) *Solution Existence.* A system of linear equations is given by $\boldsymbol{Ax} = \boldsymbol{b}$ where $\boldsymbol{A} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}$.

a) Suppose $\boldsymbol{b} = \begin{bmatrix} 8 \\ 6 \\ -2 \end{bmatrix}$. Does a solution for $\boldsymbol{x}$ exist? If so, find $\boldsymbol{x}$.

b) Suppose $\boldsymbol{b} = \begin{bmatrix} 4 \\ 6 \\ 1 \end{bmatrix}$. Does a solution for $\boldsymbol{x}$ exist? If so, find $\boldsymbol{x}$.

c) Consider the general system of linear equations $\boldsymbol{Ax} = \boldsymbol{b}$. This equation says that $\boldsymbol{b}$ is a weighted sum of the columns of $\boldsymbol{A}$. Assume $\boldsymbol{A}$ is full rank. Use the definition of linear independence to find the condition on rank$\left\{\begin{bmatrix} \boldsymbol{A} & \boldsymbol{b} \end{bmatrix}\right\}$ that guarantees a solution exists.

SOLUTION:

a) Note that the first row of $\boldsymbol{Ax} = \boldsymbol{b}$ implies the first element of $\boldsymbol{x}$ must be 8, while the last row implies the second element of $\boldsymbol{x}$ must be -2. These values work for the middle row (8-2 = 6), so $\boldsymbol{x} = \begin{bmatrix} 8 \\ -2 \end{bmatrix}$ is a solution. In this case we have rank$\boldsymbol{A}$ =rank$\begin{bmatrix} \boldsymbol{A} & \boldsymbol{b} \end{bmatrix}$.

b) Here the first row of $\boldsymbol{Ax} = \boldsymbol{b}$ implies the first element of $\boldsymbol{x}$ must be 4, while the last row implies the second element of $\boldsymbol{x}$ must be 1. These values do not satisfy the middle row $(4 + 1 \neq 6)$, so this system of equations does not have asolution. Here we have rank$\boldsymbol{A}$ <rank$\begin{bmatrix} \boldsymbol{A} & \boldsymbol{b} \end{bmatrix}$.

c) We have that

$$\boldsymbol{b} = \sum_{j=1}^{P} \boldsymbol{a}_j \alpha_j$$

where $\boldsymbol{a}_j, j = 1, 2, \ldots, P$ are the columns of $\boldsymbol{A}$. This implies that

$$\boldsymbol{b} - \sum_{j=1}^{P} \boldsymbol{a}_j \alpha_j = \boldsymbol{0}$$

so $\boldsymbol{b}$ is linearly dependent with the columns of $\boldsymbol{A}$.

3) *Non Unique Solutions.*

a) Consider $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ where $\boldsymbol{A} = \begin{bmatrix} 1 & -2 \\ -1 & 2 \\ -2 & 4 \end{bmatrix}$, $\boldsymbol{b} = \begin{bmatrix} 2 \\ -2 \\ -4 \end{bmatrix}$ and $\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

i) Does this system of equations have a solution? Justify your answer.

ii) Is the solution unique? Justify your answer.

iii) Draw the solution(s) in the x_1-x_2 plane using x_1 as the horizontal axis.

b) If the system of linear equations $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ has more than one solution, then there is at least one non zero vector $\boldsymbol{w}$ for which $\boldsymbol{x} + \boldsymbol{w}$ is also a solution. That is, $\boldsymbol{A}(\boldsymbol{x} + \boldsymbol{w}) = \boldsymbol{b}$. Use the definition of linear independence to find a condition on $\text{rank}\{\boldsymbol{A}\}$ that determines whether there is more than one solution.

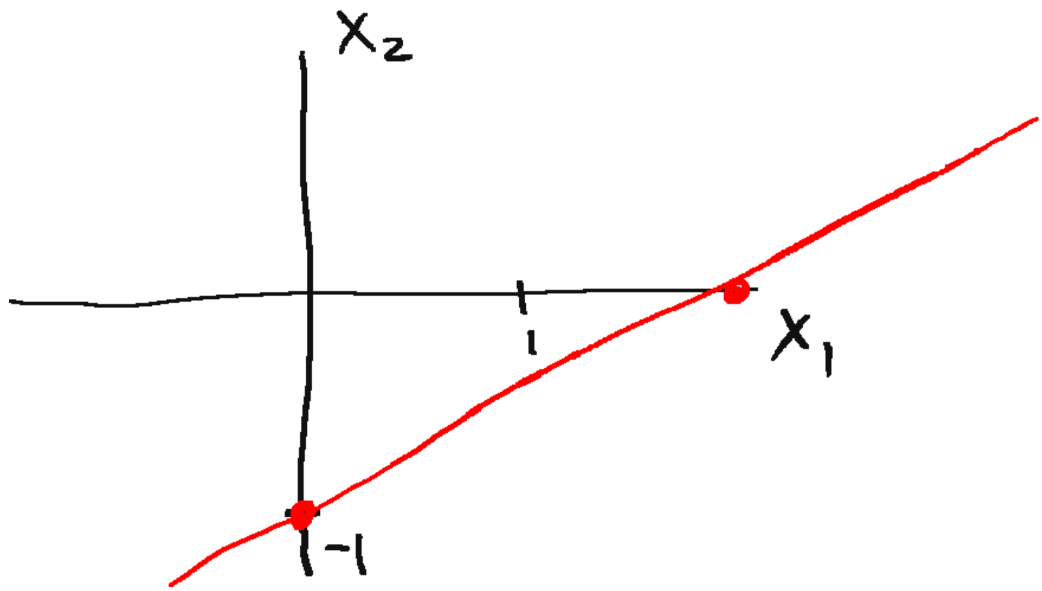
SOLUTION:

a) Here we use the results of the preceding problem and use the system of equations to specify the set of all possible solutions.

i) Note that $\text{rank}\boldsymbol{A} = \text{rank}\begin{bmatrix} \boldsymbol{A} & \boldsymbol{b} \end{bmatrix}$ so there is a solution. For example, $\boldsymbol{x} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$

ii) Note that $\boldsymbol{x} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$ is also a solution, so the solution is not unique.

iii) All solutions must satisfy $x_1 - 2x_2 = 2$ which is the equation for a line $x_2 = 0.5x_1 - 1$. This family is shown in the figure below.



b) If $\boldsymbol{x}$ and $\boldsymbol{x} + \boldsymbol{w}$ are solutions, then we must have $\boldsymbol{A}\boldsymbol{w} = \boldsymbol{0}$. This implies that a weighted sum of the columns of $\boldsymbol{A}$ is zero, that is, the columns of $\boldsymbol{A}$ are linearly dependent. Thus, if $\boldsymbol{A}$ is an N-by-P matrix, then there will be more than one solution if $\operatorname{rank}\boldsymbol{A} < P$.