Activity 23 Ayar Deep Hazra
ECE 532

1. a) $\hat{y} = x_1^2 w_1 + x_2^2 w_2 + \sqrt{2} x_1 x_2 w_3 + \sqrt{2} x_1 w_4 + \sqrt{2} x_2 w_5 + w_6$

$$\hat{y} = \begin{bmatrix} x_1^2 & x_2^2 & \sqrt{2}x_1x_2 & \sqrt{2}x_1 & \sqrt{2}x_2 & 1 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \\ w_5 \\ w_6 \end{bmatrix}$$

where w_i, $i \in \{1, 2, 3, 4, 5, 6\}$ correspond to the weights.

Thus $\phi^T(x) = \begin{bmatrix} x_1^2 & x_2^2 & \sqrt{2}x_1x_2 & \sqrt{2}x_1 & \sqrt{2}x_2 & 1 \end{bmatrix}$

$$\phi = \begin{bmatrix} x_1^2 \\ x_2^2 \\ \sqrt{2}x_1x_2 \\ \sqrt{2}x_1 \\ \sqrt{2}x_2 \\ 1 \end{bmatrix} \qquad w = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \\ w_5 \\ w_6 \end{bmatrix}$$

b) $\phi^T(x_i)\phi(x_j) = x_{i1}^2 x_{j1}^2 + x_{i2}^2 + x_{j1}^2$

$+ 2x_{i1} x_{i2} x_{j1} x_{j2} + 2x_{i1} x_{j1} +$

$2x_{i2} x_{j2} + 1$

$x_i^T x_j + 1 = x_{i1} + x_{j1} + x_{i2} x_{j1} + 1$

$(x_i^T x_j + 1)^2 = (x_{i1} x_{j1} + x_{i2} x_{j2} + 1)^2$

$= x_{i1}^2 x_{j1}^2 + x_{i2}^2 x_{j2}^2 + 2x_{i1} x_{j1} x_{i2} x_{j2}$

$+ 2x_{i1} x_{j1} + 2x_{i2} x_{j2} + 1$

Thus $(x_i^T x_j + 1)^2 = \phi^T(x_i)\,\phi(x_j)$

c) for $(x_i^T x_j + 1)^2$, we have to calculate

$\phi^T(x) = [x_1^2 \quad x_2^2 \quad \sqrt{2}x_1 x_2 \quad \sqrt{2}x_1 \quad \sqrt{2}x_2 \quad 1]$

Multiplication Multiplication Multiplication

3 Multiplications for $\phi^T(x)$ & $\phi(x)$ each.

The inner product $\phi^T(x)\,\phi(x)$ has 5 multiplications of its own (not counting 1)

Thus total Multiplications $= 3 + 3 + 5 = 11$

2. a) Given, $\hat{\omega} = (\phi^T\phi + \lambda I)^{-1}\phi^T y$

Given expression $\phi^T\phi\phi^T + \lambda\phi^T$

If we factor ϕ^T from the left, we get,

$$\phi^T(\phi\phi^T + \phi^{T^{-1}}\lambda\phi^T)$$

$$= \phi^T(\phi\phi^T + \lambda I)$$

If we factor ϕ^T from the right, we get,

$$= (\phi^T\phi + \lambda I)\phi^T$$

Since we started from the same expression, we get,

$$\phi^T(\phi\phi^T + \lambda I) = (\phi^T\phi + \lambda I)\phi^T$$

Multiplying $(\phi\phi^T + \lambda I)^{-1}$ on the right and $(\phi^T\phi + \lambda I)^{-1}$ on the left to get,

$$(\phi^T\phi + \lambda I)^{-1}\phi^T\cancel{(\phi\phi^T + \lambda I)(\phi\phi^T + \lambda I)^{-1}}$$

$$= \cancel{(\phi^T\phi + \lambda I)^{-1}(\phi^T\phi + \lambda I)}\phi^T(\phi\phi^T + \lambda I)^{-1}$$

we get

$$(\phi^T\phi + \lambda I)^{-1}\phi^T = \phi^T(\phi\phi^T + \lambda I)^{-1}$$

b) Since we proved the identity in (a),

we have,

$$\hat{w} = (\phi^T\phi + \lambda I)^{-1}\phi^T y$$

$$\hat{w} = \phi^T(\phi\phi^T + \lambda I)^{-1} y$$

c) $|K|_{i,j}$ given $K = \phi\phi^T$

If K is $N \times N$, then

$$|K|_{i,j} = \phi^T(x_i)\,\phi(x_j)$$

d) If $\phi(x)$ is defined as in Problem 1, then
$$[K]_{i,j} = (x_i^T x_j + 1)^2.$$

e) $\hat{y}(x) = \underbrace{\phi^T(x)}_{1\times p}\ \underbrace{\Phi^T}_{p\times N}\ \underbrace{(\Phi\Phi^T + \lambda I)^{-1}}_{n\times n}\ \underbrace{y}_{n\times 1}$

We have,

$$\alpha = (\Phi\Phi^T + \lambda I)^{-1} y$$

Now, $K(x, x_j) = (x^T x_j + 1)^2$

$$= \phi^T(x)\,\phi(x_j)$$

by definition.

Thus $\hat{y}(x) = \underbrace{\phi^T(x)}_{1\times p}\ \underbrace{\Phi^T}_{p\times N}\ \underbrace{\alpha}_{N\times N}$

$$\hat{y}(x) = \left[\phi^T(x)\phi(x_1)\quad \phi^T(x)\phi(x_2)\ \cdots\ \phi^T(x)\phi(x_n)\right].\alpha$$

$$= \sum_{j=1}^{N} \phi^T(x)\cdot\phi(x_j)\cdot\alpha_j$$

$$= \sum_{j=1}^{N} (x^T x_j + 1)^2 \alpha_j$$

3 a) $\phi(x) = x.$

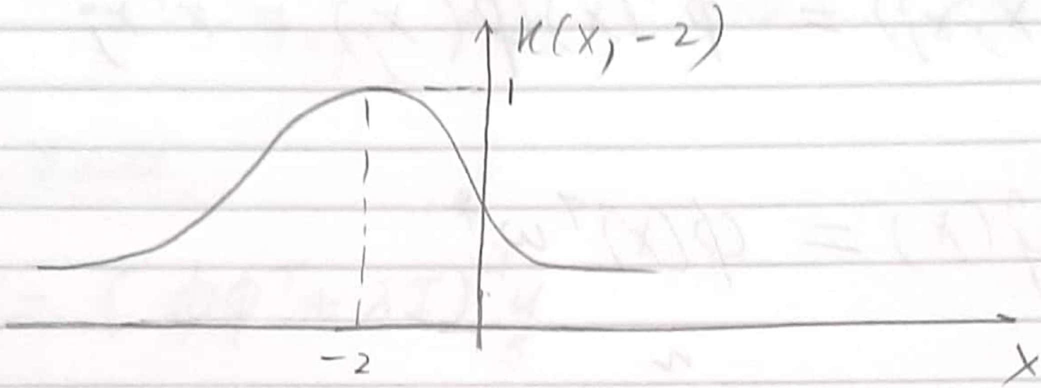
$\therefore\ k(x, x_j) = \phi^T(x)\phi(x_j) = x^T x_j$

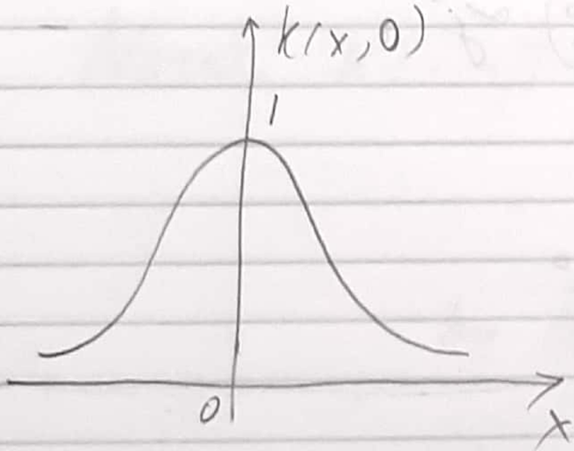
b) $\hat{y}(x) = \phi(x)^T w^*$

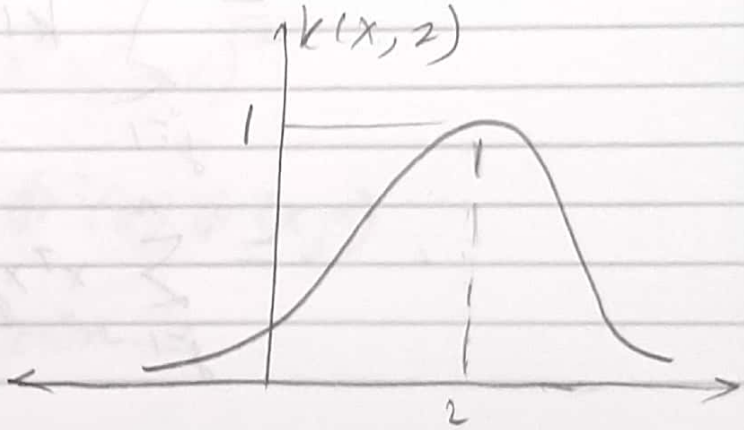
$$= \sum_{j=1}^{n} k(x, x_j)\, \alpha_j$$

$$= \sum_{j=1}^{n} x^T x_j\, \alpha_j$$

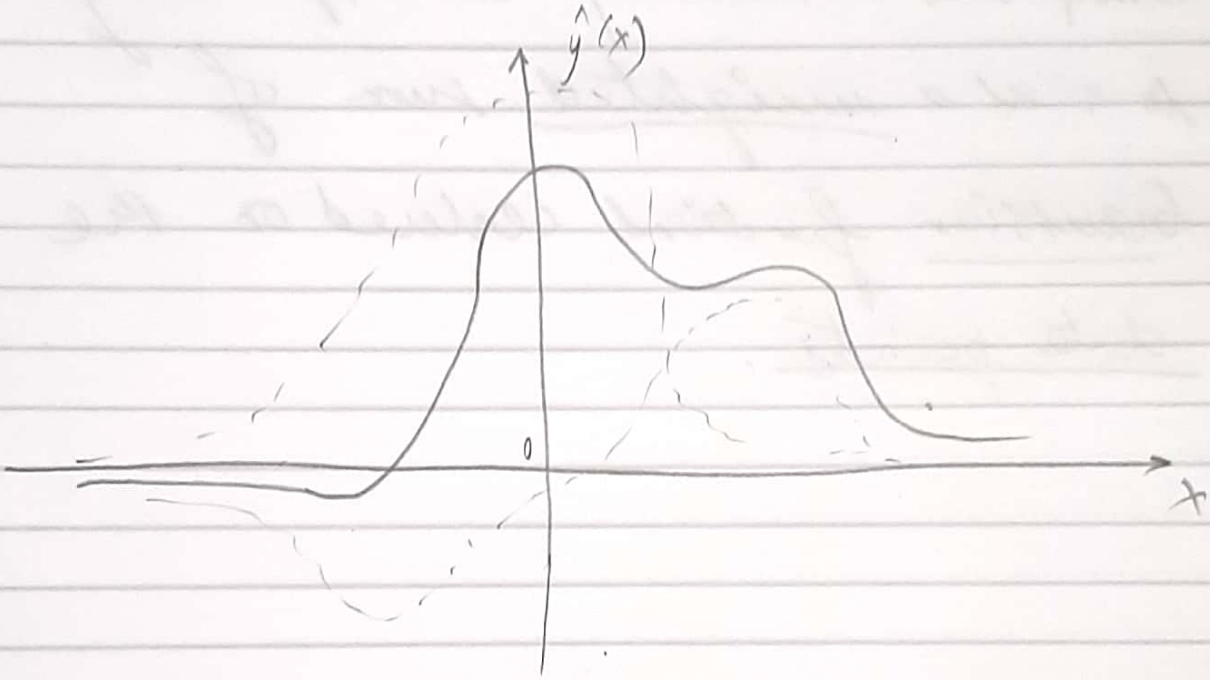
4, a) $k(x, x_j) = \exp\left\{ \dfrac{-\|x - x_j\|_2^2}{2\sigma} \right\}$







b) $\hat{y}(x) = -k(x, x_1) + 2k(x, x_2)$
$+ 8k(x, x_3)$



c) The expression $\hat{y}(x) = \sum_{j=1}^{n} K(x, x_j) \alpha_j$ interpolates a value y corresponding to x as a weighted sum of Gaussian functions centered on the data points.