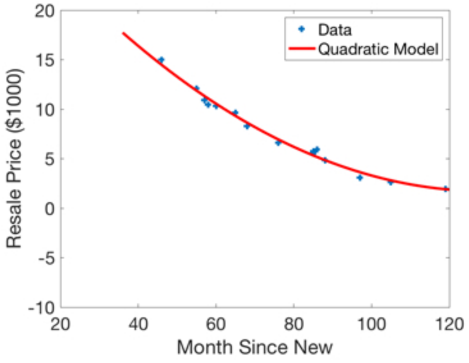
Linear Independence and Rank in Learning

Objectives

- Review the role of systems of linear equations in machine learning
- Define linear independence
- Define the rank of a matrix

Learning classifiers and data models requires solving systems of linear equations



$$\begin{bmatrix}\hat{p}_1\\ \hat{p}_2\\ \vdots \\ \hat{p}_{20}\end{bmatrix} = \begin{bmatrix}1 & t_1 & t_1^2\\ 1 & t_2 & t_2^2 \\ & \vdots & \\ 1 & t_{20} & t_{20}^2\end{bmatrix}\begin{bmatrix}w_1\\ w_2\\ w_3\end{bmatrix}$$

Model fitting

$$\underset{\sim}{A}\,\underline{w} = \underline{d}$$

(NxM)(Mx1) (Nx1)

$$\begin{bmatrix}l_1\\ l_2\\ \vdots\\ l_N\end{bmatrix} = \begin{bmatrix} \underline{x}_1^T \\ \underline{x}_2^T\\ \vdots \\ \underline{x}_N^T\end{bmatrix}\underline{w}$$

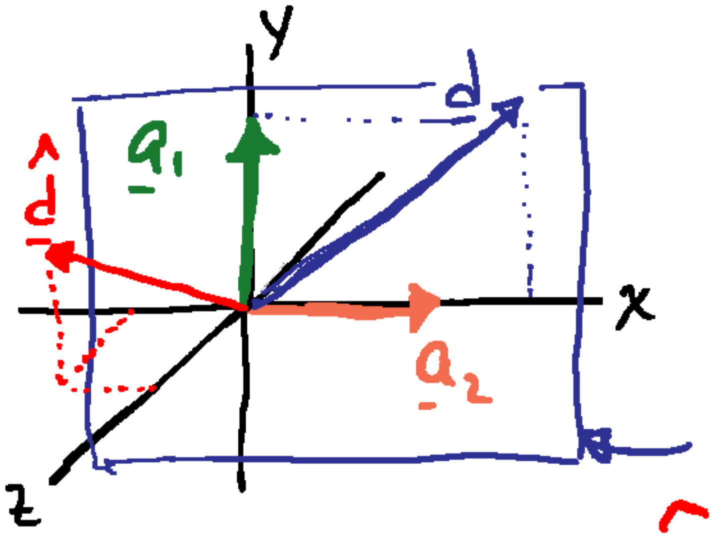
Classifier design

Important: Can we solve $\underline{A}\,\underline{w} = \underline{d}$?

Let $\underline{A} = [\underline{a}_1\ \underline{a}_2 \cdots \underline{a}_M]$ so $\underline{A}\,\underline{w} = \underline{d} \Rightarrow \underline{d} = \underbrace{\sum_{i=1}^{M} \underline{a}_i w_i}_{\text{weighted sum of } \underline{a}_i}$

$N \times 1$

Example: $\underline{A} = \begin{array}{cc} \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix} \\ \underline{a}_1 \quad \underline{a}_2 \end{array}$



$\underline{d} = \underline{a}_1 w_1 + \underline{a}_2 w_2$

set of all possible $\underline{d}$

$\hat{\underline{d}} \neq \underline{a}_1 w_1 + \underline{a}_2 w_2$

For a solution $\underline{d} - \sum_{i=1}^{M} \underline{a}_i w_i = \underline{0}$

Linear independence: A set of M vectors $\underline{v}_1, \underline{v}_2, \cdots, \underline{v}_M \in \mathbb{R}^N$ is linearly independent iff $\sum_{i=1}^{M} \underline{v}_i \alpha_i = 0 \Leftrightarrow \alpha_i = 0, i = 1, 2, \cdots, M$

otherwise "linearly dependent"

Rank of a matrix: number of linearly independent columns (or rows)

Note: row rank = column rank

Examples:

$$\underline{A} = \begin{array}{ccc} \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{bmatrix} \\ \underline{a}_1 \quad \underline{a}_2 \quad \underline{a}_3 \end{array}$$

$\underline{a}_3 = \underline{a}_2 - \underline{a}_1$

so $\alpha_1 \underline{a}_1 + \alpha_2 \underline{a}_2 + \alpha_3 \underline{a}_3 = \underline{0}$

for $\alpha_1 = 1, \alpha_2 = -1, \alpha_3 = 1 \neq 0$

$\underline{a}_1, \underline{a}_2, \underline{a}_3$ are linearly dependent!

$\alpha_1 \underline{a}_1 + \alpha_2 \underline{a}_2 = 0$, $\Updownarrow$ $\alpha_1 = \alpha_2 = 0$

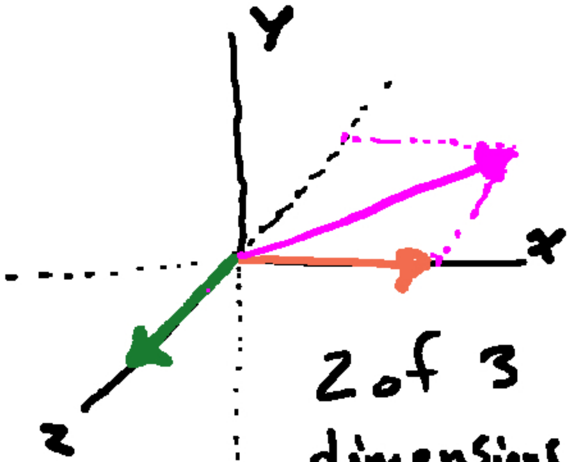
$\underline{a}_1, \underline{a}_2$ lin. indep.

$\alpha_1 \underline{a}_1 + \alpha_3 \underline{a}_3 = 0$ $\Updownarrow$ $\alpha_1 = \alpha_3 = 0$

$\underline{a}_1, \underline{a}_3$ lin indep

same for $\underline{a}_2, \underline{a}_3$

$\text{rank}(\underline{A}) = 2$



2 of 3 dimensions

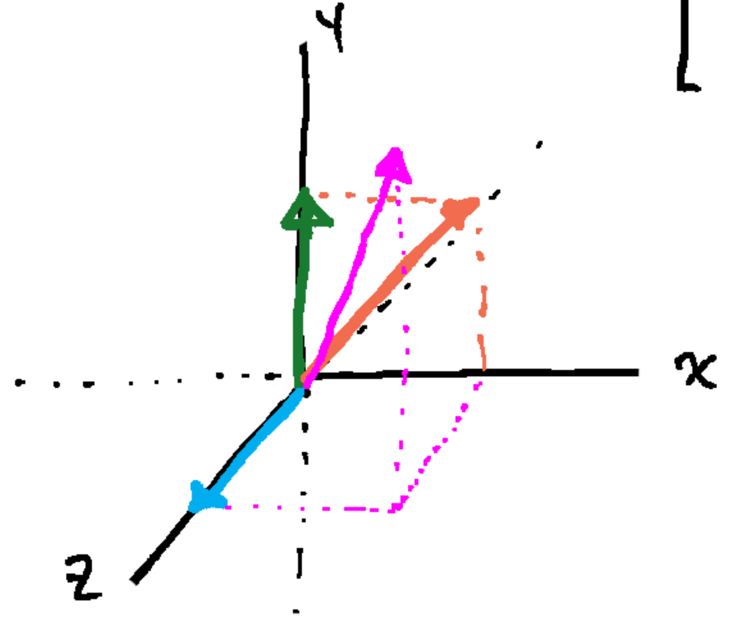
Example: $A = \begin{array}{cccc} & & & \\ \end{array}$

$$A = \left[\begin{array}{cccc} 0 & 1 & 1 & 0 \\ 1 & 1 & 2 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right]$$
$$\begin{array}{cccc} \underline{a}_1 & \underline{a}_2 & \underline{a}_3 & \underline{a}_4 \end{array}$$

$$\sum_{i=1}^{4} \alpha_i \underline{a}_i = \underline{0} \not\Leftrightarrow \alpha_i = 0$$

$$\underline{a}_1 + \underline{a}_2 - \underline{a}_3 + \underline{a}_4 = \underline{0}$$

$\underline{a}_1, \underline{a}_2, \underline{a}_3, \underline{a}_4$ lin. dep.



$$\alpha_1 \underline{a}_1 + \alpha_2 \underline{a}_2 + \alpha_3 \underline{a}_3 = 0 \Leftrightarrow \alpha_i = 0$$

any combination of 3 are lin. indep.

$\mathrm{Rank}(\underline{A}) = 3$

any 3 vectors describe all 3 dimensions

Back to $\underline{A}\underline{w} = \underline{d}$ solution:

$\underline{A}\underline{w} - \underline{d} = \underline{0}$ or

$$\sum_{i=1}^{m} \underline{a}_i w_i + (-1)\,\underline{d} = \underline{0} \Rightarrow \underline{a}_1, \dots, \underline{a}_m, \underline{d} \text{ are lin. dep.}$$

Summary

$$\underline{A}\,\underline{w} = \underline{d} \qquad \underline{A} = [\underline{a}_1\ \underline{a}_2 \cdots \underline{a}_M]$$

1) If $\underline{d}$ is a linear combination of $\underline{a}_1, \underline{a}_2, \cdots \underline{a}_M$ there is a solution

$$\text{rank}(\underline{A}) = \text{rank}([\underline{A} \,\vdots\, \underline{d}])$$

2) If $\underline{a}_1, \underline{a}_2, \cdots \underline{a}_M, \underline{d}$ are linearly independent, there is no solution

$$\text{rank}(\underline{A}) < \text{rank}([\underline{A} \,\vdots\, \underline{d}])$$

3) Unique solution?

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