Operators & Postulates

Group Theory is a branch of mathematics and abstract algebra that defines an algebraic structure named as **group**. Generally, a group comprises of a set of elements and an operation over any two elements on that set to form a third element also in that set.

In 1854, Arthur Cayley, the British Mathematician, gave the modern definition of group for the first time −

> *“A set of symbols all of them different, and such that the product of any two of them (no matter in what order), or the product of any one of them into itself, belongs to the set, is said to be a group. These symbols are not in general convertible [commutative], but are associative.”*

In this chapter, we will know about **operators and postulates** that form the basics of set theory, group theory and Boolean algebra.

Any set of elements in a mathematical system may be defined with a set of operators and a number of postulates.

A **binary operator** defined on a set of elements is a rule that assigns to each pair of elements a unique element from that set. For example, given the set $A = \{1, 2, 3, 4, 5\}$, we can say $\otimes$ is a binary operator for the operation $c = a \otimes b$, if it specifies a rule for finding c for the pair of (a, b) , such that $a, b, c \in A$.

The **postulates** of a mathematical system form the basic assumptions from which rules can be deduced. The postulates are −

Closure

A set is closed with respect to a binary operator if for every pair of elements in the set, the operator finds a unique element from that set.

Example

Let $A = \{0, 1, 2, 3, 4, 5, \dots\}$

This set is closed under binary operator into $(*)$, because for the operation $c = a * b$, for any $a, b \in A$, the product $c \in A$.

The set is not closed under binary operator divide $(\div)$, because, for the operation $c = a \div b$, for any $a, b \in A$, the product c may not be in the set A. If $a = 7, b = 2$, then $c = 3.5$. Here $a, b \in A$ but $c \notin A$.

Associative Laws

A binary operator $\otimes$ on a set A is associative when it holds the following property −

$$(x \otimes y) \otimes z = x \otimes (y \otimes z) \text{ , where } x, y, z \in A$$

Example

Let $A = \{1, 2, 3, 4\}$

The operator plus $(+)$ is associative because for any three elements, $x, y, z \in A$, the property $(x + y) + z = x + (y + z)$ holds.

The operator minus $(-)$ is not associative since

$$(x - y) - z \neq x - (y - z)$$

Commutative Laws

A binary operator $\otimes$ on a set A is commutative when it holds the following property −

$$x \otimes y = y \otimes x \text{ , where } x, y \in A$$

Example

Let $A = \{1, 2, 3, 4\}$

The operator plus $(+)$ is commutative because for any two elements, $x, y \in A$, the property $x + y = y + x$ holds.

The operator minus $(-)$ is not associative since

$$x - y \neq y - x$$

Distributive Laws

Two binary operators $\otimes$ and $\circledast$ on a set A, are distributive over operator $\circledast$ when the following property holds −

$$x \otimes (y \circledast z) = (x \otimes y) \circledast (x \otimes z) \text{ , where } x, y, z \in A$$

Example

Let $A = \{1, 2, 3, 4\}$

The operators into $(*)$ and plus $(+)$ are distributive over operator + because for any three elements, $x, y, z \in A$, the property $x * (y + z) = (x * y) + (x * z)$ holds.

However, these operators are not distributive over $*$ since

$$x + (y * z) \neq (x + y) * (x + z)$$

Identity Element

A set A has an identity element with respect to a binary operation $\otimes$ on A, if there exists an element $e \in A$, such that the following property holds −

$$e \otimes x = x \otimes e \text{ , where } x \in A$$

Example

Let $Z = \{0, 1, 2, 3, 4, 5, \ldots\}$

The element 1 is an identity element with respect to operation $*$ since for any element $x \in Z$,

$$1 * x = x * 1$$

On the other hand, there is no identity element for the operation minus $(-)$

Inverse

If a set A has an identity element e with respect to a binary operator $\otimes$, it is said to have an inverse whenever for every element $x \in A$, there exists another element $y \in A$, such that the following property holds −

$$x \otimes y = e$$

Example

Let $A = \{\cdots - 4, -3, -2, -1, 0, 1, 2, 3, 4, 5, \ldots\}$

Given the operation plus $(+)$ and $e = 0$, the inverse of any element x is $(-x)$ since $x + (x) = 0$

De Morgan's Law

De Morgan’s Laws gives a pair of transformations between union and intersection of two (or more) sets in terms of their complements. The laws are −

$$(A \cup B)' = A' \cap B'$$

$$(A \cap B)' = A' \cup B'$$

Example

Let $A = \{1, 2, 3, 4\}, B = \{1, 3, 5, 7\}$, and

Universal set $U = \{1, 2, 3, \ldots, 9, 10\}$

$$A' = \{5, 6, 7, 8, 9, 10\}$$

$$B' = \{2, 4, 6, 8, 9, 10\}$$

$$A \cup B = \{1, 2, 3, 4, 5, 7\}$$

$$A \cap B = \{1, 3\}$$

$$(A \cup B)' = \{6, 8, 9, 10\}$$

$$A' \cap B' = \{6, 8, 9, 10\}$$

Thus, we see that $(A \cup B)' = A' \cap B'$

$$(A \cap B)' = \{2, 4, 5, 6, 7, 8, 9, 10\}$$

$$A' \cup B' = \{2, 4, 5, 6, 7, 8, 9, 10\}$$

Thus, we see that $(A \cap B)' = A' \cup B'$