An one dimensional model for naturally curved composite ribbons

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CONTENT

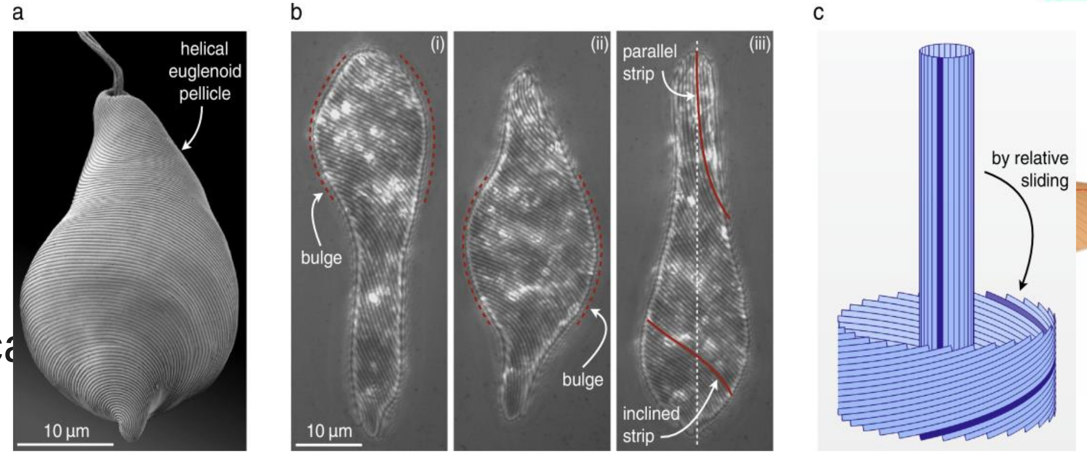
- Elastic ribbons
- Applications
- Motivation
- Ribbon model
- Benchmark problems
- Composite lattices
- Conclusions

ELASTIC RIBBONS

- Ribbons are thin elastic objects whose thickness t, width a and length ℓ are all very different, $t \ll a \ll \ell$

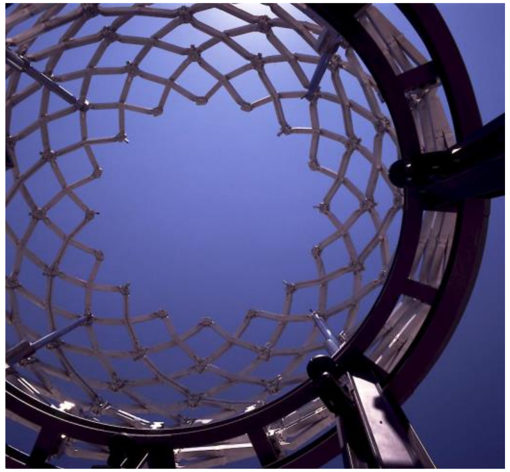
- Applications in designing lightweight structures that ca respond to actuation

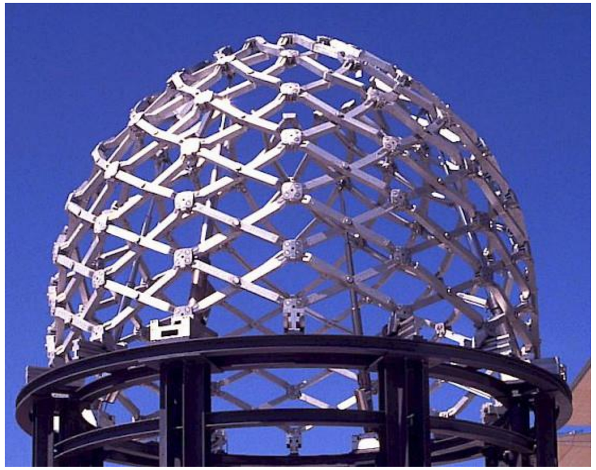
- Their flexibility and their thin geometry help turn the relatively small strain produced by actuation into a large-amplitude motion



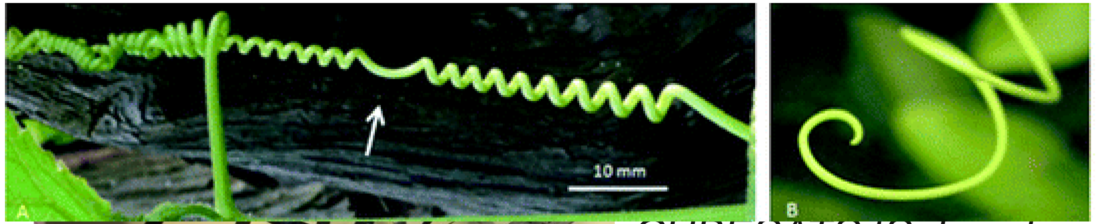
The pellicle of euglenids provides excellent example of the ingenious way nature utilises helices in a hierarchical manner for adaptive structural design. [Noselli, De Simone, JMPS 2018]

- Sadowsky ribbon model -> Inextensible
- Wunderlich ribbon model -> Extensible

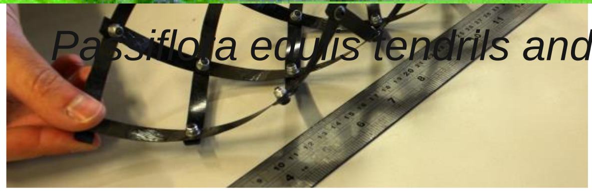




Adaptive architecture [Hobbermann 2012]



Passiflora edulis tendrils and cellulosic fiber helices

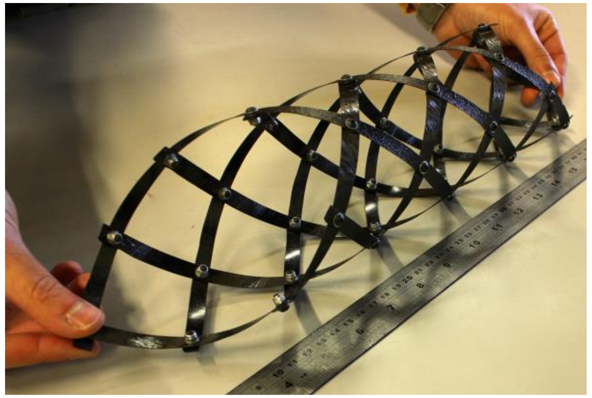


CUBESATS [Sakovsky 2023]

MULTISTABLE LATTICES

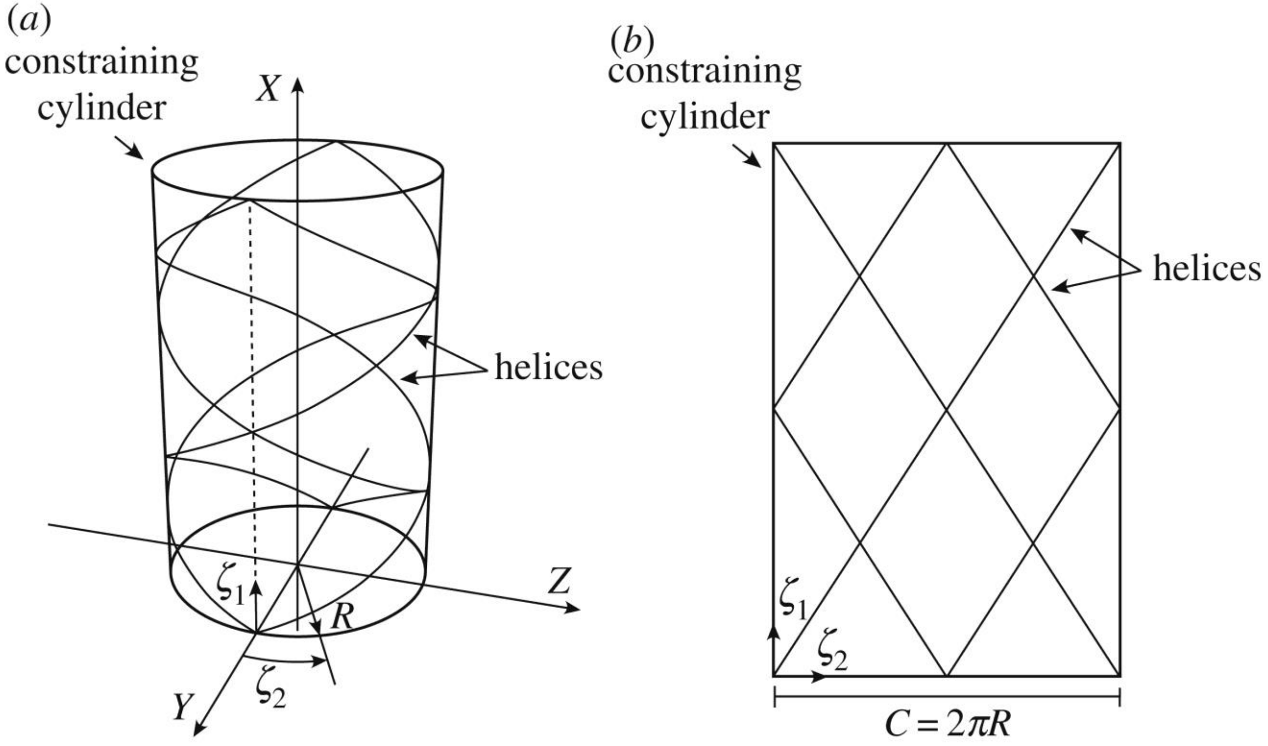
Stable

Stable



Unstable

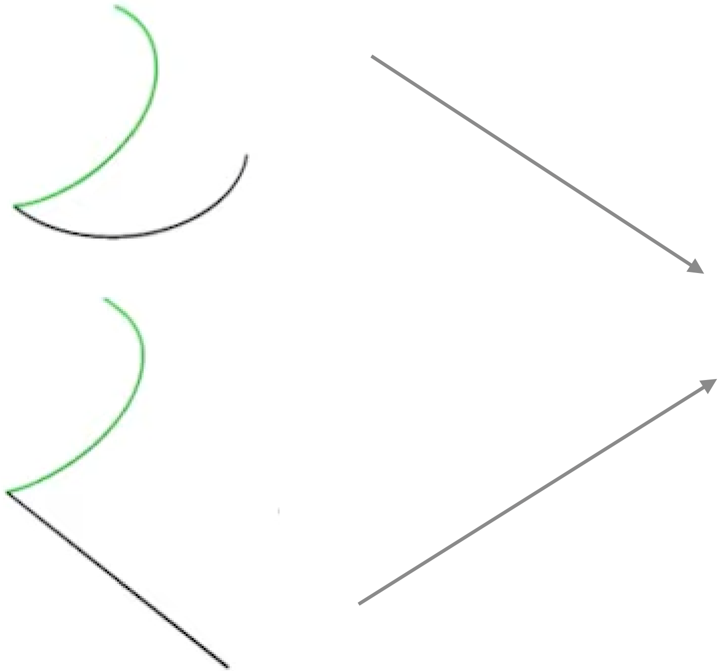
Pirrera et al. 2013



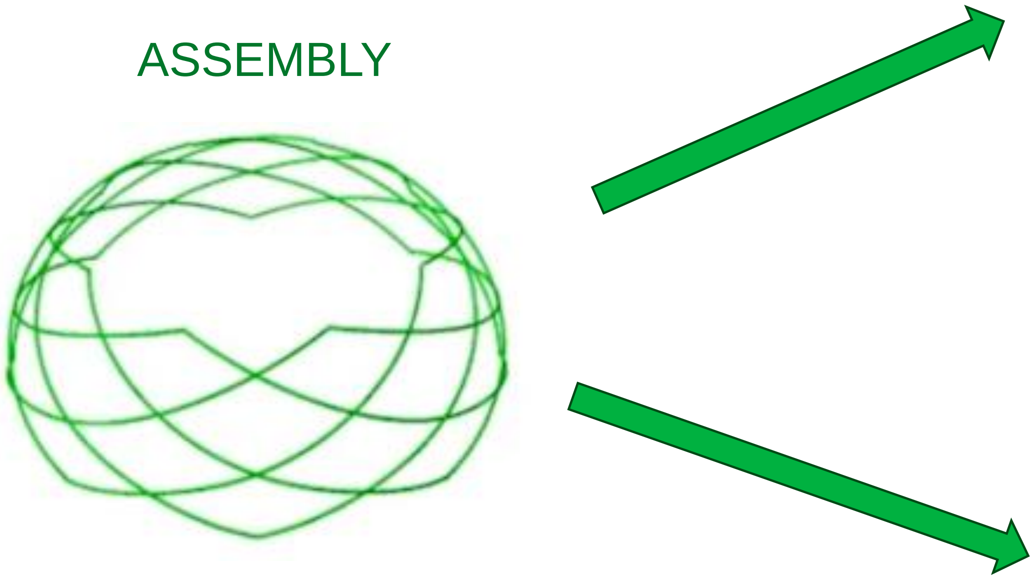
Behaviour of lattice structures from non-cylindrical geomet

GENERAL MODEL

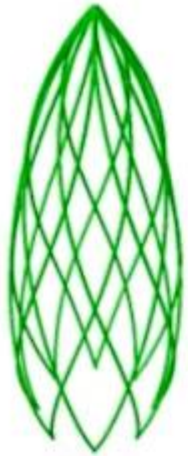
INDIVIDUAL RIBBON ELEMENTS



ASSEMBLY



OPERATION



Stable shape-1

Stable shape-2

- **Developing a general model** for lattice with arbitrary shapes allowing large deformation extendable to anisotropic materials
- Linking lattices changes the system's energy in a manner that resembles a **series expansion**, allowing us to approximate any continuous energy to desired accuracy
- Alternative geometry -> **Higher design space** and **tailored response**

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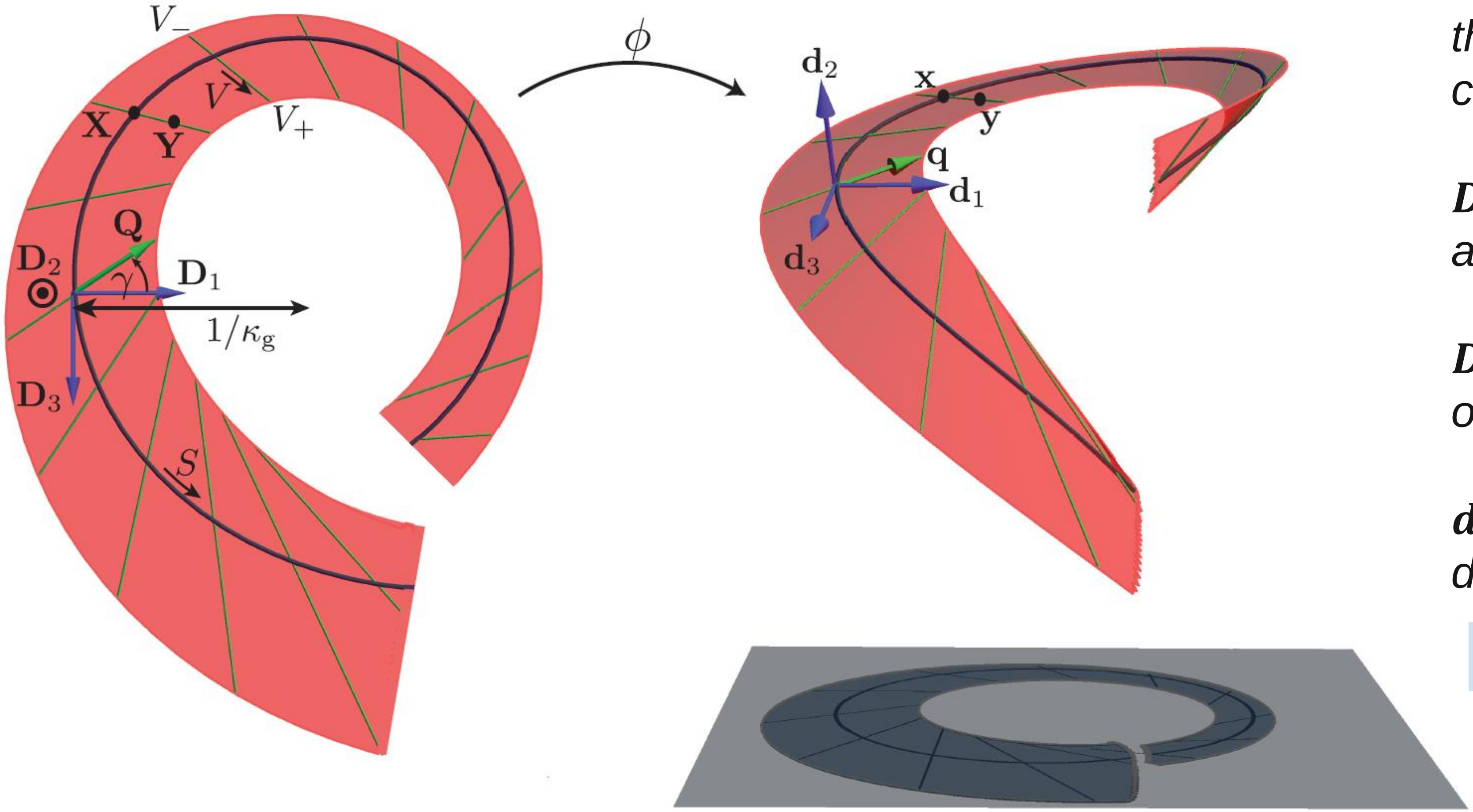
RIBBON THEORY

Undeformed Configuration *Deformed Configuration*



Considering ribbon is inextensible

S: Arc length measured along the centerline of undeformed conf

$\boldsymbol{D_3}$ = $\boldsymbol{X}$*'(S) is the unit tangent along the centerline*

$\boldsymbol{D_1}$, $\boldsymbol{D_2}$, $\boldsymbol{D_3}$ *form an orthonormal frame*

$\boldsymbol{d_3}$ = $\boldsymbol{x}$*'(s) is the tangent of the deformed configuration*

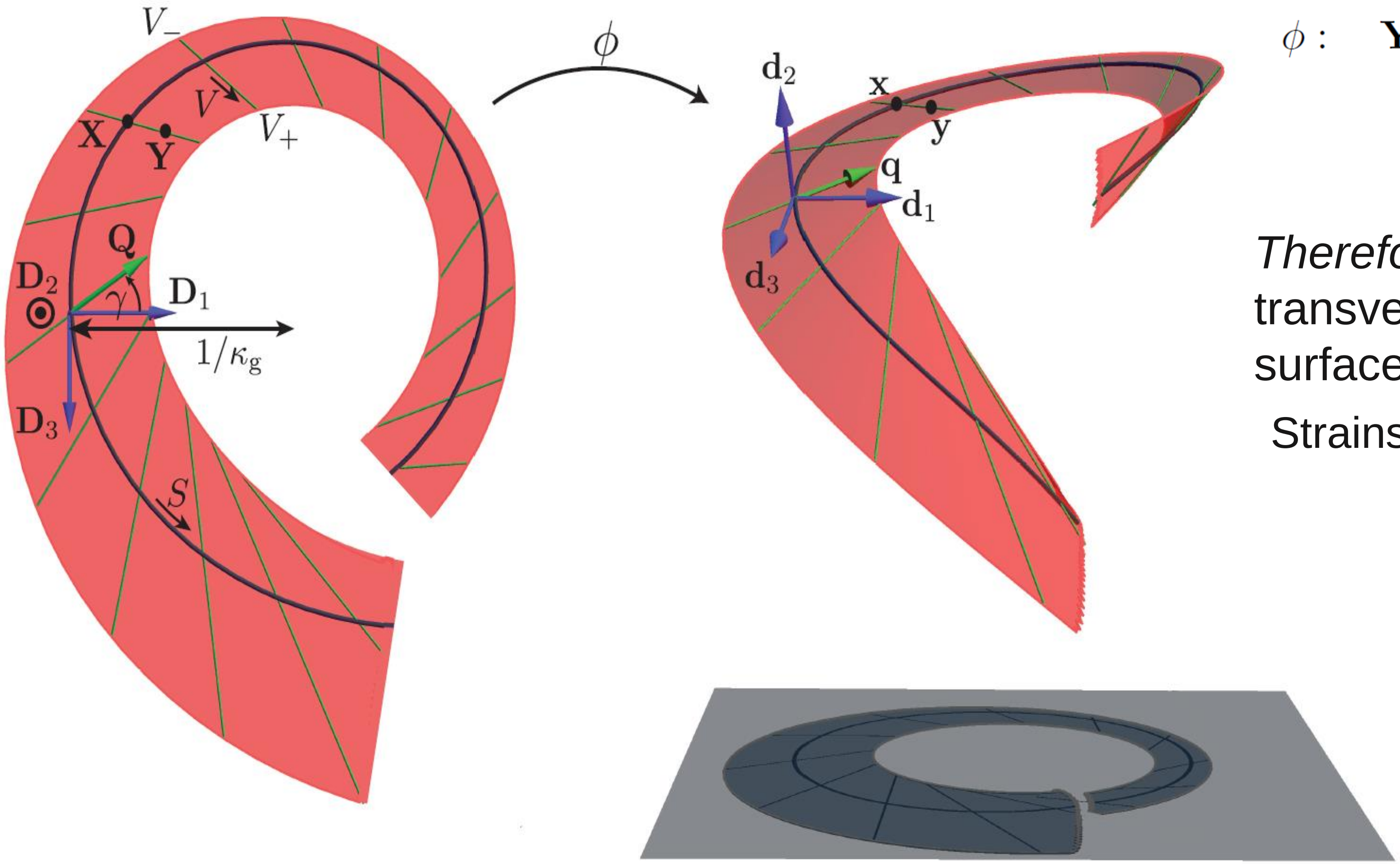
$$\boldsymbol{q}(\eta, S) = \eta(S)\boldsymbol{d_3}(S) + \boldsymbol{d_1}(S)$$

RIBBON THEORY

Transformation from reference to deformed configuration is expressed by the mapping ϕ:

Undeformed Configuration

Deformed Configuration



$$\phi : \quad \mathbf{Y} = \mathbf{X}(S) + V\,\mathbf{Q}(\eta, S) \quad \mapsto \quad \mathbf{y} = \mathbf{x}(S) + V\,\mathbf{q}(\eta, S).$$

V is a coordinate along the generatrix

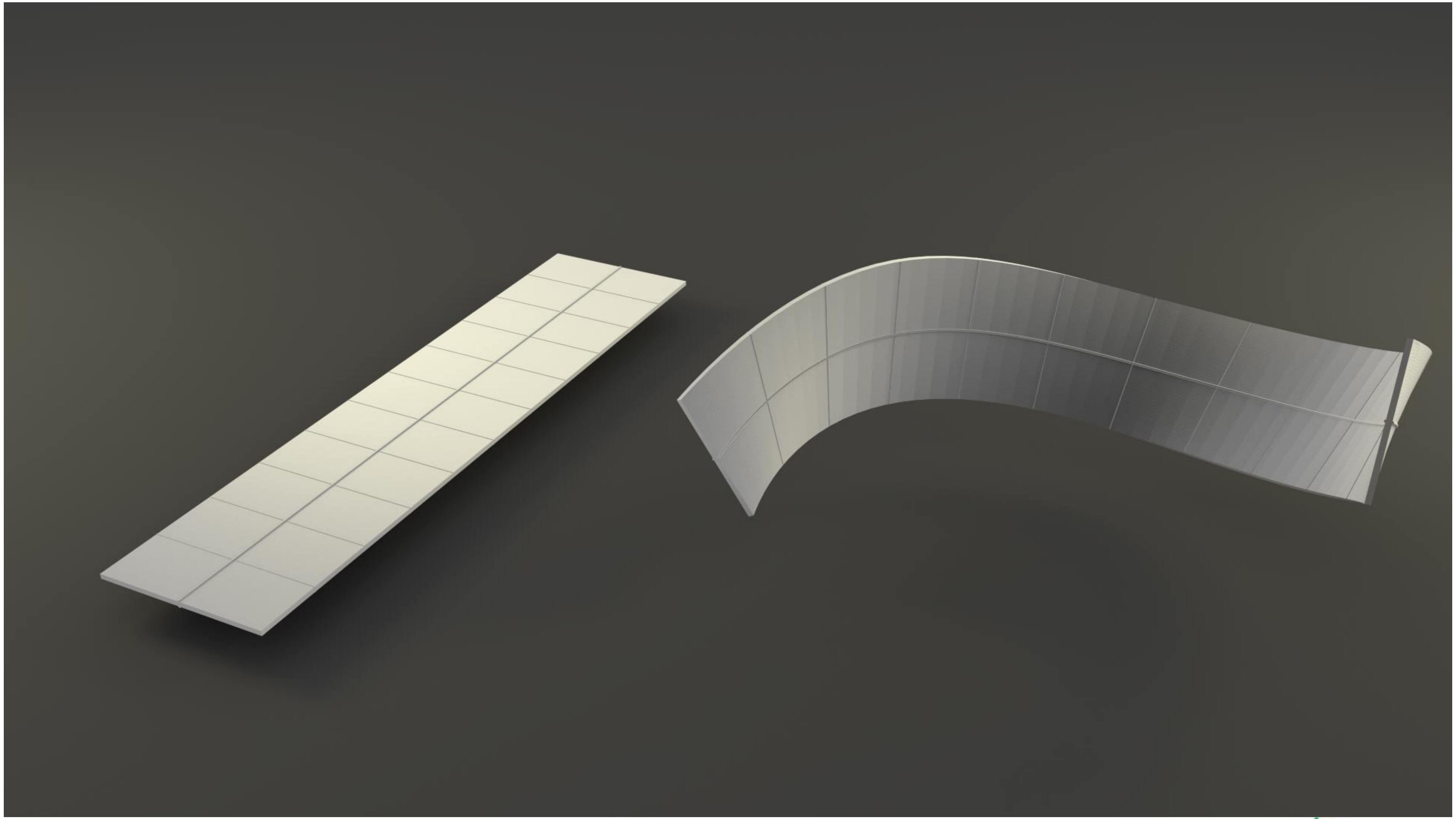
Therefore, $\boldsymbol{S}$ *and* $\boldsymbol{V}$ are the longitudinal and transverse coordinate to parameterize ribbon surface.

Strains can be calculated from the Darboux vector:

$$\mathbf{d}_i'(S) = \boldsymbol{\omega}(S) \times \mathbf{d}_i(S)$$

$\omega_1(S), \omega_2(S), \omega_3(S)$: are the normal, geodesic and the twisting curvatures respectively

DEVELOPABLE RECTANGULAR RIBBONS



$$V_{\pm}(\eta, S) = \pm\frac{w}{2} \quad \text{(rectangular ribbon)} \qquad \eta(S) = \tan\gamma = \tan\left(\mathbf{d}_1 \measuredangle \mathbf{q}\right)$$

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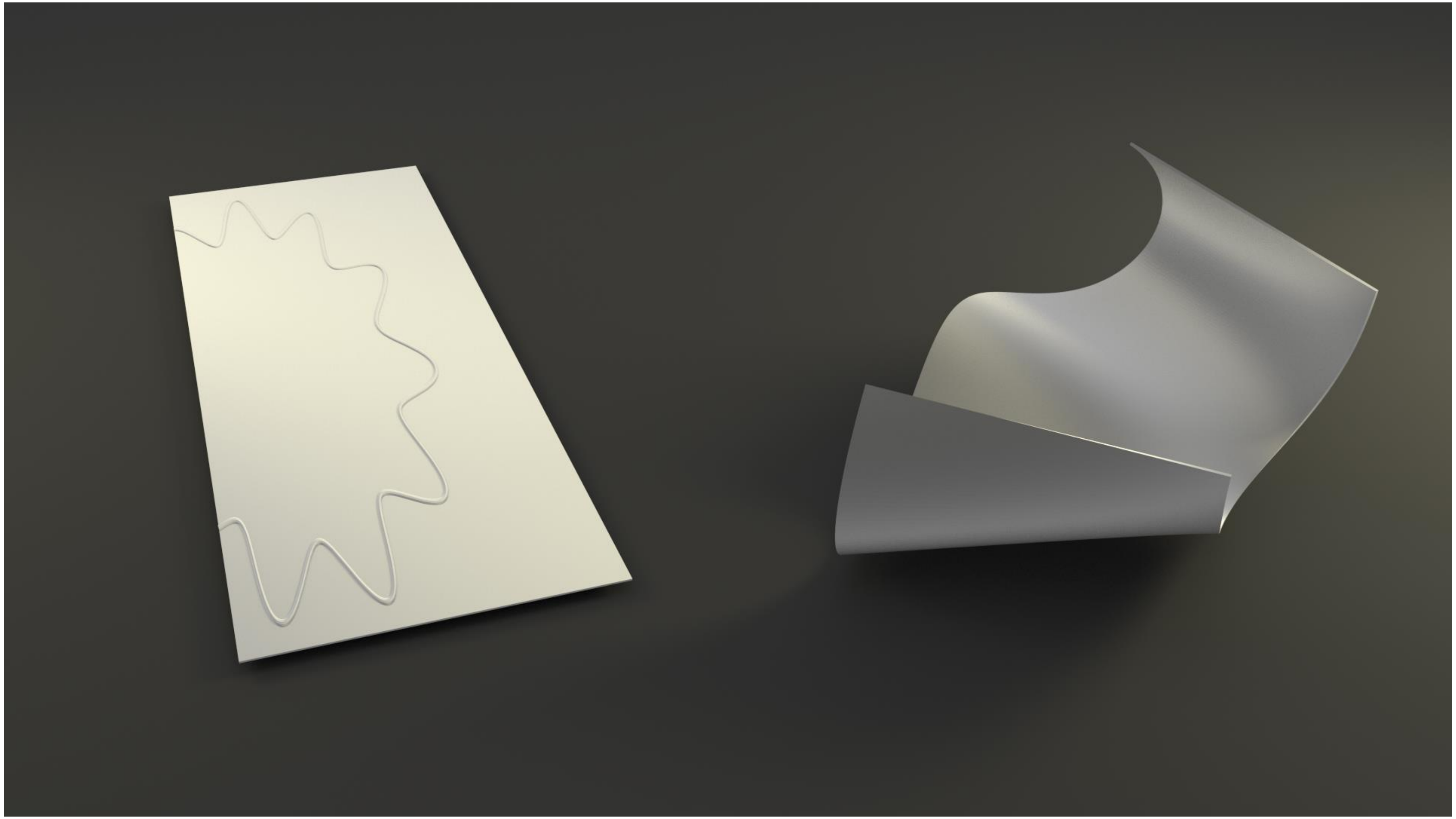
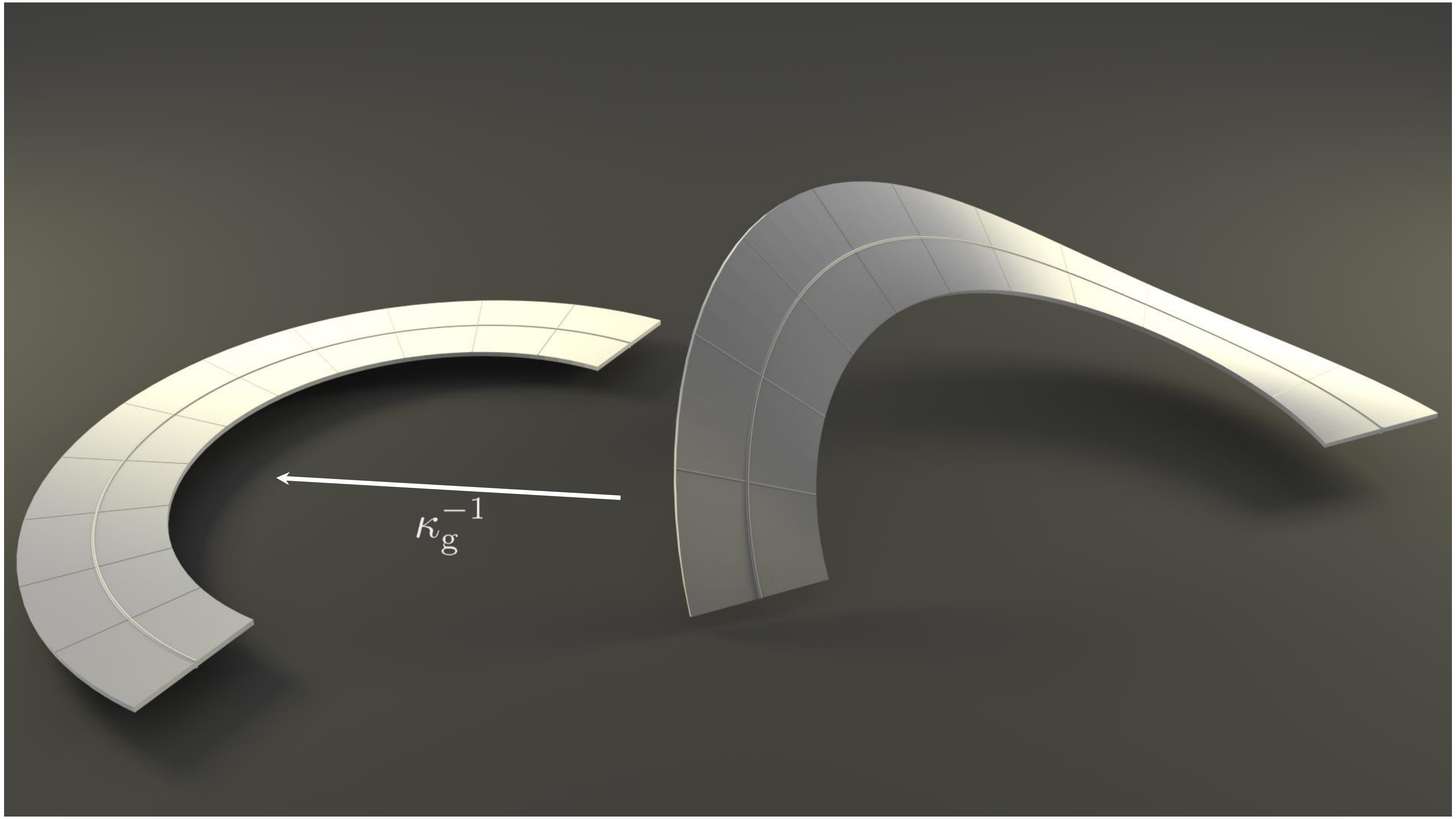
RECTANGULAR RIBBONS WITH ARBITARY CENTERLINE



$$V_{\pm}\left(\eta, S\right) = \ ?$$

DEVELOPABLE CURVED RIBBONS

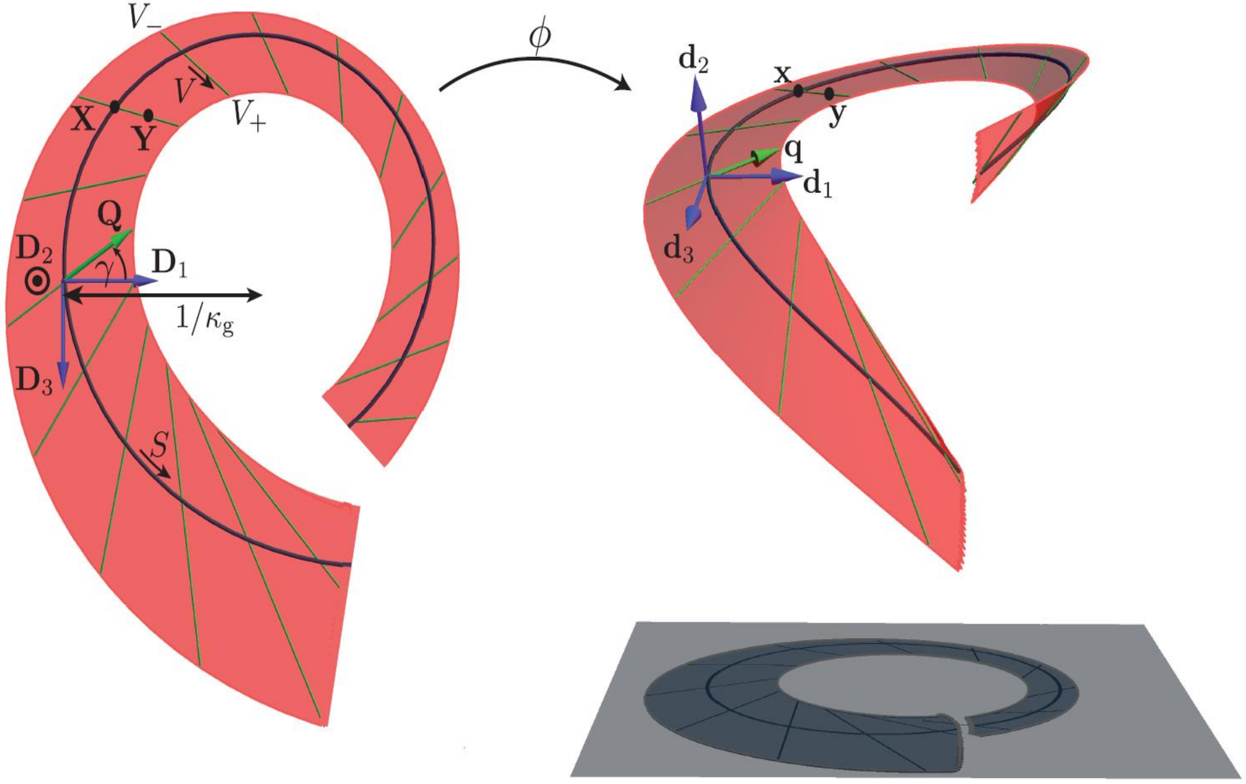


$$V_{\pm}(\eta, S) = \frac{1}{\kappa_g}\frac{1-\sqrt{1\mp(1+\eta^2)w\kappa_g(1\mp\frac{w\kappa_g}{4})}}{1+\eta^2} \quad \text{(annular ribbon)}$$

$$\eta(S) = \tan\gamma = \tan\left(\mathbf{d}_1 \measuredangle \mathbf{q}\right)$$

RIBBON THEORY

Undeformed Configuration *Deformed Configuration*



Rectangular Ribbon

$$V_{\pm}(\eta, S) = \pm\frac{w}{2}$$

Annular Ribbon

$$V_{\pm}(\eta, S) = \frac{1}{\kappa_g}\,\frac{1 - \sqrt{1 \mp (1+\eta^2) w\kappa_g (1 \mp \frac{w\kappa_g}{4})}}{1+\eta^2}$$

General Case: with geodesic curvature κ_g

$$V_c(\eta, \eta', S) = \frac{1}{(1+\eta^2)\kappa_g(S) - \eta'}$$

$$\left|\frac{V}{V_c}\right| < 1 \qquad \frac{1 - V_+/V_c}{1 - V_-/V_c} > 0$$

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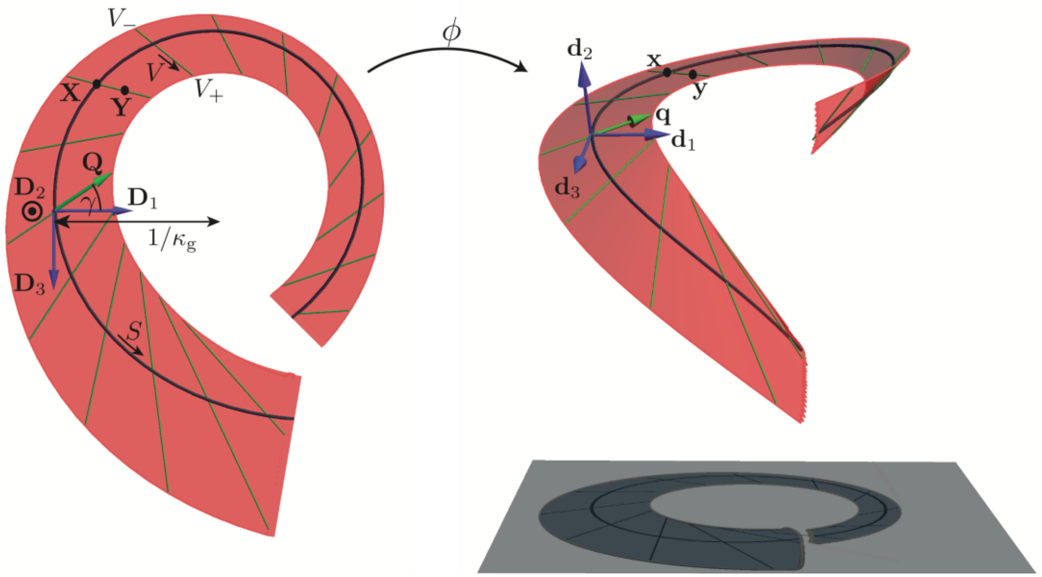
ENERGY TERM: ISOTROPIC CASE



$$\mathbf{K}(\eta, \eta', \omega_1, S, V) = \frac{\mathbf{K}_0(\eta, \omega_1)}{1 - \frac{V}{V_c(\eta, \eta', S)}}$$

$$\mathbf{K}_0(\eta, \omega_1) = -\omega_1 \begin{pmatrix} 1 & -\eta \\ -\eta & \eta^2 \end{pmatrix}_{(\mathbf{d_3}, \mathbf{d_1})}$$

$$E = \frac{D}{2} \iint \left[(1-\nu)\,\mathrm{tr}\big((\mathbf{K} - \mathbf{K}_r)^2\big) + \nu\,\mathrm{tr}^2(\mathbf{K} - \mathbf{K}_r) - (1-\nu)\,\mathrm{tr}\big(\mathbf{K}_r^2\big) - \nu\,\mathrm{tr}^2\,\mathbf{K}_r \right] \mathrm{d}a$$

$$\mathrm{d}a = |\partial_S \mathbf{y} \times \partial_V \mathbf{y}|\,\mathrm{d}S\,\mathrm{d}V = \left(1 - \frac{V}{V_c(\eta, S)}\right) \mathrm{d}S\,\mathrm{d}V$$

$$\boxed{E(\eta, \eta', \omega_1) = \frac{D}{2} \int_0^L \left[\left(-V_c \ln \frac{1 - V_+/V_c}{1 - V_-/V_c} \right) \mathrm{tr}\big(\mathbf{K}_0{}^2\big) - 2(V_+ - V_-)\mathbf{Q}_r : \mathbf{K}_0 \right] \mathrm{d}S}$$

Natural curvature: $\mathbf{Q}_r \equiv \mathbf{K}_r + \nu \operatorname{cof} \mathbf{K}_r$

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COMPOSITE RIBBONS

Considering inextensibility

$$U = \frac{1}{2}\boldsymbol{\kappa}^T \boldsymbol{D} \boldsymbol{\kappa}$$

$$D = D_{11} \begin{bmatrix} 1 & \nu & \psi \\ \nu & \beta & \phi \\ \psi & \phi & \rho \end{bmatrix}$$

Anisotropy with fully populated bending stiffness

$$U = \int\int \frac{1}{2} D_{11} \left(\kappa_{11}^2 + \beta\kappa_{22}^2 + 4\kappa_{12}\left(\kappa_{12}\rho + \kappa_{22}\phi\right) + 2\kappa_{11}\left(\kappa_{22}\nu + 2\kappa_{12}\psi\right)\right) \mathrm{d}a$$

Curvature tensor with developability constraint

$$\mathbf{K}(\eta, \eta', \omega_1, S, V) = \frac{\mathbf{K}_0(\eta, \omega_1)}{1 - \frac{V}{V_c(\eta, \eta', S)}}$$

$$\mathbf{K}_0(\eta, \omega_1) = -\omega_1 \left(\mathbf{d}_3 \otimes \mathbf{d}_3 - \eta\left(\mathbf{d}_3 \otimes \mathbf{d}_1 + \mathbf{d}_1 \otimes \mathbf{d}_3\right) + \eta^2 \mathbf{d}_1 \otimes \mathbf{d}_1\right)$$

$$U = \int\int \frac{1}{2} D_{11} \frac{\omega_1^2}{\left(1 - \frac{V}{V_c}\right)^2} \left(1 + \beta\eta^4 + 2\eta^2\left(\nu + 2\rho\right) - 4\eta^3\phi - 4\eta\psi\right) \mathrm{d}a$$

$$V_c(\eta, \eta', S) = \frac{1}{\left(1 + \eta^2\right)\kappa_g(S) - \eta'}$$

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TOTAL ENERGY: COMPOSITE RIBBONS

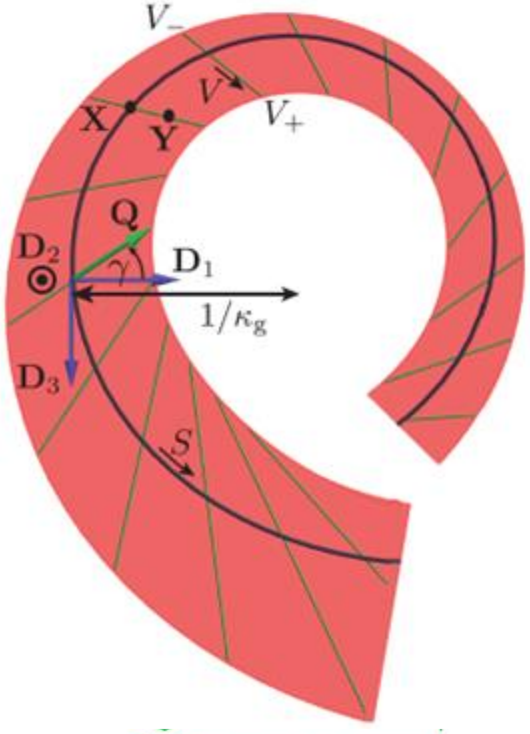
$$U = \int\int \frac{1}{2} D_{11} \frac{\omega_1^2}{\left(1 - \frac{V}{V_c}\right)^2} \left(1 + \beta\eta^4 + 2\eta^2 (\nu + 2\rho) - 4\eta^3\phi - 4\eta\psi\right) \mathrm{d}a$$

$$\mathrm{d}a = |\partial_S \mathbf{y} \times \partial_V \mathbf{y}| \,\mathrm{d}S\,\mathrm{d}V = \left(1 - \frac{V}{V_\mathrm{c}(\eta, S)}\right) \mathrm{d}S\,\mathrm{d}V,$$

$$V_\mathrm{c}(\eta, \eta', S) = \frac{1}{(1+\eta^2)\,\kappa_\mathrm{g}(S) - \eta'}.$$

$$U = \int_0^L \frac{1}{2} D_{11} \left(\kappa_{11}^2 + \beta\kappa_{22}^2 + 4\kappa_{12}\left(\kappa_{12}\rho + \kappa_{22}\phi\right) + 2\kappa_{11}\left(\kappa_{22}\nu + 2\kappa_{12}\psi\right)\right) \log \frac{V^- - V_c}{V^+ - V_c} \mathrm{d}S$$

Using virtual work principle, the weak form can be derived for solving numerically



EQUATIONS OF EQUILIBIRUM

Principle of the virtual work:

$$\mathcal{W}_\eta + \mathcal{W}_{\mathrm{i}}^{\mathrm{rod}} + \mathcal{W}_{\mathrm{e}} + \mathcal{W}_{\mathrm{cEB}} = 0$$

$$\mathcal{W}_{\mathrm{cEB}} = -\int_0^L \mathbf{N}\cdot\left(\hat{\mathbf{x}}' - \hat{\mathbf{d}}_3\right)\mathrm{d}S$$

$$\mathcal{W}_{\mathrm{e}} = \int_0^L \left(\mathbf{p}\cdot\hat{\mathbf{x}} + \mathbf{c}\cdot\hat{\boldsymbol{\psi}}\right)\mathrm{d}S$$

$$\mathcal{W}_{\mathrm{i}}^{\mathrm{rod}} = -\int_0^L \sum_{i=1}^{3}\left(\frac{\partial\mathcal{E}}{\partial\omega_i} - \sum_{j\in\{\mathrm{g},\mathrm{d}\}}\lambda_j\frac{\partial\mathcal{C}_j}{\partial\omega_i}\right)\hat{\omega}_i\,\mathrm{d}S$$

$$= -\int_0^L \mathbf{M}\cdot\sum_{i=1}^{3}(\hat{\omega}_i\,\mathbf{d}_i)\,\mathrm{d}S$$

$$\mathcal{C}_{\mathrm{g}}(\omega_2, S) = \kappa_{\mathrm{g}}(S) - \omega_2 = 0$$

$$\mathcal{C}_{\mathrm{d}}(\omega_1, \omega_3, \eta) = \eta\,\omega_1 - \omega_3 = 0$$

$$\mathcal{W}_\eta = -\int_0^L \left[\frac{\partial\mathcal{E}}{\partial\eta'}\hat{\eta}' + \left(\frac{\partial\mathcal{E}}{\partial\eta} - \lambda_{\mathrm{d}}\frac{\partial\mathcal{C}_{\mathrm{d}}}{\partial\eta}\right)\hat{\eta}\right]\mathrm{d}S$$

Balance equations:

$$\mathbf{N}'(S) + \mathbf{p}(S) = \mathbf{0},$$

$$\mathbf{M}'(S) + \mathbf{x}'(S)\times\mathbf{N}(S) + \mathbf{c}(S) = \mathbf{0}$$

Constitutive law:

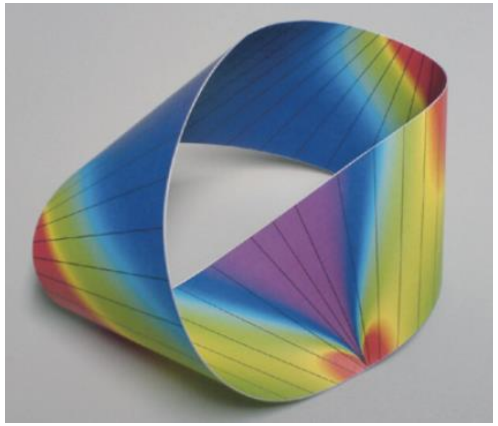
$$\mathbf{M} = \sum_{i=1}^{3}\left(\frac{\partial\mathcal{E}}{\partial\omega_i} - \lambda_{\mathrm{d}}\frac{\partial\mathcal{C}_{\mathrm{d}}}{\partial\omega_i} - \lambda_{\mathrm{g}}\frac{\partial\mathcal{C}_{\mathrm{g}}}{\partial\omega_i}\right)\mathbf{d}_i$$

LIMITING CASES

Sadowsky's Limit: Narrow Rectangular Ribbons

$$E_S = \frac{Dw}{2}\int_0^L {\omega_1}^2\left(1+\eta^2\right)^2 \mathrm{d}S$$

Sadowsky, M, Sitzungsber. Preuss. Akad. Wiss. **22**, 412–415 (1930).



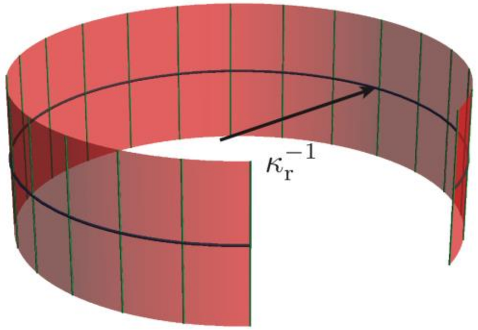
E. L. Starostin et al., Nature Materials (2007)

Wunderlich's Limit: Finite width Rectangular Ribbons

$$E_{\mathrm{W}} = \frac{Dw}{2}\int_0^L {\omega_1}^2\left(1+\eta^2\right)^2 \frac{1}{\eta' w}\ln\left(\frac{1+\eta' w/2}{1-\eta' w/2}\right) \mathrm{d}S$$

Wunderlich, W.: Über ein abwickelbares Möbiusband. Monatshefte für Mathematik 66(3), 266–289 (1962).

Starostin and van der Heijden, PRL **101** (2008).



Audoly & Seffen's Limit: Cylindrical Ribbon

$$E = \frac{Dw}{2}\int_0^L \left[{\omega_1}^2\left(1+\eta^2\right)^2 - 2\kappa_{\mathrm{r}}\omega_1\left(1+\nu\eta^2\right)\right] \mathrm{d}S$$

Basile Audoly and Keith A. Seffen, Journal of Elasticity (2015)

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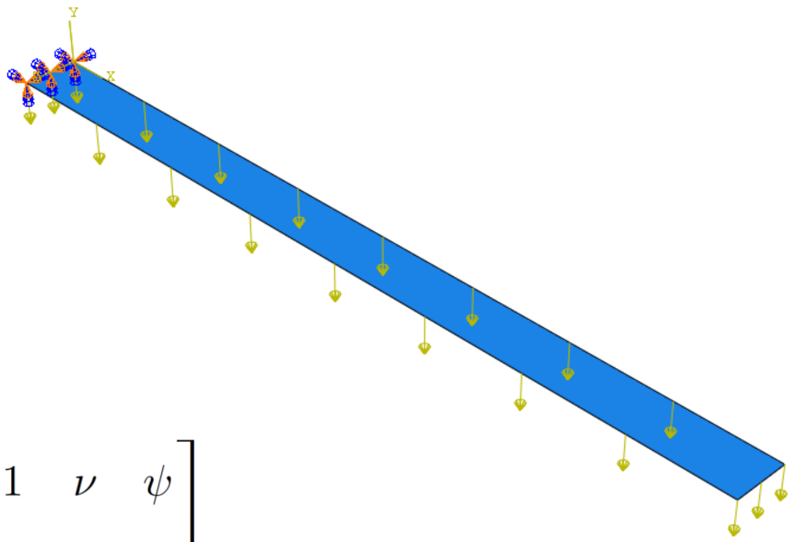
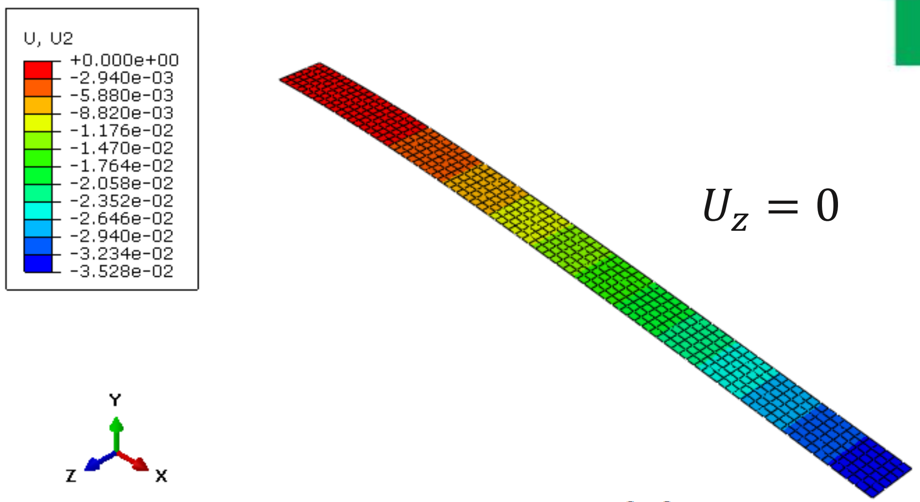
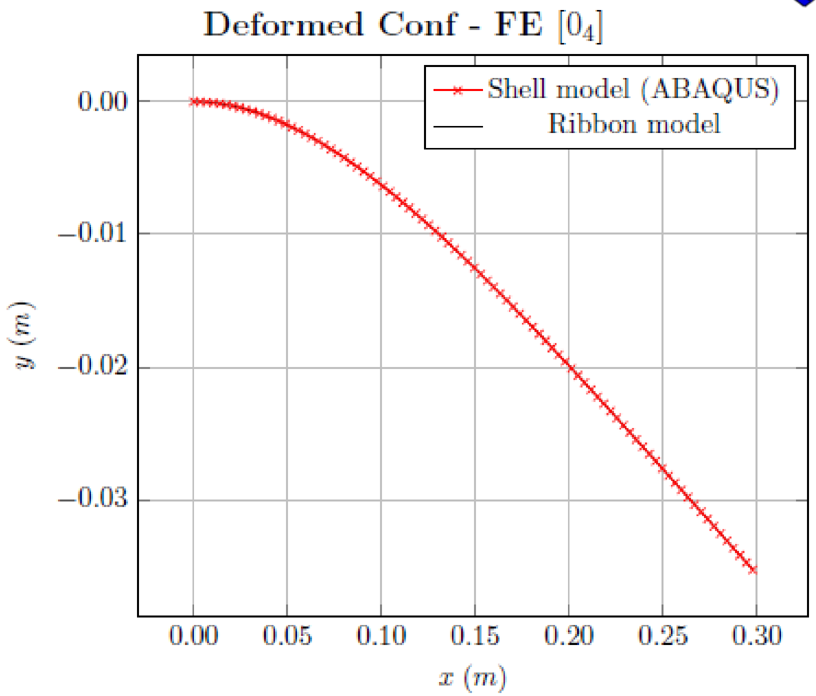
BENDING $[O_4]$





$U_z = 0$



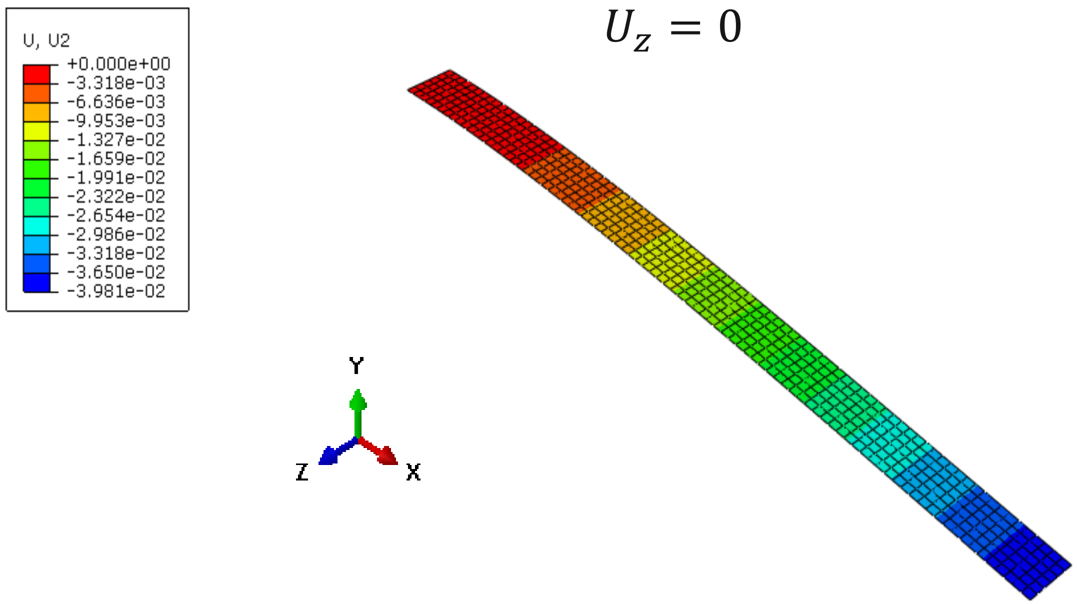
$$D = D_{11} \begin{bmatrix} 1 & \nu & \psi \\ \nu & \beta & \phi \\ \psi & \phi & \rho \end{bmatrix}$$

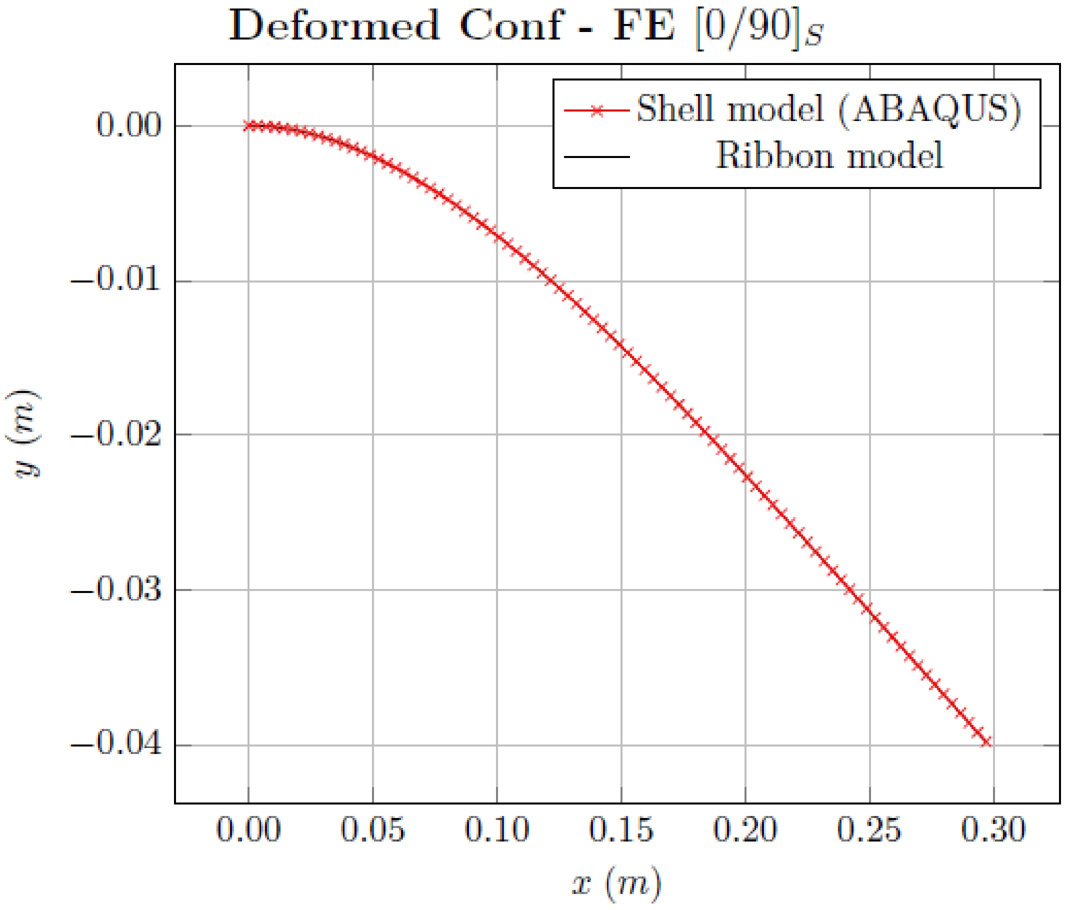
E_{11}: 157 GPa
E_{22}: 9.4 GPa
G_{12}: 5.2 GPa

$$D = D_{11} \begin{pmatrix} 1. & 0.0179 & 0. \\ 0.0179 & 0.0598 & 0. \\ 0. & 0. & 0.0329 \end{pmatrix} GPa - m$$

$$D_{11} = 0.05387, \beta = 0.0598, \nu = 0.0179,$$
$$\psi = 0,\ \ \phi = 0,\ \ \rho = 0.0331$$

BENDING [0/90/90/0]

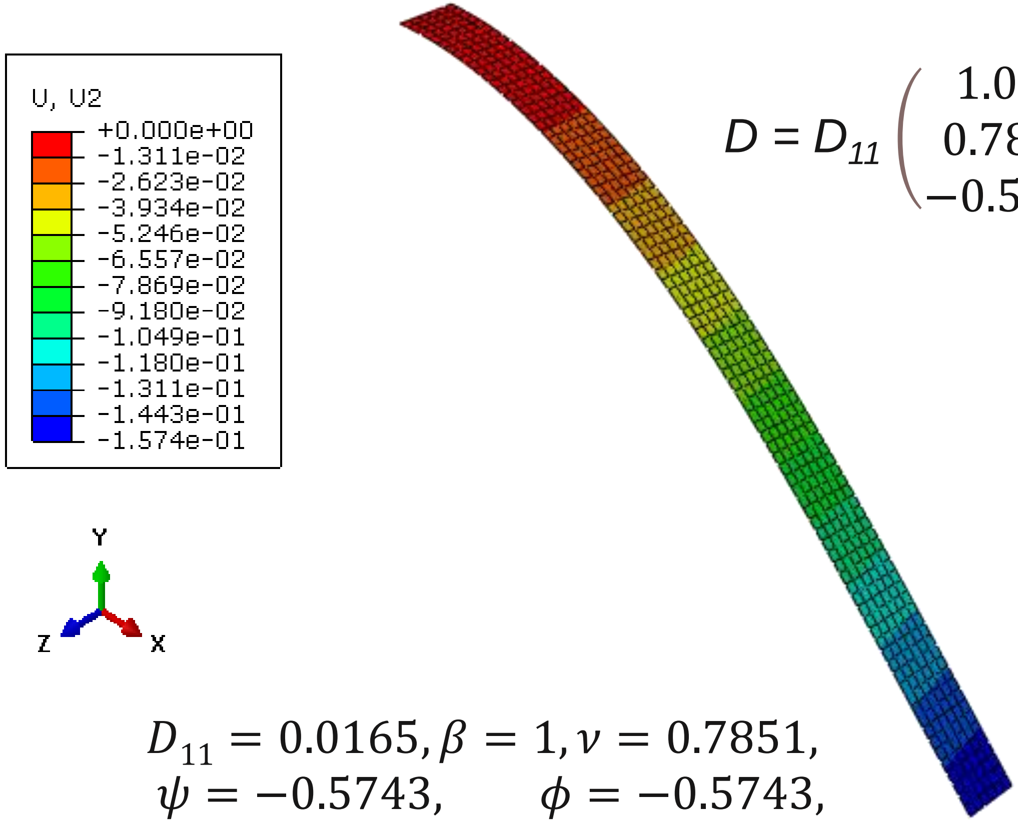




$$D = D_{11} \begin{pmatrix} 1 & 0.0203 & 0.0 \\ 0.0203 & 0.2010 & 0.0 \\ 0.0 & 0.0 & 0.0373 \end{pmatrix} GPa - m^3$$

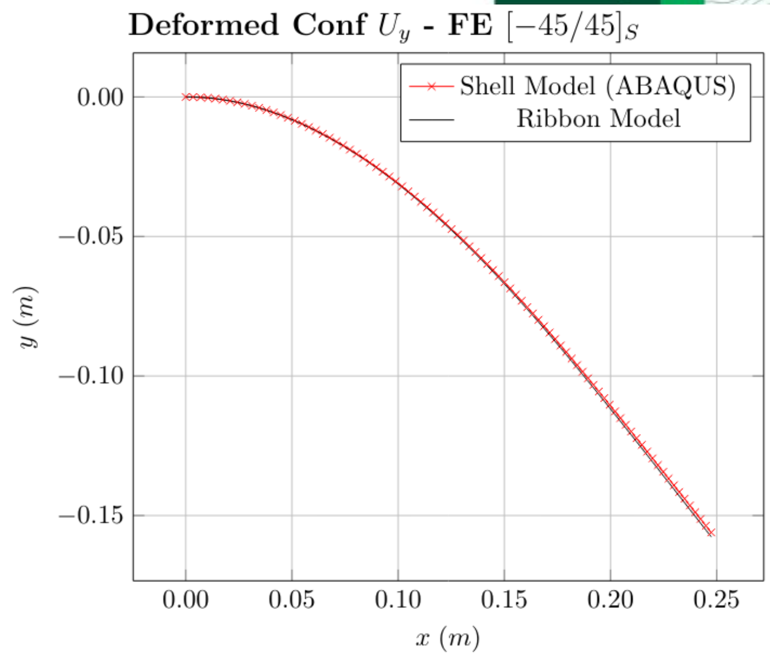
$$D_{11} = 0.0475, \beta = 0.2010, \nu = 0.0203,$$
$$\psi = 0,\ \ \phi = 0,\ \ \rho = 0.0373$$

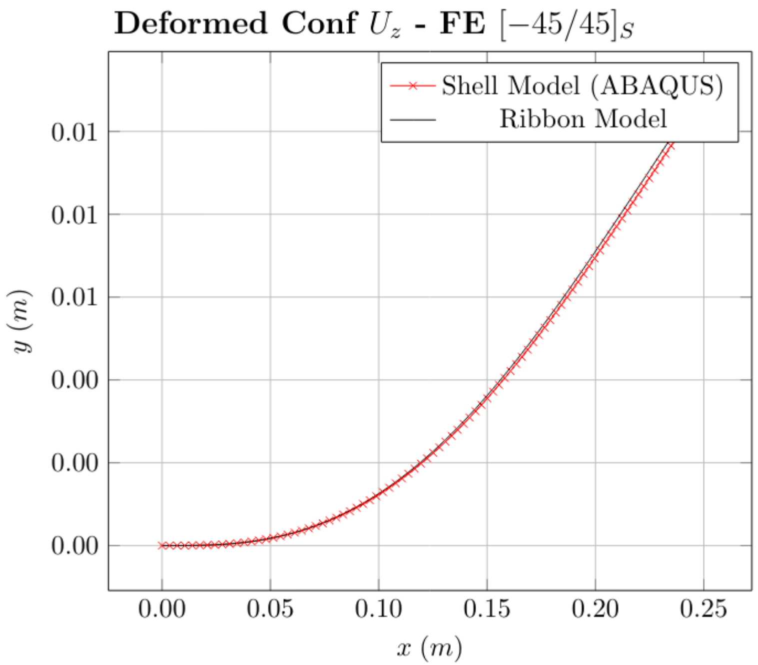
BENDING [-45/45/45/-45]



$$D = D_{11}\begin{pmatrix} 1.000 & 0.7851 & -0.5743 \\ 0.7851 & 1.000 & -0.5743 \\ -0.5743 & -0.5743 & 0.8341 \end{pmatrix}$$

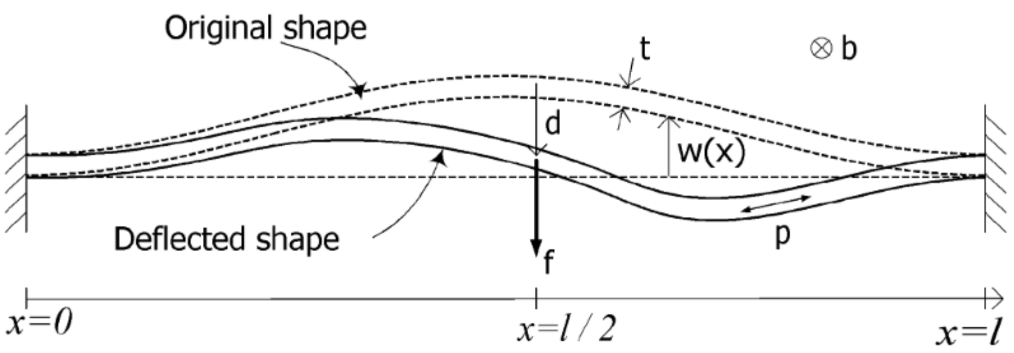
$D_{11} = 0.0165, \beta = 1, \nu = 0.7851,$
$\psi = -0.5743, \quad \phi = -0.5743,$
$\rho = 2.1762$

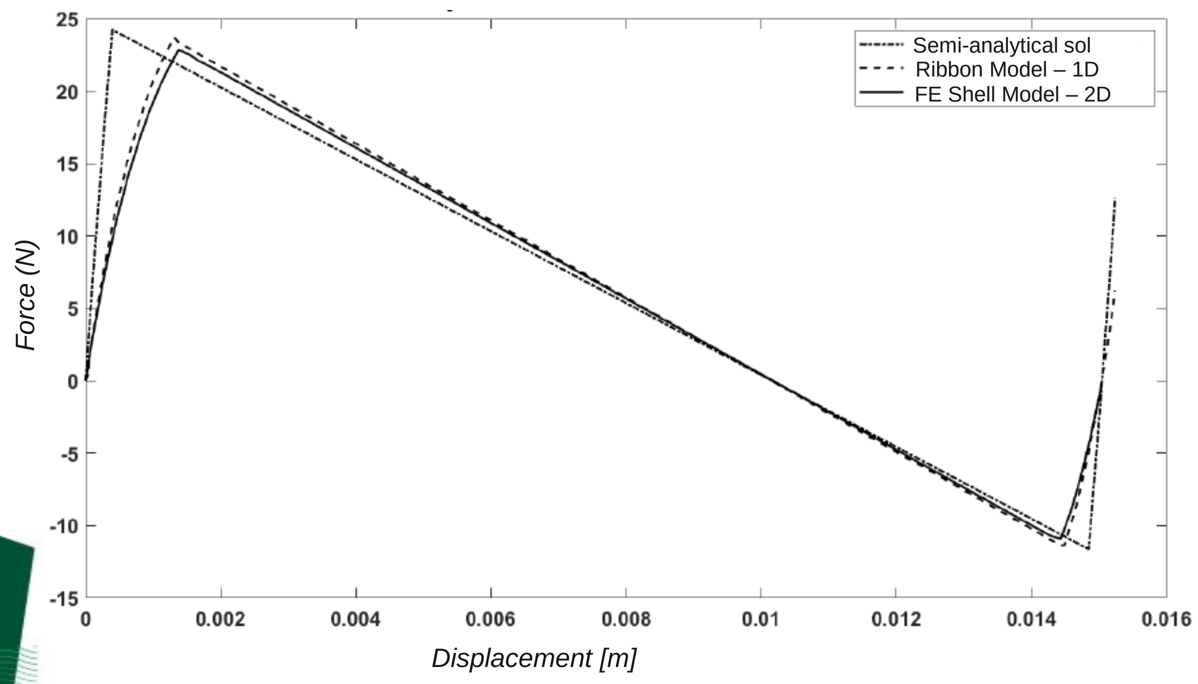


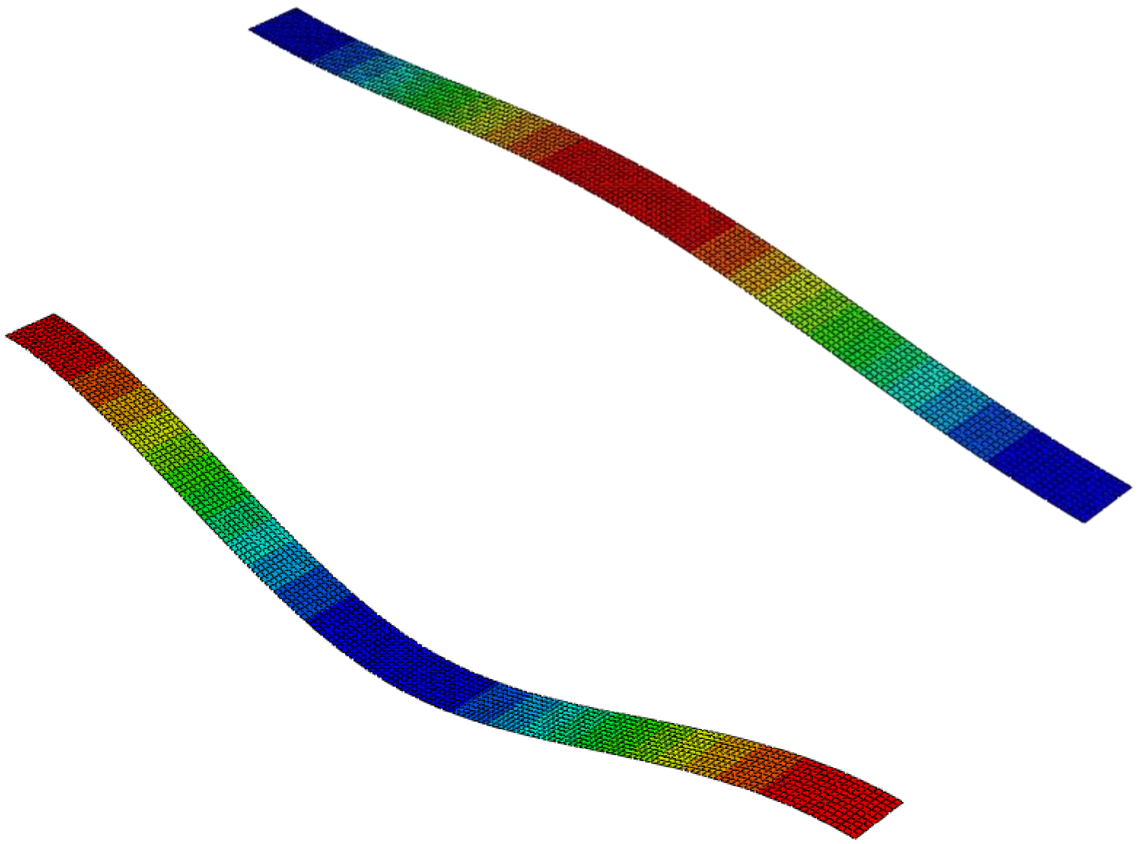


Captures well the Bending-Twisting coupling of angle plies

SNAP-THROUGH OF COMPOSITE RIBBON







Laminate: $[0]_8$

E_{11}: 161 GPa
E_{22}: 12 GPa
G_{12}: 5.2 GPa
ν_{12}: 0.3

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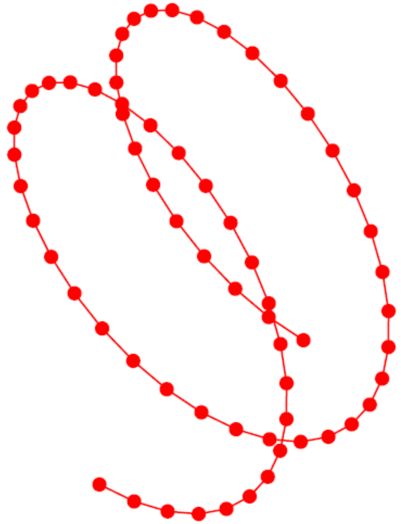
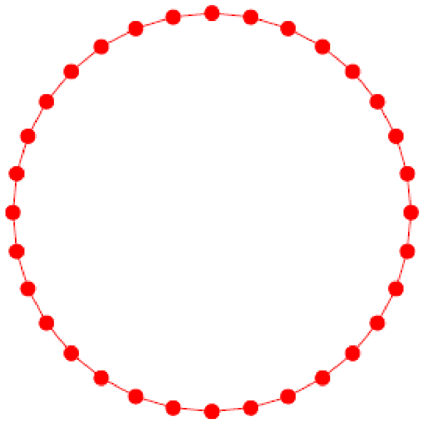
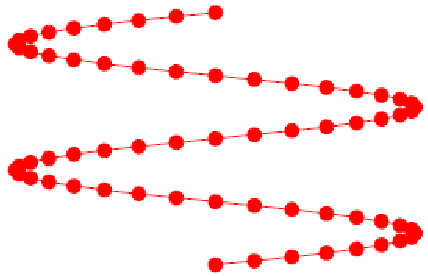
CYLINDRICAL LATTICES



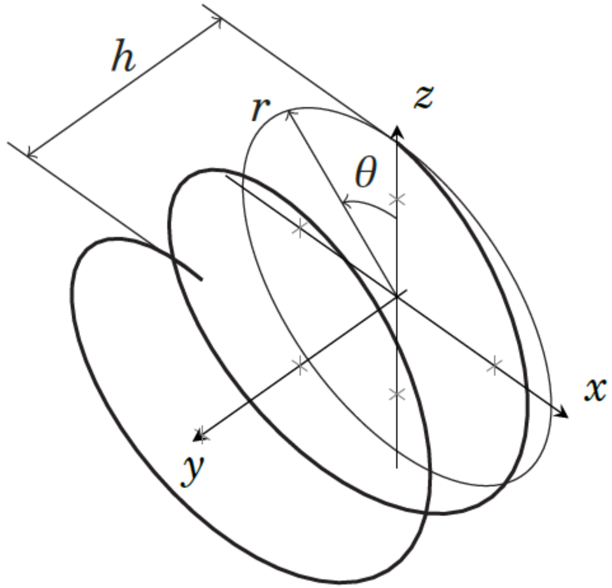
Isometric view



x-z plane

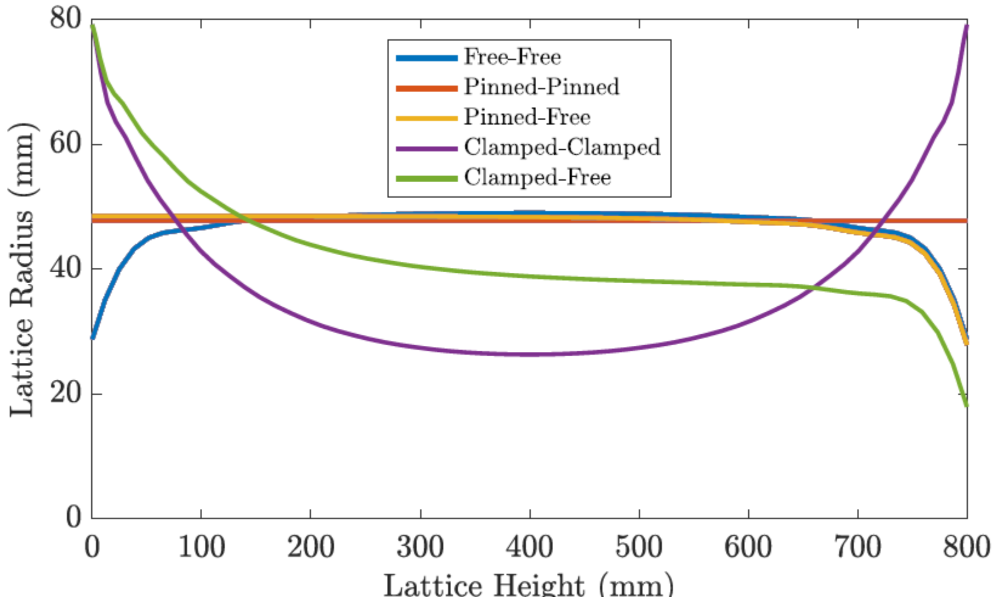


y-z plane

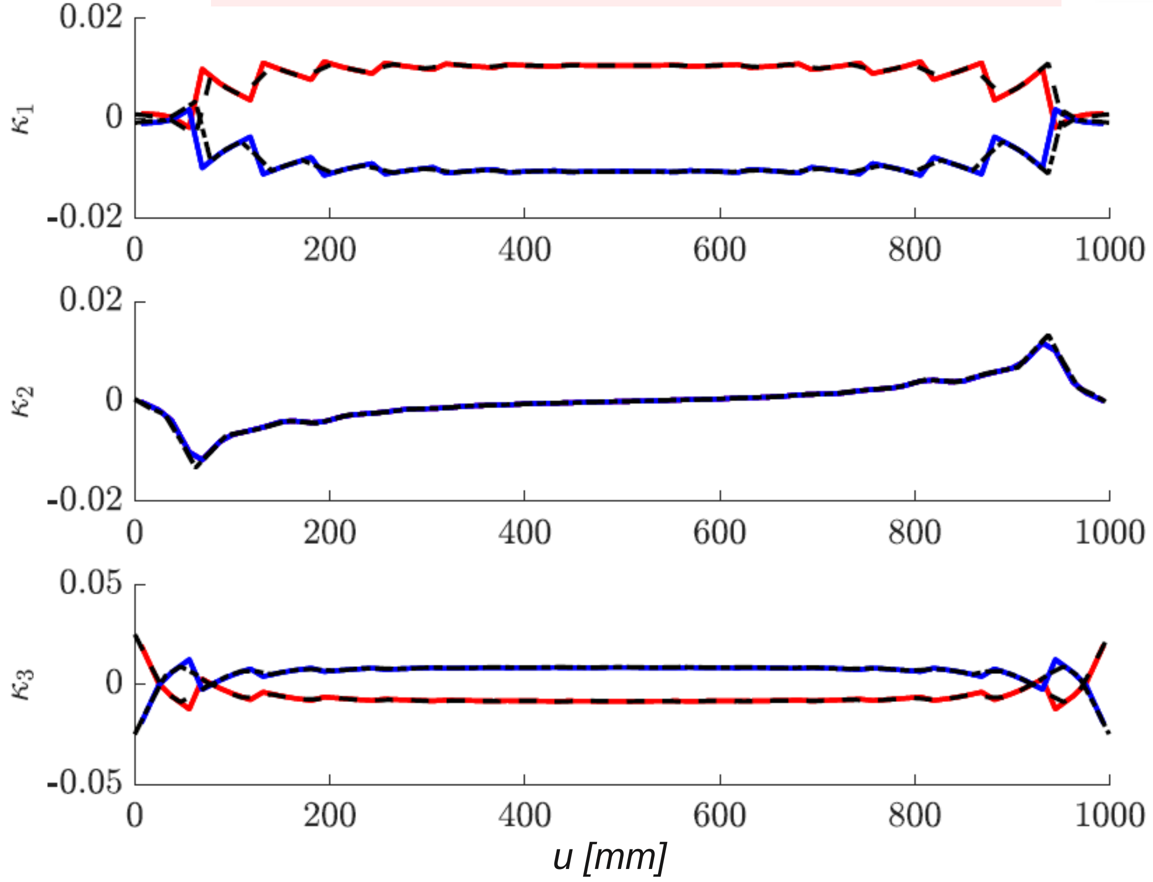


- Length = 1000 mm, r =79.2 mm, h= 100mm
- Discretization: 32 nodes .
- The helices are hinged at intersection points
- At intersection, relative angle between helices can change

EFFECT OF BOUNDARY CONDITIONS



Comparison between Ribbon and FE (2D)

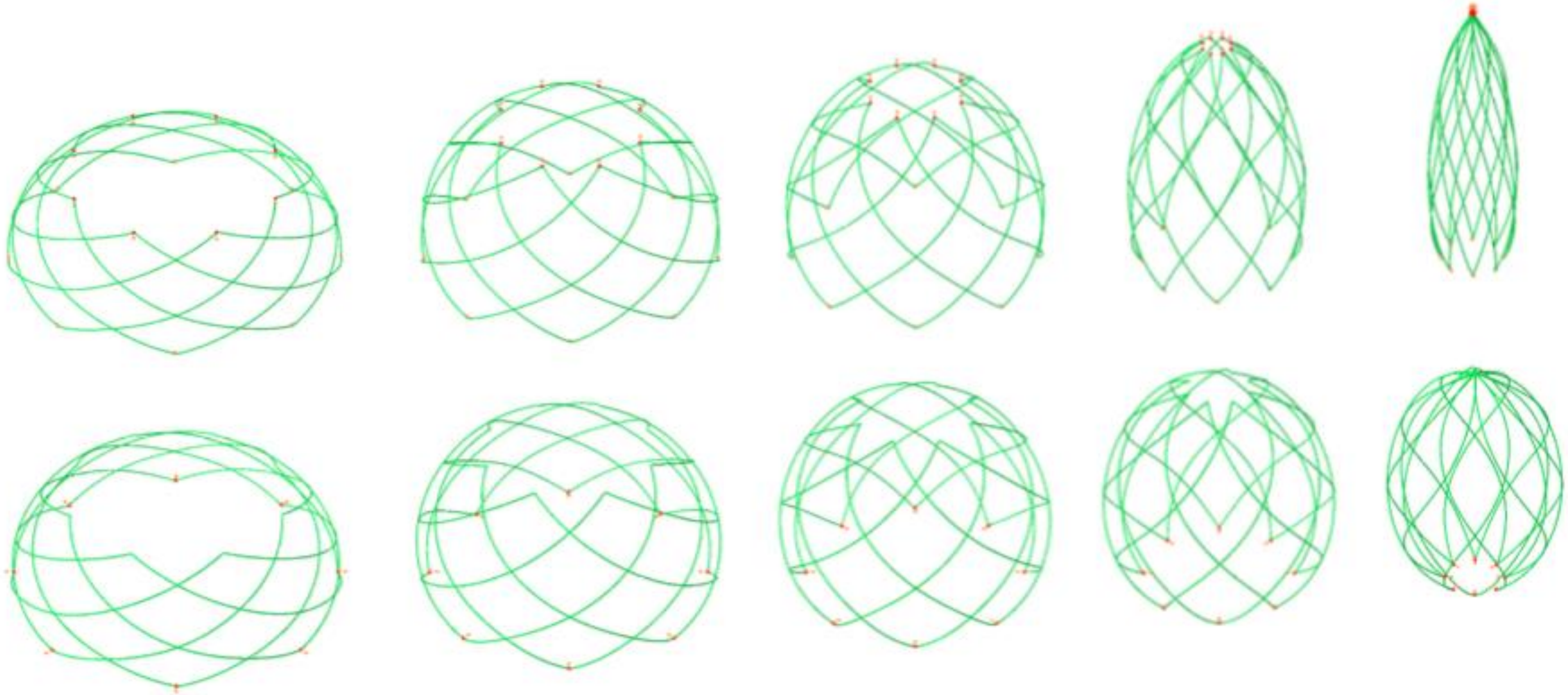


Curvature profile for the lattice at an axial extension of 700mm subjected to clamped-clamped BCs

Solid red and blue lines represent positive and negative handed helices

κ_1, κ_2 and κ_3 are the two normal and twisting curvatures respectively

Future Work: Non-cylindrical lattices



CONCLUSIONS

- A new model for anisotropic ribbons with reduced dimensionality has been developed.
- The model captures geometric nonlinearities associated with large deformations and rotations in anisotropic materials well.
- The formulation relies on two variables: η (related to vector defining the generatrix) and ω_1 (normal curvature). η is resolved after integration across the width.
- Various test problems were solved to demonstrate the model's ability to capture the anisotropic behaviour of composites
- Cylindrical lattices exhibiting non-uniform curvatures due to boundary effects were modelled, demonstrating the effectiveness of the new formulation.
- Future works will involve investigating general non-linear lattices having applications in deployable structures for satellites and spacecrafts.

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THANK YOU

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