ASSIGNMENT 1 :

Numerical Analysis (IVP AND IBVP)

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Solution 1)

$$\begin{cases} u : \overline{J} \to R^d \text{ such that} \\ \dot{u} - Au = f \text{ in J:=(0,T]} \\ u(0) = u_I \end{cases} \tag{1.1}$$

where:

$\overline{J}$=J$\cup\partial$ J

u := [u$_1$,u$_2$,...,u$_d$]^T,

u$_I$:= [u$_{I1}$,u$_{I2}$,...,u$_{Id}$]^T,

du:=du/dt=[du$_1$/dt,du_2/dt,...,du$_d$/dt]^T,

A $\in R^{d\times d}$ is independent of t

f $\in C(J;R^d)$:= Space of continuous R^d-valued functions on J.

$$\begin{cases} U^n \text{ for } n = 1,2,...,N \\ \begin{cases} U^{*n} = U^{n-1} + k[A(t_{n-1})U^{n-1} + f(t_{n-1})] \\ \frac{U^n - U^{n-1}}{k} - \frac{1}{2}[A(t_{n-1})U^{n-1} + A(t_n)U^{*n}] = \frac{1}{2}[f(t_{n-1}) + f(t_n)] \end{cases} \\ U^0 = u_I \end{cases} \tag{1.2}$$

$$\equiv \begin{cases} U^n \text{ for } n = 1,2,...,N \\ U^n = (1 + \frac{kA(t_n)}{2})(1 + \frac{kA(t_{n-1})}{2})U^{n-1} + \frac{k}{2}[(1 + kA(t_{n-1}))f(t_{n-1}) + f(t_n)] \\ U^0 = u_I \end{cases} \tag{1.3}$$

To show that the improved Euler's method for approximating the solution of IVP (1.1) is well-posed,

1. the problem (1.3) must have a unique solution, and
2. the problem (1.3) must be stable,

As the method is explicit so a solution will exist;
Using Method of contradiction
Suppose V_N:=[V^0,V^1,...,V^N] and W_N:=[W^0,W^1,...,W^N] be two distinct solutions of the IVP.
Let us denote, D^N=V^N-W^N
Applying in eqn(1.3) we get

$$\begin{cases} D^n = (1+\frac{kA(t_n)}{2})(1+\frac{kA(t_{n-1})}{2})D^{n-1} \\ D^0 = 0 \end{cases} \tag{1.4}$$

$\equiv D^n$=0 $\forall$n $\in$1,2,...,N
$\equiv D_n$=0 i.e.;
$\equiv V_n$-W_n=0 $\forall$n $\in$1,2,...,N
Which is contradiction to our assumption.

i) QED Problem has a Unique Solution

Consider a perturbed condition of problem (1.3) given by,

$$\begin{cases} \tilde{U}^n \text{ for } n = 1,2,...,N \\ \tilde{U}^n = (1+\frac{kA(t_n)}{2})(1+\frac{kA(t_{n-1})}{2})\tilde{U}^{n-1} + \frac{k}{2}[(1+kA(t_{n-1}))(f(t_{n-1})+\delta^{n-1}) + (f(t_n)+\delta^n)] \\ \tilde{U}^0 = u_I + \delta_1 \end{cases} \tag{1.5}$$

Let E^n=$\tilde{U}^n$-U^n,
So, the problem (1.4) now becomes

$$\begin{cases} E^n \text{ for } n = 1,2,...,N \\ E^n = (1+\frac{kA(t_n)}{2})(1+\frac{kA(t_{n-1})}{2})E^{n-1} + \frac{k}{2}[(1+kA(t_{n-1}))\delta^{n-1} + \delta^n] \\ E^0 = \delta_1 \end{cases} \tag{1.6}$$

$$||E^n|| = ||(1+\frac{kA(t_n)}{2})(1+\frac{kA(t_{n-1})}{2})E^{n-1} + \frac{k}{2}[(1+kA(t_{n-1}))\delta^{n-1} + \delta^n]||$$

where

$$||B|| := max||Bx||_{R^d} \text{ for } ||x||_{R^d} = 1$$

as k>0

Using inequality property,

$$||E^n|| \leq ||(1+\frac{kA(t_n)}{2})(1+\frac{kA(t_{n-1})}{2})| \times ||E^{n-1}|| + \frac{k}{2}[|(1+kA(t_{n-1}))| \times ||\delta^{n-1}|| + ||\delta^n||]$$

Let A=max ||A(t_n)|| $\forall$ n $\in \{1, 2, ..., N\}$,

$$||E^n|| \leq (1+\frac{kA}{2})(1+\frac{kA}{2}) \times ||E^{n-1}|| + \frac{k}{2}[(1+kA) \times ||\delta^{n-1}|| + ||\delta^n||]$$

$$(1+\frac{kA}{2})^2 = 1 + kA + (kA)^2$$

$$(1+\frac{kA}{2})^2 > 1 + kA$$

$$||E^n|| \leq (1+\frac{kA}{2})^2 \times ||E^{n-1}|| + \frac{k}{2}[(1+\frac{kA}{2})^2 \times ||\delta^{n-1}|| + ||\delta^n||]$$

$$||E^n|| \leq (1+\frac{kA}{2})^2 \times ||E^{n-1}|| + \frac{k}{2}[(1+\frac{kA}{2})^2 \times ||\delta^{n-1}|| + ||\delta^n||] \tag{1.7}$$

On solving the iterations till we get ||E^0|| on L.H.S.,

$$||E^n|| \leq (1+\frac{kA}{2})^{2n} \times ||E^0|| + \frac{k}{2}\sum_{j=1}^{n}[(1+\frac{kA}{2})^{2(n+1-j)} \times ||\delta^{j-1}|| + (1+\frac{kA}{2})^{2(n-j)} \times ||\delta^j||]$$

$$||E^n|| \leq (1+\frac{kA}{2})^{2n} \times \{||E^0|| + \frac{k}{2}\sum_{j=1}^{n}[(1+\frac{kA}{2})^{2-2j} \times ||\delta^{j-1}|| + (1+\frac{kA}{2})^{-2j} \times ||\delta^j||]\}$$

$$||E^n|| \leq (1+\frac{kA}{2})^{2n} \times \{||E^0|| + \frac{k}{2}\sum_{j=1}^{n}[(1+\frac{kA}{2})^{2-2j} \times ||\delta^{j-1}||] + \frac{k}{2}\sum_{j=1}^{n}[(1+\frac{kA}{2})^{-2j} \times ||\delta^j||]\}$$

As (1+kA/2)$\geq$1 and replacing p=j-1,

$$||E^n|| \leq (1+\frac{kA}{2})^{2n} \times \{||E^0|| + \frac{k}{2}\sum_{p=0}^{n-1}[(1+\frac{kA}{2})^{-2p} \times ||\delta^p||] + \frac{k}{2}\sum_{j=1}^{n}||\delta^j||\}$$

$$||E^n|| \leq (1+\frac{kA}{2})^{2n} \times \{||E^0|| + \frac{k}{2}\sum_{p=0}^{n-1}||\delta^p|| + \frac{k}{2}\sum_{j=1}^{n}||\delta^j||\}$$

Let δ^n=max ||δ^j|| over j $\in \{1,2,...,n\}$,

$$||E^n|| \leq (1+\frac{kA}{2})^{2n} \times \{||E^0|| + \frac{k}{2}(n\times\delta^n) + \frac{k}{2}(n\times\delta^n)\}$$

$$\max_{1\leq n\leq N}||E^n|| \leq (1+\frac{kA}{2})^{2n} \times \{||E^0|| + (nk\times\delta^n)\}$$

As n$\leq N$,

$$\max_{1\leq n\leq N}||E^n|| \leq (1+\frac{kA}{2})^{2N} \times \{||E^0|| + (Nk\times\delta^N)\} \tag{1.8}$$

We know that T=kN,
Also (1+x)$\leq$exp(x),
Applying both in eqn(1.7) we get

$$\max_{1\leq n\leq N}||E^n|| \leq exp(AT) \times (\delta_1 + T\times\delta^N) \tag{1.9}$$

Where right hand side will be some constant and thus problem is stable as the maximum error is bounded and independent of number of partitions taken for calculations.

ii) QED Problem is Stable

Under the assumption that u$\in$C^3($\overline{J}$,R^d) to show and convergence of the problem (1.2)
Using Taylor's Theorem,

$$\begin{cases} u(t_n) = u(t_{n-1}) + k\dot{u}(t_{n-1}) + \frac{k^2}{2}\ddot{u}(t_{n-1}) + \frac{k^3}{6}\dddot{u}(\Theta) \\ \dot{u}(t_n) = \dot{u}(t_{n-1}) + k\ddot{u}(t_{n-1}) + \frac{k^2}{2}\dddot{u}(\theta) \end{cases}$$

From above two equations we get

$$u(t_n) = u(t_{n-1}) + \frac{k}{2}(\dot{u}(t_{n-1}) + \dot{u}(t_n)) + \mathcal{O}(k^3)$$

Using eqn(1.1) for value of $\dot{u}(t_{n-1})$ and $\dot{u}(t_n)$,

$$u(t_n) = u(t_{n-1}) + \frac{k}{2}(f(t_{n-1}) - A(t_{n-1})u(t_{n-1}) + f(t_n) - A(t_n)u(t_n)) + \mathcal{O}(k^3)$$

Using Taylor's Theorem for $u(t_n)$ on R.H.S. and eqn(1.1),

$$u(t_n) = u(t_{n-1}) + \frac{k}{2}(f(t_{n-1}) - A(t_{n-1})u(t_{n-1}) + f(t_n) - A(t_n)(u(t_{n-1})$$

+k(f(t$_{n-1}$) $- A(t_{n-1})u(t_{n-1}) + \mathcal{O}(k^2)))) + \mathcal{O}(k^3)$

$\equiv$

$$u(t_n) = (1+\frac{kA(t_n)}{2})(1+\frac{kA(t_{n-1})}{2})u(t_{n-1})+\frac{k}{2}[(1+kA(t_{n-1}))f(t_{n-1})+f(t_n)]+\frac{k}{2}\mathcal{O}(k^2)+\mathcal{O}(k^3)$$

$\equiv$

$$u(t_n) = (1+\frac{kA(t_n)}{2})(1+\frac{kA(t_{n-1})}{2})u(t_{n-1}) + \frac{k}{2}[(1+kA(t_{n-1}))f(t_{n-1}) + f(t_n)] + \mathcal{O}(k^3) \tag{1.10}$$

Let us assume e^n:=u(t$_n$)-U$_n$,

From eqn(1.10) and eqn(1.10) we get,

$$e_n = (1+\frac{kA(t_n)}{2})(1+\frac{kA(t_{n-1})}{2})e_{n-1} + \tau_n \tag{1.11}$$

Hence the local truncation error is given by,

$$\tau_n = \mathcal{O}(k^3) \tag{1.12}$$

As we have e_0=0 as the initial condition

So, from eqn(1.11) and comaring with eqn(1.9) the method must be convergent.

$$\underset{1\leq n\leq N}{max}|e_n| \leq exp(AT) \times (\underset{1\leq n\leq N}{max}|\tau_n|) \leq \mathcal{C}(k^3) \tag{1.13}$$

QED the improved Euler method is convergent with rate k^3 where k is the step size.

Solution 2)

$$\begin{cases} u : \Omega \times J \to R, \Omega := (x_L, x_R), J : (0, T] \\ \mathcal{L}u = f \text{ in } \Omega \times J \\ u = u_I \text{ in } \overline{\Omega} \times \{0\} \\ u = 0 \text{ on } \partial\Omega \times \overline{J} \\ \equiv \\ \mathcal{L}u(x,t) = f(x,t)\ \forall (x,t) \in \Omega \times J \\ u(x,0) = u_I(x)\ \forall x \in \overline{\Omega} \\ u(x_L, t) = 0 = u(x_R, t) \forall t \in \overline{J} \end{cases} \tag{1.14}$$

where:

$$\mathcal{L}y := y_t + Ly;$$

$$Ly := -ay_{xx} + by_x + cy;$$

$$y_t(x,t) = \frac{\partial y}{\partial t}(x,t) := \lim_{k \to 0} \frac{f(x, t+k) - f(x,t)}{k}$$

$$y_x(x,t) = \frac{\partial y}{\partial x}(x,t) := \lim_{h \to 0} \frac{f(x, t+h) - f(x,t)}{h}$$

$$\Omega := (x_L, x_R), \overline{\Omega} := \Omega \cup \partial\Omega = [x_L, x_R]$$

$$\partial J := \{0, T\}, \overline{J} := J \cup = [0, T]$$

$$\Omega \times J := \{(x,t) \in R^2, x_L < x < x_R, 0 < t \leq T\}$$

$$\partial\Omega \times \overline{J} := \{(x_L, t) \in R^2, 0 \leq t \leq T\} \cup \{(x_R, t) \in R^2, 0 \leq t \leq T\}$$

$$\overline{\Omega} \times \{0\} := \{(x, 0) \in R^2, x_L \leq x \leq x_R\}$$

$$a \geq a_0 > 0 \text{ on } \overline{\Omega} \times \overline{J}$$

$$u_I : \overline{\Omega} \to R \text{ is continuous}$$

IBVP in terms of theta method

$$\begin{cases} U_j^n \text{ for } n = 1, 2, ..., N_t, j = 1, 2, ..., N_x - 1 \\ \begin{cases} \mathcal{L}_{hk}^{\theta} U_j^n = f_j^{n-1+\theta} \end{cases} \\ U_j^0 = u_I(x_j), \\ U_0^n = 0 = U_{N_x}^n, \end{cases} \tag{1.15}$$

where we consider a uniform partition:

$$\overline{\Omega}_h \times \overline{J}_k := \{(x_j, t_n) \in \overline{\Omega} \times \overline{J} x_j = x_L + jh, t_n = nk, h = \frac{x_R - x_L}{N_x}, k = \frac{t}{N_t}\}$$

for n=1,2,...,N$_t$, j=1,2,...,N$_x$ of $\overline{\Omega} \times \overline{J}$

$$U_j^n \approx u_j^n := u(x_j, t_n)$$

$$\mathcal{L}_{hk}^{\theta} U_j^n := \frac{U_j^n - U_j^{n-1}}{k} + L_{hk}^{\theta}(\theta U_j^n + (1-\theta) U_j^{n-1})$$

$$\text{For } V_j^n := (\theta U_j^n + (1-\theta) U_j^{n-1})$$

$$\text{For } L_{hk}^{\theta} V_j^n := -a_j^{n-1+\theta} \frac{V_{j+1}^n - 2V_j^n + V_{j-1}^n}{h^2} + b_j^{n-1+\theta} \frac{V_{j+1}^n - V_{j-1}^n}{2h} + c_j^{n-1+\theta} V_j^n$$

Solving (1.15) further by collecting different terms we get,

$$\begin{cases} \begin{cases} (\frac{1}{k} + \frac{\theta 2 a_j^{n-1+\theta}}{h^2} + \theta c_j^{n-1+\theta}) U_j^n + (\frac{-1}{k} + \frac{(1-\theta) 2 a_j^{n-1+\theta}}{h^2} + (1-\theta) c_j^{n-1+\theta}) U_j^{n-1} \\ +(-\frac{\theta a_j^{n-1+\theta}}{h^2} + \frac{\theta b_j^{n-1+\theta}}{2h}) U_{j+1}^n + (-\frac{(1-\theta) a_j^{n-1+\theta}}{h^2} + \frac{(1-\theta) b_j^{n-1+\theta}}{2h}) U_{j+1}^{n-1} \\ +(-\frac{\theta a_j^{n-1+\theta}}{h^2} - \frac{\theta b_j^{n-1+\theta}}{2h}) U_{j-1}^n + (-\frac{(1-\theta) a_j^{n-1+\theta}}{h^2} - \frac{(1-\theta) b_j^{n-1+\theta}}{2h}) U_{j-1}^{n-1} \\ = f_j^{n-1+\theta} \end{cases} \\ U_j^0 = u_I(x_j), \\ U_0^n = 0 = U_{N_x}^n, \end{cases} \tag{1.16}$$

Let C=$\underset{1 \le j \le N_x, 1 \le n \le N_t}{max}$ |c$_j^{n-1+\theta}$|= $\underset{\overline{\Omega} \times \overline{J}}{max}$ |c|
A=$\underset{\overline{\Omega} \times \overline{J}}{max}$ |a| and B=$\underset{\overline{\Omega} \times \overline{J}}{max}$ |b|
Let us consider M^n = $\underset{1 \le j \le N_x}{max}$ ||U$_j^n$|| .

Therefore,

$$(\frac{1}{k}+\frac{\theta 2a_j^{n-1+\theta}}{h^2}+\theta c_j^{n-1+\theta})M^n \leq (\frac{1}{k}-(1-\theta)c_j^{n-1+\theta})M^{n-1}+(\frac{\theta 2a_j^{n-1+\theta}}{h^2})M^n+\max_{1\leq j\leq N_x}|f_j^{n-1+\theta}| \tag{1.17}$$

which is true under the following choice of step values

$$\begin{cases} \frac{1}{k}-\frac{2}{h^2}A-C\geq 0 \\ h\times B\geq 2\times a_0 \end{cases} \tag{1.18}$$

On solving (1.17)

$$(\frac{1}{k}+\theta c_j^{n-1+\theta})M^n \leq (\frac{1}{k}-(1-\theta)c_j^{n-1+\theta})M^{n-1}+\max_{1\leq j\leq N_x}|f_j^{n-1+\theta}|$$

$$(1+k\theta c_j^{n-1+\theta})M^n \leq (1-k(1-\theta)c_j^{n-1+\theta})M^{n-1}+k\max_{1\leq j\leq N_x}|f_j^{n-1+\theta}|$$

Using $1-k\theta C\leq 1+k\theta c_j^{n-1+\theta}$ and $1+k(1-\theta)C\geq 1-k(1-\theta)c_j^{n-1+\theta}$

$$(1-k\theta C)M^n \leq (1+k(1-\theta)C)M^{n-1}+k\max_{1\leq j\leq N_x}|f_j^{n-1+\theta}|$$

$$M^n \leq \frac{1+k(1-\theta)C}{1-k\theta C}M^{n-1}+\frac{k}{(1-k\theta C)}\max_{1\leq j\leq N_x}|f_j^{n-1+\theta}|$$

On solving by iteration we get

$$M^n \leq (\frac{1+k(1-\theta)C}{1-k\theta C})^{N_t}M^0+\frac{k}{(1-k\theta C)}\sum_{p=1}^{N_t}(\max_{1\leq j\leq N_x}|f_j^{p-1+\theta}|(\frac{1+k(1-\theta)C}{1-k\theta C})^{N_t-p})$$

$$M^n \leq (\frac{1+k(1-\theta)C}{1-k\theta C})^{N_t}(M^0+\frac{k}{(1-k\theta C)}\sum_{p=1}^{N_t}(\max_{1\leq j\leq N_x}|f_j^{p-1+\theta}|(\frac{1+k(1-\theta)C}{1-k\theta C})^{-p}))$$

As $1+k(1-\theta)C\geq 1$

$$M^n \leq (\frac{1+k(1-\theta)C}{1-k\theta C})^{N_t}(M^0+\frac{k}{(1-k\theta C)}\sum_{p=1}^{N_t}(\max_{1\leq j\leq N_x}|f_j^{p-1+\theta}|(1-k\theta C)^{p}))$$

$$M^n \leq (\frac{1+k(1-\theta)C}{1-k\theta C})^{N_t}(M^0+k\sum_{p=1}^{N_t}(\max_{1\leq j\leq N_x}|f_j^{p-1+\theta}|(1-k\theta C)^{p-1}))$$

$$M^n \leq (\frac{1+k(1-\theta)C}{1-k\theta C})^{N_t}(M^0 + k\max_{j,n}|f_j^{p-1+\theta}|\sum_{p=1}^{N_t}(1-k\theta C)^{p-1})$$

As 1-k$\theta C \leq 1$

$$M^n \leq (\frac{1+k(1-\theta)C}{1-k\theta C})^{N_t}(M^0 + k\max_{j,n}|f_j^{p-1+\theta}|\sum_{p=1}^{N_t}(1))$$

$$M^n \leq (\frac{1+k(1-\theta)C}{1-k\theta C})^{N_t}(M^0 + T\max_{j,n}|f_j^{p-1+\theta}|)$$

$$M^n \leq (1+\frac{kC}{1-k\theta C})^{N_t}(M^0 + T\max_{j,n}|f_j^{p-1+\theta}|)$$

Using (1+r)$\leq e^r$

$$M^n \leq e^{\frac{TC}{1-k\theta C}}(M^0 + T\max_{j,n}|f_j^{p-1+\theta}|)$$

Now choosing k small enough that $1 - k\theta C$ does not approach zero
Or let k$\leq \frac{0.5}{\theta C}$ then under this condition

$$max|U_j^n| \leq E(max|u_I| + T \times (max|f_j^n|)) \tag{1.19}$$

where E is some constant value.

To show that the theta method for approximating the solution of IBVP (1.14) is well-posed,

1. the problem (1.16) must have a unique solution, and
2. the problem (1.16) must be stable,

Under the certain restrictions on step size the solution of the IBVP must exist according to 1.19.
Proving that the method has at most one solution by method of contradiction
Let V_j^n and W_j^n be 2 distinct solutions of the system
Denote D_j^n=V_j^n-W_j^n
From(1.15),
Clearly,D_0^n=$D_{N_x}^n$=0
D_j^0=V_j^0-W_j^0=$u_I(x_j)$-$u_I(x_j)$= 0
Consider $\mathcal{L}_{hk}^\theta$ D_j^n= $\mathcal{L}_{hk}^\theta$ (V_j^n-W_j^n)= $\mathcal{L}_{hk}^\theta$ V_j^n-$\mathcal{L}_{hk}^\theta$ W_j^n= 0 (using linearity of $\mathcal{L}$) Therefore, now for uniqueness of the solution. D_j^n must be zero which will be true if problem(1.15) has zero as the only solution under condition u_I= 0 and f=0
Now for condition u_I= 0 and f=0 the solution must exist for certain choice of step size h and k.

But according to 1.19 max|U_j^n|≤ 0

This is only possible if U_j^n=0 for all n=1,2,..,N_t and for all j=1,2,..,N_x.

Therefore, D_j^n=0 $\equiv$ U_j^n=W_j^n

Hence, the IBVP has a unique solution.

i) QED Problem has a Unique Solution

Consider a perturbed condition of problem(1.15)

$$\begin{cases} \tilde{U}_j^n \text{ for } n = 1, 2, ..., N_t, j = 1, 2, ..., N_x - 1 \\ \begin{cases} \mathcal{L}_{hk}^{\theta} \tilde{U}_j^n = f_j^{n-1+\theta} + \delta_j^n \end{cases} \\ \tilde{U}_j^0 = u_I(x_j) + \delta_j^0, \\ \tilde{U}_0^n = 0 = \tilde{U}_{N_x}^n, \end{cases} \tag{1.20}$$

Let us denote E_j^n=$\tilde{U}_j^n$-U_j^n

Subtracting 1.15 and 1.20 will give us

$$\begin{cases} E_j^n \text{ for } n = 1, 2, ..., N_t, j = 1, 2, ..., N_x - 1 \\ \mathcal{L}_{hk}^{\theta} E_j^n = \delta_j^n \\ E_j^0 = \delta_j^0, \\ E_0^n = 0 = E_{N_x}^n, \end{cases} \tag{1.21}$$

$$\equiv \begin{cases} \begin{cases} (\frac{1}{k} + \frac{\theta 2a_j^{n-1+\theta}}{h^2} + \theta c_j^{n-1+\theta})E_j^n + (\frac{-1}{k} + \frac{(1-\theta)2a_j^{n-1+\theta}}{h^2} + (1-\theta)c_j^{n-1+\theta})E_j^{n-1} \\ +(-\frac{\theta a_j^{n-1+\theta}}{h^2} + \frac{\theta b_j^{n-1+\theta}}{2h})E_{j+1}^n + (-\frac{(1-\theta)a_j^{n-1+\theta}}{h^2} + \frac{(1-\theta)b_j^{n-1+\theta}}{2h})E_{j+1}^{n-1} \\ +(-\frac{\theta a_j^{n-1+\theta}}{h^2} - \frac{\theta b_j^{n-1+\theta}}{2h})E_{j-1}^n + (-\frac{(1-\theta)a_j^{n-1+\theta}}{h^2} - \frac{(1-\theta)b_j^{n-1+\theta}}{2h})E_{j-1}^{n-1} \\ = \delta_j^n \end{cases} \\ E_j^0 = \delta_j^0, \\ E_0^n = 0 = E_{N_x}^n, \end{cases} \tag{1.22}$$

Now, 1.22 implies that,

$$|(\frac{1}{k} + \frac{\theta 2a_j^{n-1+\theta}}{h^2} + \theta c_j^{n-1+\theta})E_j^n| = |(\frac{\theta a_j^{n-1+\theta}}{h^2} - \frac{\theta b_j^{n-1+\theta}}{2h})E_{j+1}^n + (\frac{\theta a_j^{n-1+\theta}}{h^2} + \frac{\theta b_j^{n-1+\theta}}{2h})E_{j-1}^n$$

$$+(\frac{1}{k} - \frac{(1-\theta)2a_j^{n-1+\theta}}{h^2} - (1-\theta)c_j^{n-1+\theta})E_j^{n-1} + (\frac{(1-\theta)a_j^{n-1+\theta}}{h^2} - \frac{(1-\theta)b_j^{n-1+\theta}}{2h})E_{j-1}^{n-1}$$

$$+(\frac{(1-\theta)a_j^{n-1+\theta}}{h^2}+\frac{(1-\theta)b_j^{n-1+\theta}}{2h})E_{j-1}^{n-1}+\delta_j^n|$$

$\equiv$

$$|(1+\mu\theta 2a_j^{n-1+\theta}+k\theta c_j^{n-1+\theta})||E_j^n|\leq|(\mu\theta a_j^{n-1+\theta}-\tfrac{\mu h\theta}{2}b_j^{n-1+\theta})||E_{j+1}^n|+|(\mu\theta a_j^{n-1+\theta}+\tfrac{\mu h\theta}{2}b_j^{n-1+\theta})||E_{j-1}^n|$$

$$+|(1-2\mu(1-\theta)a_j^{n-1+\theta}-k(1-\theta)c_j^{n-1+\theta})||E_j^{n-1}|+|(\mu(1-\theta)a_j^{n-1+\theta}-\frac{\mu h(1-\theta)}{2}b_j^{n-1+\theta})||E_{j-1}^{n-1}|$$

$$+|(\mu(1-\theta)a_j^{n-1+\theta}+\frac{\mu h(1-\theta)}{2}b_j^{n-1+\theta})||E_{j-1}^{n-1}|+k|\delta_j^n|$$

Choose step value k and h such that,

$$a_0-\frac{h}{2}\max_{\overline{\Omega}\times\overline{J}}|b|\geq 0$$

$$2\mu\max_{\overline{\Omega}\times\overline{J}}a+k\max_{\overline{\Omega}\times\overline{J}}|c|\leq 1$$

Let $M^n := \max_{1\leq j\leq N_x-1}|E_j^n|$
On solving we get,

$$(1+\mu\theta 2a_j^{n-1+\theta}+k\theta c_j^{n-1+\theta})M^n\leq(\mu\theta a_j^{n-1+\theta}-\frac{\mu h\theta}{2}b_j^{n-1+\theta})M^n+(\mu\theta a_j^{n-1+\theta}+\frac{\mu h\theta}{2}b_j^{n-1+\theta})M^n$$

$$+(1-2\mu(1-\theta)a_j^{n-1+\theta}-k(1-\theta)c_j^{n-1+\theta})M^{n-1}+(\mu(1-\theta)a_j^{n-1+\theta}-\frac{\mu h(1-\theta)}{2}b_j^{n-1+\theta})M^{n-1}$$

$$+(\mu(1-\theta)a_j^{n-1+\theta}+\frac{\mu h(1-\theta)}{2}b_j^{n-1+\theta})M^{n-1}+k\max_{1\leq j\leq N_x-1}|\delta_j^n|$$

$\equiv$

$$(1+k\theta c_j^{n-1+\theta})M^n\leq(1-k(1-\theta)c_j^{n-1+\theta})M^{n-1}+k\max_{1\leq j\leq N_x-1}|\delta_j^n| \tag{1.23}$$

Using $1-k\theta C\leq 1+k\theta c_j^{n-1+\theta}$ and $1+k(1-\theta)C\geq 1-k(1-\theta)c_j^{n-1+\theta}$

$$(1-k\theta C)M^n\leq(1+k(1-\theta)C)M^{n-1}+k\max_{1\leq j\leq N_x-1}|\delta_j^n|$$

$$M^n\leq\frac{1+k(1-\theta)C}{1-k\theta C}M^{n-1}+\frac{k}{(1-k\theta C)}\max_{1\leq j\leq N_x-1}|\delta_j^n|$$

On solving by iteration we get

$$M^n \leq (\frac{1+k(1-\theta)C}{1-k\theta C})^{N_t} M^0 + \frac{k}{(1-k\theta C)} \sum_{p=1}^{N_t} (\max_{1\leq j\leq N_x-1} |\delta_j^n| (\frac{1+k(1-\theta)C}{1-k\theta C})^{N_t-p})$$

$$M^n \leq (\frac{1+k(1-\theta)C}{1-k\theta C})^{N_t} (M^0 + \frac{k}{(1-k\theta C)} \sum_{p=1}^{N_t} (\max_{1\leq j\leq N_x-1} |\delta_j^n| (\frac{1+k(1-\theta)C}{1-k\theta C})^{-p}))$$

As $1+k(1-\theta)C \geq 1$

$$M^n \leq (\frac{1+k(1-\theta)C}{1-k\theta C})^{N_t} (M^0 + \frac{k}{(1-k\theta C)} \sum_{p=1}^{N_t} (\max_{1\leq j\leq N_x-1} |\delta_j^n| (1-k\theta C)^p))$$

$$M^n \leq (\frac{1+k(1-\theta)C}{1-k\theta C})^{N_t} (M^0 + k \sum_{p=1}^{N_t} (\max_{n,j} |\delta_j^n| (1-k\theta C)^{p-1}))$$

$$M^n \leq (\frac{1+k(1-\theta)C}{1-k\theta C})^{N_t} (M^0 + k \max_{n,j} |\delta_j^n| \sum_{p=1}^{N_t} (1-k\theta C)^{p-1})$$

As 1-k$\theta C \leq 1$

$$M^n \leq (\frac{1+k(1-\theta)C}{1-k\theta C})^{N_t} (M^0 + k \max_{n,j} |\delta_j^n| \sum_{p=1}^{N_t} (1))$$

$$M^n \leq (\frac{1+k(1-\theta)C}{1-k\theta C})^{N_t} (M^0 + T \max_{n,j} |\delta_j^n|)$$

$$M^n \leq (1 + \frac{kC}{1-k\theta C})^{N_t} (M^0 + T \max_{n,j} |\delta_j^n|)$$

Using (1+r)$\leq e^r$

$$M^n \leq e^{\frac{TC}{1-k\theta C}} (M^0 + T \max_{n,j} |\delta_j^n|)$$

Now choosing k small enough that $1 - k\theta C$ does not approach zero
Or let k$\leq \frac{0.5}{\theta C}$ then under this condition

$$max|U_j^n| \leq E(max|u_I| + T \times (\max_{n,j} |\delta_j^n|)) \tag{1.24}$$

ii) QED Problem is stable under the following restriction of step size k and h

$$a_0 - \frac{h}{2} \max_{\overline{\Omega}\times\overline{J}} |b| \geq 0$$

$$2\mu \max_{\overline{\Omega}\times\overline{J}} a + k \max_{\overline{\Omega}\times\overline{J}} |c| \leq 1$$

And $k\theta \max_{\overline{\Omega}\times\overline{J}} |c|$ does not approach 1 or we can take $k\theta \max_{\overline{\Omega}\times\overline{J}} |c| \leq 0.5$

Part (b)

To find the truncation error under the assumption u∈ $C^{4,3}([x_L,x_R] \times [0,T])$

$$\tau^{\theta}_{hk}(x_j, t_{n-1+\theta}) := \mathcal{L}^{\theta}_{hk} u^{n-1+\theta}_j - \mathcal{L} u^{n-1+\theta}_j \tag{1.25}$$

According to 1.14

$$\mathcal{L} u^{n-1+\theta}_j = u_t(x_j, t_{n-1+\theta}) - a u_{xx}(x_j, t_{n-1+\theta}) + b u_x(x_j, t_{n-1+\theta}) + c u(x_j, t_{n-1+\theta})$$

$$\begin{cases} \mathcal{L} u^{n-1+\theta}_j = (\theta u_t(x_j, t_n) + (1-\theta) u_t(x_j, t_{n-1})) - a(\theta u_{xx}(x_j, t_n) + (1-\theta) u_{xx}(x_j, t_{n-1})) \\ +b(\theta u_x(x_j, t_n) + (1-\theta) u_x(x_j, t_{n-1})) + c(\theta u(x_j, t_n) + (1-\theta) u(x_j, t_{n-1})) \end{cases} \tag{1.26}$$

Using Taylors theorem

$$\begin{cases} u(x_j, t_n) = u(x_j, t_{n-1}) + k u_t(x_j, t_{n-1}) + \frac{k^2}{2} u_{tt}(x_j, t_{n-1}) + \mathcal{O}(k^3) \\ u_t(x_j, t_n) = u_t(x_j, t_{n-1}) + k u_{tt}(x_j, t_{n-1}) + \mathcal{O}(k^3) \end{cases}$$

From above 2 equations,

$$\theta u_t(x_j, t_n) + (1-\theta) u_t(x_j, t_{n-1}) = \frac{u(x_j, t_n) - u(x_j, t_{n-1})}{k} + (\theta - \frac{1}{2}) \frac{k}{2} u_{tt}(x_j, t_n) + \mathcal{O}(k^2) \tag{1.27}$$

Using Taylors theorem

$$\begin{cases} u(x_{j-1}, t_n) = u(x_j, t_n) - h u_x(x_j, t_{n-1}) + \frac{h^2}{2} u_{xx}(x_j, t_n) - \frac{h^3}{6} u_{xxx}(x_j, t_n) + \mathcal{O}(h^4) \\ u(x_{j+1}, t_n) = u(x_j, t_n) + h u_x(x_j, t_{n-1}) + \frac{h^2}{2} u_{xx}(x_j, t_n) + \frac{h^3}{6} u_{xxx}(x_j, t_n) + \mathcal{O}(h^4) \end{cases}$$

From above 2 equations,

$$\begin{cases} u_{xx}(x_j, t_n) = \frac{u(x_{j+1}, t_n) - 2u(x_j, t_n) + u(x_{j-1}, t_n)}{h^2} + \mathcal{O}(h^2) \\ u_x(x_j, t_n) = \frac{u(x_{j+1}, t_n) - u(x_{j-1}, t_n)}{2h} - \frac{h^2}{6} u_{xx}(x_j, t_n) + \mathcal{O}(h^3) \end{cases} \tag{1.28}$$

From (1.27) and (1.28) putting values in (1.26)

$$\begin{cases} \mathcal{L}u_j^{n-1+\theta} = \frac{u(x_j,t_n)-u(x_j,t_{n-1})}{k} + (\theta-\frac{1}{2})\frac{k}{2}u_{tt}(x_j,t_n) + \mathcal{O}(k^2) \\ -\theta a(x_j,t_n)(\frac{u(x_{j+1},t_n)-2u(x_j,t_n)+u(x_{j-1},t_n)}{h^2} + \mathcal{O}(h^2)) \\ -(1-\theta)a(x_j,t_{n-1})(\frac{u(x_{j+1},t_{n-1})-2u(x_j,t_{n-1})+u(x_{j-1},t_{n-1})}{h^2} + \mathcal{O}(h^2)) \\ +\theta b(x_j,t_n)(\frac{u(x_{j+1},t_n)-u(x_{j-1},t_n)}{2h} - \frac{h^2}{6}u_{xx}(x_j,t_n) + \mathcal{O}(h^2)) \\ +(1-\theta)b(x_j,t_{n-1})(\frac{u(x_{j+1},t_{n-1})-u(x_{j-1},t_{n-1})}{2h} - \frac{h^2}{6}u_{xx}(x_j,t_{n-1}) + \mathcal{O}(h^2)) \\ +c(\theta u(x_j,t_n) + (1-\theta)u(x_j,t_{n-1})) \end{cases} \tag{1.29}$$

According to theta method we have,

$$\begin{cases} \mathcal{L}_{hk}^{\theta}u_j^{n-1+\theta} = \frac{u(x_j,t_n)-u(x_j,t_{n-1})}{k} \\ -\theta a(x_j,t_n)(\frac{u(x_{j+1},t_n)-2u(x_j,t_n)+u(x_{j-1},t_n)}{h^2}) \\ -(1-\theta)a(x_j,t_{n-1})(\frac{u(x_{j+1},t_{n-1})-2u(x_j,t_{n-1})+u(x_{j-1},t_{n-1})}{h^2}) \\ +\theta b(x_j,t_n)(\frac{u(x_{j+1},t_n)-u(x_{j-1},t_n)}{2h}) \\ +(1-\theta)b(x_j,t_{n-1})(\frac{u(x_{j+1},t_{n-1})-u(x_{j-1},t_{n-1})}{2h}) \\ +c(\theta u(x_j,t_n) + (1-\theta)u(x_j,t_{n-1})) \end{cases} \tag{1.30}$$

Putting values in 1.25,

$$\begin{cases} \tau_{hk}^{\theta} = (\theta-\frac{1}{2})\frac{k}{2}u_{tt}(x_j,t_n) + \mathcal{O}(k^2) + \mathcal{O}(h^2) \end{cases} \tag{1.31}$$

$\equiv$

$$\tau_{hk}^{\theta} = \begin{cases} \mathcal{O}(k^2) + \mathcal{O}(h^2) & \theta = \frac{1}{2} \\ \mathcal{O}(k) + \mathcal{O}(h^2) & \theta = otherwise \end{cases}$$

$$\tau_{hk}^{\theta} = \begin{cases} \mathcal{O}(k^2 + h^2) & \theta = \frac{1}{2} \\ \mathcal{O}(k + h^2) & \theta = otherwise \end{cases} \tag{1.32}$$

Part (c)

For u$\in$ C^{4,3}([x$_L$,x$_R$] $\times$ [0,T])

Under the following restrictions on step size

$$a_0 - \frac{h}{2}\max_{\overline{\Omega}\times\overline{J}}|b| \geq 0$$

$$2\mu \max_{\overline{\Omega}\times\overline{J}} a + k \max_{\overline{\Omega}\times\overline{J}}|c| \leq 1$$

And $k\theta \max_{\overline{\Omega}\times\overline{J}}|c| << 1$ or we can take $k\theta \max_{\overline{\Omega}\times\overline{J}}|c| \leq 0.5$

Let e$_j^n$:=u(x_j, t_n)-U$_j^n$

$$\begin{cases} \mathcal{L}_{hk}^{\theta} U_j^n = f_j^{n-1+\theta} \\ U_j^0 = u_I(x_j), \\ U_0^n = 0 = U_{N_x}^n, \end{cases} \tag{1.33}$$

On application of Taylors theorem

$$\begin{cases} \mathcal{L}_{hk}^{\theta} u(x_j, t_n) = f_j^{n-1+\theta} + \tau_{hk}^{\theta} \\ u(0, j) = u_I(x_j), \\ u(m, 0) = 0 = u(n, N_x), \end{cases} \tag{1.34}$$

On Subtracting both equations we get

$$\begin{cases} \mathcal{L}_{hk}^{\theta} e_j^n = \tau_{hk}^{\theta} \\ e_j^0 = 0, \\ e_0^n = 0 = e_{N_x}^n, \end{cases} \tag{1.35}$$

Now, the problem is similar to 1.21 this implies

$$(1+\mu\theta 2a_j^{n-1+\theta}+k\theta c_j^{n-1+\theta})|e_j^n| \leq (\mu\theta a_j^{n-1+\theta}-\frac{\mu h\theta}{2}b_j^{n-1+\theta})|e_{j+1}^n|+(\mu\theta a_j^{n-1+\theta}+\frac{\mu h\theta}{2}b_j^{n-1+\theta})|e_{j-1}^n|$$

$$+(1-2\mu(1-\theta)a_j^{n-1+\theta}-k(1-\theta)c_j^{n-1+\theta})|e_j^{n-1}|+(\mu(1-\theta)a_j^{n-1+\theta}-\frac{\mu h(1-\theta)}{2}b_j^{n-1+\theta})|e_{j-1}^{n-1}|$$

$$+(\mu(1-\theta)a_j^{n-1+\theta}+\frac{\mu h(1-\theta)}{2}b_j^{n-1+\theta})|e_{j-1}^{n-1}|+k|\tau_{hk}^{\theta}|$$

Let $M^n := \underset{j}{max}\ |e_j^n|$ and on solving above equation we get,

$$(1+k\theta c_j^{n-1+\theta})M^n \leq (1-k(1-\theta)c_j^{n-1+\theta})M^{n-1}+k\max_j|\tau_{hk}^{\theta}|$$

On solving further similar to equation (1.23) we get

$$M^n \leq P(M^0+T\max_{j,n}|\tau_{hk}^{\theta}|)$$

where P is some constant. For our equation $M^0=0$ Therefore,

$$\max_n M^n \leq PT\max_{j,n}|\tau_{hk}^{\theta}|$$

From 1.31 we get

$$\max_{j,n}\ |e_j^n| \leq PT\max_{j,n}|\tau_{hk}^{\theta}| \leq \mathcal{C}((\theta-\frac{1}{2})k+k^2+h^2)$$

$\equiv$

$$\max_{j,n}\ |e_j^n| = \begin{cases} \mathcal{O}(k^2+h^2) & \theta=\frac{1}{2} \\ \mathcal{O}(k+h^2) & \theta=otherwise \end{cases}$$