

# DEPARTMENT OF COMPUTER SCIENCE & ENGINEERING

## PhD Qualifier Examination, Paper I

Total time: 2 Hours

October 28, 2009

Maximum Marks: 120

[Answer ALL the parts]

### Part A: Discrete mathematics

[Answer ALL questions]

- A.1 Let  $\Sigma$  be the alphabet  $\{a, b, c\}$ . Show that the number of words of length  $n$  in which the letter  $a$  appears an even number of times is  $(3^n + 1)/2$ . (You may use induction or any other proof technique.) [10]
- A.2 For any positive integer  $n$ , it is possible to draw  $n$  circles in the Euclidean plane such that every pair of circles intersects at two distinct points, and no three circles have a point in common. Let  $a_n$  denote the total number of regions formed by  $n$  such intersecting circles.
- (a) Compute  $a_1, a_2, a_3$  and  $a_4$ . [1]
- (b) Find a recurrence relation satisfied by  $a_n$  for  $n \geq 1$ . [4]
- (c) Solve the recurrence relation above. [5]
- A.3 (a) How many onto (surjective) functions are there from a set with six elements to a set with three elements? [5]
- (b) Let  $f : A \rightarrow B$  and  $g : B \rightarrow C$  be functions. Prove that if  $g \circ f$  is injective and  $f$  is surjective, then  $g$  is injective. [5]
- A.4 A cabinet has 3 drawers. In the first drawer, there are 2 gold balls, in the second drawer, there are two silver balls, and in the third drawer, there are one gold ball and one silver ball. A drawer is picked at random, and a ball is chosen at random from the two balls in the drawer. Given that a gold ball was drawn, what is the probability that the drawer with the two gold balls was chosen? [10]

### Part B: Algorithms

[Answer ANY FOUR questions]

- B.1 The running time of an algorithm is  $T(n)$  for an input of size  $n$ . Suppose that  $T(n)$  satisfies the following recurrence relation.

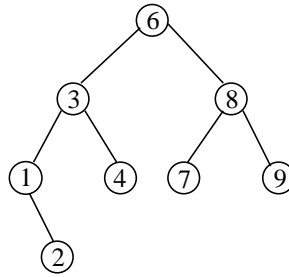
$$\begin{aligned} T(1) &= 1, \\ T(n) &= 3T(n-1) + 1 \text{ for } n \geq 2. \end{aligned}$$

Prove that  $T(n) = \Theta(3^n)$ . [10]

- B.2 Explain how the merge sort algorithm sorts the following array of eight elements. [10]

7	3	8	2	1	9	6	4
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B.3 Consider the following binary search tree.



- (a) Suppose that the new value 5 is to be inserted in this tree. Indicate which nodes of the tree are to be visited for locating the insertion position. Draw the tree after this insertion. [3+2]
- (b) After this insertion, the value 3 is to be deleted from the tree. Explain which nodes in the tree are to be visited for the deletion. Draw the tree after this deletion. [3+2]
- B.4 Explain an efficient algorithm to detect the presence of cycles in an undirected graph. What is the running time of your algorithm? [7+3]
- B.5 Let LONG-CYCLE denote the computational problem of deciding whether a given undirected graph  $G$  on  $n$  vertices contains a cycle of length  $\geq n/2$ . Prove that LONG-CYCLE is NP-complete. You may assume that the Hamiltonian cycle problem (on undirected graphs) is NP-complete. [10]

### Part C: Formal Languages and Automata Theory

[Answer ANY FOUR questions]

- C.1 Design a DFA for the language [10]  
 $L = \{w \mid w \text{ contains an equal number of occurrences of } 01 \text{ and } 10 \text{ as substrings}\}.$   
 (Note that the strings 101 and 010 are in  $L$ .)
- C.2 Let  $L = \{w \mid w \text{ contains an even number of } a\text{'s and an odd number of } b\text{'s and does not contain the substring } ab\}$ . Give a DFA that recognizes  $L$  and a regular expression that generates  $L$ . (Hint: Describe  $L$  more simply.) [8+2]
- C.3 (a) Indicate with proper justification whether the following statement is *true* or *false*: If  $N$  is an NFA that recognizes the language  $L$ , swapping the accept and nonaccept states yields a new NFA that recognizes the complement of  $L$ . [2]  
 (b) Give a recognizer for the language  $L = \{a^i b^j c^k \mid i = j \text{ or } j = k\}$ . Explain the working of the recognizer. [8]
- C.4 (a) Let  $G$  be the grammar  

$$\begin{aligned} S &\rightarrow aB \mid bA, \\ A &\rightarrow a \mid aS \mid bAA, \\ B &\rightarrow b \mid bS \mid aBB. \end{aligned}$$
  
 Is the grammar unambiguous? Explain your answer. [5]  
 (b) Consider the following decision problem: Given any CFG  $G$ , is  $\mathcal{L}(G) = \emptyset$ ? Show that it is a decidable problem by giving a decision algorithm for the problem. [5]
- C.5 Give a context-free grammar for the set of all strings over the alphabet  $\{a, b\}$  with exactly twice as many  $a$ 's as  $b$ 's. Explain the working of the grammar by characterizing the strings generated by each non-terminal. [10]