DEPARTMENT OF COMPUTER SCIENCE & ENGINEERING

PhD Qualifier Examination, Paper I

Total time: 2 Hours **March 16, 2010** **Maximum Marks: 120**

Answer ALL the parts

Part A: Discrete Mathematics

Answer ANY FOUR questions

A.1 In a round-robin tournament, every player plays every other player exactly once, and each match has a winner and a loser (that is, there is no tie). We say that a sequence of m players $p_1, p_2, \ldots, p_m$ forms a beating sequence if p_1 beats p_2, p_2 beats p_3, $\ldots$, and p_{m-1} beats p_m. Use mathematical induction to show that in every round-robin tournament of n players where $n \geq 2$, there must exist a beating sequence of length n. **(10)**

A.2 (a) Prove or disprove the following statement: In any graph $G = (V, E)$ with a finite number of vertices, there always exist two vertices with the same degree. **(5)**

(b) Prove that if a tree T has a vertex with degree at least 3, then T has at least 3 vertices with degree 1. You may make use of the facts that for a graph G with e edges, the sum of the degrees of the vertices of G is $2e$, and that a tree with n vertices has exactly $n - 1$ edges. **(5)**

A.3 (a) Determine the number of paths in the x-y plane starting at the origin $(0, 0)$ and ending at the point (m, n). Here, each path is made up of a series of steps, where each step is a move one unit to the right or a move one unit upward. **(5)**

(b) Three dice (red, green, blue) are rolled. What is the probability of getting three distinct numbers? Explain how you found your answer. **(5)**

A.4 Let $f : A \to B$ and $g : B \to C$ be functions. State whether each of the following statements is necessarily true. If a statement is true, give a proof. Otherwise, give a counterexample. **(5×2)**

(i) If the function $g \circ f : A \to C$ is injective, then f must be injective.

(ii) If the function $g \circ f$ is surjective, then f must be surjective.

A.5 Let $A = \{1, 2, 3, 4\} \times \{1, 2, 3, 4\}$, and let $\mathcal{R}$ be a binary relation on A defined by

$$\mathcal{R} = \{((x_1, y_1); (x_2, y_2)) \mid x_1^2 - y_1 = x_2^2 - y_2\}.$$

(a) Show that $\mathcal{R}$ is an equivalence relation on A. **(5)**

(b) Find the equivalence class containing $(1, 2)$. **(5)**

Part B: Algorithms

Answer ANY FOUR questions

B.1 Suppose that the running time $T(n)$ of an algorithm on an input of size n satisfies the recurrence relation

$$T(n) = T\left(\left\lceil \frac{n}{2} \right\rceil\right) + T\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + 1$$

for $n \geq 2$. Prove that $T(n) = \Theta(n)$. **(10)**

B.2 Suppose you want to sort n integers $a_1, a_2, \ldots, a_n$. To that effect, you first insert $a_1, a_2, \ldots, a_n$ in an initially empty binary search tree. What is the maximum time needed for these n insertions? Explain how you can generate the desired sorted list from this tree in $\mathrm{O}(n)$ time. **(5+5)**

B.3 The integers $A(m, n)$ for integer-valued arguments $m, n \geq 0$ are defined recursively as

$$\begin{aligned} A(0,n) &= n \text{ for all } n \geq 0, \\ A(m,0) &= m \text{ for all } m \geq 0, \\ A(m,n) &= mA(m-1,n) + nA(m,n-1) - A(m-1,n-1) \text{ for all } m,n \geq 1. \end{aligned}$$

Devise a dynamic-programming algorithm to compute $A(m, n)$ in time polynomial in m and n. **(10)**

B.4 Let $G = (V, E)$ be an undirected graph. The *eccentricity* $\epsilon(u)$ of a node $u \in V$ is the maximum of the distances $d(u, v)$ as v ranges over V (where distance is measured by the number of edges). A node u with the largest eccentricity is called a *center* of the graph G. Supply a polynomial-time algorithm to identify a center in G. Also specify the running time of your algorithm. **(7+3)**

B.5 An undirected graph G is called k*-colorable* if the nodes of G can be assigned k colors in such a way that no two adjacent nodes receive the same color. By k-COLOR, we denote the problem of deciding whether G is k-colorable. Given that 3-COLOR is an NP-Complete problem, prove that the problem 4-COLOR is NP-Complete too. **(10)**

Part C: Formal Languages and Automata Theory

Answer ALL questions

C.1 Design a DFA for the language

$$L = \{w \mid w \in \{0,1\}^*, w \text{ represents a non-negative integer in binary with } mod(w,4) = 3\}.$$ **(10)**

C.2 Design a DPDA for the language

$$L = \{w \mid w \in \{0,1\}^*, w \text{ contains as many 0's as 1's}\}$$

Give only the state transition diagram of the DPDA. **(10)**

C.3 (a) Show that the language

$$L = \{1^k y \mid y \in \{0,1\}^* \text{ and } y \text{ contains at least } k \text{ 1's, for } k \geq 1\}$$

is regular by giving a DFA recognizer for L and explaining its operation clearly. **(5)**

(b) Assume the fact that if L_c is a CFL and L_r is regular, then $L_c \cap L_r$ is a CFL. Assume also the fact that the language $\{a^n b^n c^n \mid n \geq 0\}$ is not a CFL. Show that the language

$$\{w \in \{a,b,c\}^* \mid w \text{ contains an equal number of } a\text{'s, } b\text{'s and } c\text{'s}\}$$

is not a CFL. **(5)**

C.4 (a) Give a context-free grammar for the language

$$L = \{w \mid w \in \{0,1\}^+, w \text{ contains as many 0's as 1's}\}.$$ **(7)**

(b) Assuming that the language $ALL_{CFG} = \{\langle G \rangle \mid G \text{ is a CFG and } L(G) = \Sigma^*\}$ is undecidable, show that the language $EQ_{CFG} = \{\langle G_1, G_2 \rangle \mid G_1, G_2 \text{ are CFGs and } L(G_1) = L(G_2)\}$ is undecidable. **(3)**
