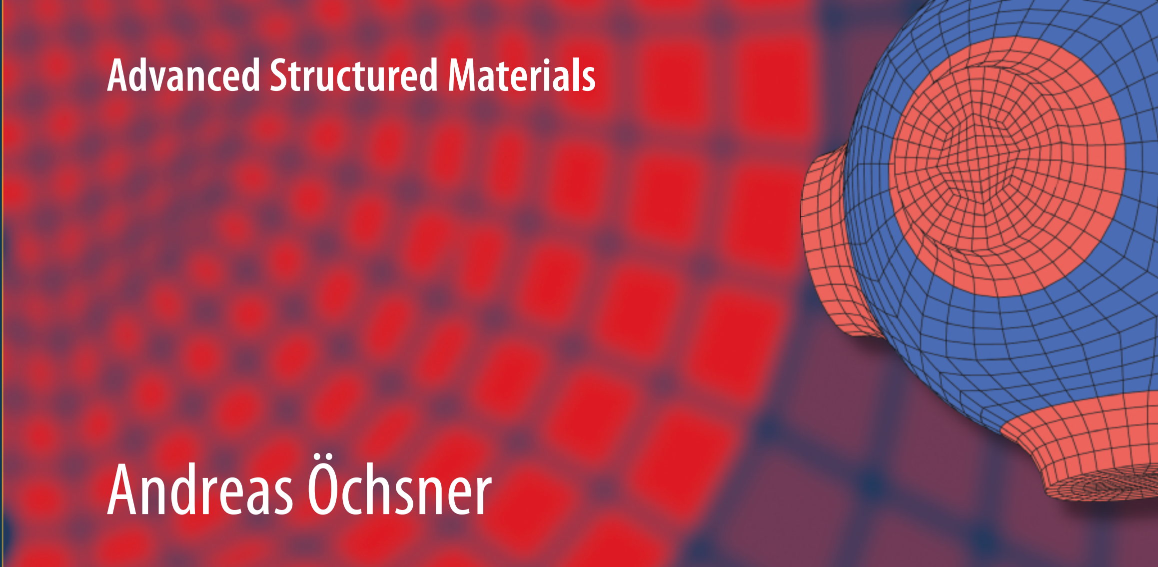


Advanced Structured Materials

Andreas Öchsner

Foundations of Classical Laminate Theory

Springer

Advanced Structured Materials

Volume 163

Series Editors

Andreas Öchsner, Faculty of Mechanical Engineering, Esslingen University of Applied Sciences, Esslingen, Germany

Lucas F. M. da Silva, Department of Mechanical Engineering, Faculty of Engineering, University of Porto, Porto, Portugal

Holm Altenbach, Faculty of Mechanical Engineering, Otto von Guericke University Magdeburg, Magdeburg, Sachsen-Anhalt, Germany

Common engineering materials reach in many applications their limits and new developments are required to fulfil increasing demands on engineering materials. The performance of materials can be increased by combining different materials to achieve better properties than a single constituent or by shaping the material or constituents in a specific structure. The interaction between material and structure may arise on different length scales, such as micro-, meso- or macroscale, and offers possible applications in quite diverse fields.

This book series addresses the fundamental relationship between materials and their structure on the overall properties (e.g. mechanical, thermal, chemical or magnetic etc.) and applications.

The topics of *Advanced Structured Materials* include but are not limited to

- classical fibre-reinforced composites (e.g. glass, carbon or Aramid reinforced plastics)
- metal matrix composites (MMCs)
- micro porous composites
- micro channel materials
- multilayered materials
- cellular materials (e.g., metallic or polymer foams, sponges, hollow sphere structures)
- porous materials
- truss structures
- nanocomposite materials
- biomaterials
- nanoporous metals
- concrete
- coated materials
- smart materials

Advanced Structured Materials is indexed in Google Scholar and Scopus.

More information about this series at http://www.springer.com/series/8611

Andreas Öchsner

Foundations of Classical Laminate Theory

Springer

Andreas Öchsner
Faculty of Mechanical Engineering
Esslingen University of Applied Sciences
Esslingen am Neckar, Germany

ISSN 1869-8433 ISSN 1869-8441 (electronic)
Advanced Structured Materials
ISBN 978-3-030-82630-7 ISBN 978-3-030-82631-4 (eBook)
https://doi.org/10.1007/978-3-030-82631-4

© The Editor(s) (if applicable) and The Author(s), under exclusive license to Springer Nature Switzerland AG 2021
This work is subject to copyright. All rights are solely and exclusively licensed by the Publisher, whether the whole or part of the material is concerned, specifically the rights of translation, reprinting, reuse of illustrations, recitation, broadcasting, reproduction on microfilms or in any other physical way, and transmission or information storage and retrieval, electronic adaptation, computer software, or by similar or dissimilar methodology now known or hereafter developed.
The use of general descriptive names, registered names, trademarks, service marks, etc. in this publication does not imply, even in the absence of a specific statement, that such names are exempt from the relevant protective laws and regulations and therefore free for general use.
The publisher, the authors and the editors are safe to assume that the advice and information in this book are believed to be true and accurate at the date of publication. Neither the publisher nor the authors or the editors give a warranty, expressed or implied, with respect to the material contained herein or for any errors or omissions that may have been made. The publisher remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

This Springer imprint is published by the registered company Springer Nature Switzerland AG
The registered company address is: Gewerbestrasse 11, 6330 Cham, Switzerland

Preface

Composite materials, especially fiber-reinforced composites, are gaining increasing importance since they can overcome the limits of many structures based on classical metals. Particularly the combination of a matrix with fibers provides far better properties than the components alone. Despite their importance, many engineering degree programs do not treat the mechanical behavior of this class of advanced structured materials in detail. Thus, some engineers are not able to thoroughly apply and introduce these modern engineering materials in their design process.

Partial differential equations lay the foundation to mathematically describe the mechanical behavior of any classical structural member known in engineering mechanics, including composite materials. Based on the three basic equations of continuum mechanics, i.e., the kinematic relationship, the constitutive law, and the equilibrium equation, these partial differential equations describe the physical problem. The so-called classical laminate theory provides a simplified stress analysis, and a subsequent failure analysis, without the solution of the system of coupled differential equations for the unknown displacements in the three coordinate directions. This theory provides solution for the statically indeterminate system based on a generalized stress–strain relationship under consideration of the constitutive relationship and the definition of the so-called stress resultants. Nevertheless, the fundamental knowledge from the first years of engineering education, i.e., higher mathematics, physics, materials science, applied mechanics, design, and programming skills, might be required to master this topic.

This monograph provides a systematic and thorough introduction to the classical laminate theory based on the theory for plane elasticity elements and classical (shear-rigid) plate elements. The focus is on unidirectional lamina which can be described based on orthotropic constitutive equations and their composition to layered laminates. In addition to the elastic behavior, failure is investigated based on the maximum stress, maximum strain, Tsai-Hill, and the Tsai-Wu criteria.

Esslingen, Germany
June 2021

Andreas Öchsner

Contents

Symbols and Abbreviations

Latin Symbols (Capital Letters)

A	Area, cross-sectional area
A	Submatrix of the generalized elasticity matrix
A'	Inverted submatrix
B	Submatrix of the generalized elasticity matrix
B'	Inverted submatrix
C	Element of elasticity matrix
$\overline{C}$	Element of transformed elasticity matrix
C	Elasticity matrix
$\overline{\boldsymbol{C}}$	Transformed elasticity matrix
C^*	Generalized elasticity matrix
D	Element of the compliance matrix
$\overline{D}$	Element of transformed compliance matrix
D	Compliance matrix, submatrix of the generalized elasticity matrix
D'	Inverted submatrix
$\overline{\boldsymbol{D}}$	Transformed compliance matrix
E	Elasticity modulus
F	Force
G	Shear modulus
I	Second moment of area
M	Moment
$\boldsymbol{M}$	Column matrix of moments
N	Normal force (internal)
$\boldsymbol{N}$	Column matrix of normal force
Q	Shear force (internal)
$\boldsymbol{Q}$	Column matrix of shear forces
R	Strength ratio

$\boldsymbol{R}$ Transformation matrix
$\boldsymbol{T}$ Transformation matrix
V Volume

Latin Symbols (Small Letters)

a Geometric dimension, coefficient
b Geometric dimension
$\boldsymbol{b}$ Column matrix of body forces
$\boldsymbol{e}$ Column matrix of generalized strains
f Body force
$\boldsymbol{g}$ Standard gravity
$k_{1\mathrm{c}}$ Longitudinal compressive strength in direction *1*
$k_{1\mathrm{t}}$ Longitudinal tensile strength in direction *1*
$k_{12\mathrm{s}}$ In-plane shear strength in plane *12*
$k_{2\mathrm{c}}$ Longitudinal compressive strength in direction *2*
$k_{2\mathrm{t}}$ Longitudinal tensile strength in direction *2*
$k^{\varepsilon}_{1\mathrm{c}}$ Longitudinal compressive failure strain in direction *1*
$k^{\varepsilon}_{1\mathrm{t}}$ Longitudinal tensile failure strain in direction *1*
$k^{\varepsilon}_{12\mathrm{s}}$ In-plane shear failure strain in plane *12*
$k^{\varepsilon}_{2\mathrm{c}}$ Longitudinal compressive failure strain in direction *2*
$k^{\varepsilon}_{2\mathrm{t}}$ Longitudinal tensile failure strain in direction *2*
m Distributed moment, mass
n Layer number
q Distributed load in thickness direction, area-specific load
$\boldsymbol{q}$ Column matrix of distributed loads
$\boldsymbol{s}$ Column matrix of stress resultants
t Lamina or laminate thickness
u Displacement
$\boldsymbol{u}$ Column matrix of displacements
x Cartesian coordinate
y Cartesian coordinate
z Cartesian coordinate

Greek Symbols (Small Letters)

α Rotation angle
γ Shear strain (engineering definition)
ε Normal strain
ε Column matrix of strain components
η Shear-extension coupling corefficient

κ Curvature
$\boldsymbol{\kappa}$ Column matrix of curvature components
ν Poisson's ratio
ρ Mass density
σ Stress
$\boldsymbol{\sigma}$ Column matrix of stress components
φ Rotation angle

Mathematical Symbols

$\mathcal{L}\{\ldots\}$ Differential operator
$\boldsymbol{\mathcal{L}}$ Matrix of differential operators

Indices, Superscripted

$\ldots^{\mathrm{n}}$ Normalized
$\ldots^{0}$ Related to the middle-surface

Indices, Subscripted

$\ldots_{1}$ Direction *1*
$\ldots_{2}$ Direction *2*
$\ldots_{12}$ Plane *12*
$\ldots_{\mathrm{c}}$ Compression
$\ldots_{\mathrm{s}}$ Shear, symmetric
$\ldots_{\mathrm{t}}$ Tension

Abbreviations

BEM Boundary element method
CLT Classical laminate theory
FDM Finite difference method
FEM Finite element method
FVM Finite volume method

Chapter 1
Introduction

Abstract The first chapter introduces the major concept of composite material, i.e., the combination of different components, to obtain in total much better properties than a single component for itself. The focus is on fiber-reinforced composites where a single layer has unidirectionally aligned reinforcing fibers. The second part of this chapter introduces to major idea and the continuum mechanical background to model structural members. It is explained that physical problems are described based on differential equations. In the context of structural mechanics, these differential equations are obtained by combining the three basic equations of continuum mechanics, i.e., the kinematics relationship, the constitutive law, and the equilibrium equation. Furthermore, some explanations on the simplifications of the chosen theory, i.e., the classical laminate theory, are provided.

1.1 Composite Materials

Composite materials refer to a wide class of advanced materials that are composed of different materials or phases. The major idea is to obtain in total much better properties than a single component for itself could provide. Typical particularities include matrices (e.g., polymers, metals, or ceramic materials) which contain a second material in the form of particles or fibers as the reinforcing elements (see Fig. 1.1 for a schematic representation) or materials which are shaped in a particular way such as cellular materials (e.g., metals foams [4] or hollow sphere structures [32]). In the case of cellular materials, the macroscopic properties are defined by the base material as well as the shape of the cells (voids, cavities, free space, or air) and their arrangement (e.g., periodic or stochastic).

For the particular group of fiber-reinforced materials, one may distinguish between classical fibers such as glass, carbon and boron [12], natural fibers [30], and nanofibers (e.g., carbon nanotubes) [44]. The following chapters will focus on an important configuration, i.e., unidirectional fiber-reinforced thin layers. Such single layers or plies, also called a lamina, provide superior mechanical properties compared to short fibers with the random arrangement. The common approach is to stack several of such lamina with different orientations with respect to some reference direction to

© The Author(s), under exclusive license to Springer Nature Switzerland AG 2021
A. Öchsner, *Foundations of Classical Laminate Theory*, Advanced Structured Materials 163, https://doi.org/10.1007/978-3-030-82631-4_1

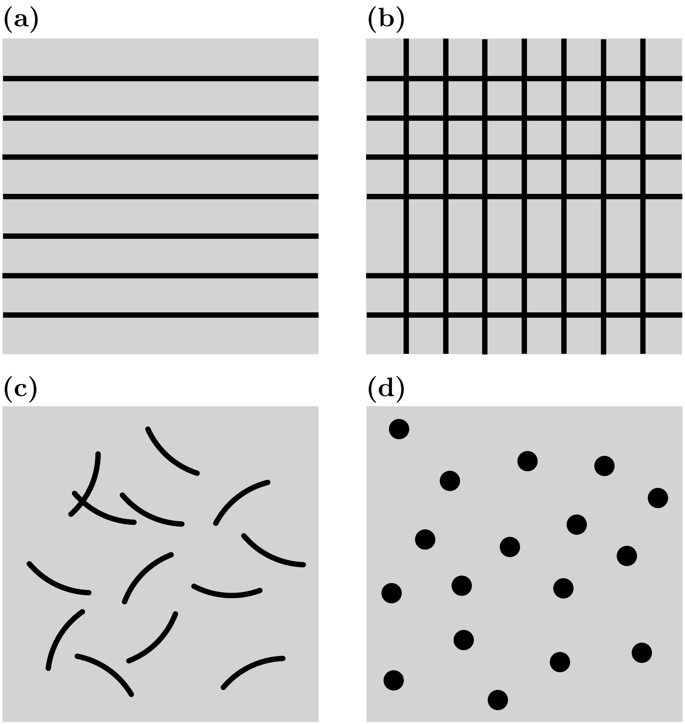


Fig. 1.1 Different types of composite materials (matrix = gray, reinforcement = black): **a** unidirectional fibers, **b** woven fibers, **c** short fibers, and **d** particles

form a so-called laminate, see Fig. 1.2 for an example. Thus, the physical properties are dependent on the matrix material, the fibers, and the number/orientation/sequence (lay-up) of the plies which allows to tailor the macroscopic properties by adjusting the different parameters. This allows much greater flexibility to adjust properties compared to classical engineering materials.

To clearly indicate the lay-up of a laminate, it is common to use the so-called laminate orientation code, i.e., some common rules to describe the different layers (laminae) of a laminate, see [29] for further details:

- Square brackets are used to indicate the beginning and the end of the laminate code: [...].
- The orientation of each lamina is indicated by the angle between the fiber direction and the global x-axis.
- Orientations of successive laminae are separated by a virgule (/). They are listed in order from the first layer up to the last toward the positive z-direction: [.../.../...].
- Adjacent laminae with the same orientation are indicated by adding a subscript to the angle, equal to the number of repetitions.

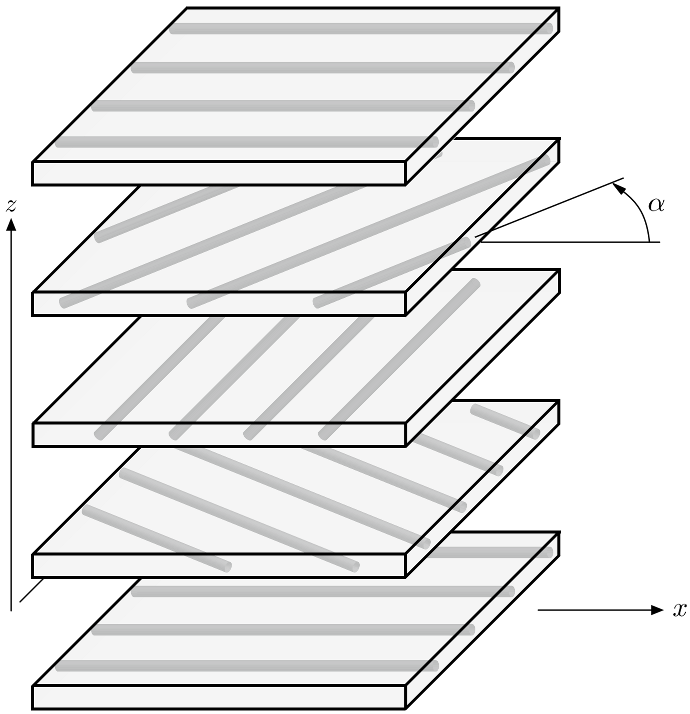


Fig. 1.2 Unbonded view of single laminae forming a 5-layer laminate

- A subscript of "s" is used if the first half of the lay-up is indicated and the second half is symmetric: $[\ldots/\ldots/\ldots]_s$. For a symmetric lay-up with an odd number of laminae, the layer which is not repeated is indicated by overlining the angle of that lamina: $[\ldots/\ldots/\overline{\ldots}]_s$.

These rules imply that each layer has the same material properties and thickness. Should this be not the case, and more information must be provided. Some examples of different laminates and the corresponding laminate codes are provided in Table 1.1. It can be seen that the same laminate may be described by different representations, e.g., due to the symmetry or particular sequences.

Some of the textbooks which cover the mechanics of classical fiber-reinforced composite materials are summarized in Table 1.2. The interested reader may choose some of them as supplementary reading material to deepen and widen the knowledge presented in this textbook. Topics which are not covered in this textbook, e.g., manufacturing [20] or the prediction of the properties of a single lamina (the so-called micromechanics) [9, 18, 22] can be found in the provided references at the end of this chapter.

Table 1.1 Example laminate orientation codes

Case	Cross Section	Laminate Code
(a)	z ↑ 8 0° 7 $+60^\circ$ 6 -60° 5 90° 4 90° 3 -45° 2 $+45^\circ$ $n=1$ 0°	$[0^\circ/45^\circ/-45^\circ/90^\circ/90^\circ/-60^\circ/60^\circ/0^\circ]$ $=[0^\circ/\pm 45^\circ/90^\circ_2/\mp 60^\circ/0^\circ]$
(b)	z ↑ 4 0° 3 90° 2 90° $n=1$ 0°	$[0^\circ/90^\circ/90^\circ/0^\circ]$ $=[0^\circ/90^\circ]_s$
(c)	z ↑ 3 90° 2 0° $n=1$ 90°	$[90^\circ/0^\circ/90^\circ]$ $=[90^\circ/\overline{0^\circ}]_s$

1.2 Continuum Mechanical Modeling

Typical thin laminates composed of orthotropic laminae (plies) as described in the previous section can be modeled under particular assumptions as a simple superposition of a plane elasticity and a classical plate element. These elements are described separately in Chap. 2 and subsequently composed to a combined plane elasticity and classical plate element. Let us highlight here that the derivations in the following chapters follow a common approach from continuum mechanics [3], see Fig. 1.3.

A combination of the kinematics equation (i.e., the relation between the strains and deformations) with the constitutive equation (i.e., the relation between the stresses and strains) and the equilibrium equation (i.e., the equilibrium between the internal reactions and the external loads) results in a partial differential equation or a corresponding system of differential equations. Limited to simple problems and configurations, *analytical* solutions are possible. These analytical solutions are then exact in the frame of the assumptions made. However, the solution of complex problems requires the application of numerical methods such as the finite element method, see [33, 34] for general details on the method and Table 1.3 for specialized literature on finite element simulation of composite materials. These *numerical* solutions are, in general, no longer exact since the numerical methods provide only an approximate solution. Thus, the major task of an engineer is then to ensure that the approximate

Table 1.2 Some of the English textbooks which cover the mechanics of classical fiber-reinforced composite materials (no claim to completeness)

Year (1st ed.)	Author	Title	Ref.
1969	J. E. Ashton, J. C. Halpin, P. H. Petit	Primer on Composite Materials: Analysis	[5]
1969	L. R. Calcote	The Analysis of Laminated Composite Structures	[11]
1970	J. E. Ashton, J. M. Whitney	Theory of Laminated Plates	[6]
1975	R. M. Jones	Mechanics of Composite Materials	[24]
1975	J. R. Vinson, T. W. Chou	Composite Materials and their Use in Structures	[41]
1979	R. M. Christensen	Mechanics of Composite Materials	[13]
1980	B. D. Agarwal, L. J. Broutman, K. Chandrashekhara	Analysis and Performance of Fiber Composites	[1]
1980	S. W. Tsai, H. T. Hahn	Introduction to Composite Materials	[38]
1981	D. Hull	An Introduction to Composite Materials	[23]
1987	K. K. Chawla	Composite Materials: Science and Engineering	[12]
1987	J. R. Vinson, R. L. Sierakowski	The Behavior of Structures Composed of Composite Materials	[42]
1987	J. M. Whitney	Structural Analysis of Laminated Anisotropic Plates	[43]
1990	V. K. S. Choo	Fundamentals of Composite Materials	[14]
1994	I. M. Daniel, O. Ishai	Engineering Mechanics of Composite Materials	[15]
1994	R. F. Gibson	Principles of Composite Material Mechanics	[19]
1994	J. N. Reddy	Mechanics of Composite Materials: Selected Works of Nicholas J. Pagano	[35]
1997	A. K. Kaw	Mechanics of Composite Materials	[27]
1998	C. T. Herakovich	Mechanics of Fibrous Composites	[21]
1999	J.-M. Berthelot	Composite Materials: Mechanical Behavior and Structural Analysis	[8]
2001	V. V. Vasilev, E. V. Morozov	Mechanics and Analysis of Composite Materials	[40]
2002	C. Decolon	Analysis of Composite Structures	[16]
2003	L. P. Kollár, G. S. Springer, W. Kissing	Mechanics of Composite Structures	[26]
2004	H. Altenbach, J. Altenbach, W. Kissing	Mechanics of Composite Structural Elements	[2]
2004	M. E. Tuttle	Structural Analysis of Polymeric Composite Materials	[39]
2008	R. de Borst, T. Sadowski	Lecture Notes on Composite Materials: Current Topics and Achievements	[10]
2010	C. Kassapoglou	Design and Analysis of Composite Structures with Applications to Aerospace Structures	[25]

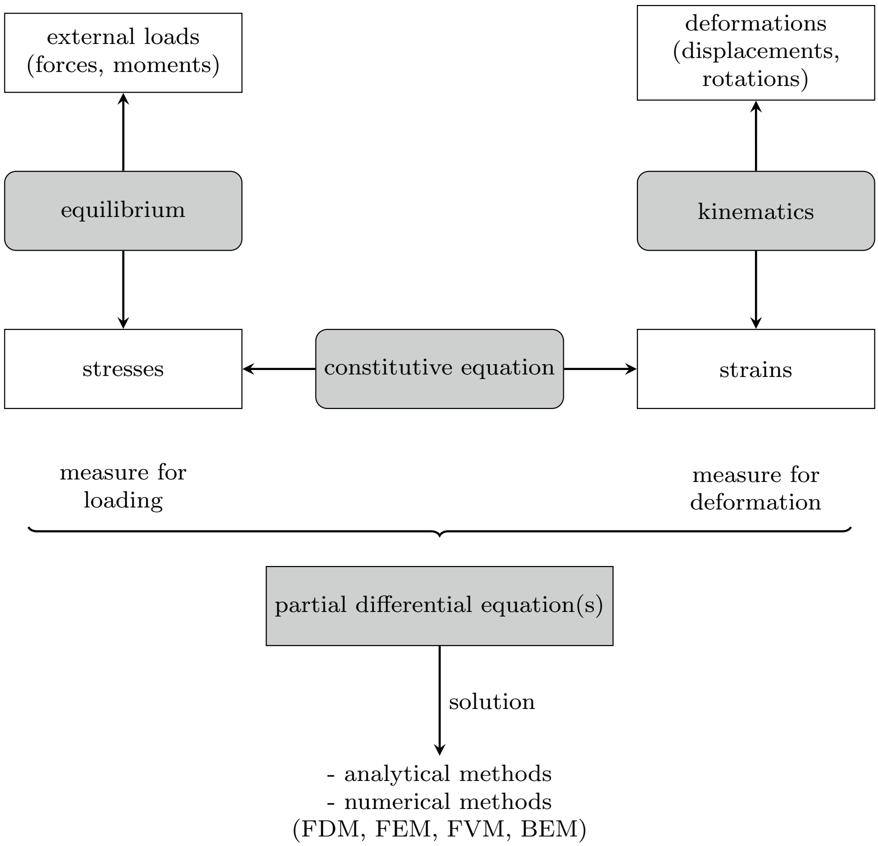


Fig. 1.3 Continuum mechanical modeling of structural members

solution is as good as possible. This requires a lot of experience and a solid foundation in the basics of applied mechanics, materials science, and mathematics.

The following chapters present a simplified approach, the so-called classical laminate theory (CLT) [17, 36], which aims to provide a stress and a subsequent strength/failure analysis without the solution of the system of coupled differential equations for the unknown displacements in the three coordinate directions. This theory provides the solution of the statically indeterminate system based on generalized stress–strain relationship under consideration of the constitutive relationship and the definition of the so-called stress resultants or generalized stresses and strains.

The common assumptions of the classical laminate theory and the derivations in the following chapters can be summarized as follows (e.g., [15]):

1. Each lamina is considered quasi-homogeneous and orthotropic (in general, the properties can range from isotropic to anisotropic).
2. Only unidirectional and flat lamina are considered in the following chapters.

Table 1.3 Some of the early textbooks which cover the finite element simulation of composite materials (no claim to completeness)

Year (1st ed.)	Author	Title	Ref.
1992	O. O. Ochoa, J.N. Reddy	Finite Element Analysis of Composite Laminates	[31]
1998	L. T. Tenek, J. Argyris	Finite Element Analysis for Composite Structures	[37]
2000	F. L. Matthews, G. A. O. Davies, D. Hitchings, C. Soutis	Finite Element Modeling of Composite Materials and Structures	[28]
2008	E. J. Barbero	Finite Element Analysis of Composite Materials	[7]

3. The laminate consists of perfectly bonded laminae and the bond lines are infinitesimally thin as well as non-shear-deformable.
4. The laminate is thin, i.e., the thickness is small compared to the lateral dimensions, and represents a state of plane stress.
5. Displacements (in thickness and lateral directions) are small compared to the thickness of the laminate.
6. Displacements are continuous throughout the laminate (non-shear-deformable bond lines).
7. In-plane displacements are linear functions of the thickness.
8. Shear strains in planes perpendicular to the middle surface are negligible.
9. Assumptions 7 and 8 imply that a line originally *straight* and *perpendicular* to the laminate middle surface remains so after deformation (Kirchhoff's hypothesis of classical plate theory).
10. Kinematics and constitutive relations are linear.
11. Normal distances from the middle surface remain constant. Thus, the transverse normal strain is negligible compared to the in-plane normal strains (Kirchhoff's hypothesis of classical plate theory).

References

1. Agarwal BD, Broutman LJ, Chandrashekhara K (1980) Analysis and performance of fiber composites. Wiley, New York
2. Altenbach H, Altenbach J, Kissing W (2004) Mechanics of composite structural elements. Springer, Berlin
3. Altenbach H, Öchsner A (eds) (2020) Encyclopedia of continuum mechanics. Springer, Berlin

4. Ashby M, Evans AG, Fleck NA, Gibson LJ, Hutchinson JW, Wadley HNG (2000) Metal foams: a design guide. Butterworth-Heinemann, Boston
5. Ashton JE, Halpin JC, Petit (PH) Primer on composite materials: analysis. Technomic Publishing Company, Stamford
6. Ashton JE, Whitney JM (1970) Theory of laminated plates. Technomic Publishing, Stamford
7. Barbero EJ (1996) Finite element analysis of composite materials. CRC Press, Boca Raton
8. Berthelot J-M (1999) Composite materials: mechanical behavior and structural analysis. Springer, Berlin
9. Buryachenko VA (2007) Micromechanics of heterogeneous materials. Springer, New York
10. de Borst R, Sadowski T (2008) Lecture notes on composite materials:current topics and achievements. Springer, Dordrecht
11. Calcote LR (1969) The analysis of laminated composite structures. Van Nostrand Reinhold Company, New York
12. Chawla KK (1987) Composite materials: science and engineering. Springer, New York
13. Christensen RM (1979) Mechanics of composite materials. Wiley, New York
14. Choo VKS (1987) Fundamentals of composite materials. Knowen Academic Press, Delaware
15. Daniel IM, Ishai O (1994) Engineering mechanics of composite materials. Oxford University Press, New York
16. Decolon C (2002) Analysis of composite structures. Hermes Penton Science, London
17. Dong SB, Pister KS, Taylor RL (1962) On the theory of laminated anisotropic shells and plates. J Aerosp Sci 29:696–975
18. Dvorak GJ (2013) Micromechanics of composite materials. Springer, New York
19. Gibson RF (1969) Principles of composite material mechanics. McGraw-Hill, New York
20. Hall W, Javanbakht Z (2021) Design and manufacture of fibre-reinforced composites. Springer, Cham
21. Herakovich CT (1998) Mechanics of fibrous composites. Wiley, New York
22. Herakovich CT (2012) Mechanics of composites: a historical review. Mech Res Commun 41:1–20
23. Hull D (1981) An introduction to composite materials. Cambridge University Press, Cambridge
24. Jones RM (1975) Mechanics of composite materials. Scripta Book Co., Washington
25. Kassapoglou C (2010) Design and analysis of composite structures with applications to aerospace structures. Wiley, Chichester
26. Kollár LP, Springer GS (2003) Mechanics of composite structures. Cambridge University Press, Cambridge
27. Kaw AK (1997) Mechanics of composite materials. CRC Press, Boca Raton
28. Matthews FL, Davies GAO, Hitchings D, Soutis C (2000) Finite element modelling of composite materials and structures. Woodhead Publishing Limited, Cambridge
29. MIL-HDBK-17, 2F, (2002) Composite materials handbook -, vol 2. Polymer matrix composites materials properties, Department of Defence, USA
30. Moudood A, Rahman A, Khanlou HM, Hall W, Öchsner A, Francucci G (2019) Environmental effects on the durability and the mechanical performance of flax fiber/bio-epoxy composites. Compos Part B-Eng 171:284–293
31. Ochoa OO, Reddy JN (1992) Finite element analysis of composite laminates. Kluwer Academic Publishers, Dordrecht
32. Öchsner A, Augustin C (2009) Multifunctional metallic hollow sphere structures: manufacturing, properties and application. Springer, Berlin
33. Öchsner A, Öchsner M (2018) A first introduction to the finite element analysis program MSC Marc/Mentat. Springer, Cham
34. Öchsner A (2020) Computational statics and dynamics: an introduction based on the finite element method. Springer, Singapore
35. Reddy JN (1994) Mechanics of composite materials: selected works of Nicholas J. Kluwer Academic Publishers, Dordrecht, Pagano
36. Stravsky Y (1961) Bending and stretching of laminated aeolotropic plates. J Eng Mech Div-ASCE 87:31–56

37. Tenek LT, Argyris J (1998) Finite element analysis for composite structures. Springer, Dordrecht
38. Tsai SW, Hahn HT (1980) Introduction to composite materials. Technomic Publishishing Company, Lancaster
39. Tuttle ME (2004) Structural analysis of polymeric composite materials. Marcel Dekker, New York
40. Vasiliev VV, Morozov EV (2001) Mechanics and analysis of composite materials. Elsevier, Oxford
41. Vinson JR, Chou TW (1975) Composite materials and their use in structures. Wiley, New York
42. Vinson JR, Sierakowski RL (1987) The behavior of structures composed of composite materials. Kluwer Academic Publisher, Dordrecht
43. Whitney JM (1980) Structural analysis of laminated anisotropic plates. Technomic Publishing Company, Lancaster
44. Yengejeh SI, Kazemi SA, Öchsner A (2017) Carbon nanotubes as reinforcement in composites: a review of the analytical, numerical and experimental approaches. Comp Mater Sci 136:85–101

Chapter 2
Macromechanics of a Lamina

Abstract This chapter covers the mechanical modeling of a single layer with unidirectionally aligned reinforcing fibers embedded in a homogeneous matrix, a so-called lamina. It is shown that a lamina can be treated as a combination of a plane elasticity element and a classical plate element. For both classical structural elements and their combination, the continuum mechanical modeling based on the three basic equations, i.e., the kinematics relationship, the constitutive law, and the equilibrium equation is presented. Combining these three questions results in the description of partial differential equations. The chapter closes with different orthotropic failure criteria, i.e., the maximum stress, the maximum strain, the Tsai–Hill, and the Tsai–Wu criterion.

2.1 Introduction

Let us consider in the following a unidirectional lamina as schematically shown in Fig. 2.1. The *1*-axis is aligned to the fibers, the *2*-axis is perpendicular to the fibers in the plane, and the *3*-axis indicates the thickness direction. Furthermore, the geometrical assumptions hold as outlined in Sect. 1.2, i.e., the lamina is assumed as flat and thin.

The general difference between a plane elasticity and a classical plate element in regards to the possible loads is illustrated in Fig. 2.2. A plane elasticity element can be only loaded by in-plane normal or shear forces, see Fig. 2.2a. Contrary to that, the classical plate element allows forces with lines of action in the thickness direction (*3*) or moments at which the moment vector lies in the *1*–*2* plane, see Fig. 2.2b. A similar distinction can be done in regards to the deformations: the plane elasticity element has only in-plane translations whereas the classical plate element reveals translations perpendicular to the *1*–*2* plane and corresponding rotations. The derivations of the basic equations of continuum mechanics, i.e., the kinematics relationship, the constitutive law, and the equilibrium equation, are presented first separately for each of these members and afterward superimposed to the combined plane elasticity and classical plate element. This approach allows for a better understanding of the con-

© The Author(s), under exclusive license to Springer Nature Switzerland AG 2021
A. Öchsner, *Foundations of Classical Laminate Theory*, Advanced Structured Materials 163, https://doi.org/10.1007/978-3-030-82631-4_2

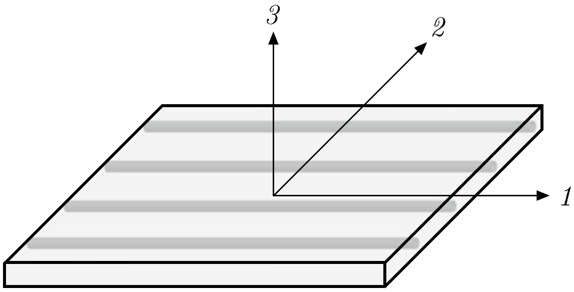


Fig. 2.1 Schematic representation of a unidirectional lamina

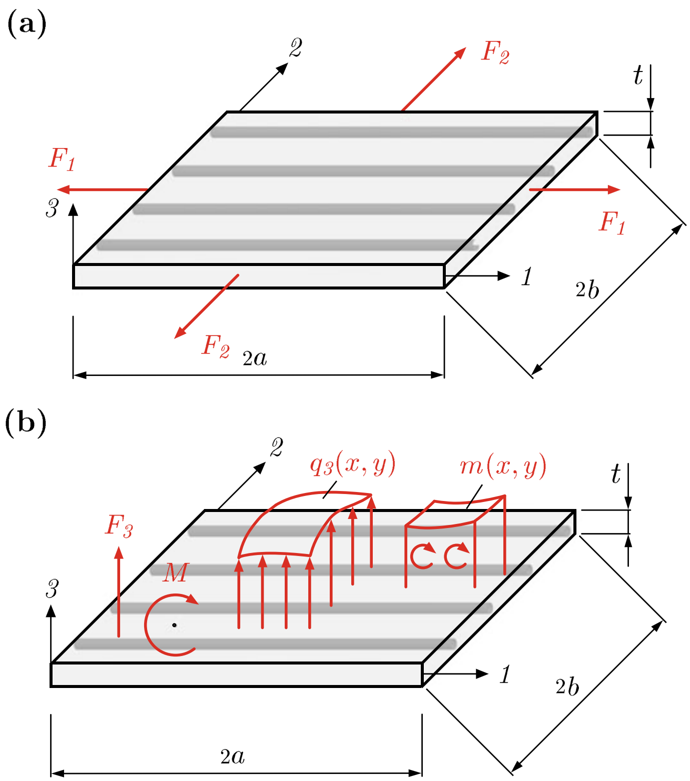


Fig. 2.2 General configuration for **a** a plane elasticity and **b** a classical plate element

tinuum mechanical basics and the procedure to combine basic structural elements. Such a combination of structural elements is normally done in the early years of degree programs in mechanical engineering by superimposing the tensile rod with a thin beam to form the so-called generalized beam element which can elongate and bend [14].

The configuration shown in Fig. 2.2 is simply the two-dimensional generalization of the rod/beam superposition.

2.2 Kinematics

The kinematics or strain–displacement relations extract the strain field contained in a displacement field.

2.2.1 Plane Elasticity Element

Using engineering definitions of strain, the following relations can be obtained [4, 7]:

$$\varepsilon_1 = \frac{\partial u_1}{\partial 1} \; ; \; \varepsilon_2 = \frac{\partial u_2}{\partial 2} \; ; \; \gamma_{12} = 2\varepsilon_{12} = \frac{\partial u_1}{\partial 2} + \frac{\partial u_2}{\partial 1} \, . \tag{2.1}$$

In matrix notation, these three kinematics relationships can be written as

$$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ 2\varepsilon_{12} \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial 1} & 0 \\ 0 & \frac{\partial}{\partial 2} \\ \frac{\partial}{\partial 2} & \frac{\partial}{\partial 1} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} , \tag{2.2}$$

or symbolically as

$$\boldsymbol{\varepsilon} = \boldsymbol{\mathcal{L}}_1 \boldsymbol{u}_{1,2} \, , \tag{2.3}$$

where $\boldsymbol{\mathcal{L}}_1$ is the differential operator matrix.

2.2.2 Classical Plate Element

Let us first derive a kinematics relation which relates the variation of u_1 across the plate thickness in terms of the displacement u_3. For this purpose, let us imagine that a plate element is bent around the 2-axis, see Fig. 2.3a. The following definition of the rotational angle φ_2 is assumed: the angle φ_2 is positive if the vector of the rotational direction is pointing in positive 2-axis.

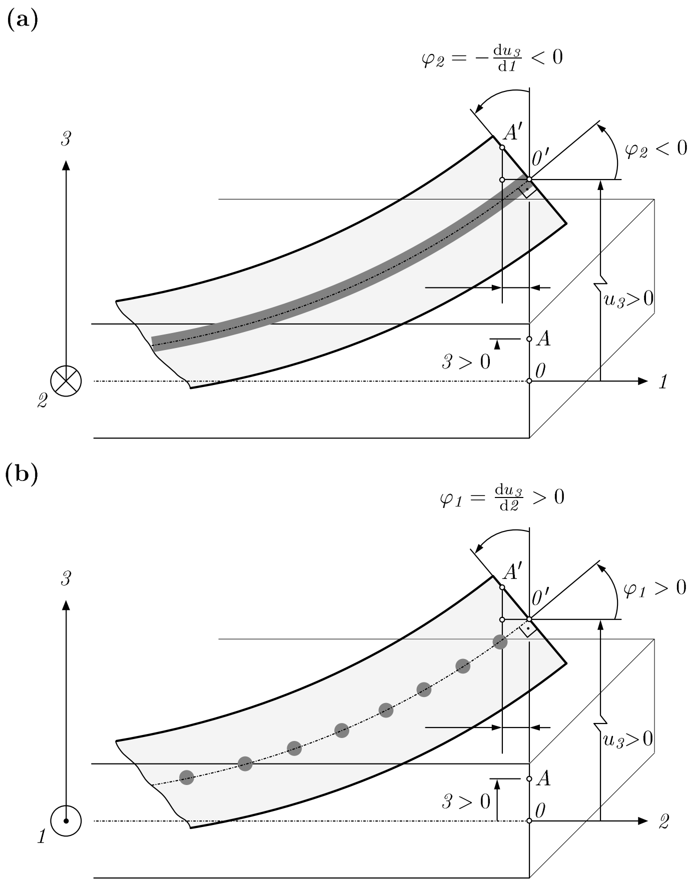


Fig. 2.3 Configurations for the derivation of kinematics relations: in **a** *1–3* plane and **b** *2–3* plane. Note that the deformation is exaggerated for better illustration

Looking at the right-angled triangle $0'A'B'$, we can state that[1]

$$\sin(-\varphi_2) = \frac{\overline{B'0'}}{\overline{0'A'}} = \frac{-u_1}{3}, \tag{2.4}$$

which results for small angles ($\sin(-\varphi_2) \approx -\varphi_2$) in

$$u_1 = +3\varphi_2 . \tag{2.5}$$

Looking at the curved center line in Fig. 2.3a, it holds that the slope of the tangent line at $0'$ equals

[1] Note that according to the assumptions of the classical *thin* plate theory, the lengths $\overline{0A}$ and $\overline{0'A'}$ remain unchanged.

$$\tan(-\varphi_2) = +\frac{\mathrm{d}u_3}{\mathrm{d}1} \approx -\varphi_2 \,. \tag{2.6}$$

If Eqs. (2.5) and (2.6) are combined, the following results:

$$u_1 = -3\frac{\mathrm{d}u_3}{\mathrm{d}1} \,. \tag{2.7}$$

Considering a plate which is bent around the *1*-axis (see Fig. 2.3b) and following the same line of reasoning (the angle φ_1 is assumed positive if the vector of the rotational direction is pointing in positive *1*-axis.), similar equations can be derived for u_2:

$$\varphi_1 \approx \frac{\mathrm{d}u_3}{\mathrm{d}2} \,, \tag{2.8}$$

$$u_2 = -3\varphi_1 \,, \tag{2.9}$$

$$u_2 = -3\frac{\mathrm{d}u_3}{\mathrm{d}2} \,. \tag{2.10}$$

One may find in the scholarly literature other definitions of the rotational angles [3, 16, 20, 21]. The angle φ_2 is introduced in the *13*-plane (see Fig. 2.3a) whereas φ_1 is introduced in the *23*-plane (see Fig. 2.3b). These definitions are closer to the classical definitions of the angles in the scope of finite elements but not conform with the definitions of the stress resultants (see M_1^{n} and M_2^{n} in Fig. 2.9). Other definitions assume, for example, that the rotational angle φ_1 (now defined in the *1*–*3* plane) is positive if it leads to a positive displacement u_1 at the positive *3*-side of the neutral axis. The same definition holds for the angle φ_2 (now defined in the *2*–*3* plane).

Using classical engineering definitions of strain, the following relations can be obtained [3, 17]:

$$\varepsilon_1 = \frac{\partial u_1}{\partial 1} \overset{(2.7)}{=} \frac{\partial}{\partial 1}\left(-3\frac{\partial u_3}{\partial 1}\right) = -3\frac{\partial^2 u_3}{\partial 1^2} = 3\kappa_1 \,, \tag{2.11}$$

$$\varepsilon_2 = \frac{\partial u_2}{\partial 2} \overset{(2.10)}{=} \frac{\partial}{\partial 2}\left(-3\frac{\partial u_3}{\partial 2}\right) = -3\frac{\partial^2 u_3}{\partial 2^2} = 3\kappa_2 \,, \tag{2.12}$$

$$\gamma_{12} = \frac{\partial u_1}{\partial 2} + \frac{\partial u_2}{\partial 1} \overset{(2.7),(2.10)}{=} -23\frac{\partial^2 u_3}{\partial 1 \partial 2} = 3\kappa_{12} \,. \tag{2.13}$$

In matrix notation, these three relationships can be written as

$$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{bmatrix} = -3 \begin{bmatrix} \frac{\partial^2}{\partial 1^2} \\ \frac{\partial^2}{\partial 2^2} \\ \frac{2\partial^2}{\partial 1 \partial 2} \end{bmatrix} u_3 = 3 \begin{bmatrix} \kappa_1 \\ \kappa_2 \\ \kappa_{12} \end{bmatrix}, \tag{2.14}$$

or symbolically as

$$\boldsymbol{\varepsilon} = -3\boldsymbol{\mathcal{L}}_2 u_3 = 3\boldsymbol{\kappa}\,. \tag{2.15}$$

2.2.3 Combined Plane Elasticity and Classical Plate Element

Combining Eqs. (2.2) and (2.14), the following kinematics description for a superposed element is obtained:

$$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial 1} & 0 \\ 0 & \frac{\partial}{\partial 2} \\ \frac{\partial}{\partial 2} & \frac{\partial}{\partial 1} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} - 3 \begin{bmatrix} \frac{\partial^2}{\partial 1^2} \\ \frac{\partial^2}{\partial 2^2} \\ \frac{2\partial^2}{\partial 1 \partial 2} \end{bmatrix} u_3 \tag{2.16}$$

$$= \begin{bmatrix} \frac{\partial}{\partial 1} & 0 \\ 0 & \frac{\partial}{\partial 2} \\ \frac{\partial}{\partial 2} & \frac{\partial}{\partial 1} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + 3 \begin{bmatrix} \kappa_1 \\ \kappa_2 \\ \kappa_{12} \end{bmatrix}, \tag{2.17}$$

or symbolically as

$$\boldsymbol{\varepsilon} = \boldsymbol{\mathcal{L}}_1 \boldsymbol{u}_{1,2} - 3\boldsymbol{\mathcal{L}}_2 u_3 \tag{2.18}$$

$$= \underbrace{\boldsymbol{\mathcal{L}}_1 \boldsymbol{u}_{1,2}}_{\boldsymbol{\varepsilon}^0} + 3\boldsymbol{\kappa}\,, \tag{2.19}$$

where $\boldsymbol{\varepsilon}^0$ collects the middle-surface strains. Alternatively, one may collect the single components of the plane elasticity and the plate contribution in matrix form as follows:

$$\begin{bmatrix} \varepsilon_1^0 \\ \varepsilon_2^0 \\ \gamma_{12}^0 \\ \hline \kappa_1 \\ \kappa_2 \\ \kappa_{12} \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial 1} & 0 & 0 \\ 0 & \frac{\partial}{\partial 2} & 0 \\ \frac{\partial}{\partial 1} & \frac{\partial}{\partial 2} & 0 \\ \hline 0 & 0 & -\frac{\partial^2}{\partial 1^2} \\ 0 & 0 & -\frac{\partial^2}{\partial 2^2} \\ 0 & 0 & -\frac{2\partial^2}{\partial 1 \partial 2} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \hline u_3 \end{bmatrix}, \tag{2.20}$$

or symbolically as

$$\boldsymbol{e} = \boldsymbol{\mathcal{L}}' \boldsymbol{u} \,. \tag{2.21}$$

The column matrix of generalized strains can be also expressed as

$$\boldsymbol{e} = \begin{bmatrix} \boldsymbol{\varepsilon}^0 \\ \hline \boldsymbol{\kappa} \end{bmatrix}. \tag{2.22}$$

2.3 Constitutive Equation

For isotropic materials, the mechanical properties are the same for all directions and it can be shown that the components of the elasticity matrix are determined by two independent constants. In mechanical engineering, many times the set of elasticity modulus E and Poisson's ration ν are chosen as the independent constants. Orthotropic materials reveal three principal, mutually orthogonal axes. These axes are also called the material principal axes.

2.3.1 Isotropic Material: Plane Elasticity Element

The two-dimensional plane stress case ($\sigma_3 = \sigma_{23} = \sigma_{13} = 0$) shown in Fig. 2.4 is commonly used for the analysis of thin, flat plates loaded in the plane of the plate (*1*–*2* plane).

It should be noted here that the normal thickness stress is zero ($\sigma_3 = 0$) whereas the thickness normal strain is present ($\varepsilon_3 \neq 0$).

The plane stress Hooke's law for a linear-elastic isotropic material based on Young's modulus E and Poisson's ratio ν can be written for constant temperature as

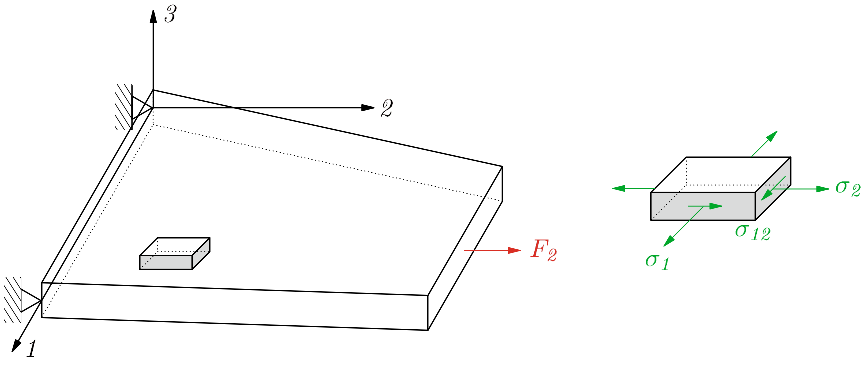


Fig. 2.4 Two-dimensional problem: plane stress case

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_{12} \end{bmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ 2\,\varepsilon_{12} \end{bmatrix}, \tag{2.23}$$

or in matrix notation as

$$\boldsymbol{\sigma} = \boldsymbol{C}\boldsymbol{\varepsilon}\,, \tag{2.24}$$

where $\boldsymbol{C}$ is the so-called elasticity matrix. It should be noted here that the engineering shear strain $\gamma_{12} = 2\varepsilon_{12}$ is used in the formulation of Eq. (2.23).

Rearranging the elastic stiffness form given in Eq. (2.23) for the strains gives the elastic compliance form

$$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ 2\,\varepsilon_{12} \end{bmatrix} = \frac{1}{E} \begin{bmatrix} 1 & -\nu & 0 \\ -\nu & 1 & 0 \\ 0 & 0 & 2(\nu+1) \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_{12} \end{bmatrix}, \tag{2.25}$$

or in matrix notation as

$$\boldsymbol{\varepsilon} = \boldsymbol{D}\boldsymbol{\sigma}\,, \tag{2.26}$$

where $\boldsymbol{D} = \boldsymbol{C}^{-1}$ is the so-called elastic compliance matrix. The general characteristic of a plane Hooke's law in the form of Eqs. (2.24) and (2.25) is that two independent material parameters are used.

It should be finally noted that the thickness strain ε_3 can be obtained based on the two in-plane normal strains ε_1 and ε_2 as

$$\varepsilon_3 = -\frac{\nu}{1-\nu} \cdot \left(\varepsilon_1 + \varepsilon_2\right). \tag{2.27}$$

The last equation can be derived from the three-dimensional formulation, see [12]. Equations (2.23) and (2.25) indicate that a plane isotropic material requires two independent material constants.[2]

2.3.2 Isotropic Material: Classical Plate Element

The classical plate theory assumes a plane stress state and the constitutive equation can be taken from Sect. 2.3.1, i.e., Eqs. (2.23) and (2.25) are still valid.

2.3.3 Isotropic Material: Combined Plane Elasticity and Classical Plate Element

Since there is no difference in the constitutive description of a plane elasticity and a classical plate element, a combined element is still described based on the the set of equations (2.23)–(2.24) and (2.25)–(2.26).

2.3.4 Orthotropic Material: Combined Plane Elasticity and Classical Plate Element

Since the plane elasticity element follows the same constitutive description as the classical plate element, it is sufficient to indicate the the relationship for the combined element. Let us start with the compliance form, i.e.,

$$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ 2\,\varepsilon_{12} \end{bmatrix} = \begin{bmatrix} \frac{1}{E_1} & -\frac{\nu_{21}}{E_2} & 0 \\ -\frac{\nu_{12}}{E_1} & \frac{1}{E_2} & 0 \\ 0 & 0 & \frac{1}{G_{12}} \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_{12} \end{bmatrix} = \begin{bmatrix} D_{11} & D_{12} & 0 \\ D_{12} & D_{22} & 0 \\ 0 & 0 & D_{44} \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_{12} \end{bmatrix}, \tag{2.28}$$

or in matrix notation as

$$\boldsymbol{\varepsilon} = \boldsymbol{D}\boldsymbol{\sigma}\,, \tag{2.29}$$

where $\boldsymbol{D} = \boldsymbol{C}^{-1}$ is again the so-called elastic compliance matrix. The constant G in Eq. (2.29) is the shear modulus in the *1–2* plane. It should be noted that in Eq. (2.28) the identity

$$-\frac{\nu_{12}}{E_1} = -\frac{\nu_{21}}{E_2} \tag{2.30}$$

[2] Three-dimensional isotropy requires as well only two independent elastic constants.

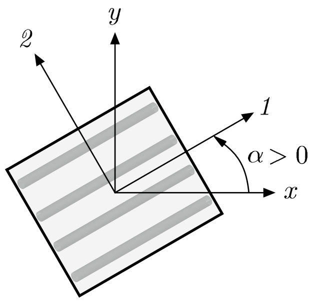


Fig. 2.5 Rotated lamina: (1, 2) principal material coordinates and (x, y) arbitrary coordinates. The rotational angle α is from the x-axis to the 1-axis (counterclockwise positive for the sketched coordinate systems)

holds and we can conclude that four independent elastic constants are required to describe a plane orthotropic material.[3] The stress–strain relationship can be obtained from Eq. (2.28) by inverting the compliance matrix to give the following representation:

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_{12} \end{bmatrix} = \begin{bmatrix} \frac{E_1}{1-\nu_{12}\nu_{21}} & \frac{\nu_{21}E_1}{1-\nu_{12}\nu_{21}} & 0 \\ \frac{\nu_{12}E_2}{1-\nu_{12}\nu_{21}} & \frac{E_2}{1-\nu_{12}\nu_{21}} & 0 \\ 0 & 0 & G_{12} \end{bmatrix} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ 2\,\varepsilon_{12} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & 0 \\ C_{12} & C_{22} & 0 \\ 0 & 0 & C_{44} \end{bmatrix} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ 2\,\varepsilon_{12} \end{bmatrix}, \quad (2.31)$$

or in matrix notation as

$$\boldsymbol{\sigma} = \boldsymbol{C}\boldsymbol{\varepsilon}\,, \quad (2.32)$$

where $\boldsymbol{C} = \boldsymbol{D}^{-1}$ is again the so-called elasticity matrix.

Let us consider in the following the rotation of the coordinate system and the corresponding transformations of stresses and strains, see Fig. 2.5. This is important in the case of laminates with different laminae at different orientations (see Chap. 3)

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{bmatrix} = \begin{bmatrix} \cos^2\alpha & \sin^2\alpha & -2\sin\alpha\cos\alpha \\ \sin^2\alpha & \cos^2\alpha & 2\sin\alpha\cos\alpha \\ \sin\alpha\cos\alpha & -\sin\alpha\cos\alpha & \cos^2\alpha-\sin^2\alpha \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_{12} \end{bmatrix}, \quad (2.33)$$

or in matrix notation as

$$\boldsymbol{\sigma}_{x,y} = \boldsymbol{T}_{\sigma}^{-1}\boldsymbol{\sigma}_{1,2}\,, \quad (2.34)$$

[3] Three-dimensional orthotropy requires nine independent elastic constants.

or in the inverse representation

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_{12} \end{bmatrix} = \begin{bmatrix} \cos^2\alpha & \sin^2\alpha & 2\sin\alpha\cos\alpha \\ \sin^2\alpha & \cos^2\alpha & -2\sin\alpha\cos\alpha \\ -\sin\alpha\cos\alpha & \sin\alpha\cos\alpha & \cos^2\alpha - \sin^2\alpha \end{bmatrix} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{bmatrix}, \tag{2.35}$$

or in matrix notation as

$$\boldsymbol{\sigma}_{1,2} = \boldsymbol{T}_{\sigma}\boldsymbol{\sigma}_{x,y}\,. \tag{2.36}$$

The corresponding transformations for the strain field read

$$\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ 2\varepsilon_{xy} \end{bmatrix} = \begin{bmatrix} \cos^2\alpha & \sin^2\alpha & -\sin\alpha\cos\alpha \\ \sin^2\alpha & \cos^2\alpha & \sin\alpha\cos\alpha \\ 2\sin\alpha\cos\alpha & -2\sin\alpha\cos\alpha & \cos^2\alpha - \sin^2\alpha \end{bmatrix} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ 2\varepsilon_{12} \end{bmatrix}, \tag{2.37}$$

or in matrix notation as

$$\boldsymbol{\varepsilon}_{x,y} = \boldsymbol{T}_{\varepsilon}^{-1}\boldsymbol{\varepsilon}_{1,2}\,, \tag{2.38}$$

or in the inverse representation

$$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ 2\varepsilon_{12} \end{bmatrix} = \begin{bmatrix} \cos^2\alpha & \sin^2\alpha & \sin\alpha\cos\alpha \\ \sin^2\alpha & \cos^2\alpha & -\sin\alpha\cos\alpha \\ -2\sin\alpha\cos\alpha & 2\sin\alpha\cos\alpha & \cos^2\alpha - \sin^2\alpha \end{bmatrix} \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ 2\varepsilon_{xy} \end{bmatrix}, \tag{2.39}$$

or in matrix notation as

$$\boldsymbol{\varepsilon}_{1,2} = \boldsymbol{T}_{\varepsilon}\boldsymbol{\varepsilon}_{x,y}\,. \tag{2.40}$$

It should be noted here that the relationship between the strain and stress transformation matrices is as follows:

$$\boldsymbol{T}_{\varepsilon} = \boldsymbol{R}\boldsymbol{T}_{\sigma}\boldsymbol{R}^{-1}\,, \tag{2.41}$$

where the matrix $\boldsymbol{R}$ has the following entries:

$$\boldsymbol{R} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}. \tag{2.42}$$

Based on these transformations of the stresses and strains, the stress–strain relationship in the arbitrary (rotated) x–y coordinate system can be obtained as

$$\boldsymbol{\sigma}_{1,2} = \boldsymbol{C}\boldsymbol{\varepsilon}_{1,2}\,, \tag{2.43}$$

$$\underbrace{\boldsymbol{T}_{\sigma}^{-1}\boldsymbol{\sigma}_{1,2}}_{\boldsymbol{\sigma}_{x,y}} = \boldsymbol{T}_{\sigma}^{-1}\boldsymbol{C}\boldsymbol{\varepsilon}_{1,2}\,, \tag{2.44}$$

$$\boldsymbol{\sigma}_{x,y} = \boldsymbol{T}_{\sigma}^{-1}\boldsymbol{C}\,\underbrace{\boldsymbol{T}_{\varepsilon}\boldsymbol{T}_{\varepsilon}^{-1}}_{\mathbf{I}}\,\boldsymbol{\varepsilon}_{1,2}\,, \tag{2.45}$$

$$= \boldsymbol{T}_{\sigma}^{-1}\boldsymbol{C}\boldsymbol{T}_{\varepsilon}\,\underbrace{\boldsymbol{T}_{\varepsilon}^{-1}\boldsymbol{\varepsilon}_{1,2}}_{\boldsymbol{\varepsilon}_{x,y}}\,, \tag{2.46}$$

$$\boldsymbol{\sigma}_{x,y} = \underbrace{\boldsymbol{T}_{\sigma}^{-1}\boldsymbol{C}\boldsymbol{T}_{\varepsilon}}_{\overline{\boldsymbol{C}}}\,\boldsymbol{\varepsilon}_{x,y}\,. \tag{2.47}$$

Thus, the transformed elasticity matrix $\overline{\boldsymbol{C}}$ can be obtained from the following triple matrix product:

$$\overline{\boldsymbol{C}} = \begin{bmatrix} \cos^2\alpha & \sin^2\alpha & -2\sin\alpha\cos\alpha \\ \sin^2\alpha & \cos^2\alpha & 2\sin\alpha\cos\alpha \\ \sin\alpha\cos\alpha & -\sin\alpha\cos\alpha & \cos^2\alpha-\sin^2\alpha \end{bmatrix} \begin{bmatrix} C_{11} & C_{12} & C_{14} \\ C_{12} & C_{22} & C_{24} \\ C_{14} & C_{24} & C_{44} \end{bmatrix}$$
$$\begin{bmatrix} \cos^2\alpha & \sin^2\alpha & \sin\alpha\cos\alpha \\ \sin^2\alpha & \cos^2\alpha & -\sin\alpha\cos\alpha \\ -2\sin\alpha\cos\alpha & 2\sin\alpha\cos\alpha & \cos^2\alpha-\sin^2\alpha \end{bmatrix} \tag{2.48}$$

$$= \begin{bmatrix} \cos^2\alpha & \sin^2\alpha & -2\sin\alpha\cos\alpha \\ \sin^2\alpha & \cos^2\alpha & 2\sin\alpha\cos\alpha \\ \sin\alpha\cos\alpha & -\sin\alpha\cos\alpha & \cos^2\alpha-\sin^2\alpha \end{bmatrix} \begin{bmatrix} \frac{E_1}{1-\nu_{12}\nu_{21}} & \frac{\nu_{21}E_1}{1-\nu_{12}\nu_{21}} & 0 \\ \frac{\nu_{12}E_2}{1-\nu_{12}\nu_{21}} & \frac{E_2}{1-\nu_{12}\nu_{21}} & 0 \\ 0 & 0 & G_{12} \end{bmatrix}$$

$$\begin{bmatrix} \cos^2\alpha & \sin^2\alpha & \sin\alpha\cos\alpha \\ \sin^2\alpha & \cos^2\alpha & -\sin\alpha\cos\alpha \\ -2\sin\alpha\cos\alpha & 2\sin\alpha\cos\alpha & \cos^2\alpha-\sin^2\alpha \end{bmatrix} \tag{2.49}$$

$$= \begin{bmatrix} \overline{C}_{11} & \overline{C}_{12} & \overline{C}_{14} \\ \overline{C}_{12} & \overline{C}_{22} & \overline{C}_{24} \\ \overline{C}_{14} & \overline{C}_{24} & \overline{C}_{44} \end{bmatrix}, \tag{2.50}$$

where the single matrix entries $\overline{C}_{ij}$ are given as follows:

$$\overline{C}_{11} = C_{11}\cos^4\alpha + 2(C_{12}+2C_{44})\sin^2\alpha\cos^2\alpha + C_{22}\sin^4\alpha\,, \tag{2.51}$$

$$\overline{C}_{12} = (C_{11}+C_{22}-4C_{44})\sin^2\alpha\cos^2 + C_{12}(\sin^4\alpha+\cos^4\alpha)\,, \tag{2.52}$$

$$\overline{C}_{22} = C_{11}\sin^4\alpha + 2(C_{12}+2C_{44})\sin^2\alpha\cos^2\alpha + C_{22}\cos^4\alpha\,, \tag{2.53}$$

$$\overline{C}_{14} = (C_{11}-C_{12}-2C_{44})\sin\alpha\cos^3\alpha + (C_{12}-C_{22}+2C_{44})\sin^3\alpha\cos\alpha\,, \tag{2.54}$$

$$\overline{C}_{24} = (C_{11}-C_{12}-2C_{44})\sin^3\alpha\cos\alpha + (C_{12}-C_{22}+2C_{44})\sin\alpha\cos^3\alpha\,, \tag{2.55}$$

$$\overline{C}_{44} = (C_{11}+C_{22}-2C_{12}-2C_{44})\sin^2\alpha\cos^2\alpha + C_{44}(\sin^4\alpha+\cos^4\alpha)\,. \tag{2.56}$$

In a similar way, the strain–stress relationship in the x–y coordinate system can be obtained as

$$\boldsymbol{\varepsilon}_{1,2} = \boldsymbol{D}\boldsymbol{\sigma}_{1,2}\,, \tag{2.57}$$

$$\underbrace{\boldsymbol{T}_{\varepsilon}^{-1}\boldsymbol{\varepsilon}_{1,2}}_{\boldsymbol{\varepsilon}_{x,y}} = \boldsymbol{T}_{\varepsilon}^{-1}\boldsymbol{D}\boldsymbol{\sigma}_{1,2}\,, \tag{2.58}$$

$$\boldsymbol{\varepsilon}_{x,y} = \boldsymbol{T}_{\varepsilon}^{-1}\boldsymbol{D}\,\underbrace{\boldsymbol{T}_{\sigma}\boldsymbol{T}_{\sigma}^{-1}}_{\mathbf{I}}\,\boldsymbol{\sigma}_{1,2}\,, \tag{2.59}$$

$$= \boldsymbol{T}_{\epsilon}^{-1}\boldsymbol{D}\boldsymbol{T}_{\sigma}\,\underbrace{\boldsymbol{T}_{\sigma}^{-1}\boldsymbol{\sigma}_{1,2}}_{\boldsymbol{\sigma}_{x,y}}\,, \tag{2.60}$$

$$\boldsymbol{\varepsilon}_{x,y} = \underbrace{\boldsymbol{T}_{\varepsilon}^{-1}\boldsymbol{D}\boldsymbol{T}_{\sigma}}_{\overline{\boldsymbol{D}}}\,\boldsymbol{\sigma}_{x,y}\,. \tag{2.61}$$

Thus, the transformed elastic compliance matrix $\overline{\boldsymbol{D}}$ can be obtained from the following triple matrix product:

$$\overline{\boldsymbol{D}} = \begin{bmatrix} \cos^2\alpha & \sin^2\alpha & -\sin\alpha\cos\alpha \\ \sin^2\alpha & \cos^2\alpha & \sin\alpha\cos\alpha \\ 2\sin\alpha\cos\alpha & -2\sin\alpha\cos\alpha & \cos^2\alpha-\sin^2\alpha \end{bmatrix} \begin{bmatrix} D_{11} & D_{12} & D_{14} \\ D_{12} & D_{22} & D_{24} \\ D_{14} & D_{24} & D_{44} \end{bmatrix} \begin{bmatrix} \cos^2\alpha & \sin^2\alpha & 2\sin\alpha\cos\alpha \\ \sin^2\alpha & \cos^2\alpha & -2\sin\alpha\cos\alpha \\ -\sin\alpha\cos\alpha & \sin\alpha\cos\alpha & \cos^2\alpha-\sin^2\alpha \end{bmatrix} \tag{2.62}$$

$$= \begin{bmatrix} \cos^2\alpha & \sin^2\alpha & -\sin\alpha\cos\alpha \\ \sin^2\alpha & \cos^2\alpha & \sin\alpha\cos\alpha \\ 2\sin\alpha\cos\alpha & -2\sin\alpha\cos\alpha & \cos^2\alpha-\sin^2\alpha \end{bmatrix} \begin{bmatrix} \frac{1}{E_1} & -\frac{\nu_{21}}{E_2} & 0 \\ -\frac{\nu_{12}}{E_1} & \frac{1}{E_2} & 0 \\ 0 & 0 & \frac{1}{G_{12}} \end{bmatrix} \begin{bmatrix} \cos^2\alpha & \sin^2\alpha & 2\sin\alpha\cos\alpha \\ \sin^2\alpha & \cos^2\alpha & -2\sin\alpha\cos\alpha \\ -\sin\alpha\cos\alpha & \sin\alpha\cos\alpha & \cos^2\alpha-\sin^2\alpha \end{bmatrix} \tag{2.63}$$

$$= \begin{bmatrix} \overline{D}_{11} & \overline{D}_{12} & \overline{D}_{14} \\ \overline{D}_{12} & \overline{D}_{22} & \overline{D}_{24} \\ \overline{D}_{14} & \overline{D}_{24} & \overline{D}_{44} \end{bmatrix}, \tag{2.64}$$

where the single matrix entries $\overline{D}_{ij}$ are given as follows:

$$\overline{D}_{11} = D_{11}\cos^4\alpha + (2D_{12}+D_{44})\sin^2\alpha\cos^2\alpha + D_{22}\sin^4\alpha\,, \tag{2.65}$$

$$\overline{D}_{12} = (D_{11}+D_{22}-D_{44})\sin^2\alpha\cos^2 + D_{12}(\sin^4\alpha+\cos^4\alpha)\,, \tag{2.66}$$

$$\overline{D}_{22} = D_{11}\sin^4\alpha + (2D_{12}+D_{44})\sin^2\alpha\cos^2\alpha + D_{22}\cos^4\alpha\,, \tag{2.67}$$

$$\overline{D}_{14} = (2D_{11}-2D_{12}-D_{44})\sin\alpha\cos^3\alpha - (2D_{22}-2D_{12}-D_{44})\sin^3\alpha\cos\alpha\,, \tag{2.68}$$

$$\overline{D}_{24} = (2D_{11}-2D_{12}-D_{44})\sin^3\alpha\cos\alpha - (2D_{22}-2D_{12}-D_{44})\sin\alpha\cos^3\alpha\,, \tag{2.69}$$

$$\overline{D}_{44} = 2(2D_{11}+2D_{22}-4D_{12}-D_{44})\sin^2\alpha\cos^2\alpha + D_{44}(\sin^4\alpha+\cos^4\alpha)\,. \tag{2.70}$$

It can be shown that the elements $\overline{D}_{ij}$ of the transformed elastic compliance matrix (see Eq. (2.64)) can be related to the apparent engineering constants of the orthotropic lamina in the rotated x–y coordinate system as follows (see [2, 5, 6, 10] for details):

$$\overline{\boldsymbol{D}} = \begin{bmatrix} \overline{D}_{11} & \overline{D}_{12} & \overline{D}_{14} \\ \overline{D}_{12} & \overline{D}_{22} & \overline{D}_{24} \\ \overline{D}_{14} & \overline{D}_{24} & \overline{D}_{44} \end{bmatrix} = \begin{bmatrix} \frac{1}{E_x} & -\frac{\nu_{xy}}{E_x} & \frac{\eta_x}{E_x} \\ -\frac{\nu_{xy}}{E_x} & \frac{1}{E_y} & \frac{\eta_y}{E_y} \\ \frac{\eta_x}{E_x} & \frac{\eta_y}{E_y} & \frac{1}{G_{xy}} \end{bmatrix}, \tag{2.71}$$

where the shear–extension coupling coefficients are generally defined as follows:

$$\eta_x = \frac{\gamma_{xy}}{\varepsilon_x} = \frac{\gamma_{xy} E_x}{\sigma_x} \quad \text{(for uniaxial tension with } \sigma_x\text{)}, \tag{2.72}$$

$$\eta_y = \frac{\gamma_{xy}}{\varepsilon_y} = \frac{\gamma_{xy} E_y}{\sigma_y} \quad \text{(for uniaxial tension with } \sigma_y\text{)}. \tag{2.73}$$

Furthermore, Poisson's ratios (extension–extension coupling coefficients) are defined as follows:

$$\nu_{xy} = -\frac{\varepsilon_y}{\varepsilon_x} \quad \text{(for uniaxial tension with } \sigma_x\text{)}, \tag{2.74}$$

$$\nu_{yx} = -\frac{\varepsilon_x}{\varepsilon_y} \quad \text{(for uniaxial tension with } \sigma_y\text{)}. \tag{2.75}$$

2.4 Equilibrium

The equilibrium equations relate the external loads (i.e., forces and moments) to the corresponding internal reactions (i.e., stresses).

2.4.1 Plane Elasticity Element

Figure 2.6 shows the normal and shear stresses which are acting on a differential volume element in the *1*-direction. All forces are drawn in their positive direction at each cut face. A positive cut face is obtained if the outward surface normal is directed in the positive direction of the corresponding coordinate axis. This means that the right-hand face in Fig. 2.6 is positive and the force $(\sigma_1 + \frac{\partial \sigma_1}{\partial x} \mathrm{d}1)\mathrm{d}2\mathrm{d}3$ is oriented in the positive *1*-direction. In a similar way, the top face is positive, i.e., the outward surface normal is directed in the positive *2*-direction, and the shear force[4] is oriented in the positive *1*-direction. Since the volume element is assumed to be in equilibrium, forces resulting from stresses on the sides of the cuboid and from the body forces f_i $(i = 1, 2, 3)$ must be balanced. These body forces are defined as forces per unit volume which can be produced by gravity,[5] acceleration, magnetic fields, and so on.

[4] In the case of a shear force σ_{ij}, the first index i indicates that the stress acts on a plane normal to the i-axis and the second index j denotes the direction in which the stress acts.

[5] If gravity is acting, the body force f results as the product of density times standard gravity: $f = \frac{F}{V} = \frac{mg}{V} = \frac{m}{V} g = \varrho g$. The units can be checked by consideration of $1\,\mathrm{N} = 1\,\frac{\mathrm{mkg}}{\mathrm{s}^2}$.

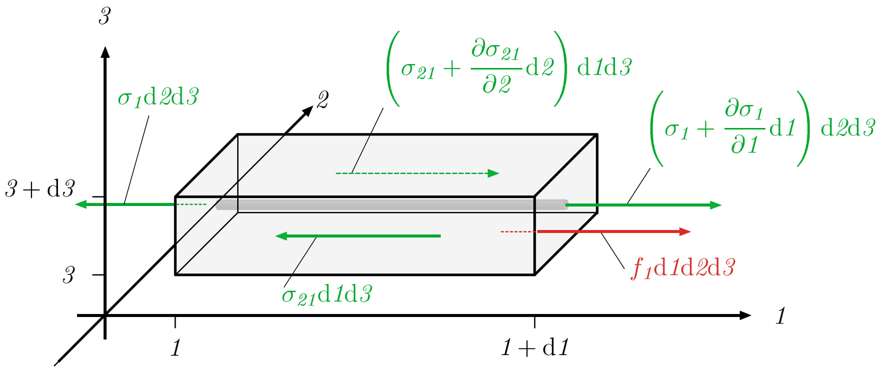


Fig. 2.6 Stresses and body forces which act on a plane differential volume element in *1*-direction (note that the three directions d*1*, d*2*, and d*3* are differently sketched to indicate the plane problem)

The static equilibrium of forces in the *1*-direction based on the five force components—two normal forces, two shear forces, and one body force—indicated in Fig. 2.6 gives

$$\left(\sigma_1 + \frac{\partial \sigma_1}{\partial 1}\right) \mathrm{d}2\mathrm{d}3 - \sigma_1 \mathrm{d}2\mathrm{d}3 + \left(\frac{\partial \sigma_{21}}{\partial 2}\right) \mathrm{d}1\mathrm{d}3$$
$$- \sigma_{21} \mathrm{d}1\mathrm{d}3 + f_1 \mathrm{d}1\mathrm{d}2\mathrm{d}3 = 0\,, \tag{2.76}$$

or after simplification and canceling with $\mathrm{d}V = \mathrm{d}1\mathrm{d}2\mathrm{d}3$:

$$\frac{\partial \sigma_1}{\partial 1} + \frac{\partial \sigma_{21}}{\partial 2} + f_1 = 0\,. \tag{2.77}$$

Based on the same approach, a similar equation can be specified in the *2*-direction:

$$\frac{\partial \sigma_2}{\partial 2} + \frac{\partial \sigma_{21}}{\partial 1} + f_2 = 0\,. \tag{2.78}$$

These two balance equations can be written in matrix notation as

$$\begin{bmatrix} \frac{\partial}{\partial 1} & 0 & \frac{\partial}{\partial 2} \\ 0 & \frac{\partial}{\partial 2} & \frac{\partial}{\partial 1} \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_{12} \end{bmatrix} + \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \tag{2.79}$$

or in symbolic notation:

$$\boldsymbol{\mathcal{L}}_1^{\mathrm{T}} \boldsymbol{\sigma} + \boldsymbol{b} = \boldsymbol{0}\,, \tag{2.80}$$

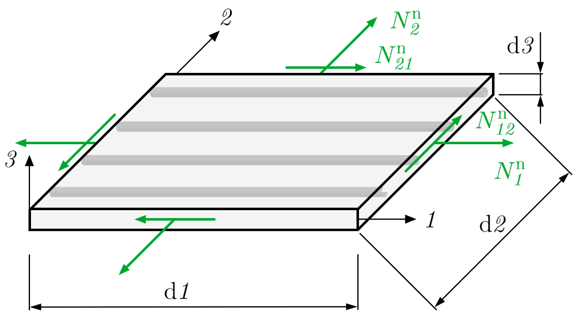


Fig. 2.7 Stress resultants for a plane elasticity element

where $\mathcal{L}_1$ is the differential operator matrix and $\boldsymbol{b}$ is the column matrix of body forces.

The internal reactions expressed as stresses as indicated in Fig. 2.6 can be alternatively stated as forces based on an integration over the corresponding reference area. These integral values are then called the stress resultants or generalized stresses.

Based on Fig. 2.7, the stress resultant N_i^n, i.e., normalized with the corresponding side length of the plate element, for the plane elasticity element can be indicated as follows:

$$N_1^n = \frac{N_1}{d2} = \int_{-t/2}^{t/2} \sigma_1 d3\,, \tag{2.81}$$

$$N_2^n = \frac{N_2}{d1} = \int_{-t/2}^{t/2} \sigma_2 d3\,, \tag{2.82}$$

$$N_{12}^n = \frac{N_{12}}{d1} = \int_{-t/2}^{t/2} \sigma_{12} d3\,, \tag{2.83}$$

or all three relations combined in matrix notation:

$$\begin{bmatrix} N_1^n \\ N_2^n \\ N_{12}^n \end{bmatrix} = \int_{-t/2}^{t/2} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_{12} \end{bmatrix} d3\,. \tag{2.84}$$

Thus, we can get the following alternative formulation of the balance equations by introducing the stress resultants in Eq. (2.79). The two balance equations can be alternatively written in matrix notation as

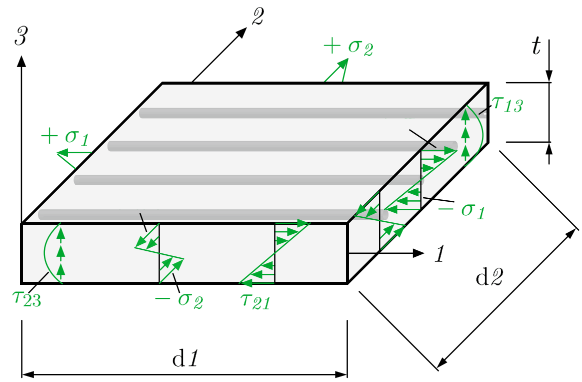


Fig. 2.8 Stresses acting on a classical plate element

$$\begin{bmatrix} \frac{\partial}{\partial 1} & 0 & \frac{\partial}{\partial 2} \\ 0 & \frac{\partial}{\partial 2} & \frac{\partial}{\partial 1} \end{bmatrix} \begin{bmatrix} N_1^{\mathrm{n}} \\ N_2^{\mathrm{n}} \\ N_{12}^{\mathrm{n}} \end{bmatrix} + \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \tag{2.85}$$

or in symbolic notation:

$$\boldsymbol{\mathcal{L}}_1^{\mathrm{T}} \boldsymbol{N}^{\mathrm{n}} + \boldsymbol{q}_{1,2} = \boldsymbol{0}, \tag{2.86}$$

where $q_1 = f_1 \mathrm{d}3$ and $q_2 = f_2 \mathrm{d}3$ are the area-specific loads.

2.4.2 Classical Plate Element

Let us first look at the stress distributions through the thickness of a classical plate element $\mathrm{d}1\mathrm{d}2t$ as shown in Fig. 2.8. Linear distributed normal stresses (σ_1, σ_2), linear distributed shear stresses (τ_{21}, τ_{12}), and parabolic distributed shear stresses (τ_{23}, τ_{13}) can be identified. These stresses can be expressed by the so-called stress resultants, i.e., bending moments and shear forces as shown in Fig. 2.9. These stress resultants are taken to be positive if they cause a tensile stress (positive) at a point with positive *3*-coordinate.

These stress resultants are obtained by integrating over the stress distributions. In the case of plates, however, the integration is only performed over the thickness, i.e., the moments and forces are given per unit length (normalized with the corresponding side length of the plate element).

The normalized (superscript "n") bending moments are obtained as follows:

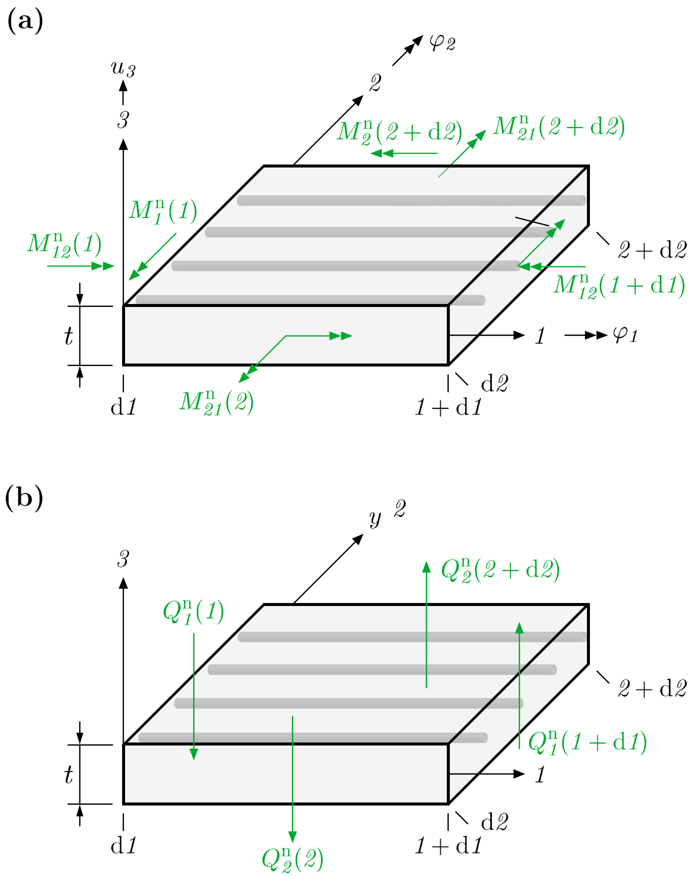


Fig. 2.9 Stress resultants acting on a classical plate element: **a** bending and twisting moments and **b** shear forces. Positive directions are drawn

$$M_1^{\mathrm{n}} = \frac{M_1}{\mathrm{d}2} = \int_{-t/2}^{t/2} 3\sigma_1 \mathrm{d}3\,, \tag{2.87}$$

$$M_2^{\mathrm{n}} = \frac{M_2}{\mathrm{d}1} = \int_{-t/2}^{t/2} 3\sigma_2 \mathrm{d}3\,. \tag{2.88}$$

The twisting moment per unit length reads

$$M_{12}^{\mathrm{n}} = M_{21}^{\mathrm{n}} = \frac{M_{12}}{\mathrm{d}2} = \frac{M_{21}}{\mathrm{d}1} = \int_{-t/2}^{t/2} 3\tau_{12} \mathrm{d}3\,, \tag{2.89}$$

or all three relations combined in matrix notation:

$$\begin{bmatrix} M_1^{\mathrm{n}} \\ M_2^{\mathrm{n}} \\ M_{12}^{\mathrm{n}} \end{bmatrix} = \int\limits_{-t/2}^{t/2} 3 \begin{bmatrix} \sigma_1 \\ \sigma_{12} \\ \sigma_{12} \end{bmatrix} \mathrm{d}3\,. \tag{2.90}$$

Furthermore, the shear forces per unit length are calculated in the following way:

$$Q_1^{\mathrm{n}} = \frac{Q_1}{\mathrm{d}2} = \int\limits_{-t/2}^{t/2} \tau_{13}\mathrm{d}3\,, \tag{2.91}$$

$$Q_2^{\mathrm{n}} = \frac{Q_2}{\mathrm{d}1} = \int\limits_{-t/2}^{t/2} \tau_{23}\mathrm{d}3\,, \tag{2.92}$$

or both relations combined in matrix notation:

$$\begin{bmatrix} Q_1^{\mathrm{n}} \\ Q_2^{\mathrm{n}} \end{bmatrix} = \int\limits_{-t/2}^{t/2} \begin{bmatrix} \tau_{13} \\ \tau_{23} \end{bmatrix} \mathrm{d}3\,. \tag{2.93}$$

It should be noted that a slightly different notation when compared to the beam problems is used here. The bending moment around the 2-axis is now called M_1^{n} (which directly corresponds to the causing stress σ_1) while in the beam notation it was M_2, see [12]. Nevertheless, the orientation remains the same. The shear force, which was in the case of the beams given as Q_3, is now either Q_1^{n} or Q_2^{n}. Thus, in the case of this plate notation, the index refers rather to the plane (check the surface normal vector) in which the corresponding resultant (vector) is located.

The equilibrium condition will be determined in the following for the vertical forces. Assuming that the distributed force is constant ($q_3(1,2) \rightarrow q_3$) and that forces in the direction of the positive 3-axis are considered positive, the following results:

$$-\,Q_1^{\mathrm{n}}(1)\mathrm{d}2 - Q_2^{\mathrm{n}}(2)\mathrm{d}1 + Q_1^{\mathrm{n}}(1+\mathrm{d}1)\mathrm{d}2 + Q_2^{\mathrm{n}}(2+\mathrm{d}2)\mathrm{d}1 + q_3\mathrm{d}1\mathrm{d}2 = 0\,. \tag{2.94}$$

Expanding the shear forces at $1+\mathrm{d}1$ and $2+\mathrm{d}2$ in Taylor's series of first order, meaning

$$Q_1^{\mathrm{n}}(1+\mathrm{d}1) \approx Q_1^{\mathrm{n}}(1) + \frac{\partial Q_1^{\mathrm{n}}}{\partial 1}\,\mathrm{d}1\,, \tag{2.95}$$

$$Q_2^{\mathrm{n}}(2+\mathrm{d}2) \approx Q_2^{\mathrm{n}}(2) + \frac{\partial Q_2^{\mathrm{n}}}{\partial 2}\,\mathrm{d}2\,, \tag{2.96}$$

Equation (2.94) results in

$$\frac{\partial Q_1^{\mathrm{n}}}{\partial 1}\,\mathrm{d}1\mathrm{d}2 + \frac{\partial Q_2^{\mathrm{n}}}{\partial 2}\,\mathrm{d}2\mathrm{d}1 + q_3\mathrm{d}1\mathrm{d}2 = 0\,, \tag{2.97}$$

or alternatively after simplification to

$$\frac{\partial Q_1^{\mathrm{n}}}{\partial 1} + \frac{\partial Q_2^{\mathrm{n}}}{\partial 2} + q_3 = 0\,. \tag{2.98}$$

The equilibrium of moments around the reference axis at $1 + \mathrm{d}1$ (positive if the moment vector is pointing in positive 2-axis) gives

$$\begin{aligned} &M_1^{\mathrm{n}}(1+\mathrm{d}1)\mathrm{d}2 - M_1^{\mathrm{n}}(1)\mathrm{d}2 + M_{21}^{\mathrm{n}}(2+\mathrm{d}2)\mathrm{d}1 - M_{21}^{\mathrm{n}}\mathrm{d}1 \\ &- Q_2^{\mathrm{n}}(2)\mathrm{d}1\tfrac{\mathrm{d}1}{2} + Q_2^{\mathrm{n}}(2+\mathrm{d}2)\mathrm{d}1\tfrac{\mathrm{d}1}{2} - Q_1^{\mathrm{n}}(1)\mathrm{d}2\mathrm{d}1 + q_3\mathrm{d}1\mathrm{d}2\tfrac{\mathrm{d}1}{2} = 0\,. \end{aligned} \tag{2.99}$$

Expanding the stress resultants at $1 + \mathrm{d}1$ and $2 + \mathrm{d}2$ into Taylor's series of first order, meaning

$$M_1^{\mathrm{n}}(1+\mathrm{d}1) = M_1^{\mathrm{n}}(1) + \frac{\partial M_1^{\mathrm{n}}}{\partial 1}\mathrm{d}1\,, \tag{2.100}$$

$$M_{21}^{\mathrm{n}}(2+\mathrm{d}2) = M_{21}^{\mathrm{n}}(2) + \frac{\partial M_{21}^{\mathrm{n}}}{\partial 2}\mathrm{d}2\,, \tag{2.101}$$

$$Q_2^{\mathrm{n}}(2+\mathrm{d}2) = Q_2^{\mathrm{n}}(2) + \frac{\partial Q_2^{\mathrm{n}}}{\partial 2}\mathrm{d}2\,, \tag{2.102}$$

Equation (2.99) results in

$$\frac{\partial M_1^{\mathrm{n}}}{\partial 1}\,\mathrm{d}1\mathrm{d}2 + \frac{\partial M_{21}^{\mathrm{n}}}{\partial 2}\,\mathrm{d}2\mathrm{d}1 + \frac{\partial Q_2^{\mathrm{n}}}{\partial 2}\,\mathrm{d}2\mathrm{d}1\frac{\mathrm{d}1}{2} - Q_1^{\mathrm{n}}(1)\mathrm{d}2\mathrm{d}1 + q_3\mathrm{d}1\mathrm{d}2\frac{\mathrm{d}1}{2} = 0\,. \tag{2.103}$$

Seeing that the terms of third order ($\mathrm{d}1\mathrm{d}2\mathrm{d}1$) are considered as infinitesimally small and because of $M_{21}^{\mathrm{n}} = M_{12}^{\mathrm{n}}$, finally the following results:

$$\frac{\partial M_1^{\mathrm{n}}}{\partial 1} + \frac{\partial M_{12}^{\mathrm{n}}}{\partial 2} - Q_1^{\mathrm{n}} = 0\,. \tag{2.104}$$

In a similar way, the equilibrium of moments around the reference axis at $2 + \mathrm{d}2$ finally gives

$$\frac{\partial M_2^{\mathrm{n}}}{\partial 2} + \frac{\partial M_{12}^{\mathrm{n}}}{\partial 1} - Q_2^{\mathrm{n}} = 0\,. \tag{2.105}$$

Thus, the three equilibrium equations can be summarized as follows:

$$\frac{\partial Q_1^{\mathrm{n}}}{\partial 1} + \frac{\partial Q_2^{\mathrm{n}}}{\partial 2} + q_3 = 0\,, \tag{2.106}$$

$$\frac{\partial M_1^{\mathrm{n}}}{\partial 1} + \frac{\partial M_{12}^{\mathrm{n}}}{\partial 2} - Q_1^{\mathrm{n}} = 0\,, \tag{2.107}$$

$$\frac{\partial M_2^{\mathrm{n}}}{\partial 2} + \frac{\partial M_{12}^{\mathrm{n}}}{\partial 1} - Q_2^{\mathrm{n}} = 0\,. \tag{2.108}$$

Let us recall here that the following equilibrium equations for the Euler–Bernoulli beam can be obtained [14]:

$$\frac{\mathrm{d}M_2(1)}{\mathrm{d}1} = Q_3(1)\,, \quad \frac{\mathrm{d}^2 M_2(1)}{\mathrm{d}1^2} = \frac{\mathrm{d}Q_3(1)}{\mathrm{d}1} = -q_3\,. \tag{2.109}$$

Rearranging Eqs. (2.107) and (2.108) for Q^{n} and introducing in Eq. (2.106) finally give the combined equilibrium equation as

$$\frac{\partial^2 M_1^{\mathrm{n}}}{\partial 1^2} + 2\frac{\partial^2 M_{12}}{\partial 1 \partial 2} + \frac{\partial^2 M_2^{\mathrm{n}}}{\partial 2^2} + q_3 = 0\,. \tag{2.110}$$

The last equation can be written in matrix notation as

$$\begin{bmatrix} \frac{\partial^2}{\partial 1^2} & \frac{\partial^2}{\partial 2^2} & \frac{2\partial^2}{\partial 1 \partial 2} \end{bmatrix} \begin{bmatrix} M_1^{\mathrm{n}} \\ M_2^{\mathrm{n}} \\ M_{12}^{\mathrm{n}} \end{bmatrix} + q_3 = 0\,, \tag{2.111}$$

or symbolically as

$$\boldsymbol{\mathcal{L}}_2^{\mathrm{T}} \boldsymbol{M}^{\mathrm{n}} + q_3 = 0\,. \tag{2.112}$$

Equations (2.107) and (2.108) can be rearranged to obtain a relationship between the moments and shear forces similar to Eq. $(2.109)_1$:

$$\boldsymbol{\mathcal{L}}_1^{\mathrm{T}} \boldsymbol{M}^{\mathrm{n}} = \boldsymbol{Q}^{\mathrm{n}}\,, \tag{2.113}$$

where the first-order differential operator matrix $\boldsymbol{\mathcal{L}}_1$ is given by Eqs. (2.79) and (2.80).

2.4.3 Combined Plane Elasticity and Classical Plate Element

Combining Eqs. (2.85)–(2.86) and (2.111)–(2.112), the following equilibrium description for a superposed element is obtained

$$\left[\begin{array}{cccccc} \frac{\partial}{\partial 1} & 0 & \frac{\partial}{\partial 2} & 0 & 0 & 0 \\ 0 & \frac{\partial}{\partial 2} & \frac{\partial}{\partial 1} & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & \frac{\partial^2}{\partial 1^2} & \frac{\partial^2}{\partial 2^2} & \frac{2\partial^2}{\partial 1 \partial 2} \end{array}\right] \left[\begin{array}{c} N_1^{\mathrm{n}} \\ N_2^{\mathrm{n}} \\ N_{12}^{\mathrm{n}} \\ \hline M_1^{\mathrm{n}} \\ M_2^{\mathrm{n}} \\ M_{12}^{\mathrm{n}} \end{array}\right] + \left[\begin{array}{c} q_1 \\ q_2 \\ \hline q_3 \end{array}\right] = \left[\begin{array}{c} 0 \\ 0 \\ \hline 0 \end{array}\right], \tag{2.114}$$

or in symbolic notation:

$$\boldsymbol{\mathcal{L}}^{\mathrm{T}} \boldsymbol{s}^{\mathrm{n}} + \boldsymbol{q} = \boldsymbol{0}, \tag{2.115}$$

where $\boldsymbol{s}^{\mathrm{n}}$ is the column matrix of the stress resultants per unit length (generalized stresses per unit length).

Let us note here that the derivation of the stress resultants N_i^{n} and M_i^{n} as given in Eqs. (2.84) and (2.90) must be revised for the combined case. This comes from the fact that the total strain in the form $\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^0 + 3\boldsymbol{\kappa}$ (see Eq. (2.19)) must be used in the derivation:

$$\begin{bmatrix} N_1^{\mathrm{n}} \\ N_2^{\mathrm{n}} \\ N_{12}^{\mathrm{n}} \end{bmatrix} = \int_{-t/2}^{t/2} \boldsymbol{\sigma} \mathrm{d}3 \overset{(2.32)}{=} \int_{-t/2}^{t/2} \boldsymbol{C}\boldsymbol{\varepsilon} \mathrm{d}3 \overset{(2.19)}{=} \int_{-t/2}^{t/2} \boldsymbol{C} \left(\boldsymbol{\varepsilon}^0 + 3\boldsymbol{\kappa}\right) \mathrm{d}3 \tag{2.116}$$

$$= \int_{-t/2}^{t/2} \boldsymbol{C}\boldsymbol{\varepsilon}^0 \mathrm{d}3 + \int_{-t/2}^{t/2} \boldsymbol{C} 3\boldsymbol{\kappa} \mathrm{d}3 = \boldsymbol{C}\boldsymbol{\varepsilon}^0 [3]_{-t/2}^{t/2} + \boldsymbol{C}\boldsymbol{\kappa} [3^2/2]_{-t/2}^{t/2}$$

$$= t\boldsymbol{C}\boldsymbol{\varepsilon}^0 + \frac{t^2}{4} \boldsymbol{C}\boldsymbol{\kappa}. \tag{2.117}$$

A corresponding derivation for the internal bending moments gives

$$\begin{bmatrix} M_1^{\mathrm{n}} \\ M_2^{\mathrm{n}} \\ M_{12}^{\mathrm{n}} \end{bmatrix} = \int_{-t/2}^{t/2} 3\boldsymbol{\sigma} \mathrm{d}3 \overset{(2.32)}{=} \int_{-t/2}^{t/2} 3\boldsymbol{C}\boldsymbol{\varepsilon} \mathrm{d}3 \overset{(2.19)}{=} \int_{-t/2}^{t/2} 3\boldsymbol{C} \left(\boldsymbol{\varepsilon}^0 + 3\boldsymbol{\kappa}\right) \mathrm{d}3 \tag{2.118}$$

$$= \int_{-t/2}^{t/2} \boldsymbol{C}\boldsymbol{\varepsilon}^0 3 \mathrm{d}3 + \int_{-t/2}^{t/2} \boldsymbol{C} 3^2 \boldsymbol{\kappa} \mathrm{d}3 = \boldsymbol{C}\boldsymbol{\varepsilon}^0 [3^2/2]_{-t/2}^{t/2} + \boldsymbol{C}\boldsymbol{\kappa} [3^3/3]_{-t/2}^{t/2}$$

$$= \frac{t^2}{4} \boldsymbol{C}\boldsymbol{\varepsilon}^0 + \frac{t^2 3}{12} \boldsymbol{C}\boldsymbol{\kappa}. \tag{2.119}$$

Equations (2.117) and (2.119) can be combined to obtain a single matrix representation:

$$\begin{bmatrix} N_1^{\text{n}} \\ N_2^{\text{n}} \\ N_{12}^{\text{n}} \\ \hline M_1^{\text{n}} \\ M_2^{\text{n}} \\ M_{12}^{\text{n}} \end{bmatrix} = \left[\begin{array}{c|c} \underbrace{t\,\boldsymbol{C}}_{\boldsymbol{A}} & \underbrace{\frac{t^2}{4}\,\boldsymbol{C}}_{\boldsymbol{B}} \\ \hline \underbrace{\frac{t^2}{4}\,\boldsymbol{C}}_{\boldsymbol{B}} & \underbrace{\frac{t^3}{12}\,\boldsymbol{C}}_{\boldsymbol{D}} \end{array}\right] \begin{bmatrix} \varepsilon_1^0 \\ \varepsilon_2^0 \\ \gamma_{12}^0 \\ \hline \kappa_1 \\ \kappa_2 \\ \kappa_{12} \end{bmatrix}, \tag{2.120}$$

or in symbolic notation:

$$\boldsymbol{s}^{\text{n}} = \boldsymbol{C}^{*}\boldsymbol{e}\,, \tag{2.121}$$

where $\boldsymbol{C}^{*}$ is the generalized elasticity matrix.

2.5 Partial Differential Equations

2.5.1 Plane Elasticity Element

Introducing the constitutive equation according to (2.32) in the equilibrium equation (2.80) gives

$$\boldsymbol{\mathcal{L}}_1^{\text{T}}\boldsymbol{C}\boldsymbol{\varepsilon} + \boldsymbol{b} = \boldsymbol{0}\,. \tag{2.122}$$

Introducing the kinematics relations in the last equation according to (2.3) finally gives the Lamé–Navier equations:

$$\boldsymbol{\mathcal{L}}_1^{\text{T}}\boldsymbol{C}\boldsymbol{\mathcal{L}}_1\boldsymbol{u}_{1,2} + \boldsymbol{b} = \boldsymbol{0}\,. \tag{2.123}$$

Alternatively, the displacements may be substituted and the differential equations are obtained in terms of stresses. This formulation is known as the Beltrami–Michell equations. If the body forces vanish ($\boldsymbol{b} = \boldsymbol{0}$), the partial differential equations in terms of stresses are called the Beltrami equations.

2.5.2 Classical Plate Element

Let us combine the three equations for the resulting moments according to Eqs. (2.87)–(2.89) in matrix notation as

$$\boldsymbol{M}^{\text{n}} = \begin{bmatrix} M_1^{\text{n}} \\ M_2^{\text{n}} \\ M_{12}^{\text{n}} \end{bmatrix} = \int_{-t/2}^{t/2} 3 \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{bmatrix} \text{d}3 = \int_{-t/2}^{t/2} 3\boldsymbol{\sigma}\,\text{d}3\,. \tag{2.124}$$

Introducing Hooke's law (2.32) and the kinematics relation (2.15) gives for a constant elasticity matrix $\boldsymbol{C}$

$$\boldsymbol{M}^{\mathrm{n}} = -\int_{-t/2}^{t/2} z^2 \boldsymbol{C}\boldsymbol{\mathcal{L}}_2 u_z \mathrm{d}z = -\boldsymbol{C}\boldsymbol{\mathcal{L}}_2 u_z \underbrace{\int_{-t/2}^{t/2} z^2 \mathrm{d}z}_{\frac{t^3}{12}} = -\frac{t^3}{12}\boldsymbol{C}\boldsymbol{\mathcal{L}}_2 u_z \,. \tag{2.125}$$

Using the kinematics relation in the curvature form (see Eq. (2.15)), it can be stated that

$$\boldsymbol{M}^{\mathrm{n}} = \frac{t^3}{12}\boldsymbol{C}\boldsymbol{\kappa} \,. \tag{2.126}$$

Introducing the moment–displacement relation (2.125) in the equilibrium equation (2.112) results in the plate bending differential equation in the form:

$$\boldsymbol{\mathcal{L}}_2^{\mathrm{T}}\left(\frac{t^3}{12}\boldsymbol{C}\boldsymbol{\mathcal{L}}_2 u_z\right) - q_z = 0 \,. \tag{2.127}$$

Assuming isotropic material behavior and the definitions for $\boldsymbol{\mathcal{L}}_2$ and $\boldsymbol{C}$ given in Eqs. (2.111) and (2.23), the following classical form of the plate bending differential equation can be obtained:

$$\frac{Et^3}{12(1-\nu^2)}\left(\frac{\partial^4 u_z}{\partial x^4} + 2\frac{\partial^4 u_z}{\partial x^2 \partial y^2} + \frac{\partial^4 u_z}{\partial y^4}\right) = q_z \,. \tag{2.128}$$

Let us recall here that the following partial differential equation for the Euler–Bernoulli beam can be obtained under the assumption of isotropic material behavior:

$$EI_y \frac{\mathrm{d}^4 u_z(x)}{\mathrm{d}x^4} = q_z(x) \,, \tag{2.129}$$

where q_z is now defined as a load per unit length.

2.5.3 Combined Plane Elasticity and Classical Plate Element

Starting from the equilibrium equation as given in Eq. (2.114) and inserting Eq. (2.120), i.e., the equation which is based on the definition of the stress resultants under considering the constitutive equation and the relation for the sum of the strain components, gives with the kinematics relation Eq. (2.20) the following set of partial differential equations:

$$
\begin{bmatrix}
\frac{\partial}{\partial 1} & 0 & \frac{\partial}{\partial 2} & 0 & 0 & 0 \\
0 & \frac{\partial}{\partial 2} & \frac{\partial}{\partial 1} & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{\partial^2}{\partial 1^2} & \frac{\partial^2}{\partial 2^2} & \frac{2\partial^2}{\partial 1 \partial 2}
\end{bmatrix}
\left[\begin{array}{c|c}
\underbrace{t\,\boldsymbol{C}}_{\boldsymbol{A}} & \underbrace{\frac{t^2}{4}\boldsymbol{C}}_{\boldsymbol{B}} \\ \hline
\underbrace{\frac{t^2}{4}\boldsymbol{C}}_{\boldsymbol{B}} & \underbrace{\frac{t^3}{12}\boldsymbol{C}}_{\boldsymbol{D}}
\end{array}\right]
\begin{bmatrix}
\frac{\partial}{\partial 1} & 0 & 0 \\
0 & \frac{\partial}{\partial 2} & 0 \\
\frac{\partial}{\partial 1} & \frac{\partial}{\partial 2} & 0 \\
0 & 0 & -\frac{\partial^2}{\partial 1^2} \\
0 & 0 & -\frac{\partial^2}{\partial 2^2} \\
0 & 0 & -\frac{2\partial^2}{\partial 1 \partial 2}
\end{bmatrix}
\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} +
$$

$$
+ \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \tag{2.130}
$$

or

$$
\begin{bmatrix}
\frac{\partial}{\partial 1} & 0 & \frac{\partial}{\partial 2} & 0 & 0 & 0 \\
0 & \frac{\partial}{\partial 2} & \frac{\partial}{\partial 1} & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{\partial^2}{\partial 1^2} & \frac{\partial^2}{\partial 2^2} & \frac{2\partial^2}{\partial 1 \partial 2}
\end{bmatrix}
\left[\begin{array}{ccc|ccc}
A_{11} & A_{12} & A_{14} & B_{11} & B_{12} & B_{14} \\
A_{12} & A_{22} & A_{24} & B_{12} & B_{22} & B_{24} \\
A_{14} & A_{24} & A_{24} & B_{44} & B_{24} & B_{44} \\ \hline
B_{11} & B_{12} & B_{14} & D_{11} & D_{12} & D_{14} \\
B_{12} & B_{22} & B_{24} & D_{12} & D_{22} & D_{24} \\
B_{14} & B_{24} & B_{44} & D_{14} & D_{24} & D_{44}
\end{array}\right]
\begin{bmatrix}
\frac{\partial}{\partial 1} & 0 & 0 \\
0 & \frac{\partial}{\partial 2} & 0 \\
\frac{\partial}{\partial 1} & \frac{\partial}{\partial 2} & 0 \\
0 & 0 & -\frac{\partial^2}{\partial 1^2} \\
0 & 0 & -\frac{\partial^2}{\partial 2^2} \\
0 & 0 & -\frac{2\partial^2}{\partial 1 \partial 2}
\end{bmatrix}
$$

$$
\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} + \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \tag{2.131}
$$

or in symbolic notation:

$$
\boldsymbol{\mathcal{L}}^{\mathrm{T}} \boldsymbol{C}^{*} \boldsymbol{\mathcal{L}}' \boldsymbol{u} + \boldsymbol{q} = \boldsymbol{0}\,. \tag{2.132}
$$

Once the solution of the displacements $\boldsymbol{u}$ in Eq. (2.132) is obtained, for example, based on the finite element method [12] or the finite difference method [13], the kinematics relationship (2.20) provides the strains from which the constitutive equation (2.31) allows the calculation of the stresses.

2.6 Failure Criteria

The following subsections present some of the common orthotropic failure criteria. The corresponding criteria for isotropic material behavior can be found, for example, in [11].

2.6.1 Maximum Stress Criterion

This criterion assumes that there is no failure, i.e., no fiber failure in the *1*-direction, no transversal stress failure in the *2*-direction, and no in-plane shear failure, as long as the stress components are in the following limits of the corresponding strength values k:

$$k_{1}\mathrm{c} < \sigma_{1} < k_{1\mathrm{t}}\,, \tag{2.133}$$

$$k_{2}\mathrm{c} < \sigma_{2} < k_{2\mathrm{t}}\,, \tag{2.134}$$

$$|\sigma_{12}| < k_{12\mathrm{s}}\,, \tag{2.135}$$

which assumes that there is no interaction between the different stress components. Some limitations of this criterion are discussed in [18].

2.6.2 Maximum Strain Criterion

The maximum strain criterion is similar to the maximum stress criterion and assumes no interaction between the strain components. No failure occurs as long as the strain components are in the following limits:

$$k_{1}\mathrm{c}^{\varepsilon} < \varepsilon_{1} < k^{\varepsilon}_{1\mathrm{t}}\,, \tag{2.136}$$

$$k_{2}\mathrm{c}^{\varepsilon} < \varepsilon_{2} < k^{\varepsilon}_{2\mathrm{t}}\,, \tag{2.137}$$

$$|\gamma_{12}| < k^{\varepsilon}_{12\mathrm{s}}\,, \tag{2.138}$$

where the k^{ε} represents the corresponding failure strains. Some limitations of this criterion are discussed in [18].

2.6.3 Tsai–Hill Criterion

The Tsai–Hill criterion is based on Hill's three-dimensional yield criterion for orthotropic materials which was applied by Tsai to lamina [8, 18]. This criterion can be stated for a plane stress state as follows:

$$\frac{\sigma_1^2}{k_1^2} - \frac{\sigma_1\sigma_2}{k_1^2} + \frac{\sigma_2^2}{k_2^2} + \frac{\sigma_{12}^2}{k_{12s}^2} < 1\,, \tag{2.139}$$

where $k_1 = k_{1\mathrm{t}}$ for $\sigma_1 > 0$ or $k_1 = k_{1\mathrm{c}}$ for $\sigma_1 < 0$. The same convention holds for direction 2. This criterion, which accounts for an interaction of the different stress components, showed a good agreement between the theoretical prediction and experimental values for E-glass-epoxy laminae at different rotational angles [18].

2.6.4 Tsai–Wu Criterion

The Tsai–Wu criterion can be expressed for orthotropic materials under plane stress states as follows [19]:

$$a_1\sigma_1 + a_2\sigma_2 + a_{11}\sigma_1^2 + a_{22}\sigma_2^2 + a_{12}\sigma_{12}^2 + 2a_{1,2}\sigma_1\sigma_2 < 1\,, \tag{2.140}$$

where the coefficients a can be related to the classical failure stresses as follows [1, 9]:

$$a_1 = \frac{1}{k_{1\mathrm{t}}} + \frac{1}{k_{1\mathrm{c}}}\,, \tag{2.141}$$

$$a_2 = \frac{1}{k_{2\mathrm{t}}} + \frac{1}{k_{2\mathrm{c}}}\,, \tag{2.142}$$

$$a_{11} = -\frac{1}{k_{1\mathrm{t}}k_{1\mathrm{c}}}\,, \tag{2.143}$$

$$a_{22} = -\frac{1}{k_{2\mathrm{t}}k_{2\mathrm{c}}}\,, \tag{2.144}$$

$$a_{12} = \frac{1}{k_{12\mathrm{s}}^2}\,, \tag{2.145}$$

$$a_{1,2} \approx -\frac{1}{2}\sqrt{\frac{1}{k_{1\mathrm{t}}k_{1\mathrm{c}}} \times \frac{1}{k_{2\mathrm{t}}k_{2\mathrm{c}}}}\,. \tag{2.146}$$

Excellent agreement between the Tsai–Wu criterion and experimental data for boron-epoxy is reported in [15].

References

1. Altenbach H, Altenbach J, Kissing W (2018) Mechanics of composite structural elements. Springer Nature, Singapore
2. Ashton JE, Whitney JM (1970) Theory of laminated plates. Technomic Publishing, Stamford
3. Blaauwendraad J (2010) Plates and FEM: surprises and pitfalls. Springer, Dordrecht
4. Chen WF, Han DJ (1988) Plasticity for structural engineers. Springer, New York
5. Clyne TW, Hull D (2019) An introduction to composite materials. Cambridge University Press, Cambridge
6. Daniel IM, Ishai O (1994) Engineering mechanics of composite materials. Oxford University Press, New York
7. Eschenauer H, Olhoff N, Schnell W (1997) Applied structural mechanics: fundamentals of elasticity, load-bearing structures, structural optimization. Springer, Berlin
8. Hill R (1950) The mathematical theory of plasticity. Oxford University Press, London
9. Kaw AK (2006) Mechanics of composite materials. CRC Press, Boca Raton
10. Jones RM (1975) Mechanics of composite materials. Scripta Book Co, Washington
11. Öchsner A (2016) Continuum damage and fracture mechanics. Springer, Singapore
12. Öchsner A (2020) Computational statics and dynamics: an introduction based on the finite element method. Springer, Singapore
13. Öchsner A (2020) Structural mechanics with a pen: a guide to solve finite difference problems. Springer, Cham
14. Öchsner A (2021) Classical beam theories of structural mechanics. Springer, Cham
15. Byron Pipes R, Cole BW (1973) On the off-axis strength test for anisotropic materials. J Compos Mater 7:246–256
16. Reddy JN (2006) An introduction to the finite element method. McGraw Hill, Singapore
17. Timoshenko S, Woinowsky-Krieger S (1959) Theory of plates and shells. McGraw-Hill Book Company, New York
18. Tsai SW (1968) Strength theories of filamentary structures. In: Schwartz RT, Schwartz HS (eds) Fundamental aspects of fiber reinforced plastic composites. Wiley Interscience, New York
19. Tsai SW, Wu EM (1971) A general theory of strength for anisotropic materials. J Compos Mater 5:58–80
20. Ventsel E, Krauthammer T (2001) Thin plates and shells: theory, analysis, and applications. Marcel Dekker, New York
21. Wang CM, Reddy JN, Lee KH (2000) Shear deformable beams and plates: relationships with classical solution. Elsevier, Oxford

Chapter 3
Macromechanics of a Laminate

Abstract This chapter covers the stacking of single laminae, generally under different angles, to a so-called laminate. Based on the approach of the classical laminate theory, a simplified stress analysis and a subsequent failure analysis are derived, without the solution of the system of coupled differential equations for the unknown displacements in the three coordinate directions. This theory provides the solution of the statically indeterminate system based on generalized stress–strain relationship under consideration of the constitutive relationship and the definition of the so-called stress resultants.

3.1 Introduction

Let us consider in the following a laminate, which is composed of n layers, see Fig. 3.1. Each layer k is a single lamina with its own orientation expressed in a local or lamina-specific coordinate system (1_k, 2_k, 3_k). Thus, the global or laminate-specific coordinate system (x, y, z) is used to describe the orientation of the laminate.

The height of a layer k ($1 \leq k \leq n$) can be expressed based on the thickness coordinate as

$$t_k = z_k - z_{k-1}\,, \tag{3.1}$$

and the total height of the laminate results as

$$t = \sum_{k=1}^{n} t_k\,. \tag{3.2}$$

© The Author(s), under exclusive license to Springer Nature Switzerland AG 2021

A. Öchsner, *Foundations of Classical Laminate Theory*, Advanced Structured Materials 163, https://doi.org/10.1007/978-3-030-82631-4_3

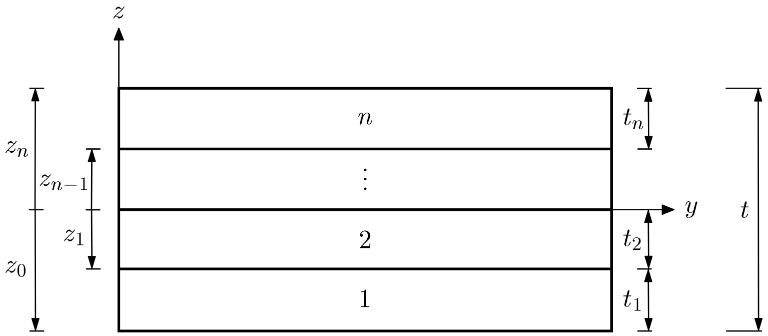


Fig. 3.1 Geometry of a laminate with n layers

3.2 Generalized Stress–Strain Relationship

Let us focus in the following on the evaluation on the stress resultants as introduced in Eqs. 2.116 and 2.118 for a single lamina. The internal normal forces can be expressed in the laminate-specific coordinate system (x, y, z) as

$$\begin{bmatrix} N_x^{\mathrm{n}} \\ N_y^{\mathrm{n}} \\ N_{xy}^{\mathrm{n}} \end{bmatrix} = \int_{-t/2}^{t/2} \boldsymbol{\sigma}\,\mathrm{d}z = \sum_{k=1}^{n} \int_{z_{k-1}}^{z_k} \boldsymbol{\sigma}_k \mathrm{d}\hat{z} = \sum_{k=1}^{n} \int_{z_{k-1}}^{z_k} \overline{\boldsymbol{C}_k} \boldsymbol{\varepsilon}_k \mathrm{d}\hat{z} \tag{3.3}$$

$$= \sum_{k=1}^{n} \int_{z_{k-1}}^{z_k} \overline{\boldsymbol{C}_k} \left(\boldsymbol{\varepsilon}^0 + \hat{z}\boldsymbol{\kappa}\right) \mathrm{d}\hat{z} \tag{3.4}$$

$$= \sum_{k=1}^{n} \left(\int_{z_{k-1}}^{z_k} \overline{\boldsymbol{C}_k} \boldsymbol{\varepsilon}^0 \mathrm{d}\hat{z} + \int_{z_{k-1}}^{z_k} \overline{\boldsymbol{C}_k} \hat{z} \boldsymbol{\kappa} \mathrm{d}\hat{z} \right) \tag{3.5}$$

$$= \sum_{k=1}^{n} \left(\underbrace{\overline{\boldsymbol{C}_k} \left(z_k - z_{k-1}\right)}_{\boldsymbol{A}_k} \boldsymbol{\varepsilon}^0 + \underbrace{\overline{\boldsymbol{C}_k} \frac{1}{2} \left((z_k)^2 - (z_{k-1})^2\right)}_{\boldsymbol{B}_k} \boldsymbol{\kappa} \right), \tag{3.6}$$

and the corresponding derivation for the internal bending moments as

$$\begin{bmatrix} M_x^{\mathrm{n}} \\ M_y^{\mathrm{n}} \\ M_{xy}^{\mathrm{n}} \end{bmatrix} = \int_{-t/2}^{t/2} z\boldsymbol{\sigma}\,\mathrm{d}z = \sum_{k=1}^{n} \int_{z_{k-1}}^{z_k} \boldsymbol{\sigma}_k \hat{z} \mathrm{d}\hat{z} = \sum_{k=1}^{n} \int_{z_{k-1}}^{z_k} \overline{\boldsymbol{C}_k} \boldsymbol{\varepsilon}_k \hat{z} \mathrm{d}\hat{z} \tag{3.7}$$

$$= \sum_{k=1}^{n} \int_{z_{k-1}}^{z_k} \overline{\boldsymbol{C}_k} \left(\boldsymbol{\varepsilon}^0 \hat{z} + \hat{z}^2 \boldsymbol{\kappa}\right) \mathrm{d}\hat{z} \tag{3.8}$$

$$= \sum_{k=1}^{n} \left(\int_{z_{k-1}}^{z_k} \overline{\boldsymbol{C}_k} \boldsymbol{\varepsilon}^0 \hat{z} \mathrm{d}\hat{z} + \int_{z_{k-1}}^{z_k} \overline{\boldsymbol{C}_k} \hat{z}^2 \boldsymbol{\kappa} \mathrm{d}\hat{z} \right) \tag{3.9}$$

$$= \sum_{k=1}^{n} \left(\underbrace{\overline{\boldsymbol{C}_k} \frac{1}{2} \left((z_k)^2 - (z_{k-1})^2 \right)}_{\boldsymbol{B}_k} \boldsymbol{\varepsilon}^0 + \underbrace{\overline{\boldsymbol{C}_k} \frac{1}{3} \left((z_k)^3 - (z_{k-1})^3 \right)}_{\boldsymbol{D}_k} \boldsymbol{\kappa} \right). \tag{3.10}$$

Equations (3.6) and (3.10) can be combined in a single matrix form to give

tension-shear coupling

tension-twist, bending-shear coupling

$$\begin{bmatrix} N_x^{\mathrm{n}} \\ N_y^{\mathrm{n}} \\ N_{xy}^{\mathrm{n}} \\ \hline M_x^{\mathrm{n}} \\ M_y^{\mathrm{n}} \\ M_{xy}^{\mathrm{n}} \end{bmatrix} = \left[\begin{array}{ccc|ccc} A_{11} & A_{12} & A_{14} & B_{11} & B_{12} & B_{14} \\ A_{12} & A_{22} & A_{24} & B_{12} & B_{22} & B_{24} \\ A_{14} & A_{24} & A_{24} & B_{44} & B_{24} & B_{44} \\ \hline B_{11} & B_{12} & B_{14} & D_{11} & D_{12} & D_{14} \\ B_{12} & B_{22} & B_{24} & D_{12} & D_{22} & D_{24} \\ B_{14} & B_{24} & B_{44} & D_{14} & D_{24} & D_{44} \end{array} \right] \begin{bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \\ \hline \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{bmatrix}, \tag{3.11}$$

bending-tension coupling

bending-twist coupling

or more symbolically as

$$\underbrace{\begin{bmatrix} \boldsymbol{N}^{\mathrm{n}} \\ \hline \boldsymbol{M}^{\mathrm{n}} \end{bmatrix}}_{\boldsymbol{s}} \underbrace{\left[\begin{array}{c|c} \boldsymbol{A} & \boldsymbol{B} \\ \hline \boldsymbol{B} & \boldsymbol{D} \end{array} \right]}_{\boldsymbol{C}^*} = \underbrace{\begin{bmatrix} \boldsymbol{\varepsilon}^0 \\ \hline \boldsymbol{\kappa} \end{bmatrix}}_{\boldsymbol{e}}, \tag{3.12}$$

where $\boldsymbol{s}$ is the column matrix of stress resultants (generalized stresses), $\boldsymbol{C}^*$ is the generalized elasticity matrix, and $\boldsymbol{e}$ is the column matrix of generalized strains. The corresponding submatrices are given as follows:

$$\boldsymbol{A} = \sum_{k=1}^{n} \boldsymbol{A}_k = \sum_{k=1}^{n} \overline{\boldsymbol{C}_k} \left(z_k - z_{k-1}\right) , \tag{3.13}$$

$$\boldsymbol{B} = \sum_{k=1}^{n} \boldsymbol{B}_k = \frac{1}{2}\sum_{k=1}^{n} \overline{\boldsymbol{C}_k} \left((z_k)^2 - (z_{k-1})^2\right) , \tag{3.14}$$

$$\boldsymbol{D} = \sum_{k=1}^{n} \boldsymbol{D}_k = \frac{1}{3}\sum_{k=1}^{n} \overline{\boldsymbol{C}_k} \left((z_k)^3 - (z_{k-1})^3\right) . \tag{3.15}$$

It should be noted here that $\boldsymbol{A}$ is called the extensional submatrix, $\boldsymbol{D}$ the bending submatrix, and $\boldsymbol{B}$ the coupling submatrix. Equation (3.12) can be inverted to obtain the strains and curvatures as function of the generalized stresses as

$$\underbrace{\left[\begin{array}{c} \boldsymbol{\varepsilon}^0 \\ \hline \boldsymbol{\kappa} \end{array}\right]}_{\boldsymbol{e}} = \left(\left[\begin{array}{c|c} \boldsymbol{A} & \boldsymbol{B} \\ \hline \boldsymbol{B} & \boldsymbol{D} \end{array}\right]\right)^{-1} \left[\begin{array}{c} \boldsymbol{N}^{\mathrm{n}} \\ \hline \boldsymbol{M}^{\mathrm{n}} \end{array}\right] = \underbrace{\left[\begin{array}{c|c} \boldsymbol{A}' & \boldsymbol{B}' \\ \hline (\boldsymbol{B}')^{\mathrm{T}} & \boldsymbol{D}' \end{array}\right]}_{(\boldsymbol{C}^*)^{-1}} \underbrace{\left[\begin{array}{c} \boldsymbol{N}^{\mathrm{n}} \\ \hline \boldsymbol{M}^{\mathrm{n}} \end{array}\right]}_{\boldsymbol{s}} , \tag{3.16}$$

where the submatrices are given as follows [1, 2]:

$$\boldsymbol{A}' = \boldsymbol{A}^{-1} + \left(-\boldsymbol{A}^{-1}\boldsymbol{B}\right)\left(\boldsymbol{D} - \boldsymbol{B}\boldsymbol{A}^{-1}\boldsymbol{B}\right)^{-1}\left(-\boldsymbol{A}^{-1}\boldsymbol{B}\right)^{\mathrm{T}} , \tag{3.17}$$

$$\boldsymbol{B}' = \left(-\boldsymbol{A}^{-1}\boldsymbol{B}\right)\left(\boldsymbol{D} - \boldsymbol{B}\boldsymbol{A}^{-1}\boldsymbol{B}\right)^{-1} , \tag{3.18}$$

$$\boldsymbol{D}' = \left(\boldsymbol{D} - \boldsymbol{B}\boldsymbol{A}^{-1}\boldsymbol{B}\right)^{-1} . \tag{3.19}$$

As can be concluded from Eq. (3.11), there is a coupling between tension and shear ($A_{14} : N_x^{\mathrm{n}} \rightarrow \gamma_{xy}^0$; $A_{24} : N_y^{\mathrm{n}} \rightarrow \gamma_{xy}^0$), bending and tension ($B_{11}, B_{12} : M_x^{\mathrm{n}} \rightarrow \varepsilon_x^0, \varepsilon_y^0$; $B_{12}, B_{22} : M_y^{\mathrm{n}} \rightarrow \varepsilon_x^0, \varepsilon_y^0$), tension and twist ($B_{14} : N_x^{\mathrm{n}} \rightarrow \kappa_{xy}$; $B_{24} : N_y^{\mathrm{n}} \rightarrow \kappa_{xy}$), bending and shear ($B_{14} : M_x^{\mathrm{n}} \rightarrow \gamma_{xy}^0$; $B_{24} : M_y^{\mathrm{n}} \rightarrow \gamma_{xy}^0$), and bending and twist ($D_{14} : M_x^{\mathrm{n}} \rightarrow \kappa_{xy}$; $D_{24} : M_y^{\mathrm{n}} \rightarrow \kappa_{xy}$).

Based on the generalized strains obtained from Eq. (3.16), it is now possible to calculate the stresses in each layer k expressed in the x–y coordinate system:

$$\boldsymbol{\sigma}_{x,y}^{k}(z) = \overline{\boldsymbol{C}_k}\left(\boldsymbol{\varepsilon}^0 + z\boldsymbol{\kappa}\right) , \tag{3.20}$$

where the vertical coordinate z ranges for the kth layer in the following boundaries: $z_{k-1} \leq z \leq z_k$. The stresses may be evaluated at the bottom ($z = z_{k-1}$), middle ($z = (z_k + z_{k-1})/2$), or top ($z = z_k$) of each layer. Based on relation (2.36), we can transform the stress values into the *1–2* coordinate system (see Fig. 2.5):

$$\boldsymbol{\sigma}_{1,2}^{k} = \boldsymbol{T}_{\sigma}^{k}\boldsymbol{\sigma}_{x,y}^{k} . \tag{3.21}$$

The obtained stress values may serve for a subsequent failure analysis of layer k (see Chap. 4).

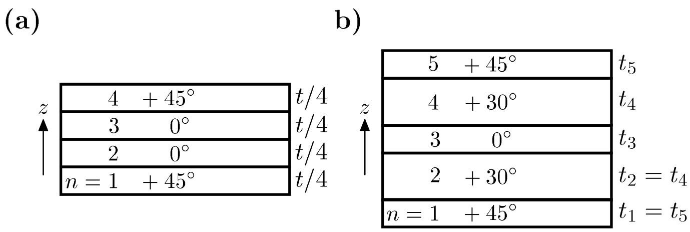


Fig. 3.2 Examples of symmetric laminates: **a** regular $[45°/0°]_s$ and **b** $[45°/30°/\overline{0°}]_s$ with different layer thicknesses

3.3 Special Cases of Laminates

The following section covers special cases of the generalized elasticity matrix $\boldsymbol{C}^*$ as provided in Eq. (3.11), see [1, 3] for details.

- Symmetric laminates:

Symmetric laminates have a symmetric buildup sequence in regards to laminae (plies) orientations, thicknesses, and material properties about the middle surface, see Fig. 3.2 for some examples.

The generalized stiffness matrix (3.11) simplifies for symmetric laminates to the following form:

$$\boldsymbol{C}^* = \left[\begin{array}{ccc|ccc} A_{11} & A_{12} & A_{14} & 0 & 0 & 0 \\ A_{12} & A_{22} & A_{24} & 0 & 0 & 0 \\ A_{14} & A_{24} & A_{24} & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & D_{11} & D_{12} & D_{14} \\ 0 & 0 & 0 & D_{12} & D_{22} & D_{24} \\ 0 & 0 & 0 & D_{14} & D_{24} & D_{44} \end{array}\right], \tag{3.22}$$

where we have now $\boldsymbol{B} = \boldsymbol{0}$.

- Symmetric laminates with isotropic layers:

Symmetric laminates with isotropic layers contain only isotropic laminae (plies), i.e., laminae where the elastic properties are independent of the direction and characterized by two independent properties (e.g., Young's modulus and Poisson's ratio), see Fig. 3.3.

The generalized stiffness matrix (3.11) simplifies for symmetric laminates with isotropic layers to the following form:

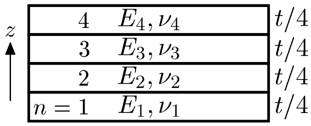


Fig. 3.3 Example of a symmetric laminate with isotropic layers

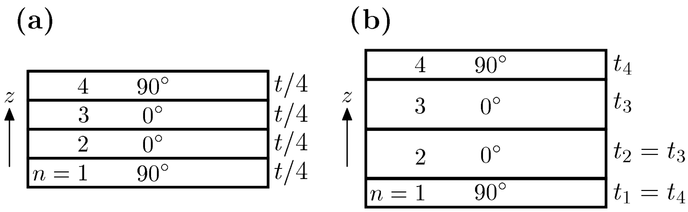


Fig. 3.4 Examples of symmetric cross-ply laminates: **a** regular $[90°/0°]_s$ and **b** $[90°/0°]_s$ with different layer thicknesses

$$
\boldsymbol{C}^* = \left[\begin{array}{ccc|ccc} A_{11} & A_{12} & 0 & 0 & 0 & 0 \\ A_{12} & A_{11} & 0 & 0 & 0 & 0 \\ 0 & 0 & A_{24} & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & D_{11} & D_{12} & 0 \\ 0 & 0 & 0 & D_{12} & D_{11} & 0 \\ 0 & 0 & 0 & 0 & 0 & D_{44} \end{array}\right]. \tag{3.23}
$$

- Symmetric cross-ply laminates:

Symmetric cross-ply laminates only contain orthotropic laminae (plies) at right angles to one another, see Fig. 3.4 for some examples. Laminae may have pair-wise (i.e., to ensure symmetry) different thicknesses and material properties.

The generalized stiffness matrix (3.11) simplifies for symmetric cross-ply laminates to the following form:

$$
\boldsymbol{C}^* = \left[\begin{array}{ccc|ccc} A_{11} & A_{12} & 0 & 0 & 0 & 0 \\ A_{12} & A_{22} & 0 & 0 & 0 & 0 \\ 0 & 0 & A_{24} & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & D_{11} & D_{12} & 0 \\ 0 & 0 & 0 & D_{12} & D_{22} & 0 \\ 0 & 0 & 0 & 0 & 0 & D_{44} \end{array}\right]. \tag{3.24}
$$

- Symmetric angle-ply laminates:

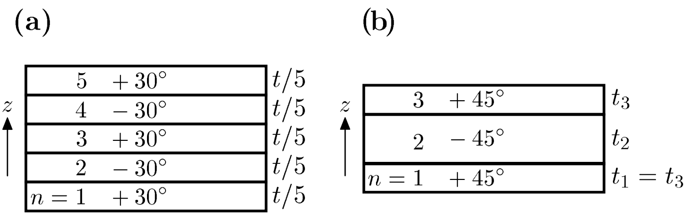


Fig. 3.5 Examples of symmetric angle-ply laminates: **a** regular $[30°/-30°/\overline{30°}]_{\mathrm{s}}$ and **b** $[45°/\overline{-45°}]_{\mathrm{s}}$ with different layer thicknesses

Symmetric angle-ply laminates only contain orthotropic laminae (plies) at which adjacent laminae have opposite orientation signs, see Fig. 3.5 for some examples. Laminae may have pair-wise (i.e., to ensure symmetry) different thicknesses and material properties.

The generalized stiffness matrix (3.11) simplifies for symmetric angle-ply laminates to the following form:

$$
\boldsymbol{C}^* = \left[\begin{array}{ccc|ccc}
A_{11} & A_{12} & A_{14} & 0 & 0 & 0 \\
A_{12} & A_{22} & A_{24} & 0 & 0 & 0 \\
A_{14} & A_{24} & A_{24} & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & D_{11} & D_{12} & D_{14} \\
0 & 0 & 0 & D_{12} & D_{22} & D_{24} \\
0 & 0 & 0 & D_{14} & D_{24} & D_{44}
\end{array}\right] . \tag{3.25}
$$

It should be noted here that we obtain for the special case $[45°/\overline{-45°}]_{\mathrm{s}}$ (see Fig. 3.5b) the further simplification $A_{22} = A_{11}$. However, $[30°/\overline{-30°}]_{\mathrm{s}}$ yields $A_{22} \neq A_{11}$.

- Symmetric balanced laminates:

Symmetric angle-ply laminates only contain orthotropic laminae (plies) at which all identical laminae at angles other than 0° and 90° occur only in ± pairs (not necessarily adjacent), see Fig. 3.6 for some examples. These laminates only contain an even number of laminae.[1]

The generalized stiffness matrix (3.11) simplifies for symmetric balanced laminates to the following form:

[1] Thus, symmetric angle-ply laminates do not belong to the group of symmetric balanced laminates.

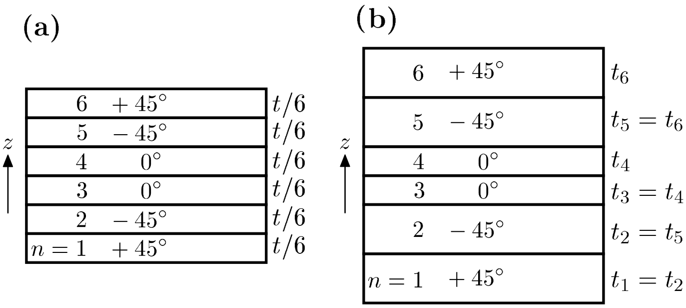
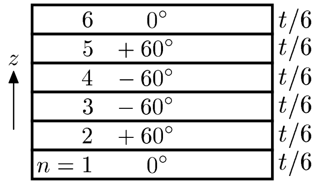


Fig. 3.6 Examples of symmetric balanced laminates: **a** regular $[45°/-45°/0°]_s$ and **b** $[45°/-45°/0°]_s$ with different layer thicknesses

Fig. 3.7 Example of a quasi-isotropic symmetric laminate: $[0°/60°/-60°]_s$



$$
\boldsymbol{C}^* = \left[\begin{array}{ccc|ccc}
A_{11} & A_{12} & 0 & 0 & 0 & 0 \\
A_{12} & A_{22} & 0 & 0 & 0 & 0 \\
0 & 0 & A_{24} & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & D_{11} & D_{12} & D_{14} \\
0 & 0 & 0 & D_{12} & D_{22} & D_{24} \\
0 & 0 & 0 & D_{14} & D_{24} & D_{44}
\end{array}\right]. \tag{3.26}
$$

• Symmetric quasi-isotropic laminates:

Symmetric quasi-isotropic laminates are balanced and only contain orthotropic laminae (plies) for which a constitutive property of interest, at a given point, displays isotropic behavior in the *plane* of the laminates, see Fig. 3.7.

The generalized stiffness matrix (3.11) simplifies for symmetric quasi-isotropic laminates to the following form:

$$
\boldsymbol{C}^* = \left[\begin{array}{ccc|ccc}
A_{11} & A_{12} & 0 & 0 & 0 & 0 \\
A_{12} & A_{11} & 0 & 0 & 0 & 0 \\
0 & 0 & A_{24} & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & D_{11} & D_{12} & D_{14} \\
0 & 0 & 0 & D_{12} & D_{22} & D_{24} \\
0 & 0 & 0 & D_{14} & D_{24} & D_{44}
\end{array}\right]. \tag{3.27}
$$

It should be noted here that symmetric quasi-isotropic laminates require that all layers have the same thickness and the same orthotropic material properties. As an example, $t_1 = t_6$ but $t_2 = t_3 = t_4 = t_5 \neq t_1$ would yield $A_{11} \neq A_{22}$. In a similar way, $\mathrm{Mat}_1 = \mathrm{Mat}_6$ but $\mathrm{Mat}_2 = \mathrm{Mat}_3 = \mathrm{Mat}_4 = \mathrm{Mat}_5 \neq \mathrm{Mat}_1$ would yield $A_{11} \neq A_{22}$. Thus, such laminates would be no longer quasi-isotropic.

Table 3.1 Recommended steps for a calculation according to the classical laminate theory. Case: internal normal forces N_i^{n} and internal bending moments M_i^{n} given

Nr.	Steps to perform
①	Define for each lamina k the material ($E_{1,k}$; $E_{2,k}$; $\nu_{12,k}$; $G_{12,k}$) and the geometrical (z_k; z_{k-1}; α_k) properties
②	Define the column matrix of generalized stresses: $\boldsymbol{s} = \left[N_x^{\mathrm{n}}\ N_y^{\mathrm{n}}\ N_{xy}^{\mathrm{n}}\quad M_x^{\mathrm{n}}\ M_y^{\mathrm{n}}\ M_{xy}^{\mathrm{n}}\right]^{\mathrm{T}}$
③	Calculate for each layer k the elasticity matrix $\boldsymbol{C}_k$ in the *1–2* lamina system according to Eq. (2.31). Transform each matrix according to Eq. (2.49) to obtain the elasticity matrix $\overline{\boldsymbol{C}_k}$ in the x–y laminate system
④	Calculate the submatrices $\boldsymbol{A}$, $\boldsymbol{B}$, and $\boldsymbol{C}$ according to Eqs. (3.13)–(3.15). Assemble the generalized elasticity matrix $\boldsymbol{C}^*$ according to Eq. (3.12)
⑤	Calculate the generalized compliance matrix $(\boldsymbol{C}^*)^{-1}$ based on Eqs. (3.17)–(3.19)
⑥	Calculate the generalized strains $\boldsymbol{e} = \left[\boldsymbol{\varepsilon}^0\ \boldsymbol{\kappa}\right]^{\mathrm{T}}$ according to Eq. (3.16)
⑦	Calculate the stresses in each layer k according to Eq. (3.20), $\boldsymbol{\sigma}^k_{x,y}(z) = \overline{\boldsymbol{C}_k}\left(\boldsymbol{\varepsilon}^0 + z\boldsymbol{\kappa}\right)$, in the x–y laminate system ($z_{k-1} \leq z \leq z_k$). The strains in the x–y laminate system are obtained from $\left(\boldsymbol{\varepsilon}^0 + z\boldsymbol{\kappa}\right)$
⑧	Transform the stresses in each layer to the *1–2* lamina system according to Eq. (3.21): $\boldsymbol{\sigma}^k_{1,2} = \boldsymbol{T}^k_{\sigma}\boldsymbol{\sigma}^k_{x,y}$. The strains in the *1–2* lamina system are obtained from Eq. (2.33): $\boldsymbol{\varepsilon}^k_{1,2} = (\boldsymbol{C}_k)^{-1}\boldsymbol{\sigma}^k_{1,2}$

Table 3.2 Recommended steps for a calculation according to the classical laminate theory. Case: generalized strains $\boldsymbol{e} = \left[\varepsilon_x^0\ \varepsilon_y^0\ \gamma_{xy}^0\ \kappa_x\ \kappa_y\ \kappa_{xy}\right]^{\mathrm{T}}$ given

Nr.	Steps to perform
①	Define for each lamina k the material ($E_{1,k}$; $E_{2,k}$; $\nu_{12,k}$; $G_{12,k}$) and the geometrical (z_k; z_{k-1}; α_k) properties
②	Define the column matrix of generalized strains: $\boldsymbol{e} = \left[\varepsilon_x^0\ \varepsilon_y^0\ \gamma_{xy}^0\ \kappa_x\ \kappa_y\ \kappa_{xy}\right]^{\mathrm{T}}$
③	Calculate for each layer k the elasticity matrix $\boldsymbol{C}_k$ in the *1–2* lamina system according to Eq. (2.31). Transform each matrix according to Eq. (2.49) to obtain the elasticity matrix $\overline{\boldsymbol{C}_k}$ in the x–y laminate system
④	Calculate the submatrices $\boldsymbol{A}$, $\boldsymbol{B}$, and $\boldsymbol{C}$ according to Eqs. (3.13)–(3.15). Assemble the generalized elasticity matrix $\boldsymbol{C}^*$ according to Eq. (3.12)
⑤	Calculate the generalized stresses $\boldsymbol{s} = \left[\boldsymbol{N}^{\mathrm{n}}\ \boldsymbol{M}^{\mathrm{n}}\right]$ according to Eq. (3.11)
⑥	In case that the stresses and strains in each layer are required, go to step ⑦ and ⑧ in Table 3.1

3.4 Failure Analysis of Laminates

Let us summarize here the recommended steps for a calculation according to the classical laminate theory (internal normal forces N_i^{n} and internal bending moments M_i^{n} given), see Tables 3.1 and 3.2.

References

1. Altenbach H, Altenbach J, Kissing W (2018) Mechanics of composite structural elements. Springer, Singapore
2. Chawla KK (1987) Composite materials: science and engineering. Springer, New York
3. MIL-HDBK-17, 2F, (2002) Composite materials handbook -, vol 2. Polymer matrix composites materials properties, Department of Defence, USA

Chapter 4
Example Problems

Abstract This chapter presents the solution of some standard calculation problems based on the classical laminate theory. Each solution is based on a step-by-step approach in order to present all the details of the solution procedure. The presented problems comprise the calculation of stresses and strains in symmetric and asymmetric laminates, the application of failure criteria, and the ply-by-ply failure of laminates. In addition, the characterization of basic properties of unidirectional laminae based on pole diagrams for the elastic properties and failure envelopes is presented.

4.1 Introduction

The following sections will cover some standard calculation problems which are solved in great detail. All the presented problems are based on the same AS4 carbon/3501-6 epoxy prepreg[1] system. The effective lamina properties can be taken from Table 4.1. Other sets of complete mechanical properties for unidirectional laminae can be found, for example, in [5, 7].

The theoretical prediction of lamina properties, i.e., the so-called micromechanics, is not covered in this textbook and the interested reader is referred to the specialized literature, e.g. [2–4].

4.2 Problem 1: Stresses and Strains in a Symmetric Laminate

Given is a symmetric 8-layer laminate of thickness $t = 8$ mm as shown in Fig. 4.1. Each lamina has the same thickness $(t/8)$ and identical material properties, which can be taken from Table 4.1. However, the orientation of each lamina changes from layer to layer as indicated in Fig. 4.1.

[1] A prepreg is a ready to mold or cure sheet which may be tow, tape, cloth, or mat impregnated with resin [6].

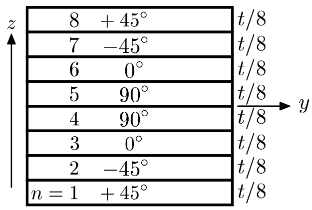
© The Author(s), under exclusive license to Springer Nature Switzerland AG 2021

A. Öchsner, *Foundations of Classical Laminate Theory*, Advanced Structured Materials 163, https://doi.org/10.1007/978-3-030-82631-4_4

Table 4.1 Selected mechanical properties of a unidirectional lamina (fibre: AS4 carbon; matrix: 3501-6 epoxy; fibre volume fraction of the prepreg: 0.6). Adapted from [7]

Property	Value
Elastic constants	
Longitudinal modulus, E_1 in MPa	126000
Transverse modulus, E_2 in MPa	11000
In-plane shear modulus, G_{12} in MPa	6600
Major Poisson's ratio, ν_{12} in	0.28
Strength parameters	
Longitudinal tensile strength, k_{1t} in MPa	1950
Longitudinal compressive strength, k_{1c} in MPa	–1480
Transverse tensile strength, k_{2t} in MPa	48
Transverse compressive strength, k_{2c} in MPa	–200
In-plane shear strength, k_{12s} in MPa	79
Longitudinal tensile failure strain, k_{1t}^{ε} in %	1.38
Longitudinal compressive failure strain, k_{1c}^{ε} in %	–1.175
Transverse tensile failure strain, k_{2t}^{ε} in %	0.436
Transverse compressive failure strain, k_{2c}^{ε} in %	–2.0
In-plane shear failure strain, k_{12s}^{ε} in %	2

Fig. 4.1 Symmetric laminate: $[\pm 45°/0°/90°]_s$



Consider (a) a pure tensile load case with $N_x^{\mathrm{n}} = 1000$ N/mm (all other forces and moments are equal to zero) and (b) a pure bending load case with $M_x^{\mathrm{n}} = 1000$ Nmm/mm (all other moments and forces are equal to zero) to determine the stresses and strains in each lamina in the global x–y coordinate system as well as in the principal material coordinates (1–2 coordinate systems) of each lamina.

- Laminate loaded by $N_x^{\mathrm{n}} = 1000$ N/mm

The solution procedure will follow the recommended steps provided in Table 3.1.

Table 4.2 Geometrical properties (z_k; z_{k-1}; α_k) of the laminate shown in Fig. 4.1

Lamina k	Coordinate z_{k-1} in mm	Coordinate z_k in mm	Angle α_k
1	–4	–3	+45°
2	–3	–2	−45°
3	–2	–1	0°
4	–1	0	90°
5	0	1	90°
6	1	2	0°
7	2	3	−45°
8	3	4	+45°

① Define for each lamina k the material ($E_{1,k}$; $E_{2,k}$; $\nu_{12,k}$; $G_{12,k}$) and the geometrical (z_k; z_{k-1}; α_k) properties.

The material properties ($E_{1,k}$; $E_{2,k}$; $\nu_{12,k}$; $G_{12,k}$) are the same for each lamina and can be taken from Table 4.1. The geometrical properties for each lamina are summarized in Table 4.2

② Define the column matrix of generalized stresses:
$\boldsymbol{s} = \left[N_x^{\text{n}}\ N_y^{\text{n}}\ N_{xy}^{\text{n}}\ \ M_x^{\text{n}}\ M_y^{\text{n}}\ M_{xy}^{\text{n}}\right]^{\text{T}}$.

The load matrix for case (a) is given as follows:

$$\boldsymbol{s} = \left[1000\ 0\ 0\ 0\ 0\ 0\right]^{\text{T}}\ \text{N/mm}\,. \tag{4.1}$$

③ Calculate for each layer k the elasticity matrix $\boldsymbol{C}_k$ in the *1–2* lamina system according to Eq. (2.31). Transform each matrix according to Eq. (2.49) to obtain the elasticity matrix $\overline{\boldsymbol{C}}_k$ in the x–y laminate system.

Since each lamina is composed of the same materials and the thickness is equal to $t/8 = 1$ mm, Eq. (2.31) yields for each lamina the same matrix

$$\boldsymbol{C}_k = \begin{bmatrix} \frac{E_1}{1-\nu_{12}\nu_{21}} & \frac{\nu_{21}E_1}{1-\nu_{12}\nu_{21}} & 0 \\ \frac{\nu_{12}E_2}{1-\nu_{12}\nu_{21}} & \frac{E_2}{1-\nu_{12}\nu_{21}} & 0 \\ 0 & 0 & G_{12} \end{bmatrix} = \begin{bmatrix} 129868.204 & 3100.729 & 0 \\ 3100.729 & 11074.033 & 0 \\ 0 & 0 & 6600.0 \end{bmatrix} \frac{\mathrm{N}}{\mathrm{mm}^2}\,. \tag{4.2}$$

The transformation to the x–y laminate system according to Eq. (2.49) gives the following matrices:

$$\overline{\boldsymbol{C}}_{[\pm 45^\circ]} = \begin{bmatrix} 43385.924 & 30185.924 & \pm 29698.543 \\ 30185.924 & 43385.924 & \pm 29698.543 \\ \pm 29698.543 & \pm 29698.543 & 33685.195 \end{bmatrix} \frac{\mathrm{N}}{\mathrm{mm}^2}\,, \tag{4.3}$$

$$\overline{\boldsymbol{C}}_{[0^\circ]} = \begin{bmatrix} 129868.204 & 3100.729 & 0.0 \\ 3100.729 & 11074.033 & 0.0 \\ 0.0 & 0.0 & 6600.0 \end{bmatrix} \frac{\mathrm{N}}{\mathrm{mm}^2}\,, \tag{4.4}$$

$$\overline{\boldsymbol{C}}_{[90^\circ]} = \begin{bmatrix} 11074.033 & 3100.729 & 0.0 \\ 3100.729 & 129868.204 & 0.0 \\ 0.0 & 0.0 & 6600.0 \end{bmatrix} \frac{\mathrm{N}}{\mathrm{mm}^2}\,. \tag{4.5}$$

④ Calculate the submatrices $\boldsymbol{A}$, $\boldsymbol{B}$, and $\boldsymbol{C}$ according to Eqs. (3.13)–(3.15). Assemble the generalized elasticity matrix $\boldsymbol{C}^*$ according to Eq. (3.12).

The evaluation of the submatrices $\boldsymbol{A}$, $\boldsymbol{B}$, and $\boldsymbol{C}$ according to Eqs. (3.13)–(3.15) gives

$$\boldsymbol{A} = \sum_{k=1}^{8} \boldsymbol{A}_k = \begin{bmatrix} 455428.170 & 133146.612 & 0.0 \\ 133146.612 & 455428.170 & 0.0 \\ 0.0 & 0.0 & 161140.779 \end{bmatrix} \frac{\text{N}}{\text{mm}}, \tag{4.6}$$

$$\boldsymbol{B} = \sum_{k=1}^{8} \boldsymbol{B}_k = \begin{bmatrix} 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 \end{bmatrix} \text{N}, \tag{4.7}$$

$$\boldsymbol{D} = \sum_{k=1}^{8} \boldsymbol{D}_k = \begin{bmatrix} 2233175.466 & 1143478.381 & 356382.514 \\ 1143478.381 & 1757998.781 & 356382.514 \\ 356382.514 & 356382.514 & 1292780.601 \end{bmatrix} \text{Nmm}, \tag{4.8}$$

from which the generalized elasticity matrix (written without units) can be assembled as follows:

$$\boldsymbol{C}^* = \begin{bmatrix} 455428.170 & 133146.612 & 0.0 & 0.0 & 0.0 & 0.0 \\ 133146.612 & 455428.170 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 161140.779 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 2233175.466 & 1143478.381 & 356382.514 \\ 0.0 & 0.0 & 0.0 & 1143478.381 & 1757998.78 & 356382.514 \\ 0.0 & 0.0 & 0.0 & 356382.514 & 356382.514 & 1292780.601 \end{bmatrix}. \tag{4.9}$$

⑤ Calculate the generalized compliance matrix $(\boldsymbol{C}^*)^{-1}$ based on Eqs. (3.17)–(3.19).

The evaluation of the submatrices $\boldsymbol{A}'$, $\boldsymbol{B}'$, and $\boldsymbol{C}'$ according to Eqs. (3.17)–(3.19) gives

$$\boldsymbol{A}' = \begin{bmatrix} 24.009482 & -7.019287 & 0.0 \\ -7.019287 & 24.009482 & 0.0 \\ 0.0 & 0.0 & 62.057538 \end{bmatrix} \frac{10^{-7}\text{mm}}{\text{N}}, \tag{4.10}$$

$$\boldsymbol{B}' = \begin{bmatrix} 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 \end{bmatrix} \frac{1}{\text{N}}, \tag{4.11}$$

$$\boldsymbol{D}' = \begin{bmatrix} 6.771890 & -4.264613 & -0.691184 \\ -4.264613 & 8.710637 & -1.225641 \\ -0.691184 & -1.225641 & 8.263678 \end{bmatrix} \frac{10^{-7}}{\text{Nmm}}, \tag{4.12}$$

from which the generalized compliance matrix (written without units) can be assembled as follows:

$$(\boldsymbol{C}^{*})^{-1} = \begin{bmatrix} 24.009482 & -7.019287 & 0.0 & 0.0 & 0.0 & 0.0 \\ -7.019287 & 24.009482 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 62.057538 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 6.771890 & -4.264613 & -0.691184 \\ 0.0 & 0.0 & 0.0 & -4.264613 & 8.710637 & -1.225641 \\ 0.0 & 0.0 & 0.0 & -0.691184 & -1.225641 & 8.263678 \end{bmatrix} 10^{-7}. \tag{4.13}$$

⑥ Calculate the generalized strains $\boldsymbol{e} = \left[\boldsymbol{\varepsilon}^{0}\ \boldsymbol{\kappa}\right]^{\mathrm{T}}$ according to Eq. (3.16).

The evaluation of Eq. (3.16) gives the generalized strains as follows:

$$\underbrace{\begin{bmatrix} \boldsymbol{\varepsilon}^0 \\ \hline \boldsymbol{\kappa} \end{bmatrix}}_{\boldsymbol{e}} = \begin{bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \\ \hline \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{bmatrix} = (\boldsymbol{C}^*)^{-1}\boldsymbol{s} = \begin{bmatrix} 2.400948 \\ -0.701929 \\ 0.0 \\ 0.0 \\ 0.0 \\ 0.0 \end{bmatrix} ‰ . \tag{4.14}$$

⑦ Calculate the stresses in each layer k according to Eq. (3.20), $\boldsymbol{\sigma}_{x,y}^k(z) = \overline{\boldsymbol{C}}_k \left(\boldsymbol{\varepsilon}^0 + z\boldsymbol{\kappa}\right)$, in the x–y laminate system ($z_{k-1} \leq z \leq z_k$). The strains in the x–y laminate system are obtained from $\left(\boldsymbol{\varepsilon}^0 + z\boldsymbol{\kappa}\right)$.

Evaluation of Eq. (3.20) for each layer k, i.e.

$$\boldsymbol{\sigma}_{x,y}^k(z) = \overline{\boldsymbol{C}}_k \left(\boldsymbol{\varepsilon}^0 + z\boldsymbol{\kappa}\right), \tag{4.15}$$

under consideration of the generalized strains given in Eq. (4.14) and the range of the z-coordinate ($z_{k-1} \leq z \leq z_k$) provided in Table 4.2 gives the stresses for each layer in the x–y laminate system. In case that the strains in the x–y laminate system are required, the evaluation of $\left(\boldsymbol{\varepsilon}^0 + z\boldsymbol{\kappa}\right)$ provides these values. A graphical representation of the normal stress (σ_x) and normal strain (ε_x) in loading direction (x) is provided in Fig. 4.2 as a function of the laminate thickness (z-coordinate). It can be seen that a uniaxial load by N_x^{n} results in the case of this symmetric laminate in symmetric distributions of the normal stress and strain with respect to $z = 0$. The strain distribution is constant (see Fig. 4.2b) across the entire thickness of the laminate. The stress distribution (see Fig. 4.2a) reveals a layer-wise constant distribution with the characteristic that the stress is proportional to the stiffness value of each layer in the loading direction. Furthermore, it can be concluded that the ±45° layers show the same behavior in the x-direction.

⑧ Transform the stresses in each layer to the *1*–*2* lamina system according to Eq. (3.21): $\boldsymbol{\sigma}_{1,2}^k = \boldsymbol{T}_\sigma^k \boldsymbol{\sigma}_{x,y}^k$. The strains in the *1*–*2* lamina system are obtained from Eq. (2.33): $\boldsymbol{\varepsilon}_{1,2}^k = (\boldsymbol{C}_k)^{-1}\boldsymbol{\sigma}_{1,2}^k$.

The transformation of the stresses and strains into the *1*–*2* lamina systems is performed according to Eqs. (3.21) and (2.33). The corresponding numerical values for the bottom (z_{k-1}), the middle ($(z_{k-1} + z_k)/2$), and the top (z_k) of each lamina

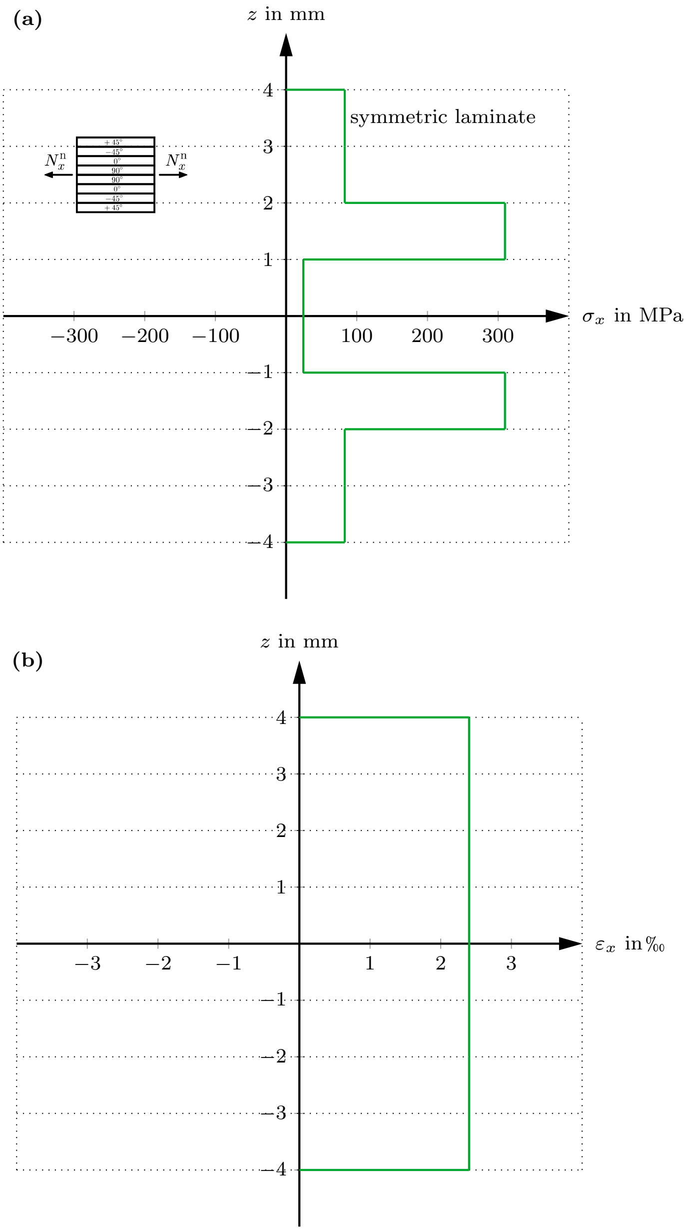


Fig. 4.2 Distribution of selected field quantities over the laminate thickness for the tensile load case: **a** normal stress $\sigma_x(z)$ and **b** normal strain $\varepsilon_x(z)$

Table 4.3 Stresses in the local *1–2* coordinate systems for an external loading of $N_x^{\mathrm{n}} = 1000$ N/mm, symmetric laminate

Lamina no.	Position	σ_1 in MPa	σ_2 in MPa	σ_{12} in MPa
1	Top	112.958	12.042	–20.479
	Center	112.958	12.042	–20.479
	Bottom	112.958	12.042	–20.479
2	Top	112.958	2.042	20.479
	Center	112.958	12.042	20.479
	Bottom	112.958	12.042	20.479
3	Top	309.630	–0.328	0.0
	Center	309.630	–0.328	0.0
	Bottom	309.630	–0.328	0.0
4	Top	–83.714	24.412	0.0
	Center	–83.714	24.412	0.0
	Bottom	–83.714	24.412	0.0
5	Top	–83.714	24.414	0.0
	Center	–83.714	24.414	0.0
	Bottom	–83.714	24.414	0.0
6	Top	309.630	–0.328	0.0
	Center	309.630	–0.328	0.0
	Bottom	309.630	–0.328	0.0
7	Top	112.958	12.042	20.479
	Center	112.958	12.042	20.479
	Bottom	112.958	12.042	20.479
8	Top	112.958	12.042	–20.479
	Center	112.958	12.042	–20.479
	Bottom	112.958	12.042	–20.479

are summarized in Tables 4.3 and 4.4. These values could be used for a subsequent failure analysis in each lamina.

- Laminate loaded by $M_x^{\mathrm{n}} = 1000$ Nmm/mm

The solution procedure will follow again the recommended steps provided in Table 3.1. However, subtasks ② and ③–⑤ are identical to load case (a) and can be omitted here. Thus, only the steps with different content/results will be presented in the following:

Table 4.4 Strains in the local *1–2* coordinate systems for an external loading of $N_x^{\mathrm{n}} = 1000$ N/mm, symmetric laminate

Lamina no.	Position	ε_1 in $^0\!/_{00}$	ε_2 in $^0\!/_{00}$	γ_{12} in $^0\!/_{00}$
1	Top	0.849510	0.849510	–3.102877
	Center	0.849510	0.849510	–3.102877
	Bottom	0.849510	0.849510	–3.102877
2	Top	0.849510	0.849510	3.102877
	Center	0.849510	0.849510	3.102877
	Bottom	0.849510	0.849510	3.102877
3	Top	2.400948	–0.701929	0.0
	Center	2.400948	–0.701929	0.0
	Bottom	2.400948	–0.701929	0.0
4	Top	–0.701929	2.400948	0.0
	Center	–0.701929	2.400948	0.0
	Bottom	–0.701929	2.400948	0.0
5	Top	–0.701929	2.400948	0.0
	Center	–0.701929	2.400948	0.0
	Bottom	–0.701929	2.400948	0.0
6	Top	2.400948	–0.701929	0.0
	Center	2.400948	–0.701929	0.0
	Bottom	2.400948	–0.701929	0.0
7	Top	0.849510	0.849510	3.102877
	Center	0.849510	0.849510	3.102877
	Bottom	0.849510	0.849510	3.102877
8	Top	0.849510	0.849510	–3.102877
	Center	0.849510	0.849510	–3.102877
	Bottom	0.849510	0.849510	–3.102877

② Define the column matrix of generalized stresses:
$\boldsymbol{s} = \left[N_x^{\mathrm{n}}\ N_y^{\mathrm{n}}\ N_{xy}^{\mathrm{n}}\quad M_x^{\mathrm{n}}\ M_y^{\mathrm{n}}\ M_{xy}^{\mathrm{n}}\right]^{\mathrm{T}}$.

The load matrix for case (b) is given as follows:

$$\boldsymbol{s} = \left[0\ 0\ 0\ 1000\ 0\ 0\right]^{\mathrm{T}}\ \text{Nmm/mm}\,. \tag{4.16}$$

⑥ Calculate the generalized strains $\boldsymbol{e} = \left[\boldsymbol{\varepsilon}^0\ \boldsymbol{\kappa}\right]^{\mathrm{T}}$ according to Eq. (3.16).

The evaluation of Eq. (3.16) gives the generalized strains as follows:

$$\underbrace{\begin{bmatrix} \boldsymbol{\varepsilon}^0 \\ \hline \boldsymbol{\kappa} \end{bmatrix}}_{\boldsymbol{e}} = \begin{bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \\ \hline \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{bmatrix} = (\boldsymbol{C}^*)^{-1}\boldsymbol{s} = \begin{bmatrix} 0.0 \\ 0.0 \\ 0.0 \\ 0.677189 \\ -0.426461 \\ -0.069118 \end{bmatrix} ‰ . \tag{4.17}$$

⑦ Calculate the stresses in each layer k according to Eq. (3.20), $\boldsymbol{\sigma}_{x,y}^k(z) = \overline{\boldsymbol{C}}_k \left(\boldsymbol{\varepsilon}^0 + z\boldsymbol{\kappa}\right)$, in the x–y laminate system ($z_{k-1} \leq z \leq z_k$). The strains in the x–y laminate system are obtained from $\left(\boldsymbol{\varepsilon}^0 + z\boldsymbol{\kappa}\right)$.

Evaluation of Eq. (3.20) for each layer k, i.e.

$$\boldsymbol{\sigma}_{x,y}^k(z) = \overline{\boldsymbol{C}}_k \left(\boldsymbol{\varepsilon}^0 + z\boldsymbol{\kappa}\right) , \tag{4.18}$$

under consideration of the generalized strains given in Eq. (4.17) and the range of the z-coordinate ($z_{k-1} \leq z \leq z_k$) provided in Table 4.2 gives the stresses for each layer in the x–y laminate system. In case that the strains in the x–y laminate system are required, the evaluation of $\left(\boldsymbol{\varepsilon}^0 + z\boldsymbol{\kappa}\right)$ provides these values. A graphical representation of the normal stress (σ_x) and normal strain (ε_x) in loading direction (x) is provided in Fig. 4.3 as a function of the laminate thickness (z-coordinate). It can be seen that a pure bending load by M_x^{n} results in the case of this symmetric laminate in a perfect linear distribution of the strain across the entire thickness of the laminate (see Fig. 4.3b). The stress distribution (see Fig. 4.3a) reveals a layer-wise linear distribution with the characteristic that the stress is proportional to the stiffness value of each layer in the loading direction. Again, it can be concluded that the ±45° layers show the same behavior in the x-direction. Furthermore, the stress distribution shows the classical characteristics known from the isotropic bending problem, i.e., the neural fiber ($\sigma_x = 0$) is in the middle of the cross section and the classical tension ($z > 0$) and compression parts ($z < 0$) are as well obtained.

⑧ Transform the stresses in each layer to the *1–2* lamina system according to Eq. (3.21): $\boldsymbol{\sigma}_{1,2}^k = \boldsymbol{T}_{\sigma}^k \boldsymbol{\sigma}_{x,y}^k$. The strains in the *1–2* lamina system are obtained from Eq. (2.33): $\boldsymbol{\varepsilon}_{1,2}^k = (\boldsymbol{C}_k)^{-1} \boldsymbol{\sigma}_{1,2}^k$.

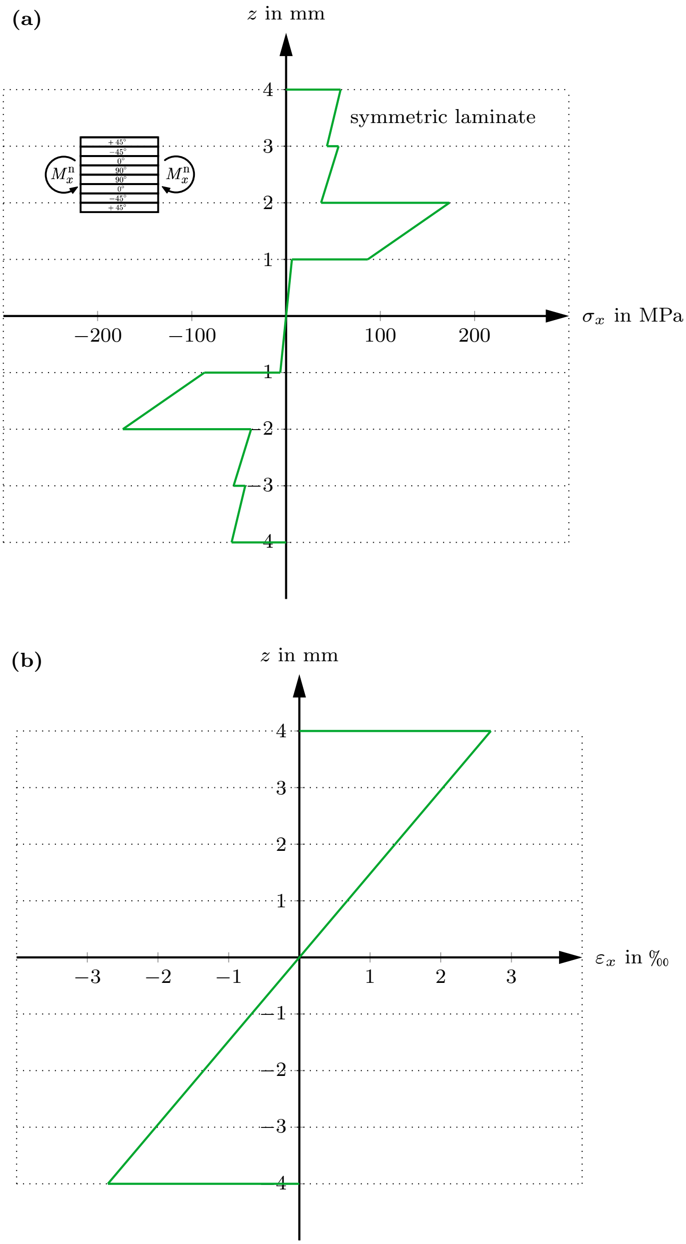


Fig. 4.3 Distribution of selected field quantities over the laminate thickness for the bending load case: **a** normal stress $\sigma_x(z)$ and **b** normal strain $\varepsilon_x(z)$

Table 4.5 Stresses in the local *1–2* coordinate systems for an external loading of $M_x^{\mathrm{n}} = 1000$ Nmm/mm, symmetric laminate

Lamina no.	Position	σ_1 in MPa	σ_2 in MPa	σ_{12} in MPa
1	Top	–36.866	–6.158	21.852
	Center	–43.010	–7.184	25.494
	Bottom	–49.154	–8.210	29.136
2	Top	–42.101	–3.003	–14.568
	Center	–52.626	–3.754	–18.210
	Bottom	–63.151	–4.504	–21.852
3	Top	–86.623	2.623	0.456
	Center	–129.934	3.934	0.684
	Bottom	–173.246	5.246	0.912
4	Top	0.0	0.0	0.0
	Center	26.642	–3.088	–0.228
	Bottom	53.284	–6.177	–0.456
5	Top	–53.284	6.177	0.456
	Center	–26.642	3.088	0.228
	Bottom	0.0	0.0	0.0
6	Top	173.246	–5.246	–0.912
	Center	129.934	–3.934	–0.684
	Bottom	86.623	–2.623	–0.456
7	Top	63.151	4.504	21.852
	Center	52.626	3.754	18.210
	Bottom	42.101	3.003	14.568
8	Top	49.154	8.210	–29.136
	Center	43.010	7.184	–25.494
	Bottom	36.866	6.158	–21.852

The transformation of the stresses and strains into the *1–2* lamina systems is performed according to Eqs. (3.21) and (2.33). The corresponding numerical values for the bottom (z_{k-1}), the middle ($(z_{k-1} + z_k)/2$), and the top (z_k) of each lamina are summarized in Tables 4.5 and 4.6. These values could be used for a subsequent failure analysis in each lamina.

4.3 Problem 2: Stresses and Strains in an Asymmetric Laminate

Given is an asymmetric 8-layer laminate of thickness $t = 8$ mm as shown in Fig. 4.4. Each lamina has the same thickness ($t/8$) and identical material properties, which can be taken from Table 4.1. However, the orientation of each lamina changes from layer to layer as indicated in Fig. 4.4.

Table 4.6 Strains in the local *1–2* coordinate systems for an external loading of $M_x^{\mathrm{n}} = 1000$ Nmm/mm, symmetric laminate

Lamina no.	Position	ε_1 in ‰	ε_2 in ‰	γ_{12} in ‰
1	Top	–0.272414	–0.479769	3.310951
	Center	–0.317816	–0.559731	3.862776
	Bottom	–0.363219	–0.639692	4.414601
2	Top	–0.319846	–0.181609	–2.207300
	Center	–0.399808	–0.227012	–2.759126
	Bottom	–0.479769	–0.272414	–3.310951
3	Top	–0.677189	0.426461	0.069118
	Center	–1.015783	0.639692	0.103678
	Bottom	–1.354378	0.852923	0.138237
4	Top	0.0	0.0	0.0
	Center	0.213231	–0.338594	–0.034559
	Bottom	0.426461	–0.677189	–0.069118
5	Top	–0.426461	0.677189	0.069118
	Center	–0.213231	0.338594	0.034559
	Bottom	0.0	0.0	0.0
6	Top	1.354378	–0.852923	–0.138237
	Center	1.015783	–0.639692	–0.103678
	Bottom	0.677189	–0.426461	–0.069118
7	Top	0.479769	0.272414	3.310951
	Center	0.399808	0.227012	2.759126
	Bottom	0.319846	0.181609	2.207300
8	Top	0.363219	0.639692	–4.414601
	Center	0.317816	0.559731	–3.862776
	Bottom	0.272414	0.479769	–3.310951

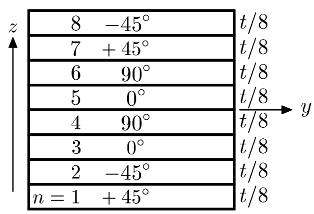


Fig. 4.4 Asymmetric laminate: [±45°/0°/90°/0°/90°/±45°]

Table 4.7 Geometrical properties (z_k; z_{k-1}; α_k) of the laminate shown in Fig. 4.4

Lamina k	Coordinate z_{k-1} in mm	Coordinate z_k in mm	Angle α_k
1	–4	–3	+45°
2	–3	–2	−45°
3	–2	–1	0°
4	–1	0	90°
5	0	1	0°
6	1	2	90°
7	2	3	+45°
8	3	4	−45°

Consider (a) a pure tensile load case with $N_x^{\mathrm{n}} = 1000$ N/mm (all other forces and moments are equal to zero) and (b) a pure bending load case with $M_x^{\mathrm{n}} = 1000$ Nmm/mm (all other moments and forces are equal to zero) to determine the stresses and strains in each lamina in the global x–y coordinate system as well as in the principal material coordinates (*1*–*2* coordinate systems) of each lamina.

- Laminate loaded by $N_x^{\mathrm{n}} = 1000$ N/mm

The solution procedure will follow the recommended steps provided in Table 3.1.

① Define for each lamina k the material ($E_{1,k}$; $E_{2,k}$; $\nu_{12,k}$; $G_{12,k}$) and the geometrical (z_k; z_{k-1}; α_k) properties.

The material properties ($E_{1,k}$; $E_{2,k}$; $\nu_{12,k}$; $G_{12,k}$) are the same for each lamina and can be taken from Table 4.1. The geometrical properties for each lamina are summarized in Table 4.7

② Define the column matrix of generalized stresses:
$\boldsymbol{s} = \left[N_x^{\mathrm{n}}\ N_y^{\mathrm{n}}\ N_{xy}^{\mathrm{n}}\ \ M_x^{\mathrm{n}}\ M_y^{\mathrm{n}}\ M_{xy}^{\mathrm{n}}\right]^{\mathrm{T}}$.

The load matrix for case (a) is given as follows:

$$\boldsymbol{s} = \left[1000\ 0\ 0\ 0\ 0\ 0\right]^{\mathrm{T}}\ \text{N/mm}\,. \tag{4.19}$$

③ Calculate for each layer k the elasticity matrix $\boldsymbol{C}_k$ in the 1–2 lamina system according to Eq. (2.31). Transform each matrix according to Eq. (2.49) to obtain the elasticity matrix $\overline{\boldsymbol{C}}_k$ in the x–y laminate system.

Since each lamina is composed of the same materials and the thickness is equal to $t/8 = 1$ mm, Eq. (2.31) yields for each lamina the same matrix

$$\boldsymbol{C}_k = \begin{bmatrix} \frac{E_1}{1-\nu_{12}\nu_{21}} & \frac{\nu_{21}E_1}{1-\nu_{12}\nu_{21}} & 0 \\ \frac{\nu_{12}E_2}{1-\nu_{12}\nu_{21}} & \frac{E_2}{1-\nu_{12}\nu_{21}} & 0 \\ 0 & 0 & G_{12} \end{bmatrix} = \begin{bmatrix} 129868.204 & 3100.729 & 0 \\ 3100.729 & 11074.033 & 0 \\ 0 & 0 & 6600.0 \end{bmatrix} \frac{\mathrm{N}}{\mathrm{mm}^2}\,. \tag{4.20}$$

The transformation to the x–y laminate system according to Eq. (2.49) gives the following matrices:

$$\overline{\boldsymbol{C}}_{[\pm 45°]} = \begin{bmatrix} 43385.924 & 30185.924 & \pm 29698.543 \\ 30185.924 & 43385.924 & \pm 29698.543 \\ \pm 29698.543 & \pm 29698.543 & 33685.195 \end{bmatrix} \frac{\mathrm{N}}{\mathrm{mm}^2}\,, \tag{4.21}$$

$$\overline{\boldsymbol{C}}_{[0°]} = \begin{bmatrix} 129868.204 & 3100.729 & 0.0 \\ 3100.729 & 11074.033 & 0.0 \\ 0.0 & 0.0 & 6600.0 \end{bmatrix} \frac{\mathrm{N}}{\mathrm{mm}^2}\,, \tag{4.22}$$

$$\overline{\boldsymbol{C}}_{[90°]} = \begin{bmatrix} 11074.033 & 3100.729 & 0.0 \\ 3100.729 & 129868.204 & 0.0 \\ 0.0 & 0.0 & 6600.0 \end{bmatrix} \frac{\mathrm{N}}{\mathrm{mm}^2}\,. \tag{4.23}$$

④ Calculate the submatrices $\boldsymbol{A}$, $\boldsymbol{B}$, and $\boldsymbol{C}$ according to Eqs. (3.13)–(3.15). Assemble the generalized elasticity matrix $\boldsymbol{C}^*$ according to Eq. (3.12).

The evaluation of the submatrices $\boldsymbol{A}$, $\boldsymbol{B}$, and $\boldsymbol{C}$ according to Eqs. (3.13)–(3.15) gives

$$\boldsymbol{A} = \sum_{k=1}^{8} \boldsymbol{A}_k = \begin{bmatrix} 455428.170 & 133146.612 & 0.0 \\ 133146.612 & 455428.170 & 0.0 \\ 0.0 & 0.0 & 161140.779 \end{bmatrix} \frac{\text{N}}{\text{mm}}, \tag{4.24}$$

$$\boldsymbol{B} = \sum_{k=1}^{8} \boldsymbol{B}_k = \begin{bmatrix} -118794.171 & 1.455\ 10^{-11} & -59397.086 \\ 1.456\ 10^{-11} & 118794.171 & -59397.086 \\ -59397.086 & -59397.086 & 0.0 \end{bmatrix} \text{N}, \tag{4.25}$$

$$\boldsymbol{D} = \sum_{k=1}^{8} \boldsymbol{D}_k = \begin{bmatrix} 1995587.124 & 1143478.381 & 0.0 \\ 1143478.381 & 1995587.124 & 0.0 \\ 0.0 & 0.0 & 1292780.601 \end{bmatrix} \text{Nmm}, \tag{4.26}$$

from which the generalized elasticity matrix (written without units) can be assembled as follows:

$$\boldsymbol{C}^* = \begin{bmatrix} 455428.170 & 133146.612 & 0.0 & -118794.171 & 1.455\ 10^{-11} & -59397.086 \\ 133146.612 & 455428.170 & 0.0 & 1.455\ 10^{-11} & 118794.171 & -59397.086 \\ 0.0 & 0.0 & 161140.779 & -59397.086 & -59397.086 & 0.0 \\ -118794.171 & 1.455\ 10^{-11} & -59397.086 & 1995587.124 & 1143478.381 & 0.0 \\ 1.455\ 10^{-11} & 118794.171 & -59397.086 & 1143478.381 & 1995587.124 & 0.0 \\ -59397.086 & -59397.086 & 0.0 & 0.0 & 0.0 & 1292780.601 \end{bmatrix}. \tag{4.27}$$

⑤ Calculate the generalized compliance matrix $(\boldsymbol{C}^*)^{-1}$ based on Eqs. (3.17)–(3.19).

The evaluation of the submatrices $\boldsymbol{A}'$, $\boldsymbol{B}'$, and $\boldsymbol{C}'$ according to Eqs. (3.17)–(3.19) gives

$$\boldsymbol{A}' = \begin{bmatrix} 24.562273 & -6.911749 & 0.445254 \\ -6.911749 & 24.562273 & -0.445254 \\ 0.445254 & -0.445254 & 62.948045 \end{bmatrix} \frac{10^{-7}\text{mm}}{\text{N}}, \tag{4.28}$$

$$\boldsymbol{B}' = \begin{bmatrix} 1.834321 & -0.626374 & 0.810957 \\ 0.626374 & -1.834321 & 0.810957 \\ 1.207947 & 1.207947 & -2.481542\ 10^{-17} \end{bmatrix} \frac{10^{-7}}{\text{N}}, \tag{4.29}$$

$$\boldsymbol{D}' = \begin{bmatrix} 7.677865 & -4.400777 & 0.113057 \\ -4.400777 & 7.677865 & -0.113057 \\ 0.113057 & -0.113057 & 7.809784 \end{bmatrix} \frac{10^{-7}}{\text{Nmm}}, \tag{4.30}$$

from which the generalized compliance matrix (written without units) can be assembled as follows:

$$(\boldsymbol{C}^*)^{-1} = \begin{bmatrix} 24.562273 & -6.911749 & 0.44525 & 1.83432 & -0.626374 & 0.810957 \\ -6.911749 & 24.562273 & -0.445254 & 0.626374 & -1.834321 & 0.810957 \\ 0.445254 & -0.445254 & 62.948045 & 1.207947 & 1.207947 & -1.042212\ 10^{-17} \\ 1.834321 & 0.626374 & 1.207947 & 7.677865 & -4.400777 & 0.113057 \\ -0.626374 & -1.834321 & 1.207947 & -4.400777 & 7.677865 & -0.113057 \\ 0.810957 & 0.810957 & -6.636555\ 10^{-18} & 0.113057 & -0.113057 & 7.809784 \end{bmatrix} 10^{-7}. \tag{4.31}$$

⑥ Calculate the generalized strains $\boldsymbol{e} = \left[\boldsymbol{\varepsilon}^0 \ \boldsymbol{\kappa}\right]^{\mathrm{T}}$ according to Eq. (3.16).

The evaluation of Eq. (3.16) gives the generalized strains as follows:

$$\underbrace{\begin{bmatrix} \boldsymbol{\varepsilon}^0 \\ \hline \boldsymbol{\kappa} \end{bmatrix}}_{\boldsymbol{e}} = \begin{bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \\ \hline \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{bmatrix} = (\boldsymbol{C}^*)^{-1}\boldsymbol{s} = \begin{bmatrix} 2.456227 \\ -0.691175 \\ 0.044525 \\ 0.183432 \\ -0.062637 \\ 0.081096 \end{bmatrix} ‰ . \tag{4.32}$$

⑦ Calculate the stresses in each layer k according to Eq. (3.20), $\boldsymbol{\sigma}_{x,y}^{k}(z) = \overline{\boldsymbol{C}}_k \left(\boldsymbol{\varepsilon}^0 + z\boldsymbol{\kappa}\right)$, in the x–y laminate system ($z_{k-1} \leq z \leq z_k$). The strains in the x–y laminate system are obtained from $\left(\boldsymbol{\varepsilon}^0 + z\boldsymbol{\kappa}\right)$.

Evaluation of Eq. (3.20) for each layer k, i.e.

$$\boldsymbol{\sigma}_{x,y}^{k}(z) = \overline{\boldsymbol{C}}_k \left(\boldsymbol{\varepsilon}^0 + z\boldsymbol{\kappa}\right), \tag{4.33}$$

under consideration of the generalized strains given in Eq. (4.32) and the range of the z-coordinate ($z_{k-1} \leq z \leq z_k$) provided in Table 4.7 gives the stresses for each layer in the x–y laminate system. In case that the strains in the x–y laminate system are required, the evaluation of $\left(\boldsymbol{\varepsilon}^0 + z\boldsymbol{\kappa}\right)$ provides these values. A graphical representation of the normal stress (σ_x) and normal strain (ε_x) in loading direction (x) is provided in Fig. 4.5 as a function of the laminate thickness (z-coordinate). It can be seen that a uniaxial load by N_x^{n} results in the case of this asymmetric laminate in an asymmetric distribution of the normal stress and strain with respect to $z = 0$. The strain distribution is linear and continuous (see Fig. 4.5b) across the entire thickness of the laminate. The stress distribution (see Fig. 4.5a) reveals a layer-wise linear distribution with the characteristic that the stress is proportional to the stiffness value of each layer in the loading direction. It can be seen that we have a clear tension-bending coupling, i.e., a superposition of a constant and linear distribution for the stresses and strains. This is also manifested in the $\boldsymbol{B}$-matrix where the bending-tension entries are now unequal to zero; see Eq. (4.25).

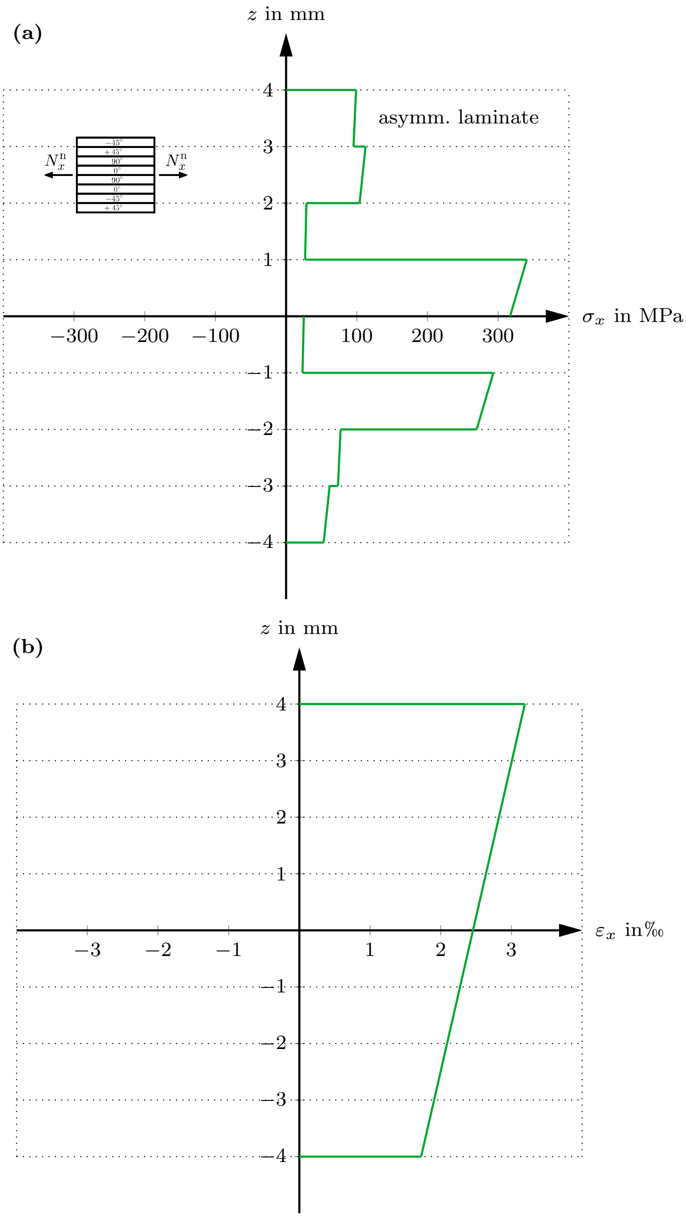


Fig. 4.5 Distribution of selected field quantities over the laminate thickness for the tensile load case: **a** normal stress $\sigma_x(z)$ and **b** normal strain $\varepsilon_x(z)$

Table 4.8 Stresses in the local *1–2* coordinate systems for an external loading of $N_x^{\mathrm{n}} = 1000$ N/mm, asymmetric laminate

Lamina no.	Position	σ_1 in MPa	σ_2 in MPa	σ_{12} in MPa
1	Top	80.657	10.734	–15.901
	Center	74.072	10.467	–15.089
	Bottom	67.486	10.201	–14.277
2	Top	108.745	10.328	17.525
	Center	107.299	9.739	16.713
	Bottom	105.854	9.149	15.901
3	Top	293.215	0.087	–0.241
	Center	281.401	0.149	–0.509
	Bottom	269.587	0.212	–0.777
4	Top	–82.146	25.057	–0.294
	Center	–78.363	24.139	–0.026
	Bottom	–74.580	23.220	0.241
5	Top	340.470	–0.163	0.829
	Center	328.657	–0.100	0.561
	Bottom	316.843	–0.038	0.294
6	Top	–97.277	28.731	–1.364
	Center	–93.494	27.813	–1.097
	Bottom	–89.711	26.894	–0.829
7	Top	159.684	13.931	–25.645
	Center	153.099	13.664	–24.833
	Bottom	146.513	13.398	–24.021
8	Top	126.090	17.405	27.269
	Center	124.644	16.815	26.457
	Bottom	123.199	16.225	25.645

⑧ Transform the stresses in each layer to the *1–2* lamina system according to Eq. (3.21): $\boldsymbol{\sigma}^k_{1,2} = \boldsymbol{T}^k_{\sigma}\boldsymbol{\sigma}^k_{x,y}$. The strains in the *1–2* lamina system are obtained from Eq. (2.33): $\boldsymbol{\varepsilon}^k_{1,2} = (\boldsymbol{C}_k)^{-1}\boldsymbol{\sigma}^k_{1,2}$.

The transformation of the stresses and strains into the *1–2* lamina systems is performed according to Eqs. (3.21) and (2.33). The corresponding numerical values for the bottom (z_{k-1}), the middle ($(z_{k-1} + z_k)/2$), and the top (z_k) of each lamina are summarized in Tables 4.8 and 4.9. These values could be used for a subsequent failure analysis in each lamina.

Table 4.9 Strains in the local *1–2* coordinate systems for an external loading of $N_x^{\mathrm{n}} = 1000$ N/mm, asymmetric laminate

Lamina no.	Position	ε_1 in ‰	ε_2 in ‰	γ_{12} in ‰
1	Top	0.601953	0.800715	–2.409194
	Center	0.551481	0.790790	–2.286159
	Bottom	0.501008	0.780866	–2.163124
2	Top	0.820565	0.702898	2.655263
	Center	0.810640	0.652426	2.532229
	Bottom	0.800715	0.601953	2.409194
3	Top	2.272795	–0.628538	–0.036570
	Center	2.181079	–0.597219	–0.077118
	Bottom	2.089363	–0.565900	–0.117666
4	Top	–0.691175	2.456227	–0.044525
	Center	–0.659856	2.364511	–0.003978
	Bottom	–0.628538	2.272795	0.036570
5	Top	2.639659	–0.753812	0.125621
	Center	2.547943	–0.722494	0.085073
	Bottom	2.456227	–0.691175	0.044525
6	Top	–0.816450	2.823091	–0.206717
	Center	–0.785131	2.731375	–0.166169
	Bottom	–0.753812	2.639659	–0.125621
7	Top	1.207624	0.919812	–3.885611
	Center	1.157152	0.909887	–3.762576
	Bottom	1.106679	0.899962	–3.639541
8	Top	0.939661	1.308570	4.131680
	Center	0.929737	1.258097	4.008645
	Bottom	0.919812	1.207624	3.885611

- Laminate loaded by $M_x^{\mathrm{n}} = 1000$ Nmm/mm

The solution procedure will follow again the recommended steps provided in Table 3.1. However, subtasks ② and ③–⑤ are identical to load case (a) and can be omitted here. Thus, only the steps with different content/results will be presented in the following:

② Define the column matrix of generalized stresses:
$$\boldsymbol{s} = \left[N_x^{\mathrm{n}}\ N_y^{\mathrm{n}}\ N_{xy}^{\mathrm{n}}\ \ M_x^{\mathrm{n}}\ M_y^{\mathrm{n}}\ M_{xy}^{\mathrm{n}}\right]^{\mathrm{T}}.$$

The load matrix for case (b) is given as follows:

$$\boldsymbol{s} = \left[0\ 0\ 0\ 1000\ 0\ 0\right]^{\mathrm{T}}\ \text{Nmm/mm}\,. \tag{4.34}$$

⑥ Calculate the generalized strains $\boldsymbol{e} = \left[\boldsymbol{\varepsilon}^0\ \boldsymbol{\kappa}\right]^{\mathrm{T}}$ according to Eq. (3.16).

The evaluation of Eq. (3.16) gives the generalized strains as follows:

$$\underbrace{\begin{bmatrix} \boldsymbol{\varepsilon}^0 \\ \hline \boldsymbol{\kappa} \end{bmatrix}}_{\boldsymbol{e}} = \begin{bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \\ \hline \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{bmatrix} = \left(\boldsymbol{C}^*\right)^{-1}\boldsymbol{s} = \begin{bmatrix} 0.183432 \\ 0.062637 \\ 0.120795 \\ 0.767786 \\ -0.440078 \\ 0.011306 \end{bmatrix} ‰\,. \tag{4.35}$$

⑦ Calculate the stresses in each layer k according to Eq. (3.20), $\boldsymbol{\sigma}^k_{x,y}(z) = \overline{\boldsymbol{C}}_k \left(\boldsymbol{\varepsilon}^0 + z\boldsymbol{\kappa}\right)$, in the x–y laminate system ($z_{k-1} \leq z \leq z_k$). The strains in the x–y laminate system are obtained from $\left(\boldsymbol{\varepsilon}^0 + z\boldsymbol{\kappa}\right)$.

Evaluation of Eq. (3.20) for each layer k, i.e.

$$\boldsymbol{\sigma}^k_{x,y}(z) = \overline{\boldsymbol{C}}_k \left(\boldsymbol{\varepsilon}^0 + z\boldsymbol{\kappa}\right), \tag{4.36}$$

under consideration of the generalized strains given in Eq. (4.35) and the range of the z-coordinate ($z_{k-1} \leq z \leq z_k$) provided in Table 4.7 gives the stresses for each layer in the x–y laminate system. In case that the strains in the x–y laminate system are required, the evaluation of $\left(\boldsymbol{\varepsilon}^0 + z\boldsymbol{\kappa}\right)$ provides these values. A graphical representation of the normal stress (σ_x) and normal strain (ε_x) in loading direction (x) is provided in Fig. 4.6 as a function of the laminate thickness (z-coordinate). It can be seen that a pure bending load by M_x^{n} results in the case of this asymmetric laminate in a perfect linear distribution of the strain across the entire thickness of the laminate (see Fig. 4.6b). The stress distribution (see Fig. 4.6a) reveals a layer-wise linear distribution with the characteristic that the stress is proportional to the stiffness value of each layer in the loading direction. We have again a tension-bending coupling, i.e., a superposition of a constant and linear distribution for the stresses and strains. This is also manifested in the $\boldsymbol{B}$-matrix where the bending-tension entries are now unequal to zero; see Eq. (4.25).

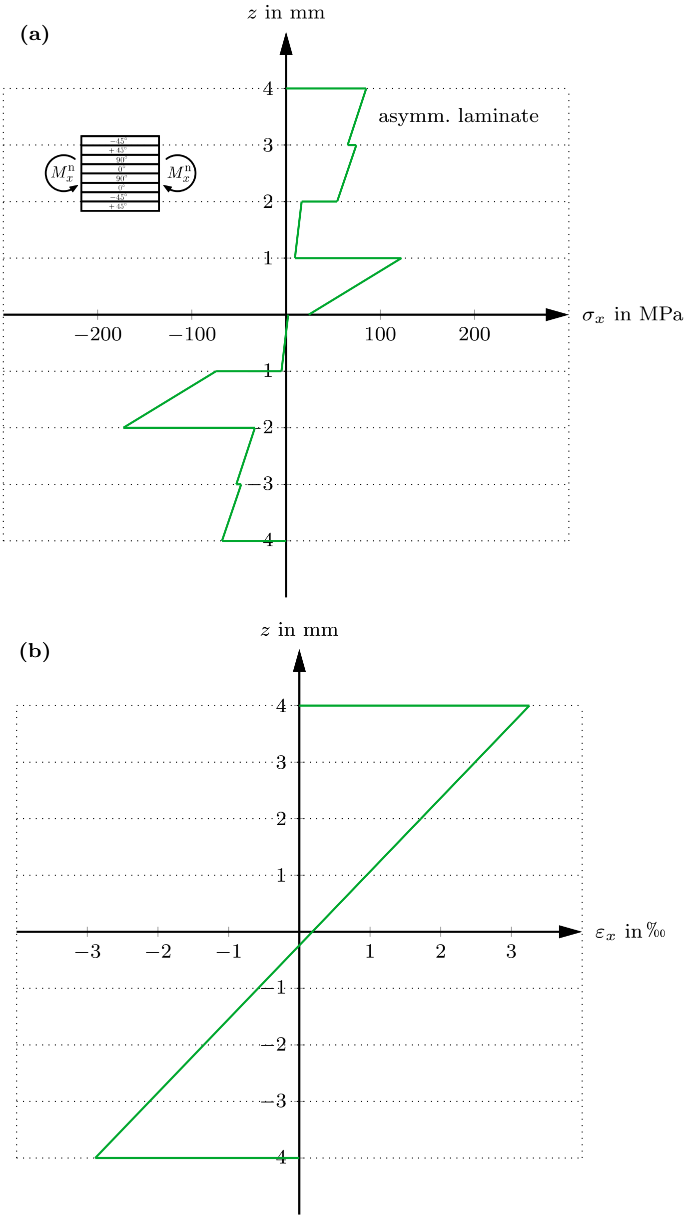


Fig. 4.6 Distribution of selected field quantities over the laminate thickness for the bending load case: **a** normal stress $\sigma_x(z)$ and **b** normal strain $\varepsilon_x(z)$

Table 4.10 Stresses in the local *1–2* coordinate systems for an external loading of $M_x^{\mathrm{n}} = 1000$ Nmm/mm, asymmetric laminate

Lamina no.	Position	σ_1 in MPa	σ_2 in MPa	σ_{12} in MPa
1	Top	–43.496	–5.570	23.118
	Center	–54.748	–6.709	27.104
	Bottom	–66.000	–7.848	31.090
2	Top	–33.439	–2.510	–15.147
	Center	–43.974	–3.694	–19.133
	Bottom	–54.509	–4.877	–23.118
3	Top	–74.330	3.755	0.723
	Center	–123.504	5.002	0.685
	Bottom	–172.677	6.248	0.648
4	Top	8.703	2.226	–0.797
	Center	36.089	–1.343	–0.760
	Bottom	63.475	–4.912	–0.723
5	Top	122.363	–1.230	0.872
	Center	73.189	0.016	0.835
	Bottom	24.016	1.262	0.797
6	Top	–100.839	16.501	–0.946
	Center	–73.454	12.932	–0.909
	Bottom	–46.068	9.363	–0.872
7	Top	91.529	8.095	–24.713
	Center	80.277	6.956	–20.727
	Bottom	69.025	5.817	–16.741
8	Top	92.987	11.696	32.685
	Center	82.452	10.512	28.699
	Bottom	71.916	9.329	24.713

⑧ Transform the stresses in each layer to the *1–2* lamina system according to Eq. (3.21): $\boldsymbol{\sigma}^k_{1,2} = \boldsymbol{T}^k_{\sigma}\boldsymbol{\sigma}^k_{x,y}$. The strains in the *1–2* lamina system are obtained from Eq. (2.33): $\boldsymbol{\varepsilon}^k_{1,2} = (\boldsymbol{C}_k)^{-1}\boldsymbol{\sigma}^k_{1,2}$.

The transformation of the stresses and strains into the *1–2* lamina systems is performed according to Eqs. (3.21) and (2.33). The corresponding numerical values for the bottom (z_{k-1}), the middle ($(z_{k-1} + z_k)/2$), and the top (z_k) of each lamina are summarized in Tables 4.10 and 4.11. These values could be used for a subsequent failure analysis in each lamina.

Table 4.11 Strains in the local *1–2* coordinate systems for an external loading of $M_x^{\mathrm{n}} = 1000$ Nmm/mm, asymmetric laminate

Lamina no.	Position	ε_1 in ‰	ε_2 in ‰	γ_{12} in ‰
1	Top	–0.325090	–0.411967	3.502798
	Center	–0.409843	–0.491068	4.106730
	Bottom	–0.494597	–0.570169	4.710662
2	Top	–0.253766	–0.155582	–2.294934
	Center	–0.332866	–0.240336	–2.898866
	Bottom	–0.411967	–0.325090	–3.502798
3	Top	–0.584354	0.502715	0.109489
	Center	–0.968248	0.722754	0.103836
	Bottom	–1.352141	0.942793	0.098183
4	Top	0.062637	0.183432	–0.120795
	Center	0.282676	–0.200461	–0.115142
	Bottom	0.502715	–0.584354	–0.109489
5	Top	0.951219	–0.377440	0.132100
	Center	0.567325	–0.157401	0.126448
	Bottom	0.183432	0.062637	0.120796
6	Top	–0.817518	1.719005	–0.143406
	Center	–0.597479	1.335112	–0.137753
	Bottom	–0.377440	0.951219	–0.132100
7	Top	0.691954	0.537242	–3.744387
	Center	0.607200	0.458141	–3.140455
	Bottom	0.522447	0.379040	–2.536523
8	Top	0.695444	0.861461	4.952251
	Center	0.616343	0.776707	4.348319
	Bottom	0.537242	0.691954	3.744387

4.4 Problem 3: Failure Criteria

Given is an asymmetric 4-layer laminate of thickness $t = 8$ mm as shown in Fig. 4.7. Each lamina has the same thickness ($t/4$) and identical material properties, which can be taken from Table 4.1. However, the orientation of each lamina changes from layer to layer as indicated in Fig. 4.7.

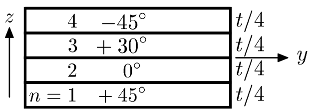


Fig. 4.7 Asymmetric laminate: [+45°/0°/+30°/−45°]

Table 4.12 Geometrical properties (z_k; z_{k-1}; α_k) of the laminate shown in Fig. 4.7

Lamina k	Coordinate z_{k-1} in mm	Coordinate z_k in mm	Angle α_k
1	–4	–2	+45°
2	–2	0	0°
3	0	2	+30°
4	2	4	−45°

Consider (a) a biaxial tensile load case with $N_x^n = 1000$ N/mm and $N_y^n = 1000$ N/mm (all other forces and moments are equal to zero) and (b) a biaxial bending load case with $M_x^n = 1000$ Nmm/mm and $M_y^n = 500$ Nmm/mm (all other moments and forces are equal to zero) to determine the stresses and strains in each lamina in the global x–y coordinate system as well as in the principal material coordinates (*1*–*2* coordinate systems) of each lamina. Use these stress and strain values to perform a failure analysis based on the maximum stress, maximum strain, Tsai-Hill, and Tsai-Wu failure criteria.

- Laminate loaded by $N_x^n = 1000$ N/mm and $N_y^n = 500$ N/mm

The solution procedure will follow the recommended steps provided in Table 3.1.

① Define for each lamina k the material ($E_{1,k}$; $E_{2,k}$; $\nu_{12,k}$; $G_{12,k}$) and the geometrical (z_k; z_{k-1}; α_k) properties.

The material properties ($E_{1,k}$; $E_{2,k}$; $\nu_{12,k}$; $G_{12,k}$) are the same for each lamina and can be taken from Table 4.1. The geometrical properties for each lamina are summarized in Table 4.7

② Define the column matrix of generalized stresses:
$\boldsymbol{s} = \left[N_x^n \; N_y^n \; N_{xy}^n \quad M_x^n \; M_y^n \; M_{xy}^n\right]^T$.

The load matrix for case (a) is given as follows:

$$\boldsymbol{s} = \left[1000\ 500\ 0\ 0\ 0\ 0\right]^T \text{ N/mm}\,. \tag{4.37}$$

③ Calculate for each layer k the elasticity matrix $\boldsymbol{C}_k$ in the *1–2* lamina system according to Eq. (2.31). Transform each matrix according to Eq. (2.49) to obtain the elasticity matrix $\overline{\boldsymbol{C}}_k$ in the x–y laminate system.

Since each lamina is composed of the same materials and the thickness is equal to $t/4 = 2$ mm, Eq. (2.31) yields for each lamina the same matrix

$$\boldsymbol{C}_k = \begin{bmatrix} \frac{E_1}{1-\nu_{12}\nu_{21}} & \frac{\nu_{21}E_1}{1-\nu_{12}\nu_{21}} & 0 \\ \frac{\nu_{12}E_2}{1-\nu_{12}\nu_{21}} & \frac{E_2}{1-\nu_{12}\nu_{21}} & 0 \\ 0 & 0 & G_{12} \end{bmatrix} = \begin{bmatrix} 129868.204 & 3100.729 & 0 \\ 3100.729 & 11074.033 & 0 \\ 0 & 0 & 6600.0 \end{bmatrix} \frac{\text{N}}{\text{mm}^2} . \tag{4.38}$$

The transformation to the x–y laminate system according to Eq. (2.49) gives the following matrices:

$$\overline{\boldsymbol{C}}_{[\pm 45^\circ]} = \begin{bmatrix} 43385.924 & 30185.924 & \pm 29698.543 \\ 30185.924 & 43385.924 & \pm 29698.543 \\ \pm 29698.543 & \pm 29698.543 & 33685.195 \end{bmatrix} \frac{\text{N}}{\text{mm}^2} , \tag{4.39}$$

$$\overline{\boldsymbol{C}}_{[0^\circ]} = \begin{bmatrix} 79855.765 & 23414.625 & 37447.926 \\ 23414.625 & 20458.680 & 13991.459 \\ 37447.926 & 13991.459 & 26913.896 \end{bmatrix} \frac{\text{N}}{\text{mm}^2} , \tag{4.40}$$

$$\overline{\boldsymbol{C}}_{[+30^\circ]} = \begin{bmatrix} 129868.204 & 3100.729 & 0.0 \\ 3100.729 & 11074.033 & 0.0 \\ 0.0 & 0.0 & 6600.0 \end{bmatrix} \frac{\text{N}}{\text{mm}^2} . \tag{4.41}$$

④ Calculate the submatrices $\boldsymbol{A}$, $\boldsymbol{B}$, and $\boldsymbol{C}$ according to Eqs. (3.13)–(3.15). Assemble the generalized elasticity matrix $\boldsymbol{C}^*$ according to Eq. (3.12).

The evaluation of the submatrices $\boldsymbol{A}$, $\boldsymbol{B}$, and $\boldsymbol{C}$ according to Eqs. (3.13)–(3.15) gives

$$\boldsymbol{A} = \sum_{k=1}^{8} \boldsymbol{A}_k = \begin{bmatrix} 592991.635 & 173774.404 & 74895.852 \\ 173774.404 & 236609.121 & 27982.918 \\ 74895.852 & 27982.918 & 201768.571 \end{bmatrix} \frac{\text{N}}{\text{mm}}, \tag{4.42}$$

$$\boldsymbol{B} = \sum_{k=1}^{8} \boldsymbol{B}_k = \begin{bmatrix} -100024.878 & 40627.792 & -281486.662 \\ 40627.792 & 18769.294 & -328399.595 \\ -281486.662 & -328399.595 & 40627.792 \end{bmatrix} \text{N}, \tag{4.43}$$

$$\boldsymbol{D} = \sum_{k=1}^{8} \boldsymbol{D}_k = \begin{bmatrix} 2179005.077 & 1197648.770 & 99861.136 \\ 1197648.770 & 1703828.392 & 37310.558 \\ 99861.136 & 37310.558 & 1346950.990 \end{bmatrix} \text{Nmm}, \tag{4.44}$$

from which the generalized elasticity matrix (written without units) can be assembled as follows:

$$\boldsymbol{C}^* = \begin{bmatrix} 592991.635 & 173774.404 & 74895.852 & -100024.878 & 40627.792 & -281486.662 \\ 173774.404 & 236609.121 & 27982.918 & 40627.792 & 18769.294 & -328399.595 \\ 74895.852 & 27982.918 & 201768.571 & -281486.662 & -328399.595 & 40627.792 \\ -100024.878 & 40627.792 & -281486.662 & 2179005.077 & 1197648.770 & 99861.136 \\ 40627.792 & 18769.294 & -328399.595 & 1197648.770 & 1703828.392 & 37310.558 \\ -281486.662 & -328399.595 & 40627.792 & 99861.136 & 37310.558 & 1346950.990 \end{bmatrix}. \tag{4.45}$$

⑤ Calculate the generalized compliance matrix $(\boldsymbol{C}^*)^{-1}$ based on Eqs. (3.17)–(3.19).

The evaluation of the submatrices $\boldsymbol{A}'$, $\boldsymbol{B}'$, and $\boldsymbol{C}'$ according to Eqs. (3.17)–(3.19) gives

$$\boldsymbol{A}' = \begin{bmatrix} 23.753280 & -13.399145 & -11.078569 \\ -13.399145 & 79.276421 & -17.770619 \\ -11.078569 & -17.770619 & 87.481158 \end{bmatrix} \frac{10^{-7}\text{mm}}{\text{N}}, \tag{4.46}$$

$$\boldsymbol{B}' = \begin{bmatrix} 2.029707 & -4.024442 & 1.992293 \\ -4.548157 & -1.163740 & 17.433633 \\ 3.160567 & 15.317403 & -9.945142 \end{bmatrix} \frac{10^{-7}}{\text{N}}, \tag{4.47}$$

$$\boldsymbol{D}' = \begin{bmatrix} 7.954633 & -4.953572 & -1.232578 \\ -4.953572 & 12.446429 & -1.564292 \\ -1.232578 & -1.564292 & 12.525699 \end{bmatrix} \frac{10^{-7}}{\text{Nmm}}, \tag{4.48}$$

from which the generalized compliance matrix (written without units) can be assembled as follows:

$$(\boldsymbol{C}^*)^{-1} = \begin{bmatrix} 23.753280 & -13.399145 & -11.078569 & 2.029707 & -4.024442 & 1.992293 \\ -13.399145 & 79.276421 & -17.770619 & -4.548157 & -1.163740 & 17.433633 \\ -11.078569 & -17.770619 & 87.481158 & 3.160567 & 15.317403 & -9.945142 \\ 2.029707 & -4.548157 & 3.160567 & 7.954633 & -4.953572 & -1.232578 \\ -4.024442 & -1.163740 & 15.317403 & -4.953572 & 12.446429 & -1.564292 \\ 1.992293 & 17.433633 & -9.945142 & -1.232578 & -1.564292 & 12.525699 \end{bmatrix} 10^{-7}. \tag{4.49}$$

⑥ Calculate the generalized strains $\boldsymbol{e} = \left[\boldsymbol{\varepsilon}^0 \ \boldsymbol{\kappa}\right]^{\mathrm{T}}$ according to Eq. (3.16).

The evaluation of Eq. (3.16) gives the generalized strains as follows:

$$\underbrace{\begin{bmatrix} \boldsymbol{\varepsilon}^0 \\ \hline \boldsymbol{\kappa} \end{bmatrix}}_{\boldsymbol{e}} = \begin{bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \\ \hline \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{bmatrix} = (\boldsymbol{C}^*)^{-1}\boldsymbol{s} = \begin{bmatrix} 1.705371 \\ 2.623907 \\ -1.996388 \\ -0.024437 \\ -0.460631 \\ 0.001071 \end{bmatrix} ‰\,. \tag{4.50}$$

⑦ Calculate the stresses in each layer k according to Eq. (3.20), $\boldsymbol{\sigma}^k_{x,y}(z) = \overline{\boldsymbol{C}}_k \left(\boldsymbol{\varepsilon}^0 + z\boldsymbol{\kappa}\right)$, in the x–y laminate system ($z_{k-1} \leq z \leq z_k$). The strains in the x–y laminate system are obtained from $\left(\boldsymbol{\varepsilon}^0 + z\boldsymbol{\kappa}\right)$.

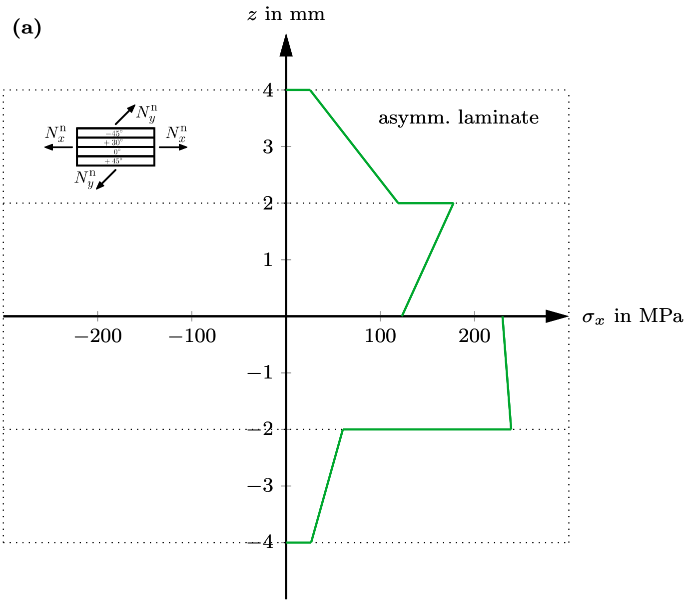
Evaluation of Eq. (3.20) for each layer k, i.e.

$$\boldsymbol{\sigma}^k_{x,y}(z) = \overline{\boldsymbol{C}}_k \left(\boldsymbol{\varepsilon}^0 + z\boldsymbol{\kappa}\right), \tag{4.51}$$

under consideration of the generalized strains given in Eq. (4.50) and the range of the z-coordinate ($z_{k-1} \leq z \leq z_k$) provided in Table 4.12 gives the stresses for each layer in the x–y laminate system. In case that the strains in the x–y laminate system are required, the evaluation of $\left(\boldsymbol{\varepsilon}^0 + z\boldsymbol{\kappa}\right)$ provides these values. Graphical representations of the normal stresses (σ_x, σ_y) and the normal strains (ε_x, ε_y) in the x- and y-direction are provided in Figs. 4.8 and 4.9 as a function of the laminate thickness (z-coordinate). It can be seen that a biaxial load by N_x^{n} and N_y^{n} results in the case of this asymmetric laminate in an asymmetric distribution of the normal stress and strain with respect to $z = 0$. The strain distributions are both linear and continuous (see Fig. 4.9) across the entire thickness of the laminate. The stress distributions (see Fig. 4.8) reveal a layer-wise linear distribution with the characteristic that the stress is proportional to the stiffness value of each layer in the corresponding loading direction. It can be seen that we have a clear tension-bending coupling, i.e., a superposition of a constant and linear distribution for the stresses and strains. This is also manifested in the $\boldsymbol{B}$-matrix where the bending-tension entries are unequal to zero; see Eq. (4.43).

⑧ Transform the stresses in each layer to the *1–2* lamina system according to Eq. (3.21): $\boldsymbol{\sigma}^k_{1,2} = \boldsymbol{T}^k_{\sigma}\boldsymbol{\sigma}^k_{x,y}$. The strains in the *1–2* lamina system are obtained from Eq. (2.33): $\boldsymbol{\varepsilon}^k_{1,2} = (\boldsymbol{C}_k)^{-1}\boldsymbol{\sigma}^k_{1,2}$.

The transformation of the stresses and strains into the *1–2* lamina systems is performed according to Eqs. (3.21) and (2.33). The corresponding numerical values for



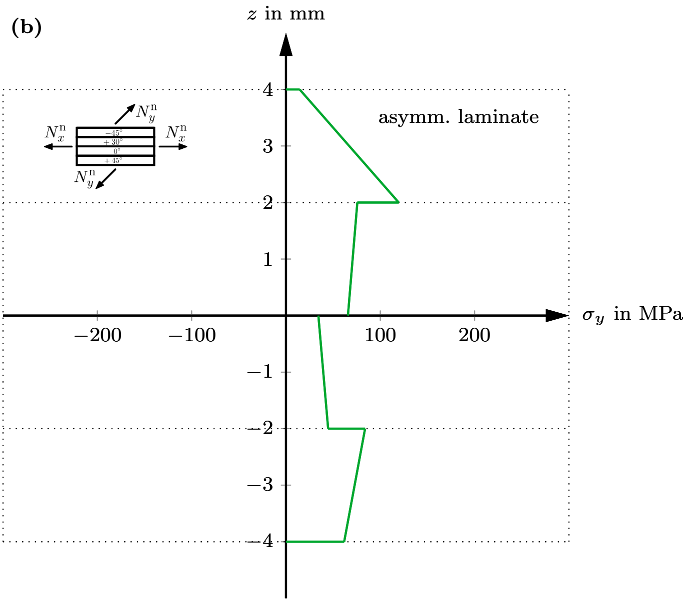


Fig. 4.8 Distribution of selected field quantities over the laminate thickness for the biaxial tensile load case: **a** normal stress $\sigma_x(z)$ and **b** normal stress $\sigma_y(z)$

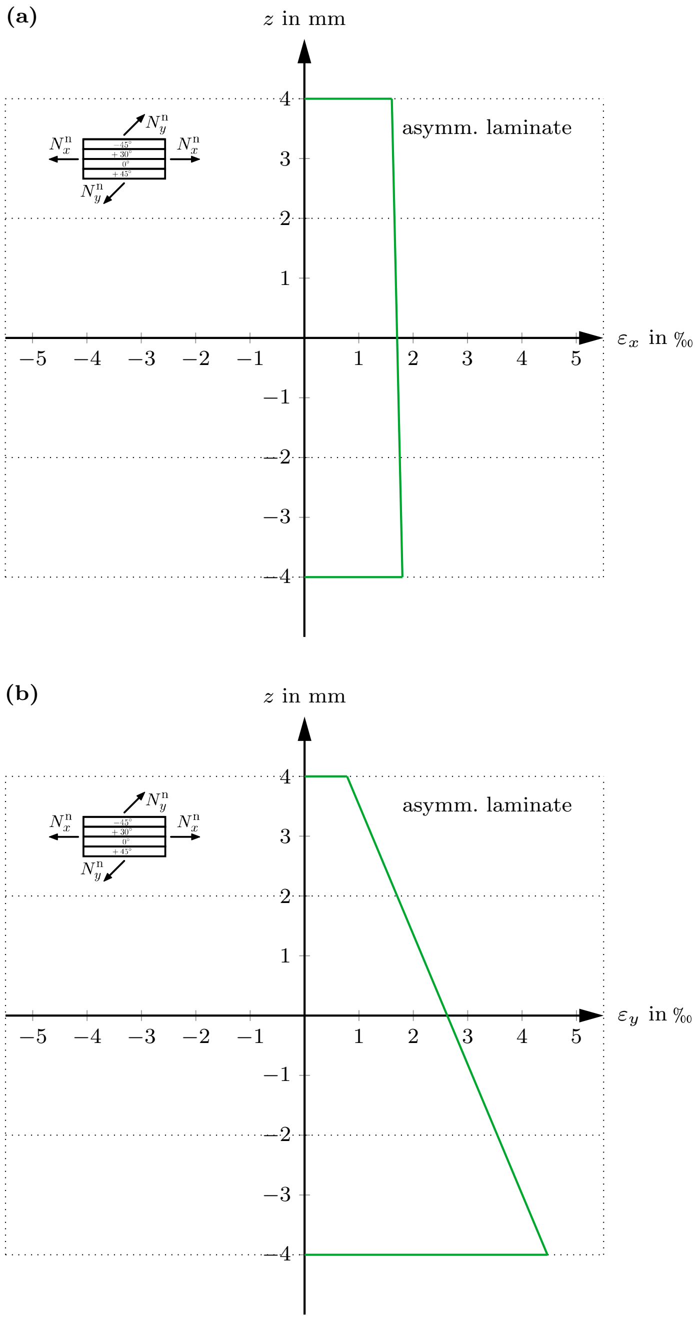


Fig. 4.9 Distribution of selected field quantities over the laminate thickness for the biaxial tensile load case: **a** normal strain $\varepsilon_x(z)$ and **b** normal strain $\varepsilon_y(z)$

Table 4.13 Stresses in the local *1–2* coordinate systems for an external loading of $N_x^{\mathrm{n}} = 1000$ N/mm and $N_y^{\mathrm{n}} = 500$ N/mm, asymmetric laminate

Lamina no.	Position	σ_1 in MPa	σ_2 in MPa	σ_{12} in MPa
1	Top	18.776	54.057	11.820
	Center	54.405	61.764	14.699
	Bottom	18.776	69.471	17.578
2	Top	229.609	34.345	–13.176
	Center	234.211	39.522	–20.244
	Bottom	238.813	44.699	–27.312
3	Top	229.851	23.397	0.744
	Center	189.493	31.402	–0.297
	Bottom	149.134	39.407	–1.338
4	Top	13.857	26.050	5.453
	Center	113.985	25.219	2.574
	Bottom	214.113	24.387	–0.305

Table 4.14 Strains in the local *1–2* coordinate systems for an external loading of $N_x^{\mathrm{n}} = 1000$ N/mm and $N_y^{\mathrm{n}} = 500$ N/mm, asymmetric laminate

Lamina no.	Position	ε_1 in ‰	ε_2 in ‰	γ_{12} in ‰
1	Top	0.580602	4.718812	1.790924
	Center	0.287681	5.496801	2.227118
	Bottom	–0.005241	6.274791	2.663312
2	Top	1.705371	2.623907	–1.996388
	Center	1.729808	3.084538	–3.067299
	Bottom	1.754245	3.545169	–4.138210
3	Top	1.731008	0.001628	0.112682
	Center	0.001401	0.002443	–0.045018
	Bottom	0.001071	0.003259	–0.202719
4	Top	0.050874	2.338130	0.826241
	Center	0.828864	0.002045	0.390046
	Bottom	1.606853	0.001752	–0.046148

the bottom (z_{k-1}), the middle ($(z_{k-1} + z_k)/2$), and the top (z_k) of each lamina are summarized in Tables 4.13 and 4.14. These values will be used for the subsequent failure analysis in each lamina.

Failure analysis based on the maximum stress, maximum strain, Tsai-Hill, and Tsai-Wu failure criteria.

To easier compare the different failure criteria, we determine the limit loads by applying loads as $N_x^{\mathrm{n}} = R \times 1000$ N/mm and $N_y^{\mathrm{n}} = R \times 500$ N/mm, i.e., we multiply the loads with the strength ratio R. In case of $R > 1$, there is no failure, whereas $R \leq 1$ results in the failure of the respective lamina.

Considering the stresses from Table 4.13 in the conditions of the maximum stress criterion according to Eqs. (2.133)–(2.135) gives

$$(k_{I\mathrm{c}} <)\ R \times \sigma_1 < k_{I\mathrm{t}}\,, \tag{4.52}$$

$$(k_{2\mathrm{c}} <)\ R \times \sigma_2 < k_{2\mathrm{t}}\,, \tag{4.53}$$

$$|R \times \sigma_{I2}| < k_{I2\mathrm{s}}\,, \tag{4.54}$$

from which the load ratio R can be identified; see Table 4.15.

Considering the strains from Table 4.14 in the conditions of the maximum strain criterion according to Eqs. (2.136)–(2.138) gives

$$k_{I\mathrm{c}}^{\varepsilon} < R \times \varepsilon_1 < k_{I\mathrm{t}}^{\varepsilon}\,, \tag{4.55}$$

$$(k_{2\mathrm{c}}^{\varepsilon} <)\ R \times \varepsilon_2 < k_{2\mathrm{t}}^{\varepsilon}\,, \tag{4.56}$$

$$|R \times \gamma_{I2}| < k_{I2\mathrm{s}}^{\varepsilon}\,, \tag{4.57}$$

from which the load ratio R can be identified; see Table 4.15.

Considering the stresses from Table 4.13 in the Tsai-Hill criterion according to Eq. (2.139) gives

$$\frac{(R \times \sigma_I)^2}{k_I^2} - \frac{(R \times \sigma_I)(R \times \sigma_2)}{k_1^2} + \frac{(R \times \sigma_2)^2}{k_2^2} + \frac{(R \times \sigma_{I2})^2}{k_{I2s}^2} < 1\,, \tag{4.58}$$

from which the load ratio R can be identified; see Table 4.15.

Considering the stresses from Table 4.13 in the Tsai-Wu criterion according to Eq. (2.140) gives

$$\begin{aligned} & a_1(R \times \sigma_I) + a_2(R \times \sigma_2) + a_{11}(R \times \sigma_I)^2 + a_{22}(R \times \sigma_2)^2 \\ & + a_{12}(R \times \sigma_{I2})^2 + 2a_{1,2}(R \times \sigma_I)(R \times \sigma_2) < 1\,. \end{aligned} \tag{4.59}$$

This is a classical quadratic equation in R, from which the value of R can be identified; see Table 4.15.

Table 4.15 Strength ratio R in the local *1–2* coordinate systems for an external loading of $N_x^{\mathrm{n}} = R \times 1000$ N/mm and $N_y^{\mathrm{n}} = R \times 500$ N/mm, asymmetric laminate

Lamina no.	Position	Direction	Max. stress	Max. Strain	Tsai-Hill	Tsai-Wu
1	Top	*1*	21.659	23.768		
		2	0.888	0.924	0.880	0.901
		12	6.684	11.167		
	Center	*1*	35.843	47.970		
		2	0.777	0.793	0.769	0.776
		12	5.375	8.980		
	Bottom	*1*	103.857	2242.144		
		2	0.691	0.695	0.683	0.681
		12	4.494	7.509		
2	Top	*1*	8.493	8.092		
		2	1.398	1.662	1.346	1.467
		12	5.996	10.018		
	Center	*1*	8.326	7.978		
		2	1.215	1.414	1.150	1.222
		12	3.902	6.520		
	Bottom	*1*	8.165	7.867		
		2	1.074	1.230	1.001	1.041
		12	2.892	4.833		
3	Top	*1*	8.484	7.972		
		2	2.052	2.678	1.999	2.325
		12	106.225	177.491		
	Center	*1*	10.291	9.852		
		2	1.529	1.784	1.515	1.676
		12	265.885	444.264		
	Bottom	*1*	13.075	12.891		
		2	1.218	1.338	1.214	1.296
		12	59.046	98.659		
4	Top	*1*	140.725	271.258		
		2	1.843	1.865	1.828	1.836
		12	14.487	24.206		
	Center	*1*	17.108	16.649		
		2	1.903	2.132	1.891	2.040
		12	30.688	51.276		
	Bottom	*1*	9.107	8.588		
		2	1.968	2.488	1.929	2.216
		12	259.378	433.392		

It can be seen from Table 4.15 that *all* failure criteria predict the failure of lamina 1 ($R < 1$). The Tsai-Hill and the Tsai-Wu criteria require only a single evaluation for each layer and a differentiation according to the directions is not required. Finally, it can be stated that all criteria provide strength ratios in a similar order of magnitude if the maximum stress is compared with the maximum strain criterion, respectively, if the Tsai-Hill criterion is compared with the Tsai-Wu approach.

- Laminate loaded by $M_x^{\mathrm{n}} = 1000\,\mathrm{Nmm/mm}$ and $M_y^{\mathrm{n}} = 500\,\mathrm{Nmm/mm}$

The solution procedure will follow again the recommended steps provided in Table 3.1. However, subtasks ② and ③–⑤ are identical to load case (a) and can be omitted here. Thus, only the steps with different content/results will be presented in the following:

② Define the column matrix of generalized stresses:
$\boldsymbol{s} = \left[N_x^{\mathrm{n}}\ N_y^{\mathrm{n}}\ N_{xy}^{\mathrm{n}}\ \ M_x^{\mathrm{n}}\ M_y^{\mathrm{n}}\ M_{xy}^{\mathrm{n}}\right]^{\mathrm{T}}$.

The load matrix for case (b) is given as follows:

$$\boldsymbol{s} = \left[0\ 0\ 0\ 1000\ 500\ 0\right]^{\mathrm{T}}\ \mathrm{Nmm/mm}\,. \tag{4.60}$$

⑥ Calculate the generalized strains $\boldsymbol{e} = \left[\boldsymbol{\varepsilon}^0\ \boldsymbol{\kappa}\right]^{\mathrm{T}}$ according to Eq. (3.16).

The evaluation of Eq. (3.16) gives the generalized strains as follows:

$$\underbrace{\begin{bmatrix} \boldsymbol{\varepsilon}^0 \\ \hline \boldsymbol{\kappa} \end{bmatrix}}_{\boldsymbol{e}} = \begin{bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \\ \hline \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{bmatrix} = (\boldsymbol{C}^*)^{-1}\boldsymbol{s} = \begin{bmatrix} 0.001749 \\ -0.513003 \\ 1.081927 \\ 0.547785 \\ 0.126964 \\ -0.201472 \end{bmatrix} ‰\,. \tag{4.61}$$

⑦ Calculate the stresses in each layer k according to Eq. (3.20), $\boldsymbol{\sigma}^k_{x,y}(z) = \overline{\boldsymbol{C}}_k\left(\boldsymbol{\varepsilon}^0 + z\boldsymbol{\kappa}\right)$, in the x–y laminate system ($z_{k-1} \leq z \leq z_k$). The strains in the x–y laminate system are obtained from $\left(\boldsymbol{\varepsilon}^0 + z\boldsymbol{\kappa}\right)$.

Evaluation of Eq. (3.20) for each layer k, i.e.

$$\boldsymbol{\sigma}^k_{x,y}(z) = \overline{\boldsymbol{C}}_k\left(\boldsymbol{\varepsilon}^0 + z\boldsymbol{\kappa}\right), \tag{4.62}$$

under consideration of the generalized strains given in Eq. (4.61) and the range of the z-coordinate ($z_{k-1} \leq z \leq z_k$) provided in Table 4.12 gives the stresses for each layer in the x–y laminate system. In case that the strains in the x–y laminate system are required, the evaluation of $\left(\boldsymbol{\varepsilon}^0 + z\boldsymbol{\kappa}\right)$ provides these values. Graphical representations of the normal stresses (σ_x, σ_y) and the normal strains ($\varepsilon_x, \varepsilon_y$) in the x- and y-direction are provided in Figs. 4.10 and 4.11 as a function of the laminate thickness (z-coordinate). It can be seen that a biaxial load by M^{n}_x and M^{n}_y results in case of this asymmetric laminate in an asymmetric distribution of the normal stress and strain with respect to $z = 0$. The strain distributions are both linear and continuous (see Fig. 4.11) across the entire thickness of the laminate. The stress distributions (see Fig. 4.10) reveal a layer-wise linear distribution with the characteristic that the stress is proportional to the stiffness value of each layer in the corresponding loading direction. It can be seen that we have a clear tension-bending coupling, i.e., a superposition of a constant and linear distribution for the stresses and strains; see Fig. 4.11b. This is also manifested in the $\boldsymbol{B}$-matrix where the bending-tension entries are unequal to zero; see Eq. (4.43).

⑧ Transform the stresses in each layer to the *1–2* lamina system according to Eq. (3.21): $\boldsymbol{\sigma}^k_{1,2} = \boldsymbol{T}^k_\sigma \boldsymbol{\sigma}^k_{x,y}$. The strains in the *1–2* lamina system are obtained from Eq. (2.33): $\boldsymbol{\varepsilon}^k_{1,2} = (\boldsymbol{C}_k)^{-1}\boldsymbol{\sigma}^k_{1,2}$.

The transformation of the stresses and strains into the *1–2* lamina systems is performed according to Eqs. (3.21) and (2.33). The corresponding numerical values for the bottom (z_{k-1}), the middle ($(z_{k-1} + z_k)/2$), and the top (z_k) of each lamina are summarized in Tables 4.16 and 4.17. These values will be used for the subsequent failure analysis in each lamina.

Failure analysis based on the maximum stress, maximum strain, Tsai-Hill, and Tsai-Wu failure criteria.

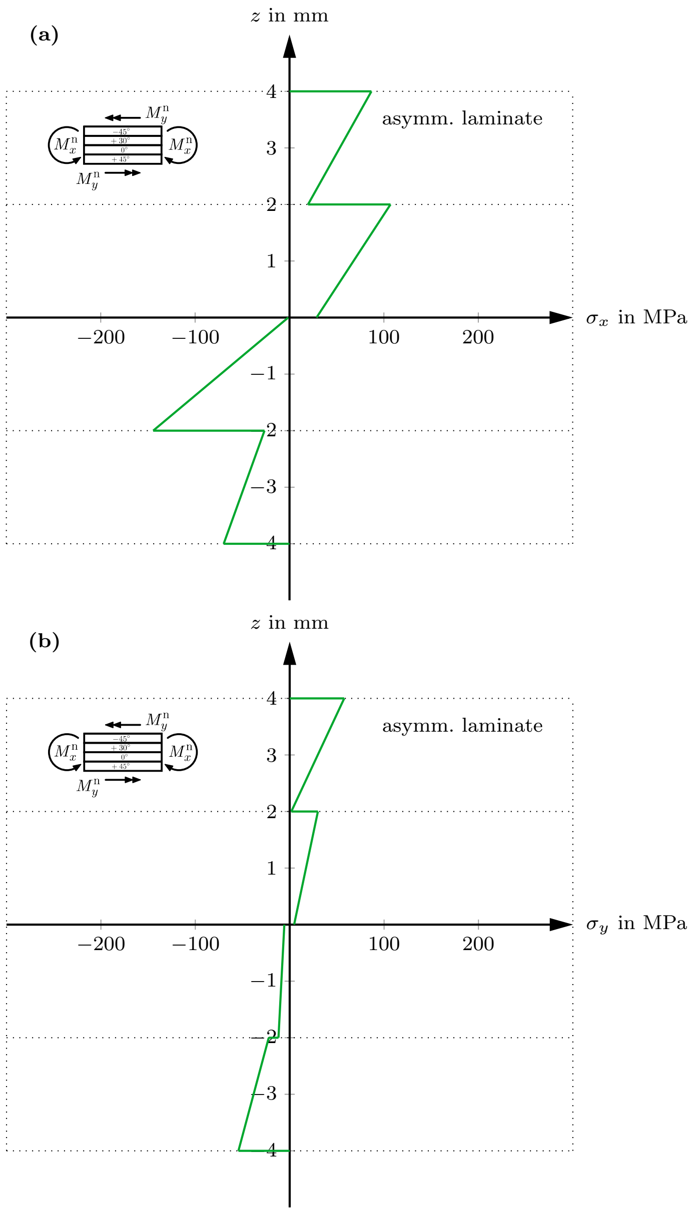


Fig. 4.10 Distribution of selected field quantities over the laminate thickness for the biaxial bending load case: **a** normal stress $\sigma_x(z)$ and **b** normal stress $\sigma_y(z)$

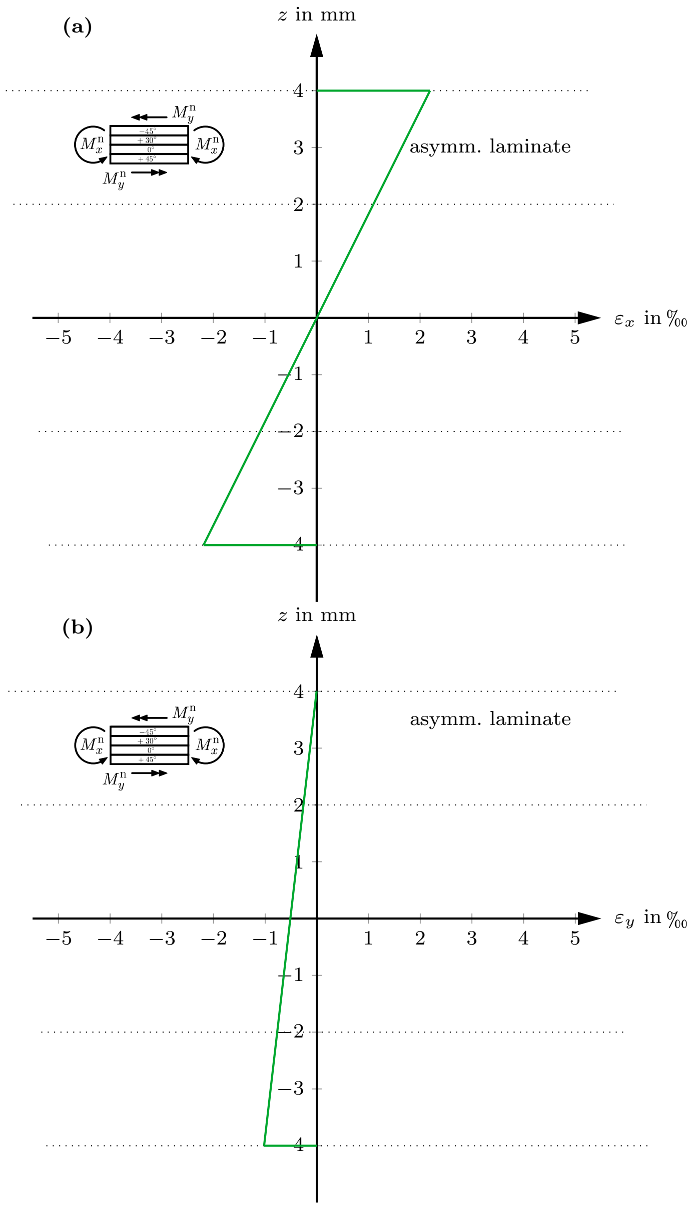


Fig. 4.11 Distribution of selected field quantities over the laminate thickness for the biaxial bending load case: **a** normal strain $\varepsilon_x(z)$ and **b** normal stress $\varepsilon_y(z)$

Table 4.16 Stresses in the local *1–2* coordinate systems for an external loading of $M_x^{\mathrm{n}} = 1000$ Nmm/mm and $M_y^{\mathrm{n}} = 500$ Nmm/mm, asymmetric laminate

Lamina no.	Position	σ_1 in MPa	σ_2 in MPa	σ_{12} in MPa
1	Top	–29.594	–19.108	2.157
	Center	–61.685	–24.693	4.935
	Bottom	–93.775	–30.278	7.712
2	Top	–1.364	–5.676	7.141
	Center	–72.897	–8.780	8.470
	Bottom	–144.431	–11.885	9.800
3	Top	135.987	0.893	–5.512
	Center	88.850	–3.746	–2.442
	Bottom	41.712	–8.385	0.628
4	Top	127.955	16.606	1 4.507
	Center	70.324	12.627	11.730
	Bottom	12.694	8.648	8.952

Table 4.17 Strains in the local *1–2* coordinate systems for an external loading of $M_x^{\mathrm{n}} = 1000$ Nmm/mm und $M_y^{\mathrm{n}} = 500$ Nmm/mm, asymmetric laminate

Lamina no.	Position	ε_1 in ‰	ε_2 in ‰	γ_{12} in ‰
1	Top	–0.187940	–1.672812	0.326890
	Center	–0.424578	–2.110923	0.747710
	Bottom	–0.661217	–2.549033	1.168531
2	Top	0.001749	–0.513003	1.081927
	Center	–0.546036	–0.639967	1.283399
	Bottom	–1.093821	–0.766931	1.484872
3	Top	1.052228	–0.213984	–0.835179
	Center	0.696888	–0.533393	–0.370002
	Bottom	0.341549	–0.852803	0.095176
4	Top	0.955852	1.231890	2.198033
	Center	0.517742	0.995251	1.777213
	Bottom	0.079631	0.758613	1.356392

To easier compare the different failure criteria, we determine the limit loads by applying loads as $M_x^{\mathrm{n}} = R \times 1000$ Nmm/mm and $M_y^{\mathrm{n}} = R \times 500$ Nmm/mm, i.e., we multiply the loads with the strength ratio R. In case of $R > 1$, there is no failure, whereas $R \leq 1$ results in the failure of the respective lamina. The evaluation of the different failure criteria is exactly done as for load case (a) and the corresponding results are summarized in Table 4.18.

Table 4.18 Strength ratio R in the local *1–2* coordinate systems for an external loading of $M_x^{\mathrm{n}} = R \times 1000$ Nmm/mm and $M_y^{\mathrm{n}} = R \times 500$ Nmm/mm, asymmetric laminate

Lamina no.	Position	Direction	Max. stress	Max. Strain	Tsai-Hill	Tsai-Wu
1	Top	*1*	48.996	62.520		
		2	10.467	11.956	9.990	10.909
		12	36.617	61.183		
	Center	*1*	23.507	27.675		
		2	8.099	9.475	7.031	8.389
		12	16.008	26.748		
	Bottom	*1*	15.463	17.770		
		2	6.605	7.846	5.324	6.727
		12	10.243	17.116		
2	Top	*1*	1063.369	7891.877		
		2	35.239	38.986	10.557	14.021
		12	11.063	18.486		
	Center	*1*	19.891	21.519		
		2	22.779	31.252	7.994	11.968
		12	9.327	15.584		
	Bottom	*1*	10.039	10.742		
		2	16.828	26.078	5.973	9.786
		12	8.061	13.469		
3	Top	*1*	14.340	13.115		
		2	53.751	93.465	9.977	10.114
		12	14.332	23.947		
	Center	*1*	21.947	19.802		
		2	53.391	37.496	16.976	18.044
		12	32.350	54.054		
	Bottom	*1*	46.749	40.404		
		2	23.852	23.452	20.539	19.030
		12	125.764	210.138		
4	Top	*1*	15.240	14.437		
		2	2.891	3.539	2.523	2.602
		12	5.446	9.099		
	Center	*1*	27.729	26.654		
		2	3.801	4.381	3.291	3.300
		12	6.735	11.254		
	Bottom	*1*	153.618	173.299		
		2	5.551	5.747	4.698	4.463
		12	8.825	14.745		

It can be seen from Table 4.18 that *all* failure criteria predict that not a single lamina will fail ($R > 1$). In addition, it can be stated that all criteria provide strength ratios in a similar order of magnitude if the maximum stress is compared with the maximum strain criterion, respectively, if the Tsai-Hill criterion is compared with the Tsai-Wu approach.

4.5 Problem 4: Ply-By-Ply Failure Loads

Given is a symmetric 3-layer laminate of thickness $t = 9$ mm as shown in Fig. 4.12. Each lamina has the same thickness ($t/3$) and identical material properties, which can be taken from Table 4.1. However, the orientation of each lamina changes from layer to layer as indicated in Fig. 4.12.

Consider a pure tensile load case with $N_x^{\mathrm{n}} > 0$ (all other forces and moments are equal to zero) to investigate the ply-by-ply failure of this laminate based on the Tsai-Wu criterion.

The solution procedure will follow the recommended steps provided in Table 3.1.

① Define for each lamina k the material ($E_{1,k}$; $E_{2,k}$; $\nu_{12,k}$; $G_{12,k}$) and the geometrical (z_k; z_{k-1}; α_k) properties.

The material properties ($E_{1,k}$; $E_{2,k}$; $\nu_{12,k}$; $G_{12,k}$) are the same for each lamina and can be taken from Table 4.1. The geometrical properties for each lamina are summarized in Table 4.19.

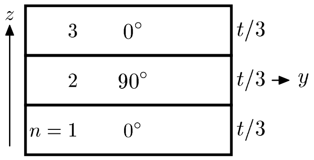


Fig. 4.12 Symmetric laminate: $[0°/\overline{90°}]_s$

Table 4.19 Geometrical properties (z_k; z_{k-1}; α_k) of the laminate shown in Fig. 4.12

Lamina k	Coordinate z_{k-1} in mm	Coordinate z_k in mm	Angle α_k
1	–4.5	–1.5	0°
2	–1.5	1.5	90°
3	1.5	4.5	0°

② Define the column matrix of generalized stresses:
$\boldsymbol{s} = \left[N_x^{\mathrm{n}} \; N_y^{\mathrm{n}} \; N_{xy}^{\mathrm{n}} \;\; M_x^{\mathrm{n}} \; M_y^{\mathrm{n}} \; M_{xy}^{\mathrm{n}}\right]^{\mathrm{T}}$.

Let us assume a unit load which facilitates the scaling of any failure loads

$$\boldsymbol{s} = \left[1\ 0\ 0\ 0\ 0\ 0\right]^{\mathrm{T}} \ \mathrm{N/mm}. \tag{4.63}$$

③ Calculate for each layer k the elasticity matrix $\boldsymbol{C}_k$ in the *1–2* lamina system according to Eq. (2.31). Transform each matrix according to Eq. (2.49) to obtain the elasticity matrix $\overline{\boldsymbol{C}}_k$ in the x–y laminate system.

Since each lamina is composed of the same materials and the thickness is equal to $t/3 = 3$ mm, Eq. (2.31) yields for each lamina the same matrix

$$\boldsymbol{C}_k = \begin{bmatrix} \frac{E_1}{1-\nu_{12}\nu_{21}} & \frac{\nu_{21}E_1}{1-\nu_{12}\nu_{21}} & 0 \\ \frac{\nu_{12}E_2}{1-\nu_{12}\nu_{21}} & \frac{E_2}{1-\nu_{12}\nu_{21}} & 0 \\ 0 & 0 & G_{12} \end{bmatrix} = \begin{bmatrix} 129868.204 & 3100.729 & 0 \\ 3100.729 & 11074.033 & 0 \\ 0 & 0 & 6600.0 \end{bmatrix} \frac{\mathrm{N}}{\mathrm{mm}^2}. \tag{4.64}$$

The transformation to the x–y laminate system according to Eq. (2.49) gives the following matrices:

$$\overline{\boldsymbol{C}}_{[0°]} = \begin{bmatrix} 129868.204 & 3100.729 & 0.0 \\ 3100.729 & 11074.033 & 0.0 \\ 0.0 & 0.0 & 6600.0 \end{bmatrix} \frac{\mathrm{N}}{\mathrm{mm}^2}, \tag{4.65}$$

$$\overline{\boldsymbol{C}}_{[90°]} = \begin{bmatrix} 11074.033 & 3100.729 & 0.0 \\ 3100.729 & 129868.204 & 0.0 \\ 0.0 & 0.0 & 6600.0 \end{bmatrix} \frac{\mathrm{N}}{\mathrm{mm}^2}. \tag{4.66}$$

④ Calculate the submatrices $\boldsymbol{A}$, $\boldsymbol{B}$, and $\boldsymbol{C}$ according to Eqs. (3.13)–(3.15). Assemble the generalized elasticity matrix $\boldsymbol{C}^*$ according to Eq. (3.12).

The evaluation of the submatrices $\boldsymbol{A}$, $\boldsymbol{B}$, and $\boldsymbol{C}$ according to Eqs. (3.13)–(3.15) gives

$$\boldsymbol{A} = \sum_{k=1}^{8} \boldsymbol{A}_k = \begin{bmatrix} 812431.324 & 27906.563 & 0.0 \\ 27906.563 & 456048.810 & 0.0 \\ 0.0 & 0.0 & 59400.0 \end{bmatrix} \frac{\text{N}}{\text{mm}}, \tag{4.67}$$

$$\boldsymbol{B} = \sum_{k=1}^{8} \boldsymbol{B}_k = \begin{bmatrix} 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 \end{bmatrix} \text{N} \tag{4.68}$$

$$\boldsymbol{D} = \sum_{k=1}^{8} \boldsymbol{D}_k = \begin{bmatrix} 7622206.519 & 188369.300 & 0.0 \\ 188369.300 & 940034.385 & 0.0 \\ 0.0 & 0.0 & 400950.0 \end{bmatrix} \text{Nmm}, \tag{4.69}$$

from which the generalized elasticity matrix (written without units) can be assembled as follows:

$$\boldsymbol{C}^* = \begin{bmatrix} 812431.324 & 27906.563 & 0.0 & 0.0 & 0.0 & 0.0 \\ 27906.563 & 456048.810 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 59400.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 7622206.519 & 188369.300 & 0.0 \\ 0.0 & 0.0 & 0.0 & 188369.300 & 940034.385 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 400950.0 \end{bmatrix}. \tag{4.70}$$

⑤ Calculate the generalized compliance matrix $(\boldsymbol{C}^*)^{-1}$ based on Eqs. (3.17)–(3.19).

The evaluation of the submatrices $\boldsymbol{A}'$, $\boldsymbol{B}'$, and $\boldsymbol{C}'$ according to Eqs. (3.17)–(3.19) gives

$$\boldsymbol{A}' = \begin{bmatrix} 12.334659 & -0.754783 & 0.0 \\ -0.754783 & 21.973664 & 0.0 \\ 0.0 & 0.0 & 168.350168 \end{bmatrix} \frac{10^{-7}\text{mm}}{\text{N}} \tag{4.71}$$

$$\boldsymbol{B}' = \begin{bmatrix} 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 \end{bmatrix} \frac{1}{\text{N}} \tag{4.72}$$

$$\boldsymbol{D}' = \begin{bmatrix} 1.318485 & -0.264205 & 0.0 \\ -0.264205 & 10.690852 & 0.0 \\ 0.0 & 0.0 & 24.940766 \end{bmatrix} \frac{10^{-7}}{\text{Nmm}}, \tag{4.73}$$

from which the generalized compliance matrix (written without units) can be assembled as follows:

$$(\boldsymbol{C}^*)^{-1} = \begin{bmatrix} 12.334659 & -0.754783 & 0.0 & 0.0 & 0.0 & 0.0 \\ -0.754783 & 21.973664 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 168.350168 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 1.318485 & -0.264205 & 0.0 \\ 0.0 & 0.0 & 0.0 & -0.264205 & 10.690852 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 24.940766 \end{bmatrix} 10^{-7}. \tag{4.74}$$

⑥ Calculate the generalized strains $\boldsymbol{e} = \left[\boldsymbol{\varepsilon}^0 \; \boldsymbol{\kappa}\right]^{\mathrm{T}}$ according to Eq. (3.16).

The evaluation of Eq. (3.16) gives the generalized strains as follows:

$$\underbrace{\begin{bmatrix} \boldsymbol{\varepsilon}^0 \\ \hline \boldsymbol{\kappa} \end{bmatrix}}_{\boldsymbol{e}} = \begin{bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \\ \hline \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{bmatrix} = (\boldsymbol{C}^*)^{-1}\boldsymbol{s} = \begin{bmatrix} 1.233466 \times 10^{-3} \\ -7.547831 \times 10^{-5} \\ 0.0 \\ 0.0 \\ 0.0 \\ 0.0 \end{bmatrix} {}^0\!/_{00}\,. \tag{4.75}$$

⑦ Calculate the stresses in each layer k according to Eq. (3.20), $\boldsymbol{\sigma}_{x,y}^k(z) = \overline{\boldsymbol{C}}_k\left(\boldsymbol{\varepsilon}^0 + z\boldsymbol{\kappa}\right)$, in the x–y laminate system ($z_{k-1} \le z \le z_k$). The strains in the x–y laminate system are obtained from $\left(\boldsymbol{\varepsilon}^0 + z\boldsymbol{\kappa}\right)$.

Evaluation of Eq. (3.20) for each layer k, i.e.

$$\boldsymbol{\sigma}_{x,y}^k(z) = \overline{\boldsymbol{C}}_k\left(\boldsymbol{\varepsilon}^0 + z\boldsymbol{\kappa}\right), \tag{4.76}$$

under consideration of the generalized strains given in Eq. (4.75) and the range of the z-coordinate ($z_{k-1} \le z \le z_k$) provided in Table 4.19 gives the stresses for each layer in the x–y laminate system. In case that the strains in the x–y laminate system are required, the evaluation of $\left(\boldsymbol{\varepsilon}^0 + z\boldsymbol{\kappa}\right)$ provides these values. A graphical representation of the normal stress (σ_x) and normal strain (ε_x) in loading direction (x) is provided in Fig. 4.13 as a function of the laminate thickness (z-coordinate). It can be seen that a uniaxial load by N_x^{n} results in the case of this symmetric laminate in a symmetric distribution of the normal stress and strain with respect to $z = 0$. The strain distribution is constant (see Fig. 4.13b) across the entire thickness of the laminate. The stress distribution (see Fig. 4.13a) reveals a layer-wise constant distribution with the characteristic that the stress is proportional to the stiffness value of each layer in the loading direction.

⑧ Transform the stresses in each layer to the *1–2* lamina system according to Eq. (3.21): $\boldsymbol{\sigma}_{1,2}^k = \boldsymbol{T}_\sigma^k \boldsymbol{\sigma}_{x,y}^k$. The strains in the *1–2* lamina system are obtained from Eq. (2.33): $\boldsymbol{\varepsilon}_{1,2}^k = (\boldsymbol{C}_k)^{-1}\boldsymbol{\sigma}_{1,2}^k$.

The transformation of the stresses into the *1–2* lamina systems is performed according to Eqs. (3.21) and (2.33). The corresponding numerical values for the bottom (z_{k-1}), the middle ($(z_{k-1} + z_k)/2$), and the top (z_k) of each lamina are summarized

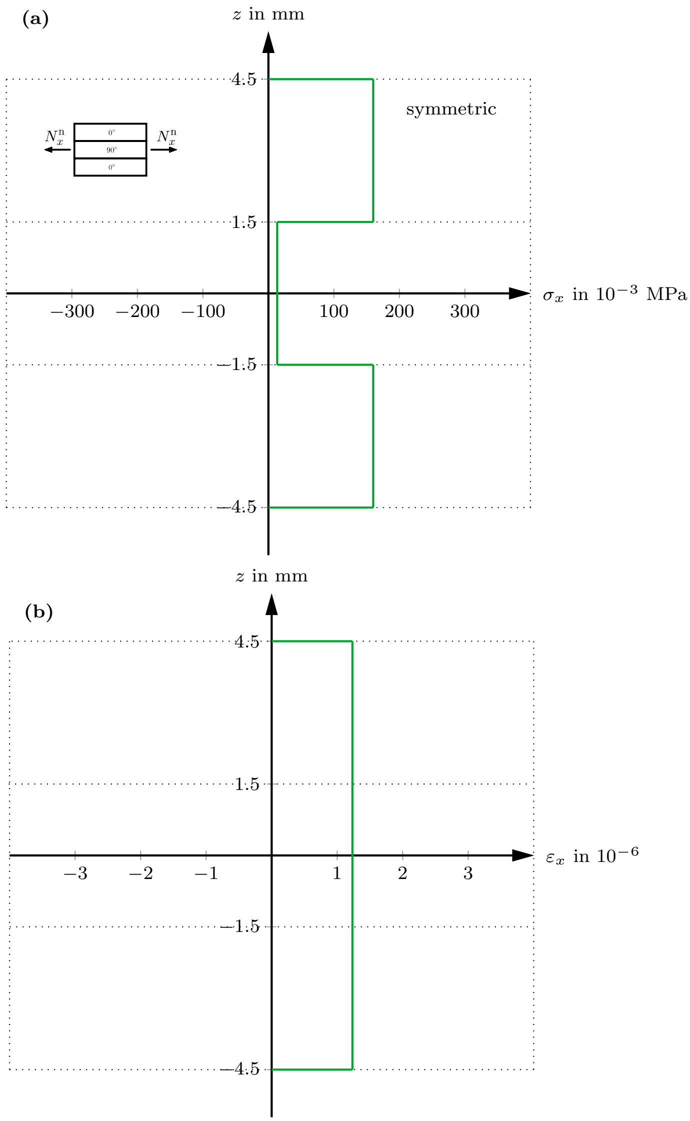


Fig. 4.13 Distribution of selected field quantities over the laminate thickness for the tensile load case: **a** normal stress $\sigma_x(z)$ and **b** normal strain $\varepsilon_x(z)$

Table 4.20 Stresses in the local *1–2* coordinate systems for an external loading of $N_x^{\text{n}} = 1$ N/mm, symmetric laminate

Lamina no.	Position	σ_1 in MPa	σ_2 in MPa	σ_{12} in MPa
1	Top	0.159954	0.002989	0.0
	Center	0.159954	0.002989	0.0
	Bottom	0.159954	0.002989	0.0
2	Top	–0.005978	0.013425	0.0
	Center	–0.005978	0.013425	0.0
	Bottom	–0.005978	0.013425	0.0
3	Top	0.159954	0.002989	0.0
	Center	0.159954	0.002989	0.0
	Bottom	0.159954	0.002989	0.0

in Table 4.20. These values will be used for the subsequent failure analysis based on the Tsai-Wu criterion.

Failure analysis based on the Tsai-Wu failure criterion.

To easier evaluate the Tsai-Wu failure criterion, we determine the limit loads by applying loads as $N_x^{\text{n}} = R \times 1$ N/mm, i.e., we multiply the load with the strength ratio R.

Considering the stresses from Table 4.20 in the Tsai-Wu criterion according to Eq. (2.140) gives

$$\begin{aligned} & a_1(R \times \sigma_1) + a_2(R \times \sigma_2) + a_{11}(R \times \sigma_1)^2 + a_{22}(R \times \sigma_2)^2 \\ & + a_{12}(R \times \sigma_{12})^2 + 2a_{1,2}(R \times \sigma_1)(R \times \sigma_2) < 1\,. \end{aligned} \tag{4.77}$$

This is a classical quadratic equation in R, from which the value of R can be identified; see Table 4.21.

It can be concluded from Table 4.21 that the minimum strength ratio R is obtained for lamina 2, i.e., the 90° layer. Thus, we obtain the maximum load as

$$N_x^{\text{n}} = R \times 1\ \text{N/mm} = 3546.336 \times 1\ \text{N/mm} = 3546.336\ \text{N/mm}\,, \tag{4.78}$$

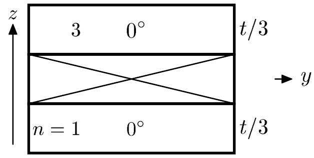
which corresponds to a maximum stress of

$$\frac{N_x^{\text{n}}}{t} = \frac{3546.336\ \text{N/mm}}{9\ \text{mm}} = 394.037\ \frac{\text{N}}{\text{mm}^2}\,. \tag{4.79}$$

Table 4.21 Strength ratio R in the local *1–2* coordinate systems for an external loading of $N_x^{\mathrm{n}} = R \times 1$ N/mm, symmetric laminate

Lamina no.	Position	Tsai-Wu
1	Top	10617.584
	Center	10617.584
	Bottom	10617.584
2	Top	3546.336
	Center	3546.336
	Bottom	3546.336
3	Top	10617.584
	Center	10617.584
	Bottom	10617.584

Fig. 4.14 Symmetric laminate: $[0^\circ/\overline{90^\circ}]_{\mathrm{s}}$ after first ply failure



The corresponding maximum normal strain in the x-direction for this load is obtained as

$$R \times \varepsilon_x^0 = 3546.336 \times 1.233466 \times 10^{-3}‰ = 4.374285‰\,. \tag{4.80}$$

For the next iteration, we consider the laminate without lamina 2, i.e., the 90° layer; see Fig. 4.14.

Steps ② to ③ remain unchanged and only the modified steps, i.e., without the 90° layer, will be presented in the following:

④ Calculate the submatrices $\boldsymbol{A}$, $\boldsymbol{B}$, and $\boldsymbol{C}$ according to Eqs. (3.13)–(3.15). Assemble the generalized elasticity matrix $\boldsymbol{C}^*$ according to Eq. (3.12).

The evaluation of the submatrices $\boldsymbol{A}$, $\boldsymbol{B}$, and $\boldsymbol{C}$ according to Eqs. (3.13)–(3.15) gives

$$A = \sum_{k=1}^{8} A_k = \begin{bmatrix} 779209.225 & 18604.375 & 0.0 \\ 18604.375 & 66444.197 & 0.0 \\ 0.0 & 0.0 & 39600.0 \end{bmatrix} \frac{\text{N}}{\text{mm}}, \tag{4.81}$$

$$B = \sum_{k=1}^{8} B_k = \begin{bmatrix} 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 \end{bmatrix} \text{N}, \tag{4.82}$$

$$D = \sum_{k=1}^{8} D_k = \begin{bmatrix} 7597289.945 & 181392.659 & 0.0 \\ 181392.659 & 647830.926 & 0.0 \\ 0.0 & 0.0 & 386100.0 \end{bmatrix} \text{Nmm}, \tag{4.83}$$

from which the generalized elasticity matrix (written without units) can be assembled as follows:

$$C^* = \begin{bmatrix} 779209.225 & 18604.375 & 0.0 & 0.0 & 0.0 & 0.0 \\ 18604.375 & 66444.197 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 39600.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 7597289.945 & 181392.659 & 0.0 \\ 0.0 & 0.0 & 0.0 & 181392.659 & 647830.926 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 386100.0 \end{bmatrix}. \tag{4.84}$$

⑤ Calculate the generalized compliance matrix $(C^*)^{-1}$ based on Eqs. (3.17)–(3.19).

The evaluation of the submatrices A', B', and C' according to Eqs. (3.17)–(3.19) gives

$$
\boldsymbol{A}' = \begin{bmatrix} 12.919897 & -3.617571 & 0.0 \\ -3.617571 & 151.515152 & 0.0 \\ 0.0 & 0.0 & 252.525253 \end{bmatrix} \frac{10^{-7}\text{mm}}{\text{N}}, \tag{4.85}
$$

$$
\boldsymbol{B}' = \begin{bmatrix} 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 \end{bmatrix} \frac{1}{\text{N}}, \tag{4.86}
$$

$$
\boldsymbol{D}' = \begin{bmatrix} 1.325118 & -0.371033 & 0.0 \\ -0.371033 & 15.540016 & 0.0 \\ 0.0 & 0.0 & 25.900026 \end{bmatrix} \frac{10^{-7}}{\text{Nmm}}, \tag{4.87}
$$

from which the generalized compliance matrix (written without units) can be assembled as follows:

$$
(\boldsymbol{C}^*)^{-1} = \begin{bmatrix} 12.919897 & -3.617571 & 0.0 & 0.0 & 0.0 & 0.0 \\ -3.617571 & 151.515152 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 252.525253 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 1.325118 & -0.371033 & 0.0 \\ 0.0 & 0.0 & 0.0 & -0.371033 & 15.540016 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 25.900026 \end{bmatrix} 10^{-7}. \tag{4.88}
$$

⑥ Calculate the generalized strains $\boldsymbol{e} = \left[\boldsymbol{\varepsilon}^0 \ \boldsymbol{\kappa}\right]^{\mathrm{T}}$ according to Eq. (3.16).

The evaluation of Eq. (3.16) gives the generalized strains as follows:

$$\underbrace{\begin{bmatrix} \boldsymbol{\varepsilon}^0 \\ \hline \boldsymbol{\kappa} \end{bmatrix}}_{\boldsymbol{e}} = \begin{bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \\ \hline \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{bmatrix} = (\boldsymbol{C}^*)^{-1}\boldsymbol{s} = \begin{bmatrix} 1.291990 \times 10^{-3} \\ -3.617571 \times 10^{-4} \\ 0.0 \\ 0.0 \\ 0.0 \\ 0.0 \end{bmatrix} ‰ . \tag{4.89}$$

⑦ Calculate the stresses in each layer k according to Eq. (3.20), $\boldsymbol{\sigma}_{x,y}^k(z) = \overline{\boldsymbol{C}}_k \left(\boldsymbol{\varepsilon}^0 + z\boldsymbol{\kappa}\right)$, in the x–y laminate system ($z_{k-1} \leq z \leq z_k$). The strains in the x–y laminate system are obtained from $\left(\boldsymbol{\varepsilon}^0 + z\boldsymbol{\kappa}\right)$.

Evaluation of Eq. (3.20) for each layer k, i.e.

$$\boldsymbol{\sigma}_{x,y}^k(z) = \overline{\boldsymbol{C}}_k \left(\boldsymbol{\varepsilon}^0 + z\boldsymbol{\kappa}\right), \tag{4.90}$$

under consideration of the generalized strains given in Eq. (4.89) and the range of the z-coordinate ($z_{k-1} \leq z \leq z_k$) provided in Table 4.19 gives the stresses for each layer in the x–y laminate system. In case that the strains in the x–y laminate system are required, the evaluation of $\left(\boldsymbol{\varepsilon}^0 + z\boldsymbol{\kappa}\right)$ provides these values. A graphical representation of the normal stress (σ_x) and normal strain (ε_x) in loading direction (x) is provided in Fig. 4.15 as a function of the laminate thickness (z-coordinate). It can be seen that a uniaxial load by N_x^{n} results in the case of this symmetric laminate in a symmetric distribution of the normal stress and strain with respect to $z = 0$. The strain distribution is constant (see Fig. 4.15b) across the entire thickness of the laminate. The stress distribution (see Fig. 4.15a) reveals a layer-wise constant distribution with the characteristic that the stress is proportional to the stiffness value of each layer in the loading direction.

⑧ Transform the stresses in each layer to the *1–2* lamina system according to Eq. (3.21): $\boldsymbol{\sigma}_{1,2}^k = \boldsymbol{T}_\sigma^k \boldsymbol{\sigma}_{x,y}^k$. The strains in the *1–2* lamina system are obtained from Eq. (2.33): $\boldsymbol{\varepsilon}_{1,2}^k = (\boldsymbol{C}_k)^{-1}\boldsymbol{\sigma}_{1,2}^k$.

The transformation of the stresses into the *1–2* lamina systems is performed according to Eqs. (3.21) and (2.33). The corresponding numerical values for the bottom (z_{k-1}), the middle ($(z_{k-1} + z_k)/2$), and the top (z_k) of each lamina are summarized in Table 4.22. These values will be used for the subsequent failure analysis based on the Tsai-Wu criterion.

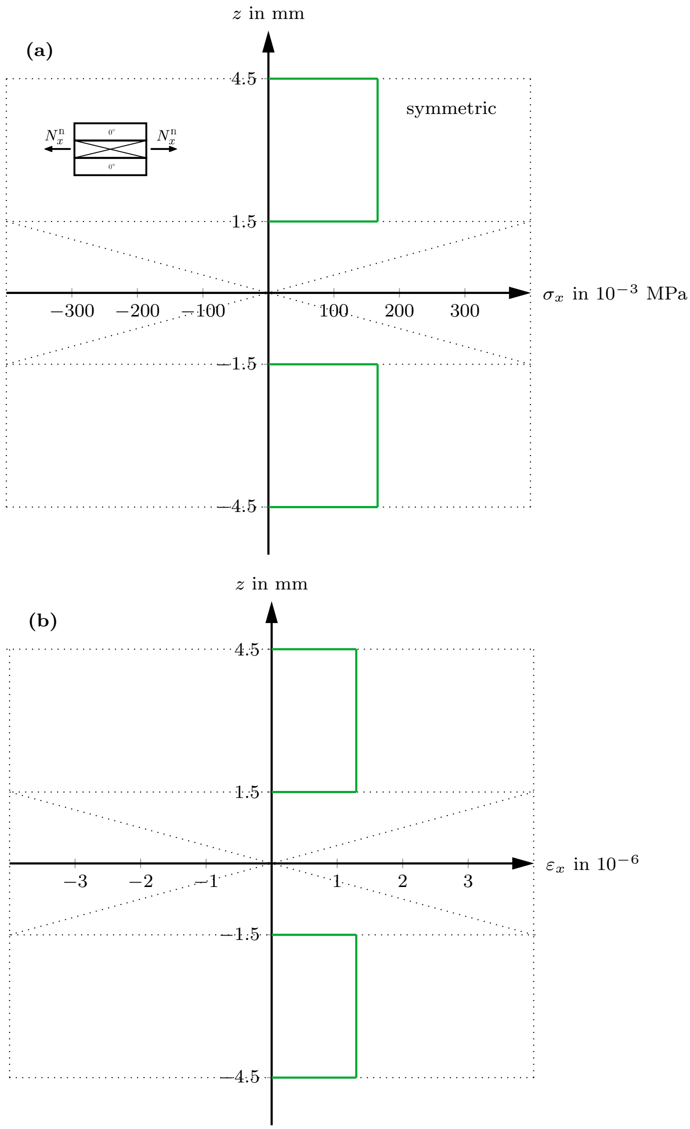


Fig. 4.15 Distribution of selected field quantities over the laminate thickness for the tensile load case after the first ply failure: **a** normal stress $\sigma_x(z)$ and **b** normal strain $\varepsilon_x(z)$

Table 4.22 Stresses in the local *1–2* coordinate systems for an external loading of $N_x^{\mathrm{n}} = 1$ N/mm after first ply failure, symmetric laminate

Lamina no.	Position	σ_1 in MPa	σ_2 in MPa	σ_{12} in MPa
1	Top	0.166667	0.0	0.0
	Center	0.166667	0.0	0.0
	Bottom	0.166667	0.0	0.0
2	Top	–	–	–
	Center	–	–	–
	Bottom	–	–	–
3	Top	0.166667	0.0	0.0
	Center	0.166667	0.0	0.0
	Bottom	0.166667	0.0	0.0

Failure analysis based on the Tsai-Wu failure criterion.

To easier evaluate the Tsai-Wu failure criterion, we determine the limit loads by applying loads as $N_x^{\mathrm{n}} = R \times 1$ N/mm, i.e., we multiply the load with the strength ratio R.

Considering the stresses from Table 4.22 in the Tsai-Wu criterion according to Eq. (2.140) gives

$$a_1(R \times \sigma_1) + a_2(R \times \sigma_2) + a_{11}(R \times \sigma_1)^2 + a_{22}(R \times \sigma_2)^2 \\ + a_{12}(R \times \sigma_{12})^2 + 2a_{1,2}(R \times \sigma_1)(R \times \sigma_2) < 1 \,. \tag{4.91}$$

This is a classical quadratic equation in R, from which the value of R can be identified; see Table 4.23.

It can be concluded from Table 4.23 that the minimum strength ratio R is obtained for lamina 1 and 3, i.e., the 0° layers. Thus, we obtain the maximum load as

$$N_x^{\mathrm{n}} = R \times 1\ \mathrm{N/mm} = 11700.0 \times 1\ \mathrm{N/mm} = 11700.0\ \mathrm{N/mm}\,, \tag{4.92}$$

which corresponds to a maximum stress of

$$\frac{N_x^{\mathrm{n}}}{t} = \frac{11700.0\ \mathrm{N/mm}}{9\ \mathrm{mm}} = 1300.0\ \frac{\mathrm{N}}{\mathrm{mm}^2}\,. \tag{4.93}$$

Table 4.23 Strength ratio R in the local *1–2* coordinate systems for an external loading of $N_x^{\mathrm{n}} = R \times 1$ N/mm after first ply failure, symmetric laminate

Lamina no.	Position	Tsai-Wu
1	Top	11700.0
	Center	11700.0
	Bottom	11700.0
2	Top	–
	Center	–
	Bottom	–
3	Top	11700.0
	Center	11700.0
	Bottom	11700.0

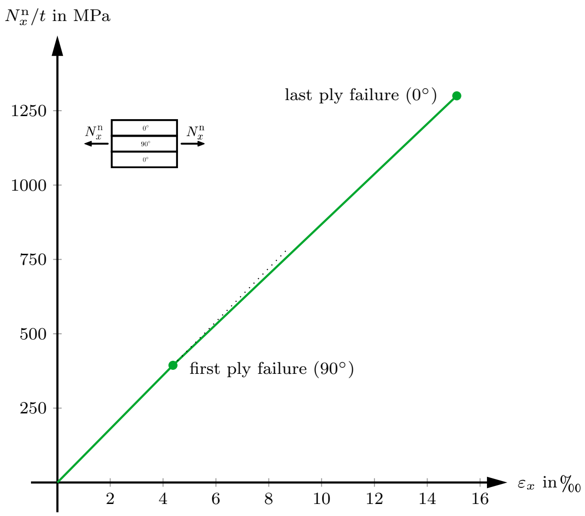


Fig. 4.16 Stress–strain curve showing the ply-by-ply failure of the $[0°/\overline{90°}]_{\mathrm{s}}$ lamiante under uniaxial loading with N_x^{n}

The corresponding maximum normal strain in the x-direction for this load is obtained as

$$R \times \varepsilon_x^0 = 11700.0 \times 1.291990 \times 10^{-3}‰ = 15.116283‰\,. \tag{4.94}$$

The macroscopic stress–strain diagram in x-direction is shown in Fig. 4.16. It can be seen that a bi-linear behavior is obtained in the elastic range for the entire laminate.

4.6 Problem 5: Pole Diagrams of Elastic Properties for Unidirectional Laminae

Given is a unidirectional lamina, i.e., a AS4 carbon/3501-6 epoxy prepreg, of 1 mm thickness; see Fig. 4.17.

Calculate the apparent elastic properties E_x, E_y, G_{xy}, ν_{xy}, and ν_{yx} as a function of the angle of rotation. Represent the elastic properties in the form of pole diagrams ($0° \leq \alpha \leq 360°$). In addition, provide a classical representation in a Cartesian coordinate system ($0° \leq \alpha \leq 90°$). The material properties of the lamina can be taken from Table 4.1.

The apparent elastic properties can be obtained from the elements $\overline{D}_{ij}$ of the transformed elastic compliance matrix according to Eq. (2.71)

$$\overline{\boldsymbol{D}} = \begin{bmatrix} \overline{D}_{11} & \overline{D}_{12} & \overline{D}_{14} \\ \overline{D}_{12} & \overline{D}_{22} & \overline{D}_{24} \\ \overline{D}_{14} & \overline{D}_{24} & \overline{D}_{44} \end{bmatrix} = \begin{bmatrix} \frac{1}{E_x} & -\frac{\nu_{xy}}{E_x} & \frac{\eta_x}{E_x} \\ -\frac{\nu_{xy}}{E_x} & \frac{1}{E_y} & \frac{\eta_y}{E_y} \\ \frac{\eta_x}{E_x} & \frac{\eta_y}{E_y} & \frac{1}{G_{xy}} \end{bmatrix}, \tag{4.95}$$

where the compliance coefficients $\overline{D}_{ij}$ can be evaluated based Eqs. (2.65)–(2.70)

$$\overline{D}_{11} = D_{11}\cos^4\alpha + (2D_{12} + D_{44})\sin^2\alpha\cos^2\alpha + D_{22}\sin^4\alpha\,, \tag{4.96}$$

$$\overline{D}_{12} = (D_{11} + D_{22} - D_{44})\sin^2\alpha\cos^2 + D_{12}(\sin^4\alpha + \cos^4\alpha)\,, \tag{4.97}$$

$$\overline{D}_{22} = D_{11}\sin^4\alpha + (2D_{12} + D_{44})\sin^2\alpha\cos^2\alpha + D_{22}\cos^4\alpha\,, \tag{4.98}$$

$$\overline{D}_{14} = (2D_{11} - 2D_{12} - D_{44})\sin\alpha\cos^3\alpha - (2D_{22} - 2D_{12} - D_{44})\sin^3\alpha\cos\alpha\,, \tag{4.99}$$

$$\overline{D}_{24} = (2D_{11} - 2D_{12} - D_{44})\sin^3\alpha\cos\alpha - (2D_{22} - 2D_{12} - D_{44})\sin\alpha\cos^3\alpha\,, \tag{4.100}$$

$$\overline{D}_{44} = 2(2D_{11} + 2D_{22} - 4D_{12} - D_{44})\sin^2\alpha\cos^2\alpha + D_{44}(\sin^4\alpha + \cos^4\alpha)\,. \tag{4.101}$$

or after replacing the D_{ij} (see Eq. (2.63)) as

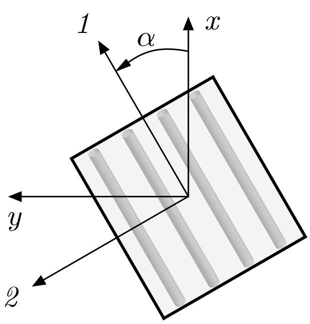


Fig. 4.17 Unidirectional lamina rotated under the angle α

$$\overline{D}_{11} = \frac{1}{E_1}\cos^4\alpha + (2(-\frac{\nu_{12}}{E_1}) + \frac{1}{G_{12}})\sin^2\alpha\cos^2\alpha + \frac{1}{E_2}\sin^4\alpha\,, \tag{4.102}$$

$$\overline{D}_{12} = (\frac{1}{E_1} + \frac{1}{E_2} - \frac{1}{G_{12}})\sin^2\alpha\cos^2 + (-\frac{\nu_{12}}{E_1})(\sin^4\alpha + \cos^4\alpha)\,, \tag{4.103}$$

$$\overline{D}_{22} = \frac{1}{E_1}\sin^4\alpha + (2(-\frac{\nu_{12}}{E_1}) + \frac{1}{G_{12}})\sin^2\alpha\cos^2\alpha + \frac{1}{E_2}\cos^4\alpha\,,$$

$$\overline{D}_{14} = (2\frac{1}{E_1} - 2(-\frac{\nu_{12}}{E_1}) - \frac{1}{G_{12}})\sin\alpha\cos^3\alpha \tag{4.104}$$

$$- (2\frac{1}{E_2} - 2(-\frac{\nu_{12}}{E_1}) - \frac{1}{G_{12}})\sin^3\alpha\cos\alpha\,, \tag{4.105}$$

$$\overline{D}_{24} = (2\frac{1}{E_1} - 2(-\frac{\nu_{12}}{E_1}) - \frac{1}{G_{12}})\sin^3\alpha\cos\alpha$$

$$- (2\frac{1}{E_2} - 2(-\frac{\nu_{12}}{E_1}) - \frac{1}{G_{12}})\sin\alpha\cos^3\alpha\,, \tag{4.106}$$

$$\overline{D}_{44} = 2(2\frac{1}{E_1} + 2\frac{1}{E_2} - 4(-\frac{\nu_{12}}{E_1}) - \frac{1}{G_{12}})\sin^2\alpha\cos^2\alpha + \frac{1}{G_{12}}(\sin^4\alpha + \cos^4\alpha)\,. \tag{4.107}$$

Thus, we can state the apparent elastic properties as follows[2]

$$\frac{1}{E_x} = \frac{1}{E_1}\cos^4\alpha + \left(\frac{1}{G_{12}} - \frac{2\nu_{12}}{E_1}\right)\sin^2\alpha\cos^2\alpha + \frac{1}{E_2}\sin^4\alpha\,, \tag{4.108}$$

$$\nu_{xy} = E_x\left[-\left(\frac{1}{E_1} + \frac{1}{E_2} - \frac{1}{G_{12}}\right)\sin^2\alpha\cos^2 + \frac{\nu_{12}}{E_1}\left(\sin^4\alpha + \cos^4\alpha\right)\right], \tag{4.109}$$

$$\frac{1}{E_y} = \frac{1}{E_1}\sin^4\alpha + \left(\frac{1}{G_{12}} - \frac{2\nu_{12}}{E_1}\right)\sin^2\alpha\cos^2\alpha + \frac{1}{E_2}\cos^4\alpha\,, \tag{4.110}$$

$$\frac{1}{G_{xy}} = 2\left(\frac{2}{E_1} + \frac{2}{E_2} + \frac{4\nu_{12}}{E_1} - \frac{1}{G_{12}}\right)\sin^2\alpha\cos^2\alpha + \frac{1}{G_{12}}(\sin^4\alpha + \cos^4\alpha)\,, \tag{4.111}$$

$$\eta_x = E_x\left[\left(\frac{2}{E_1} + \frac{2\nu_{12}}{E_1} - \frac{1}{G_{12}}\right)\sin\alpha\cos^3\alpha - \left(\frac{2}{E_2} + \frac{2\nu_{12}}{E_1} - \frac{1}{G_{12}}\right)\sin^3\alpha\cos\alpha\right], \tag{4.112}$$

$$\eta_y = E_y\left[\left(\frac{2}{E_1} + \frac{2\nu_{12}}{E_1} - \frac{1}{G_{12}}\right)\sin^3\alpha\cos\alpha - \left(\frac{2}{E_2} + \frac{2\nu_{12}}{E_1} - \frac{1}{G_{12}}\right)\sin\alpha\cos^3\alpha\right]. \tag{4.113}$$

The graphical representations of the apparent elastic properties for the AS4 carbon/3501-6 epoxy prepreg are shown in Figs. 4.18, 4.19, 4.20, 4.21 as pole diagrams and in Fig. 4.22 in a classical Cartesian representation.

---

[2] It should be noted here that some textbooks present the right-hand sides of Eqs. (4.112) and (4.113) in a slightly different manner by considering $E_x \to E_1$, $E_y \to E_2$ or $E_x, E_y \to E_1$ see [1, 5].

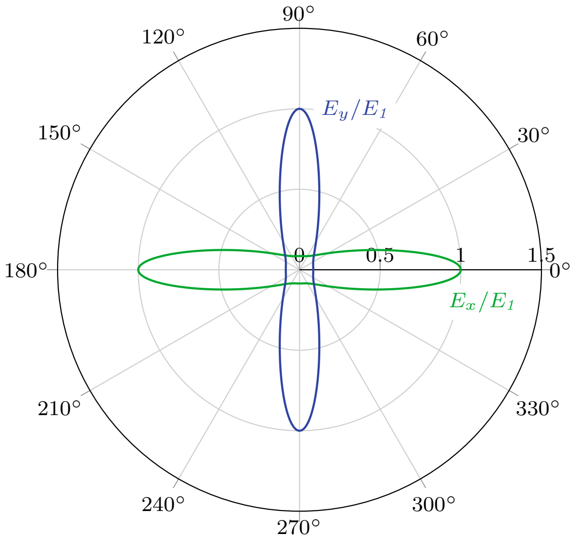


Fig. 4.18 Pole diagram of the elastic properties E_x/E_1 and E_y/E_1 for AS4 carbon/3501-6 epoxy

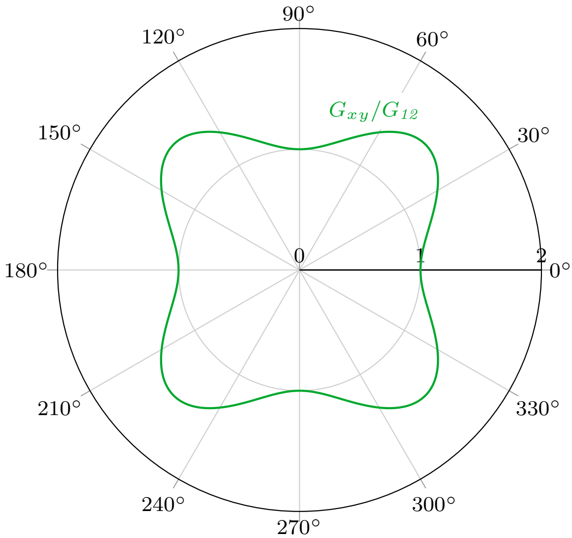


Fig. 4.19 Pole diagram of the elastic property G_{xy}/G_{12} for AS4 carbon/3501-6 epoxy

Fig. 4.20 Pole diagram of the elastic property ν_{xy} for AS4 carbon/3501-6 epoxy

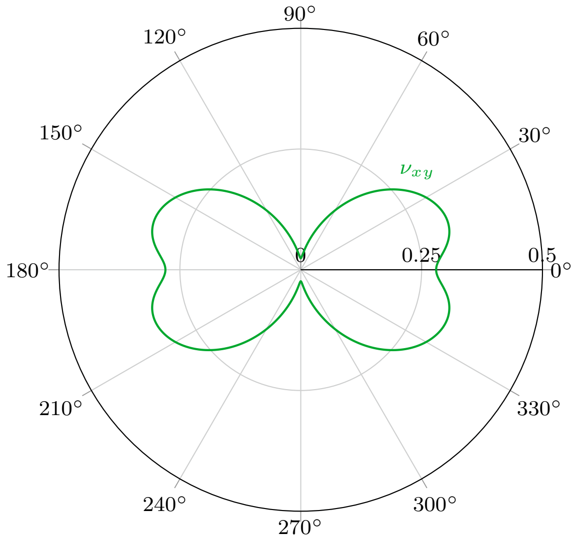
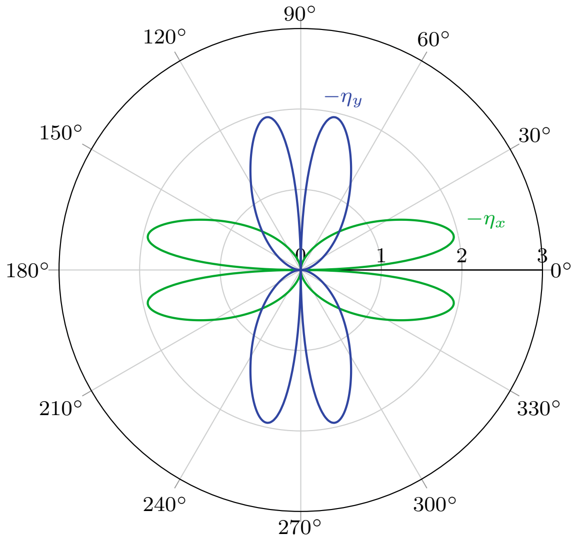


Fig. 4.21 Pole diagram of the elastic properties $-\eta_x$ and $-\eta_y$ for AS4 carbon/3501-6 epoxy



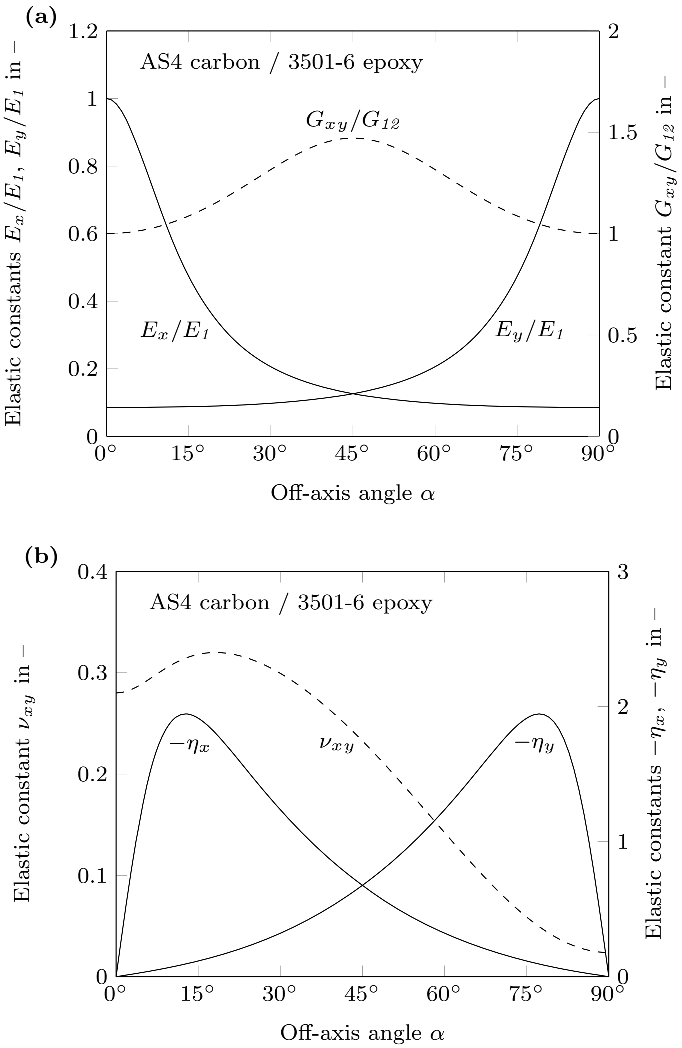


Fig. 4.22 Elastic constants as a function of the off-axis angle: **a** E_x/E_1, E_y/E_1, G_{xy}/G_{12} and **b** ν_{xy}, $-\eta_x$, $-\eta_y$

4.7 Problem 6: Failure Envelopes for Unidirectional Laminae

Given is a unidirectional lamina, i.e., a AS4 carbon/3501-6 epoxy prepreg, of 1 mm thickness, which is loaded by a normal stress σ_x in the x-direction; see Fig. 4.23.

Calculate and sketch the failure envelope, i.e., the failure stress in the x-direction as a function of the angle of rotation ($0° \leq \alpha \leq 90°$), based on the maximum stress, maximum strain, Tsai-Hill, and Tsai-Wu failure criteria. The material properties of the lamina can be taken from Table 4.1.

The stresses in principal material coordinates (*1–2* coordinate system) resulting from a single off-axis normal stress (σ_x) can be calculated from transformation (2.35)

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_{12} \end{bmatrix} = \begin{bmatrix} \cos^2\alpha & \sin^2\alpha & 2\sin\alpha\cos\alpha \\ \sin^2\alpha & \cos^2\alpha & -2\sin\alpha\cos\alpha \\ -\sin\alpha\cos\alpha & \sin\alpha\cos\alpha & \cos^2\alpha - \sin^2\alpha \end{bmatrix} \begin{bmatrix} \sigma_x \\ 0 \\ 0 \end{bmatrix}, \tag{4.114}$$

or based on this result the strains in principal material coordinates (see Eq. (2.28))

$$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ 2\,\varepsilon_{12} \end{bmatrix} = \begin{bmatrix} \frac{1}{E_1} & -\frac{\nu_{21}}{E_2} & 0 \\ -\frac{\nu_{12}}{E_1} & \frac{1}{E_2} & 0 \\ 0 & 0 & \frac{1}{G_{12}} \end{bmatrix} \begin{bmatrix} \sigma_1(\sigma_x) \\ \sigma_2(\sigma_x) \\ \sigma_{12}(\sigma_x) \end{bmatrix}. \tag{4.115}$$

This means that the single off-axis stress (σ_x) results in a plane stress state ($\sigma_1, \sigma_2, \sigma_{12}$). To judge if this combined stress or strain state results in the failure

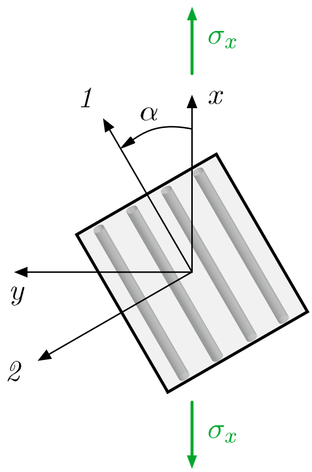


Fig. 4.23 Unidirectional lamina loaded by a single normal stress σ_x

Table 4.24 Different failure modes of the unidirectional lamina (see Table 4.1 for details) according to the maximum stress criterion

Angle range	Failure mode
Tension ($\sigma_x > 0$)	
$0° \leq \alpha \leq 2°$	Longitudinal tensile failure (k_{1}t)
$3° \leq \alpha \leq 30°$	Shear failure (k_{12}s)
$35° \leq \alpha \leq 90°$	Transverse tensile failure (k_2t)
Compression ($\sigma_x < 0$)	
$0° \leq \alpha \leq 3°$	Longitudinal compressive failure (k_{1}c)
$4° \leq \alpha \leq 65°$	Shear failure (k_{12}s)
$70° \leq \alpha \leq 90°$	Transverse compressive failure (k_2c)

of the lamina, the maximum stress, maximum strain, Tsai-Hill, and Tsai-Wu failure criteria will be applied in the following:
To determine the limit load, one may apply a unit load, i.e., $\sigma_x = 1$ (or $\sigma_x = -1$ in the case of compression), and multiply this unit load with the strength ratio R, i.e., $\sigma_x = R \times 1$. The minimum load factor is then equal to the limit load.

- Maximum Stress Criterion:

Considering the stresses from Eq. (4.114) in the conditions of the maximum stress criterion according to Eqs. (2.133)–(2.135) gives

$$k_{1\text{c}} < c_1 R \times (\pm 1) < k_{1\text{t}} , \tag{4.116}$$

$$k_{2\text{c}} < c_2 R \times (\pm 1) < k_{2\text{t}} , \tag{4.117}$$

$$|c_3 R \times (\pm 1)| < k_{12\text{s}} , \tag{4.118}$$

from which the minimum value of R can be identified as the ultimate load (the c_i are scalar values resulting from the evaluation of Eq. (4.114)). The graphical representation of the failure (ultimate) load as a function of the off-axis angle is provided in Fig. 4.24 for the maximum stress criterion. It can be seen that small off-axis angles result in a significant reduction of the failure load ($0° \leq \alpha \leq 30°$). For larger angles ($30° \leq \alpha \leq 90°$), there is a clear difference between the tension and compression case. This difference is magnified in Fig. 4.24b based on a logarithmic stress axis.

It should be noted that different failure modes occur depending on the off-axis angle; see Table 4.24. These different failure modes result in different sections of the failure envelope without a theoretical smooth transition.

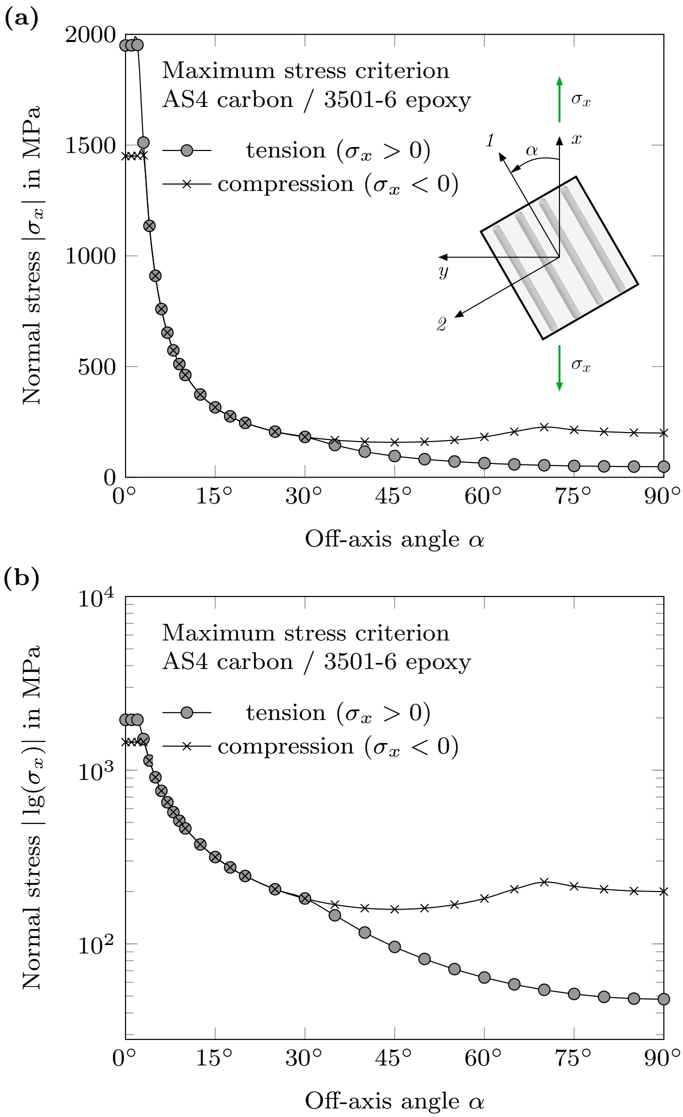


Fig. 4.24 Failure envelope for a unidirectional prepreg (see Table 4.1 for details) according to the maximum stress criterion: **a** linear division of the axes and **b** logarithmic vertical axis

Table 4.25 Different failure modes of the unidirectional lamina (see Table 4.1 for details) according to the maximum strain criterion

Angle range	Failure mode
Tension ($\sigma_x > 0$)	
$0° \leq \alpha \leq 4°$	Longitudinal tensile failure (k_{1t})
$5° \leq \alpha \leq 20°$	Shear failure (k_{12}s)
$25° \leq \alpha \leq 90°$	Transverse tensile failure (k_{2t})
Compression ($\sigma_x < 0$)	
$0° \leq \alpha \leq 5°$	Longitudinal compressive failure (k_1c)
$6° \leq \alpha \leq 55°$	Shear failure (k_{12}s)
$60° \leq \alpha \leq 90°$	Transverse compressive failure (k_2c)

• Maximum Strain Criterion:

Considering the strains from Eq. (4.115) in the conditions of the maximum strain criterion according to Eqs. (2.136)–(2.138) gives

$$k^{\varepsilon}_{1c} < c^{\varepsilon}_1 R \times (\pm 1) < k^{\varepsilon}_{1t}\,, \tag{4.119}$$
$$k^{\varepsilon}_{2c} < c^{\varepsilon}_2 R \times (\pm 1) < k^{\varepsilon}_{2t}\,, \tag{4.120}$$
$$\left| c^{\varepsilon}_3 R \times (\pm 1) \right| < k^{\varepsilon}_{12s}\,, \tag{4.121}$$

from which the minimum value of R can be identified as the ultimate load (the c^{ε}_i are scalar values resulting from the evaluation of Eq. (4.115)). The graphical representation of the failure (ultimate) load as a function of the off-axis angle is provided in Fig. 4.25 for the maximum stress criterion. The characteristics of the failure envelopes are quite similar to the ones observed for the maximum stress criterion.

As in the case of the maximum stress criterion, different failure modes can be distinguished; see Table 4.25 for details.

• Tsai-Hill Criterion:

Considering the stresses from Eq. (4.114) in the Tsai-Hill criterion according to Eq. (2.139) gives

$$\frac{(c_1 R \times (\pm 1))^2}{k_1^2} - \frac{(c_1 R \times (\pm 1))(c_2 R \times (\pm 1))}{k_1^2} + \frac{(c_2 R \times (\pm 1))^2}{k_2^2} + \frac{(c_3 R \times (\pm 1))^2}{k_{12s}^2} < 1\,, \tag{4.122}$$

from which the value of R can be identified as the ultimate load (the c_i are scalar values resulting from the evaluation of Eq. (4.114)). The graphical representation of the failure (ultimate) load as a function of the off-axis angle is provided in Fig. 4.26

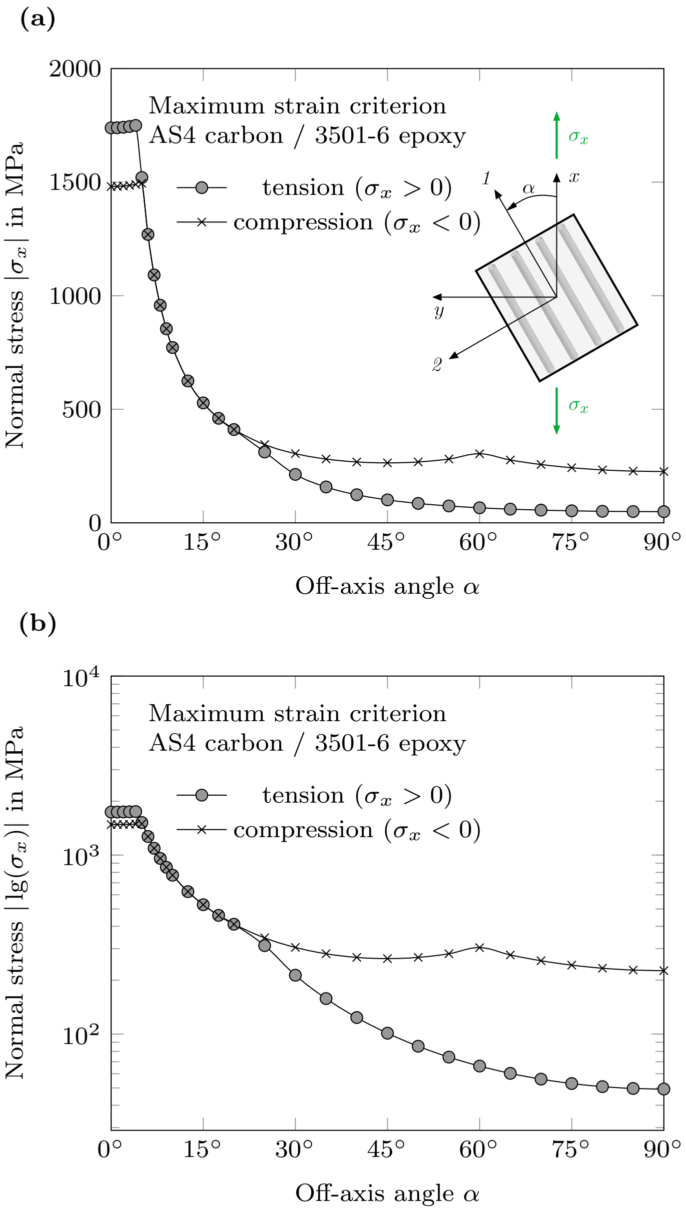


Fig. 4.25 Failure enevelope for a unidirectional prepreg (see Table 4.1 for details) according to the maximum strain criterion: **a** linear division of the axes and **b** logarithmic vertical axis

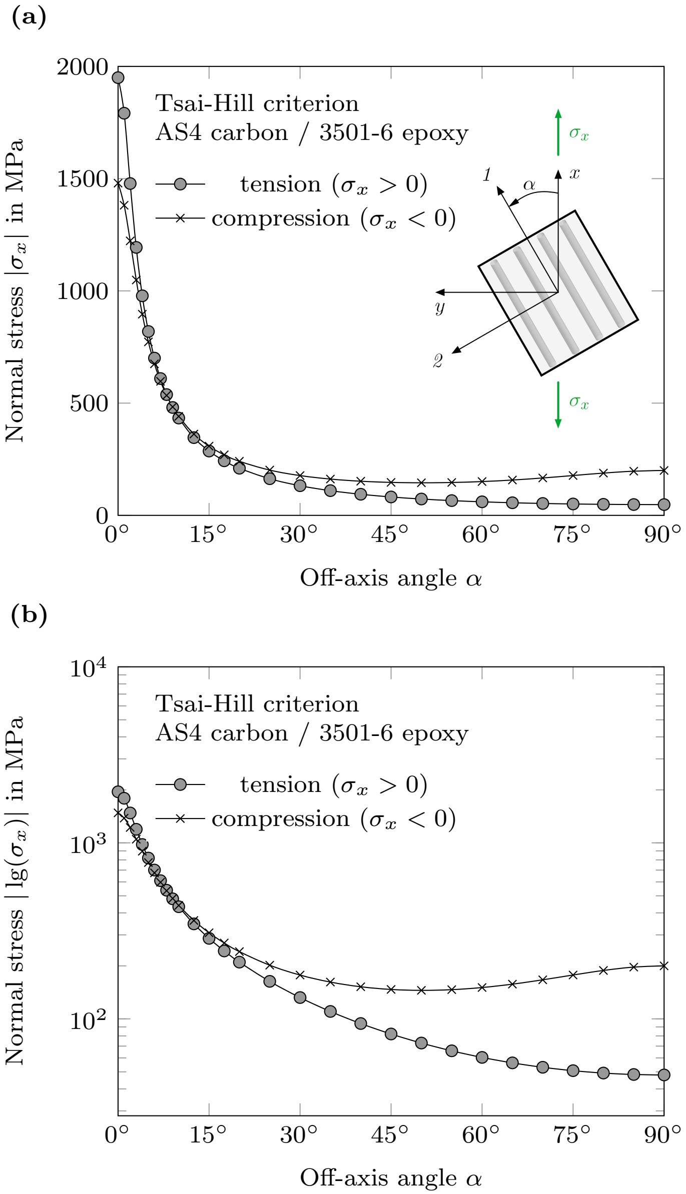


Fig. 4.26 Failure enevelope for a unidirectional prepreg (see Table 4.1 for details) according to the Tsai-Hill criterion: **a** linear division of the axes and **b** logarithmic vertical axis

for the maximum stress criterion. Different envelopes are again obtained for tension and compression but the envelopes are much smoother.

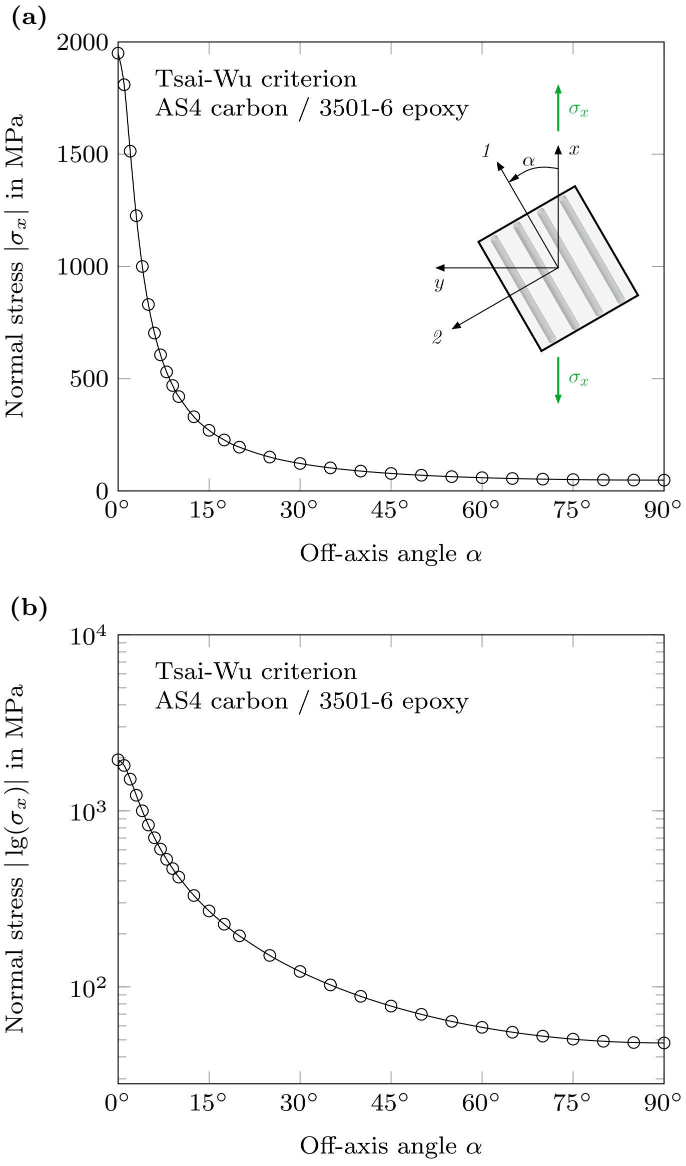


Fig. 4.27 Failure enevelope for a unidirectional prepreg (see Table 4.1 for details) according to the Tsai-Wu criterion: **a** linear division of the axes and **b** logarithmic vertical axis

• Tsai-Wu Criterion:

Considering the stresses from Eq. (4.114) in the Tsai-Wu criterion according to Eq. (2.140) gives

$$
\begin{aligned}
& a_1(c_1 R \times (\pm 1)) + a_2(c_2 R \times (\pm 1)) + a_{11}(c_1 R \times (\pm 1))^2 + a_{22}(c_2 R \times (\pm 1))^2 \\
& + a_{12}(c_3 R \times (\pm 1))^2 + 2a_{1,2}(c_1 R \times (\pm 1))(c_2 R \times (\pm 1)) < 1\,,
\end{aligned}
\tag{4.123}
$$

i.e., a classical quadratic equation in R, from which the value of R can be identified as the ultimate load (the c_i are scalar values resulting from the evaluation of Eq. (4.114)). A single and smooth failure envelope is obtained which facilitates the application compared to the previously mentioned criteria (Fig. 4.27).

References

1. Altenbach H, Altenbach J, Kissing W (2018) Mechanics of composite structural elements. Springer Nature, Singapore
2. Buryachenko VA (2007) Micromechanics of heterogeneous materials. Springer, New York
3. Dvorak GJ (2013) Micromechanics of composite materials. Springer, New York
4. Herakovich CT (2012) Mechanics of composites: a historical review. Mech Res Commun 41:1–20
5. Kaw AK (2006) Mechanics of composite materials. CRC Press, Boca Raton
6. MIL-HDBK-17, 1F, (2002) Composite materials handbook -, vol 1. Polymer matrix composites guidelines for characterization of structural materials, Department of Defence, USA
7. Soden PD, Hinton MJ, Kaddour AS (1998) Lamina properties, lay-up configurations and loading conditions for a range of fibre-reinforced composite laminates. Compos Sci Technol 58:1011–1022

Index

© The Editor(s) (if applicable) and The Author(s), under exclusive license to Springer Nature Switzerland AG 2021

A. Öchsner, *Foundations of Classical Laminate Theory*, Advanced Structured Materials 163, https://doi.org/10.1007/978-3-030-82631-4