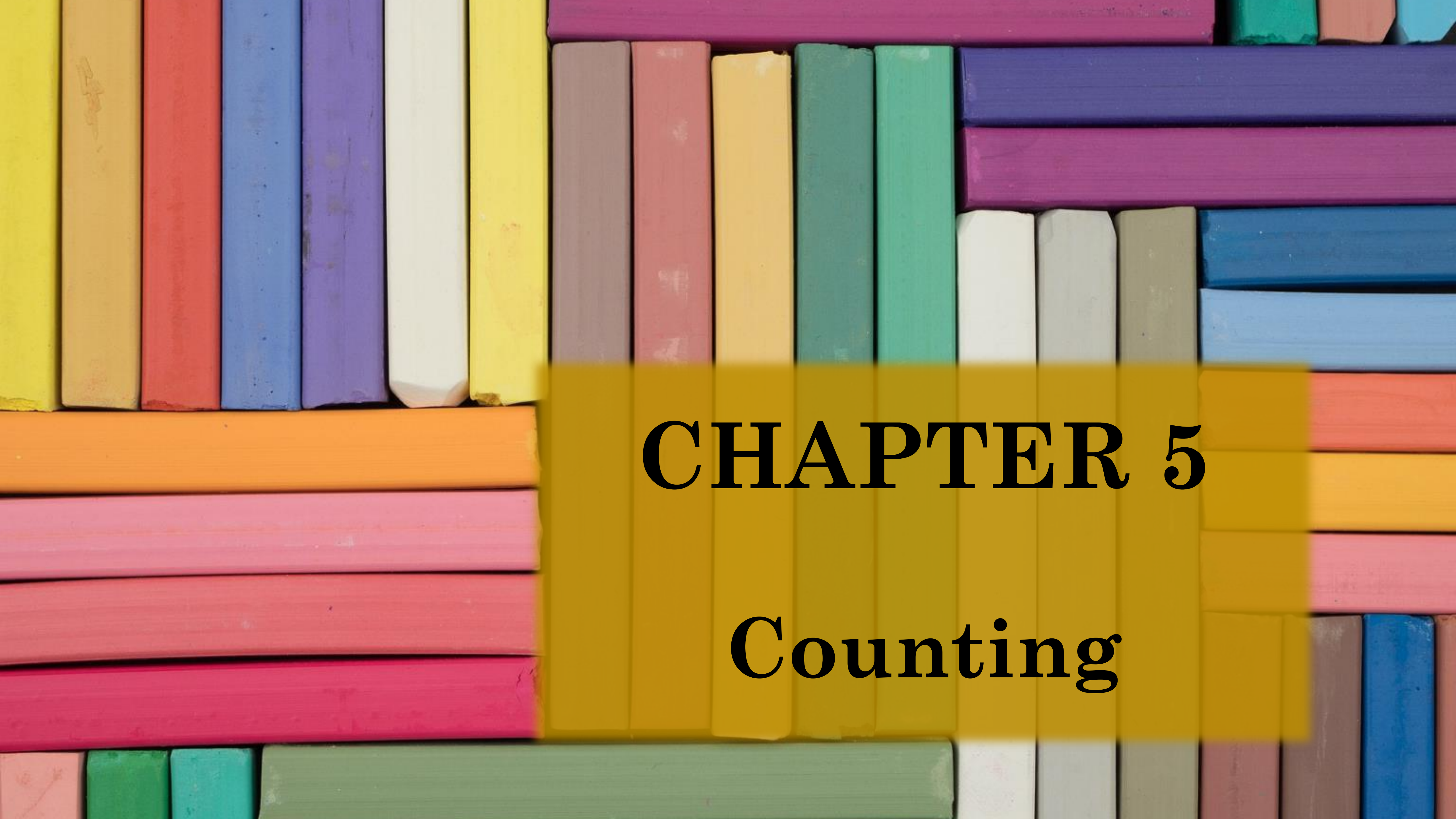

CHAPTER 5

Counting

LEARNING OBJECTIVES

1) Do basic counting using the Sum and Product rule.

2) Understand and apply Permutation and Combination theories.

3) Use the Inclusive-Exclusive Principle to do counting.

COUNTING

1) The Sum Rules
2) The Product Rules
3) Permutation
4) Combination
5) Permutation With Repetition
6) Combination With Repetition
7) Inclusive-Exclusive Principle
8) The Generalized Pigeonhole Principle

INTRODUCTION

- Techniques for counting are important in mathematics and in computer science, especially in the analysis of algorithms.
- Assume we have a set of **objects with certain properties**, ***counting*** is used to determine **the number of these objects**.
- Example: The number of possible phone number with 10 digits in the local calling area.

1) The Sum Rule

- If a **first task** can be done in **n1 ways** and a **second task** can be done in **n2 ways**, and if these tasks cannot be done at the same time, then there are **n1 + n2 ways** to do **either task**.

If A and B are finite sets that are **disjoint** (meaning $A \cap B = \emptyset$), then

$$|A \cup B| = |A| + |B|$$

- **Recall:** For set A, |A| is the **cardinality** of A.

Example

- Suppose there are **5 chicken dishes** and **8 beef dishes**. How many selections does a customer have if they can only **choose one dish**?

- **Solution:**

- There are 5 outcomes for the chicken event and 8 outcomes for the beef event. According to the addition principle there are

$$5 + 8 = \mathbf{13} \text{ possible selections.}$$

Example

- How many items of menu are available in Sedap Catering if there are 2 appetizers, 3 main course items and 4 beverages?

- **Solution:**

Example

- There are 3 math subjects and 5 programming subjects, how many ways are there to pick 1 math subject or 1 programming subject?

- **Solution:**

-

2) The Product Rule

- Suppose that a procedure can be broken down into **two tasks** that are to be **performed in sequence**. If there are **n1 ways** to do the **first task** and **n2 ways** to do the **second tasks** after the first task has been done, then there are **n1n2 ways** to do the procedure.

If A and B are finite sets, then:

$$|A \times B| = |A| \cdot |B|$$

Example

- There are 3 math subjects and 5 programming subjects. How many ways are there to pick one math subject **and** one programming subject?

- **Solution:**

- $3 \times 5 = 15$ ways

Example

- In how many different orders can 3 married couples be seated in a row of 6 chairs under the following conditions

  a) Anyone may sit in any chair.

  b) Men must occupy the first three chairs and women the last three.

- **Solution:**

Example

- A man has 8 shirts, 4 pairs of pants and 5 pairs of shoes. How many different outfits (combination of shirts, pants & shoes) are possible?

- **Solution:**

-

Example

- How many license plates can be made if the first 3 entries must be letters, followed by 4 numbers, under the following condition:
  a) With repetition
  b) Without repetition

- **Solution:**

Example

- A club has 15 members. How many ways are there to choose a president, vice president, secretary and treasurer of the club?

- **<u>Solution:</u>**

-

3) Permutation

- A permutation of a set of distinct objects is an **ordered arrangement** of these objects.

- **Theorem:** There are $\boldsymbol{n}!$ Permutation of n elements

- **<u>Example:</u>**
- Password - the order is important.
- 123 and 321 is different

Example

- How many arrangement are there for set {a,b,c}?

- **<u>Solution:</u>**
- n! = 3!
- 3 x 2 x 1 = 6 arrangement

{a,b,c}	{b,a,c}	{c,a,b}
{a,c,b}	{b,c,a}	{c,b,a}

R-Permutation

- Ordered arrangements of ***r* element** (**some** of the elements) of a set.

Theorem:

The number of *r-permutations* of set of *n* distinct objects is

$$P(n, r) = n(n-1)(n-2)\cdots(n-r+1)$$

$$= \frac{n!}{(n-r)!} \qquad , r \leq n.$$

Example

- What is 2 permutations of the set {p, q, r}?

- **Solution:**
- $P(3,2) = 6$

Example

- We have 10 letters and want to create a password that consist of 5 letters.
  How many arrangement are there?

- **Solution:**

Example

- Supposed there is a class of 20 students and elections are being made for class president and class vice president. How many different ways could the candidates be picked?

- **Solution:**

4) Combination

- A selection of objects **without regard to order** is called a **combination.**

- An ***r*-combination** of elements of a set is an **unordered selection of *r* elements** from the sets.

Theorem:

The number of *r*-combination of *n* distinct objects is

$$C(n, r) = \frac{n!}{r!(n-r)!} = \frac{P(n,r)}{r!}, \qquad r \leq n$$

4) Combination

- My fruit salad is a combination of apple, kiwi and grape.
- The **order is not important**.
- In the salad bowl,
  - **Apple , kiwi , grape** is the same as **kiwi, apple, grape**

Example

- In how many ways can we select a committee of three from a group of 10 people?

- **Solution:**

- Since a committee is an unordered group of people, then there are

$$C(10, 3) = 120 \text{ ways}$$

Example

- From among a group of six men and nine women, how many three-member committees contain only men OR only women?

- **Solution:**

5) Permutation with Repetition

Theorem:

Suppose that a sequence S of n items has

n_1 identical objects of type 1

n_2 identical objects of type 2

⋮

⋮

n_t identical objects of type t

Then, the number of orderings of S is

$$\frac{n!}{n_1!n_2!\ldots n_t!}$$

Example

- Each member of a 9-member committee must be assigned to exactly one of three subcommittees which are the executive subcommittee, the finance subcommittee or the rules committee. If these subcommittees are to contain 3, 4 and 2 members respectively, how many different subcommittee appointments can be made?

- **Solution:**

- Number of ways = $\frac{9!}{3!\cdot 4!\cdot 2!} = 1260$ ways

- OR other way to calculate these:

$$C(9,3) * C(6,4) * C(2,2) = 1260$$

Example

- Supposed there are 3 red marbles, 3 blue marbles and 4 yellow marbles. Place this marbles in a specific order. How many ways are there to arrange those marbles?

- **Solution:**

Example

- Find the number of different ways the letters of the word **b,a,n,a,n,a** can be arranged.

- **Solution:**

6) Combination with Repetition

Theorem:

- If X is a set containing t elements, the number of unordered, k-element selections from X, repetitions allowed, is

$$\boldsymbol{C(k+t-1, t-1) = C(k+t-1, k)}$$

- Note: use this when you need to select **the element** from **varieties or different choices of the elements** (quantity for each element is not stated).

Example

- How many different assortments of one dozen donuts can be purchased from a bakery that makes donuts with chocolate, vanilla, cinnamon, powdered sugar and glazed icing?

- **Solution:**
- t = 5 flavors (chocolate, vanilla, cinnamon, powdered sugar and glazed icing)
- k = 12 (one dozen donuts)

- Number of ways $= \ C(12 + 5 - 1, 12) \ = \ 1820$

Example

- How many ways can we select 15 cans of soda from a cooler containing large quantities of Coke, Pepsi, Diet Coke, Root Beer and Sprite?

- **Solution:**

Example

- In how many ways can 12 identical mathematics books be distributed among 4 students A, B, C and D.

- **Solution:**

Example

- A bakery makes six different varieties of donuts. Carl wants to buy ten donuts. How many different ways can he do it?

- **Solution:**

7) Inclusive-Exclusive Principle

- When two tasks can be done at the same time, do not use the sum rule to count the number of ways to do one of the tasks. This will result to an over count.
- Therefore, represent the counting principle in terms of sets

Theorem:

- For any finite sets A and B (not necessarily disjoint),

$$|A \cup B| = |A| + |B| - |A \cap B|$$

Example

- Among a group of children, **88** like pizza and **27** like Chinese food. If **13** of these children like pizza and Chinese food, how many liked pizza or Chinese food?

- **Solution:**

Example

- In a particular dormitory, there are 350 university students. Of these, 312 are taking an English course and 108 are taking a mathematics course. If 95 of these freshmen are taking course in both English and mathematics, how many are taking in neither English nor mathematics?

- **Solution:**

8) The Generalized Pigeonhole Principle

Concepts: If there are more pigeons than pigeonhole, then there must be at least one pigeonhole with at least two pigeons.

Pigeonhole Principle

- For any positive integer k, if k + 1 objects (pigeons) are placed in k boxes (pigeonholes), then at least one box contains two or more objects.

Generalized Pigeonhole Principle (GPP)

- If $N \geq 0$ objects are placed in $k \geq 1$ boxes, then at least one box contains at least $\left\lceil \frac{N}{k} \right\rceil$ objects.

Example

- There are 400 students in a programming class. Show that at least 2 of them were born on the same day of the month.

- **Solution:**
- 365 → days in a year & as the pigeonhole (k)
- 400 students → Pigeons (N)
-

Example

- In the mid term examination taken by 70 students, the scores range from 60 to 88. At least how many students must have the same score?

- **Solution:**

Example

- How many people should be chosen from a group, so that at least two of them have been born in the same day of a week？

- **Solution:**

-

Review Question

Sum & Product Rule

1) Supposed there are 435 members of the House of Representatives and 100 members of the Senate. How many selections do you have if:
   a. If you can choose to speak to one member only
   b. If you can choose to speak to one member of each body

2) To fulfill the requirements for a degree, a student must take one course each from the following groups: health, civics, critical thinking and elective. If there are 4 healths, 3 civics, 6 critical thinking and 10 elective courses available, how many different options for fulfilling the requirements does a student have?

3) How many different postcodes are available (if we assume all numbers can be used)

Permutations and Combinations

4) In a test, a student must select 6 out of 10 questions. In how many ways can this be done?

5) Supposed 10 horses run a race, how many different ways could a horse come in 1st place, 2nd place and 3rd place?

6) A club has 25 members. How many ways are there to choose 4 members of the club to serve on an executive committee?

7) Supposed a club of 17 members is to select a leader and an assistant leader. In how many ways can this be done?

Permutations and Combinations

8) Find the number of different ways the letter of the word M I S S I S S I P P I can be arranged.

9) How many ways can we fill a box holding 25 pieces of candy from 10 different types of candy?

Inclusive-Exclusive Principle

10) Among 100 students in MMU, 25 of them play football, 16 of them play bowling and 12 of them play hockey. If 9 of them play football and bowling, 3 play bowling and hockey, 6 play football and hockey and 1 play football, bowling and hockey. How many of these students who are not playing football, bowling or hockey?

11) Among 50 children, 14 of them like to read and 25 like to watch tv. If 12 of them like to read and watch tv, how many of them like to read or watch tv?

THANK YOU

End of Chapter 5