Simulation and Modelling

Spring 2023
CS4056

Muhammad Shahid Ashraf

Notes

Statistics

Notes

Uniform Distributions

- a random variable with any thinkable distribution can be generated from a Uniform random variable
- Uniform distribution is used in any situation when a value is picked "at random" from a given interval; that is, without any preference to lower, higher, or medium values.
- Whenever the probability is proportional to the length of the interval, the random variable is uniformly distributed

Notes

Uniform Distributions

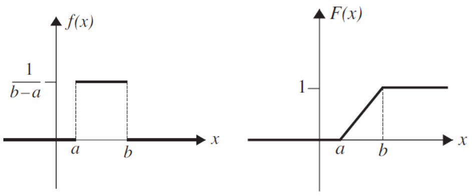
- Suppose that X is the value of the random point selected from an interval (a, b). Then X is called a uniform random variable over (a, b). Let F and f be distribution and probability density functions of X, respectively

$$F(t) = \begin{cases} 0 & t \leq a \\ \dfrac{t-a}{b-a} & a \leq t \leq b \\ 1 & t \geq b \end{cases}$$

$$f(t) = F'(t) = \begin{cases} \dfrac{1}{b-a} & a < t < b \\ 0 & \text{otherwise} \end{cases}$$

Notes

Uniform Distributions



Notes

Uniform Distributions

- In random selections of a large number of points from (a, b), we expect that the average of the values of the points will be approximately $\frac{a+b}{2}$, the midpoint of (a, b).

$$E(X) = \int_a^b x \, \frac{1}{b-a} dx$$

$$= \frac{1}{b-a}\left[\frac{1}{2}x^2\right]_a^b$$

$$= \frac{1}{b-a}\left(\frac{1}{2}b^2 - \frac{1}{2}a^2\right)$$

$$= \frac{(b-a)(b+a)}{2(b-a)}$$

$$= \frac{a+b}{2}.$$

Notes

Uniform Distributions

- To find $Var(X)$, we have

$$E(X^2) = \int_a^b x^2 \frac{1}{b-a} dx$$

$$= \frac{1}{3}\frac{b^3 - a^3}{b-a}$$

$$= \frac{1}{3}(a^2 + ab + b^2)$$

Hence

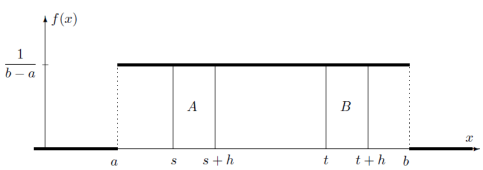
$$\mathrm{Var}(X) = E(X^2) - [E(X)]^2$$

$$= \frac{1}{3}(a^2 + ab + b^2) - \left(\frac{a+b}{2}\right)^2$$

$$= \frac{(b-a)^2}{12}$$

Notes

Uniform Distributions



(a, b)	$=$	range of values
$f(x)$	$=$	$\frac{1}{b-a}, \quad a < x < b$
$\mathbf{E}(X)$	$=$	$\frac{a+b}{2}$
$\mathrm{Var}(X)$	$=$	$\frac{(b-a)^2}{12}$

Notes

Uniform Distributions

- Starting at 5:00 A.M., every half hour there is a flight from Karachi International airport to Islamabad International airport. Suppose that none of these planes is completely sold out and that they always have room for passengers. A person who wants to fly to L.A. arrives at the airport at a random time between 8:45 A.M. and 9:45 A.M. Find the probability that he waits
  1. at most 10 minutes
  2. at least 15 minutes

Notes

Uniform Distributions

- Starting at 5:00 A.M., every half hour there is a flight from Karachi International airport to Islamabad International airport. Suppose that none of these planes is completely sold out and that they always have room for passengers. A person who wants to fly to L.A. arrives at the airport at a random time between 8:45 A.M. and 9:45 A.M. Find the probability that he waits
- at most 10 minutes
- The passenger arrive at the airport X minutes past 8:45.
- Then X is a uniform random variable over the interval $(0, 60)$.
- Hence the probability density function of X is given by f

$$f(x = \begin{cases} \frac{1}{60} & 0 < x < 60 \\ 0 & \text{otherwise} \end{cases}$$

Notes

Uniform Distributions

- Starting at 5:00 A.M., every half hour there is a flight from Karachi International airport to Islamabad International airport. Suppose that none of these planes is completely sold out and that they always have room for passengers. A person who wants to fly to L.A. arrives at the airport at a random time between 8:45 A.M. and 9:45 A.M. Find the probability that he waits
- at most 10 minutes
- Now the passenger waits at most 10 minutes if she arrives between 8:50 and 9:00 or 9:20 and 9:30; that is, if $5 < X < 15 \ or \ 35 < X < 45$

$$P(5 < X < 15) + P(35 < X < 45) = \int_5^{15} \frac{1}{60} + \int_{35}^{45} \frac{1}{60}$$

Notes

Uniform Distributions

- A person arrives at a bus station every day at 7:00 A.M. If a bus arrives at a random time between 7:00 A.M. and 7:30 A.M., what is the average time spent waiting
- It takes a professor a random time between 20 and 27 minutes to walk from his home to school every day. If he has a class at 9:00 A.M. and he leaves home at 8:37 A.M. , find the probability that he reaches his class on time.
- The time at which a bus arrives at a station is uniform over an interval (a, b) with mean 2:00 P.M. and standard deviation $\sqrt{12}$ minutes. Determine the values of a and b

Notes

Normal Distributions

- Normal distribution is often found to be a good model for physical variables like weight, height, temperature, voltage, pollution level, and for instance, household incomes or student grades.
- Normal distribution has a density

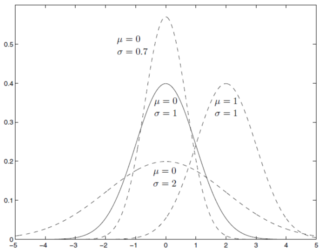
$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{\frac{-(x-\mu)^2}{2\sigma^2}\right\}$$

- where parameters μ and σ have a simple meaning of the expectation $E(X)$ and the standard deviation Std(X).
- This density is known as the bell-shaped curve, symmetric and centered at μ, its spread being controlled by σ.

Notes

Normal Distributions

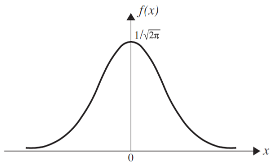
- A normal probability distribution, when plotted gives a bell shaped curve such that
  - The total area under the curve is 1
  - The curve is symmetric about mean
  - The two tails of the curve extend indefinitely



Notes

Normal Distributions

- Normal distribution with “standard parameters” $\mu = 0$ and $\sigma = 1$ is called Standard Normal distribution.



$$\begin{array}{lcl} Z & = & \text{Standard Normal random variable} \\ \phi(x) & = & \frac{1}{\sqrt{2\pi}} e^{-x^2/2}, \text{ Standard Normal pdf} \\ \Phi(x) & = & \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz, \text{ Standard Normal cdf} \end{array}$$

Notes

Normal Distributions

- Suppose that of all the clouds that are seeded with silver iodide, 58% show splendid growth. If 60 clouds are seeded with silver iodide, what is the probability that exactly 35 show splendid growth

Notes

Normal Distributions

- Suppose that of all the clouds that are seeded with silver iodide, 58% show splendid growth. If 60 clouds are seeded with silver iodide, what is the probability that exactly 35 show splendid growth?
- Let X be the number of cloud showed splendid growth
- $E(x) = np = 34.80$
- $\sigma_X = \sqrt{npq} = 3.82$
- $P(X = 35) \approx P(34.5 < X < 35.5)$

$$
\begin{aligned}
P(X = 35) \approx & P(34.5 < X < 35.5) \\
= & P\left(\frac{34.5 - 34.8}{3.82} < \frac{X - 34.8}{3.82} < \frac{35.5 - 34.8}{3.82}\right)
\end{aligned}
$$

Notes

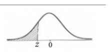
Normal Distributions

$$
\begin{aligned}
P(X = 35) \approx & P(34.5 < X < 35.5) \\
= & P\left(\frac{34.5 - 34.8}{3.82} < \frac{X - 34.8}{3.82} < \frac{35.5 - 34.8}{3.82}\right) \\
= & P\left(-0.08 < \frac{X - 34.8}{3.82} < 0.18\right) \\
= & \frac{1}{\sqrt{2\pi}} \int_{-0.08}^{0.18} exp(-\frac{x^2}{2})dx \\
= & \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0.18} exp(-\frac{x^2}{2})dx - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-0.08} exp(-\frac{x^2}{2})dx \\
= & \Phi(0.18) - \Phi(-.08)
\end{aligned}
$$

Notes

Table 1 Area under the Standard Normal Distribution to the Left of z: Negative z

$$\Phi(z) = P(Z \leq z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-x^2/2}\, dx$$

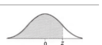


Note that for $z \leq -3.90, \Phi(z) = P(Z \leq z) \approx 0$

z	0	1	2	3	4	5	6	7	8	9
−3.8	.0001	0001	0001	0001	0001	0001	0001	0001	0001	0001
−3.7	.0001	0001	0001	0001	0001	0001	0001	0001	0001	0001
−3.6	.0002	0002	0001	0001	0001	0001	0001	0001	0001	0001
−3.5	0002	0002	0002	0002	0002	0002	0002	0002	0002	0002
−3.4	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0002
−3.3	.0005	.0005	.0005	.0004	.0004	.0004	.0004	.0004	.0004	.0003
−3.2	.0007	.0007	.0006	.0006	.0006	.0006	.0006	.0005	.0005	.0005
−3.1	.0010	.0009	.0009	.0009	.0008	.0008	.0008	.0008	.0007	.0007
−3.0	.0013	.0013	.0013	.0012	.0012	.0011	.0011	.0011	.0010	.0010
−2.9	.0019	.0018	.0018	.0017	.0016	.0016	.0015	.0015	.0014	.0014
−2.8	.0026	.0025	.0024	.0023	.0023	.0022	.0021	.0021	.0020	.0019
−2.7	.0035	.0034	.0033	.0032	.0031	.0030	.0029	.0028	.0027	.0026
−2.6	.0047	.0045	.0044	.0043	.0041	.0040	.0039	.0038	.0037	.0036
−2.5	.0062	.0060	.0059	.0057	.0055	.0054	.0052	.0051	.0049	.0048
−2.4	.0082	.0080	.0078	.0075	.0073	.0071	.0069	.0068	.0066	.0064
−2.3	.0107	.0104	.0102	.0099	.0096	.0094	.0091	.0089	.0087	.0084
−2.2	.0139	.0136	.0132	.0129	.0125	.0122	.0119	.0116	.0113	.0110
−2.1	.0179	.0174	.0170	.0166	.0162	.0158	.0154	.0150	.0146	.0143
−2.0	.0228	.0222	.0217	.0212	.0207	.0202	.0197	.0192	.0188	.0183
−1.9	.0287	.0281	.0274	.0268	.0262	.0256	.0250	.0244	.0239	.0233
−1.8	.0359	.0351	.0344	.0336	.0329	.0322	.0314	.0307	.0301	.0294
−1.7	.0446	.0436	.0427	.0418	.0409	.0401	.0392	.0384	.0375	.0367
−1.6	.0548	.0537	.0526	.0516	.0505	.0495	.0485	.0475	.0465	.0455
−1.5	.0668	.0655	.0643	.0630	.0618	.0606	.0594	.0582	.0571	.0559
−1.4	.0808	.0793	.0778	.0764	.0749	.0735	.0721	.0708	.0694	.0681
−1.3	.0968	.0951	.0934	.0918	.0901	.0885	.0869	.0853	.0838	.0823
−1.2	.1151	.1131	.1112	.1093	.1075	.1056	.1038	.1020	.1003	.0985
−1.1	.1357	.1335	.1314	.1292	.1271	.1251	.1230	.1210	.1190	.1170
−1.0	.1587	.1562	.1539	.1515	.1492	.1469	.1446	.1423	.1401	.1379
−0.9	.1841	.1814	.1788	.1762	.1736	.1711	.1685	.1660	.1635	.1611
−0.8	.2119	.2090	.2061	.2033	.2005	.1977	.1949	.1922	.1894	.1867
−0.7	.2420	.2389	.2358	.2327	.2296	.2266	.2236	.2206	.2177	.2148
−0.6	.2743	.2709	.2676	.2643	.2611	.2578	.2546	.2514	.2483	.2451
−0.5	.3085	.3050	.3015	.2981	.2946	.2912	.2877	.2843	.2810	.2776
−0.4	.3446	.3409	.3372	.3336	.3300	.3264	.3228	.3192	.3156	.3121
−0.3	.3821	.3783	.3745	.3707	.3669	.3632	.3594	.3557	.3520	.3483
−0.2	.4207	.4168	.4129	.4090	.4052	.4013	.3974	.3936	.3897	.3859
−0.1	.4602	.4562	.4522	.4483	.4443	.4404	.4364	.4325	.4286	.4247
−0.0	.5000	.4960	.4920	.4880	.4840	.4801	.4761	.4721	.4681	.4641

Table 2 Area under the Standard Normal Distribution to the Left of z: Positive z

$$\Phi(z) = P(Z \leq z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-x^2/2}\, dx$$



z	0	1	2	3	4	5	6	7	8	9
.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
.7	.7580	.7611	.7642	.7673	.7703	.7734	.7764	.7794	.7823	.7852
.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9889	.9889	.9889	.9990	.9990
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995
3.3	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9997
3.4	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9998
3.5	.9998	.9998	.9998	.9998	.9998	.9998	.9998	.9998	.9998	.9998
3.6	.9998	.9998	.9999	.9999	.9999	.9999	.9999	.9999	.9999	.9999
3.7	.9999	.9999	.9999	.9999	.9999	.9999	.9999	.9999	.9999	.9999
3.8	.9999	.9999	.9999	.9999	.9999	.9999	.9999	.9999	.9999	.9999

Note that for $z > 3.89$, $\Phi(z) = P(Z \leq z) \approx 1$.

Notes

Normal Distributions

- Suppose that the average household income in some country is 900 coins, and the standard deviation is 200 coins. Assuming the Normal distribution of incomes, compute the proportion of "the middle class," whose income is between 600 and 1200 coins.
- The government of the country decides to issue food stamps to the poorest 3% of households. Below what income will families receive food stamps?

Notes