Simulation and Modelling

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CS4056

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Notes

Random-Variate Generation

Notes

Overview

- Develop understanding of generating samples from a specified distribution as input to a simulation model.
- Illustrate some widely-used techniques for generating random variates:
  - Inverse-transform technique
  - Acceptance-rejection technique
  - Special properties

Notes

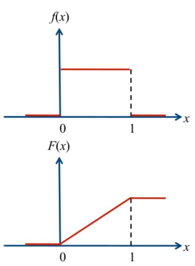
Preparation

- It is assumed that a source of uniform [0,1] random numbers exists. Linear Congruential Method (LCM)
- Random numbers $R, R_1, R_2, \ldots$ with
  - PDF

$$f_R(x) = \begin{cases} 1 & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

  - CDF

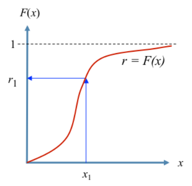
$$F_R(x) = \begin{cases} 0 & x < 0 \\ x & 0 \le x \le 1 \\ 1 & x > 1 \end{cases}$$

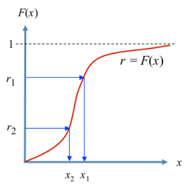


Notes

Inverse-transform Technique

- The concept:
  - For CDF function: $r = F(x)$
  - Generate r from uniform $(0,1)$, a.k.a $U(0,1)$
  - Find x, $x = F^{-1}(r)$





Notes

Inverse-transform Technique

- The inverse-transform technique can be used in principle for any distribution.
- Most useful when the CDF F(x) has an inverse $F^{-1}(x)$ which is easy to compute.
- Required steps
  - Compute the CDF of the desired random variable X.
  - Set F(X) = R on the range of X.
  - Solve the equation F(X) = R for X in terms of R.
  - Generate uniform random numbers $R_1, R_2, R_3, \ldots$ and compute the desired random variate by $X_i = F^{-1}(R_i)$

Notes

Inverse-transform Technique

- Exponential Distribution
- PDF

$$f(x) = \lambda exp(-\lambda x)$$

- CDF

$$F(x) = 1 - exp(-\lambda x)$$

- Simplification

$$X = -\frac{\ln R}{\lambda}$$

- Since R and (1-R) are uniformly distributed on $[0, 1]$

- To generate $X_1, X_2, X_3 \ldots$

$$1 - exp(-\lambda X) = R$$

$$exp(-\lambda X) = 1 - R$$

$$-\lambda X = \ln(1 - R)$$

$$X = -\frac{1}{\lambda}\ln(1 - R)$$

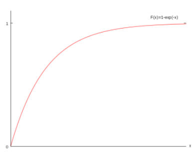
$$X = F^{-1}(R)$$

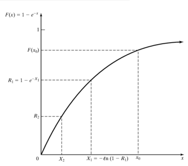
Notes

Inverse-transform Technique

Generation of Exponential Variates X_i with Mean 1, given Random Numbers R_i

i	1	2	3	4	5
R_i	0.1306	0.0422	0.6597	0.7965	0.7696
X_i	0.1400	0.0431	1.078	1.592	1.468



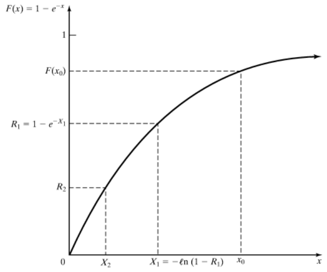


Notes

Inverse-transform Technique

Check: Does the random variable X_1 have the desired distribution?

$$P(X_1 \le x_0) = P(R_1 \le F(x_0)) = F(x_0)$$



Notes

Inverse-transform Technique

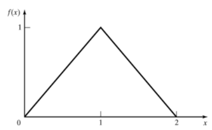
- Examples of other distributions for which inverse CDF works are:
  - Uniform distribution
  - Weibull distribution
  - Triangular distribution
- Random variable X uniformly distributed over $[a, b]$

$$\begin{aligned} F(X) &= R \\ \frac{X-a}{b-a} &= R \\ X - a &= R(b-a) \\ X &= R(b-a) + a \end{aligned}$$

Notes

Inverse-transform Technique

The CDF of a Triangular Distribution with endpoints (0, 2) is given by



$$F(x) = \begin{cases} 0 & x \le 0 \\ \frac{x^2}{2} & 0 < x \le 1 \\ 1 - \frac{(2-x)^2}{2} & 1 < x \le 2 \\ 1 & x > 2 \end{cases} \qquad R(X) = \begin{cases} \frac{X^2}{2} & 0 \le X \le 1 \\ 1 - \frac{(2-X)^2}{2} & 1 \le X \le 2 \end{cases}$$

X is generated by

$$X = \begin{cases} \sqrt{2R} & 0 \le R \le \frac{1}{2} \\ 2 - \sqrt{2(1-R)} & \frac{1}{2} < R \le 1 \end{cases}$$

Notes

Inverse-transform Technique

The variate is

- The Weibull Distribution is described by
  - PDF

$$f(x) = \frac{\beta}{\alpha^{\beta}} x^{\beta-1} e^{-\left(\frac{x}{\alpha}\right)^{\beta}}$$

  - CDF

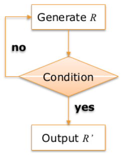
$$F(X) = 1 - e^{-\left(\frac{x}{\alpha}\right)^{\beta}}$$

Notes

Acceptance-Rejection Technique

- Useful particularly when inverse CDF does not exist in closed form
  - Thinning
- Illustration: To generate random variates, $X \sim U(1/4,1)$

Procedure:
Step 1. Generate $R \sim U(0,1)$
Step 2. If $R \geq 1/4$, accept X=R.
Step 3. If $R < 1/4$, reject R, return to Step 1



- R does not have the desired distribution, but R conditioned (R') on the event $\{R \geq 1/4\}$ does.
- Efficiency: Depends heavily on the ability to minimize the number of rejections.

Notes

Acceptance-Rejection Technique

- Probability mass function of a Poisson Distribution

$$P(N = n) = \frac{\alpha^n}{n!}e^{-\alpha}$$

- Exactly n arrivals during one time unit

$$A_1 + A_2 + \cdots + A_n \leq 1 < A_1 + A_2 + \cdots + A_n + A_{n+1}$$

- Since interarrival times are exponentially distributed we can set

$$A_i = \frac{-\ln(R_i)}{\alpha}$$

  - Well known, we derived this generator in the beginning of the class
- Procedure of generating a Poisson random variate N is as follows
  1. Set n=0, P=1
  2. Generate a random number R_{n+1}, and replace P by P x R_{n+1}
  3. If $P < \exp(-\alpha)$, then accept N=n
     - Otherwise, reject the current n, increase n by one, and return to step 2.

Notes

Acceptance-Rejection Technique

- Example: Generate three Poisson variates with mean α=0.2
  - $\exp(-0.2) = 0.8187$
- Variate 1
  - Step 1: Set $n = 0$, $P = 1$
  - Step 2: $R1 = 0.4357$, $P = 1$ x 0.4357
  - Step 3: Since $P = 0.4357 < \exp(-0.2)$, accept $N = 0$
- Variate 2
  - Step 1: Set $n = 0$, $P = 1$
  - Step 2: $R1 = 0.4146$, $P = 1$ x 0.4146
  - Step 3: Since $P = 0.4146 < \exp(-0.2)$, accept $N = 0$
- Variate 3
  - Step 1: Set $n = 0$, $P = 1$
  - Step 2: $R1 = 0.8353$, $P = 1$ x 0.8353
  - Step 3: Since $P = 0.8353 > \exp(-0.2)$, reject $n = 0$ and return to Step 2 with $n = 1$
  - Step 2: $R2 = 0.9952$, $P = 0.8353$ x $0.9952 = 0.8313$
  - Step 3: Since $P = 0.8313 > \exp(-0.2)$, reject $n = 1$ and return to Step 2 with $n = 2$
  - Step 2: $R3 = 0.8004$, $P = 0.8313$ x $0.8004 = 0.6654$
  - Step 3: Since $P = 0.6654 < \exp(-0.2)$, accept $N = 2$

Notes

Acceptance-Rejection Technique

- It took five random numbers to generate three Poisson variates
- In long run, the generation of Poisson variates requires some overhead!

N	R_{n+1}	P	Accept/Reject		Result
0	0.4357	0.4357	P < exp(- α)	Accept	N=0
0	0.4146	0.4146	P < exp(- α)	Accept	N=0
0	0.8353	0.8353	P ≥ exp(- α)	Reject	
1	0.9952	0.8313	P ≥ exp(- α)	Reject	
2	0.8004	0.6654	P < exp(- α)	Accept	N=2

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