

Successive Convexification of Non-Convex Optimal Control Problems and Its Convergence Properties

Yuanqi Mao, Michael Szmuk, and Behçet Açıkmeşe

Abstract—This paper presents an algorithm to solve non-convex optimal control problems, where non-convexity can arise from nonlinear dynamics, and non-convex state and control constraints. This paper assumes that the state and control constraints are already convex or convexified, the proposed algorithm convexifies the nonlinear dynamics, via a linearization, in a successive manner. Thus at each succession, a convex optimal control subproblem is solved. Since the dynamics are linearized and other constraints are convex, after a discretization, the subproblem can be expressed as a finite dimensional convex programming subproblem. Since convex optimization problems can be solved very efficiently, especially with custom solvers, this subproblem can be solved in time-critical applications, such as real-time path planning for autonomous vehicles. Several safe-guarding techniques are incorporated into the algorithm, namely *virtual control* and *trust regions*, which add another layer of algorithmic robustness. A convergence analysis is presented in continuous-time setting. By doing so, our convergence results will be independent from any numerical schemes used for discretization. Numerical simulations are performed for an illustrative trajectory optimization example.

I. INTRODUCTION

In this paper, we present a Successive Convexification (SCvx) algorithm to solve non-convex optimal control problems. The problems involve continuous-time nonlinear dynamics, and possibly non-convex state and control constraints. A large number of real-world problems fall into this category. An example in aerospace applications is the planetary landing problem [1], [2], [3]. In [1], [2], non-convexity arises from the minimum thrust constraints, while in [3] nonlinear, time-varying gravity fields and aerodynamic forces consist of additional sources of non-convexity. State constraints can render the problem non-convex as well. One example is the optimal path planning of autonomous vehicles in the presence of obstacles [4]. Another instance can be found in the highly constrained spacecraft rendezvous and proximity operations [5].

Such optimal control problems have been solved by a variety of approaches [6], [7]. Most methods first discretize the problem [8] and then use general nonlinear programming solvers to obtain a solution. Unfortunately, general nonlinear optimization have some challenges. First, there is few known bounds on the computational effort needed. Secondly, they can become intractable in the sense that a bad initial guess could result in divergence of the numerical algorithm. These two major drawbacks make it unsuitable

for automated solutions and real-time applications, where computation speed and guaranteed convergence are exactly the two main concerns. On the other hand, convex programming problems can be reliably solved in polynomial time to global optimality [9]. More importantly, recent advances have shown that these problems can be solved in real-time by both generic Second Order Cone Programming (SOCP) solvers [10], and by customized solvers which take advantage of specific problem structures [11], [12]. This motivates researchers to formulate optimal control problems in a convex programming framework for real-time purposes, e.g. real-time Model Predictive Control (MPC) [13], [14], [15].

Although convex programming already suits itself well in solving some optimal control problems with linear dynamics and convex constraints, most real-world problems are not such inherently convex, and thus not readily to be solved. For certain non-convex control constraints, however, recent results have proven that they can be posed as convex ones without loss of generality via a procedure known as *lossless convexification* [1], [2], [3], [16], [17]. Certain non-convex state constraints can also be convexified using a successive method as suggested in [5], provided that they are concave.

This paper aims to tackle a specific source of non-convexity: the nonlinear dynamics, which will later be extended (as immediate future work) to non-convexities in state and control constraints that cannot be handled via the methods mentioned earlier. The basic idea is to successively linearize the dynamic equations, and solve a sequence of convex subproblems, specifically SOCPs, as we iterate. While similar ideas in both finite dimensional optimization problems [18], [19] and optimal control problems [20], [21], [22] have long been tried, few convergence results were reported, and they may suffer from poor performance. Also, different heuristics have been utilized along the way.

Our SCvx algorithm, on the other hand, presents a systematic way. More importantly, through a continuous-time convergence analysis, we guarantee that the SCvx algorithm will converge, and the solution it converges to will recover optimality for the original problem. To facilitate convergence, *virtual control* and *trust regions* are incorporated into our algorithm. The former acts like an exact penalty function [19], [22], [23], but with additional controllability features. The latter is similar to a standard trust-region-type updating rules [24], but the distinction lies in that we solve each convex subproblem to full

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optimality to take advantage of powerful SOCP solvers. To the best of our knowledge, the main contributions of this work are: 1) A novel Successive Convexification (SCvx) algorithm to solve non-convex optimal control problems; 2) A convergence proof in continuous-time.

II. SUCCESSIVE CONVEXIFICATION

A. Problem Formulation

The following nonlinear system dynamics are assumed,

$$\dot{x}(t) = f(x(t), u(t), t) \quad a.e. \quad 0 \leq t \leq T, \quad (1)$$

where $x : [0, T] \rightarrow \mathbb{R}^n$, is the state trajectory, $u : [0, T] \rightarrow \mathbb{R}^m$, is the control input, and $f : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R} \rightarrow \mathbb{R}^n$, is the control-state mapping, which is Fréchet differentiable with respect to all arguments. The state is initialized as x_0 at time $t = 0$, i.e. $x(0) = x_0$, and the time horizon, T , is finite. Note that, though T is fixed here, free final time problems can be handled under this framework with no additional difficulties. The control input u is assumed to be Lebesgue integrable on $[0, T]$, e.g., u can be measurable and essentially bounded (i.e. bounded almost everywhere) on $[0, T]$: $u \in L_\infty[0, T]^m$, with the ∞ -norm defined as

$$\|u\|_\infty := \text{ess sup}_{t \in [0, T]} \|u(t)\|,$$

where $\|\cdot\|$ is the Euclidean vector norm on \mathbb{R}^m , and ess sup means the essential supremum. As a result of the differentiability of function f , x is continuous on $[0, T]$, and $x \in W_{1,\infty}[0, T]^n$, i.e. the space of absolute continuous functions on $[0, T]$ with measurable and essentially bounded (first order) time derivatives. The 1-norm of this space is defined by

$$\|x\|_{1,\infty} := \max\{\|x\|_\infty, \|\dot{x}\|_\infty\}.$$

It can be shown that, equipped with these two norms, both $L_\infty[0, T]^m$ and $W_{1,\infty}[0, T]^n$ are Banach spaces.

Besides the dynamics, most real-world problems include control and state constraints, that must hold for all $t \in [0, T]$. For simplicity, we assume these constraints are time invariant, i.e., $u(t) \in U$ and $x(t) \in X$. Here U and X are non-convex sets in general, but as we mentioned in I, there are several methods proposed to transform these constraints into convex ones. Further methods of their convexification will be the subject of a future paper. In following, therefore, we assume that they have already been convexified. Another element is an objective functional (cost), which is also assumed to be convex. Note that any non-convexity in the cost can be transferred into constraints and convexified afterwards. Hence the following Non-Convex Optimal Control Problem (NCOCP) is considered:

Problem 1 (NCOCP). Determine a control function $u^* \in L_\infty[0, T]^m$, and a state trajectory $x^* \in W_{1,\infty}[0, T]^n$, which minimize the functional

$$C(x, u) := \phi(x(T), T) + \int_0^T L(x(t), u(t), t) dt, \quad (2a)$$

subject to the constraints:

$$\dot{x}(t) = f(x(t), u(t), t) \quad a.e. \quad 0 \leq t \leq T, \quad (2b)$$

$$u(t) \in U \quad a.e. \quad 0 \leq t \leq T, \quad (2c)$$

$$x(t) \in X \quad 0 \leq t \leq T, \quad (2d)$$

where $\phi : \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}$, is the terminal cost, $L : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R} \rightarrow \mathbb{R}$, is the running cost, and both are convex and Fréchet differentiable; $U \subset \mathbb{R}^m$, and $X \subset \mathbb{R}^n$. Both are convex and compact sets with nonempty interior.

B. Algorithm Description

The only non-convexity remaining in Problem 1 lies in the nonlinear dynamics (2b). Since it is an equality constraint, a natural way to convexify it is linearization by using its first order Taylor approximation. The solution to the convexified problem, however, won't necessarily be the same as its non-convex counterpart. To recover optimality, we need to come up with an algorithm that can find a solution which satisfies at least first order optimality condition of the original problem. A natural thought would be doing this linearization successively, i.e. at k^{th} succession, we linearize the dynamics about the trajectory and the corresponding control computed in the $(k-1)^{\text{th}}$ succession. This procedure is repeated until convergence. This process essentially forms the basic idea behind our Successive Convexification (SCvx) algorithm.

1) *Linearization:* Assume the $(i-1)^{\text{th}}$ succession gives us a solution $(x^{i-1}(t), u^{i-1}(t))$. Let

$$A(t) = \frac{\partial}{\partial x} f(x^{i-1}(t), u^{i-1}(t), t),$$

$$B(t) = \frac{\partial}{\partial u} f(x^{i-1}(t), u^{i-1}(t), t),$$

$$D(t) = f(x^{i-1}(t), u^{i-1}(t), t),$$

$d(t) = x(t) - x^{i-1}(t)$, and $w(t) = u(t) - u^{i-1}(t)$, then the first order Taylor expansion about that solution will be

$$\dot{d}(t) = A(t)d(t) + B(t)w(t) + D(t) + H.O.T.. \quad (3)$$

This is a linear system with respect to $d(t)$ and $w(t)$, which are our new states and control respectively. The linearization procedure gets us the benefit of convexity, but it also introduces two new issues, namely *artificial infeasibility* and *approximation error*. We will address them in the following two subsections.

2) *Virtual Control:* At various points in the solution space, the above method can generate an infeasible problem, even if the original nonlinear problem itself is feasible. That is the *artificial infeasibility* introduced by linearization. In such scenarios, the undesirable infeasibility obstructs the iteration process and prevents convergence. To prevent this *artificial infeasibility*, we introduce an additional control input $v(t)$, called *virtual control*, to the linear dynamics (3) (without the higher order terms):

$$\dot{d}(t) = A(t)d(t) + B(t)w(t) + E(t)v(t) + D(t), \quad (4)$$

where $E(t)$ can be chosen based on $A(t)$ such that the pair $A(\cdot), E(\cdot)$ is controllable. Then, since $v(t)$ is unconstrained,

any state in the feasible region can be reachable in finite time. This is why this *virtual control* can eliminate the *artificial infeasibility*. For example, on autonomous vehicles, the *virtual control* can be understood as a synthetic acceleration that acts on the vehicle, which can drive the vehicle virtually anywhere in the feasible area.

Since we want to resort to this *virtual control* as needed, it will be heavily penalized via an additional term $\lambda \gamma(Ev)$ in the cost, where λ is the penalty weight, and $\gamma(\cdot)$ is the penalty function, defined by

$$\gamma(\cdot) := \text{ess sup}_{t \in [0, T]} \|\cdot(t)\|_1,$$

where $\|\cdot\|_1$ is the L_1 norm on \mathbb{R}^n . For example, $\|x(t)\|_1 = \sum_{i=1}^n |x(t)|$. Thus we have

$$\gamma(Ev) := \text{ess sup}_{t \in [0, T]} \|E(t)v(t)\|_1.$$

Now the penalized cost after linearizing will be defined as

$$L(d, w) := C(x, u) + \lambda \gamma(Ev), \quad (5)$$

while the penalized cost before linearizing can be formulated in a similar fashion:

$$J(x, u) := C(x, u) + \lambda \gamma(\dot{x} - f). \quad (6)$$

3) *Trust Regions*: Another concern when linearizing is potentially rendering the problem unbounded. A simple example will be linearizing the cost $y_1(x) = 0.5x^2$ at $x = 1$ to get $y_2(x) = x - 0.5$. Now if going left is a feasible direction, then the linearized problem could potentially be unbounded while the nonlinear problem will definitely find its minimum at $x = 0$. The reason behind is: when large deviation is allowed and occurred, the linear approximation sometimes fails to capture the distinction made by nonlinearity, for instance $y_1(x)$ attains its stationary point at $x = 0$, while $y_2(x)$ certainly does not.

To mitigate this risk, we ensure that the linearized trajectory does not deviate significantly from the nominal one obtained in the previous succession, via a *trust region* on our new control input,

$$\|w\|_\infty \leq \Delta, \quad (7)$$

and thus our new state will be restricted as well due to the dynamic equations. The rationale is that we only trust the linear approximation in the trust region. Fig. 1 shows the typical convergence process of this trust-region type algorithm in solving a 2-D problem. The algorithm can start from virtually anywhere, and manages to converge to a feasible point. Note that the figure also demonstrates *virtual control*, as the trajectory deviates from the constraint in the first few successions.

4) *The SCvx Algorithm*: The final problem formulation and the SCvx algorithm can now be presented. Considering the *virtual control* and the *trust regions*, a Convex Optimal Control Problem (COCp) is solved at k^{th} succession:

Problem 2 (COCp). Determine $w^* \in L_\infty[0, T]^m$, and $d^* \in W_{1,\infty}[0, T]^n$, which minimize the functional $L(d, w)$ defined

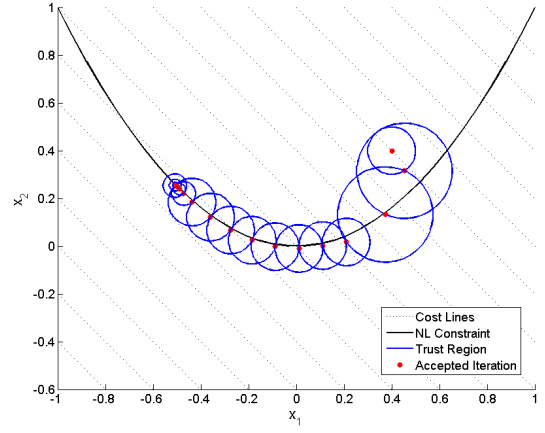


Fig. 1. Typical convergence process with *virtual control* and *trust regions*.

in (5), subject to constraints (4), (7), as well as $u^k(t) + w(t) \in U, x^k(t) + d(t) \in X$.

With this convex subproblem, SCvx algorithm is given as in Algorithm 1. This algorithm is of trust region type, and follows standard trust region radius update rules with some modifications.

One important distinction lies in the subproblem to be solved at each succession. Conventional trust region algorithms usually perform a line search along the Cauchy arc to achieve a "sufficient" reduction [24]. In our algorithms, however, a full convex optimization problem is solved to speed up the process, i.e. we solve the subproblem to its full optimality. As a result, the number of successions can be significantly less, by achieving more cost reduction at each succession. Admittedly, computational effort may slightly increase at each succession, but thanks to the algorithm customization techniques [12], we are able to solve each convex subproblem fast enough to outweigh the negative impact of solving it to full optimality.

In Step 2, the ratio r^k is used as a metric for the quality of linear approximations. A desirable scenario is when ΔJ^k agrees with ΔL^k , i.e. r^k is close to 1. Hence if r^k is above ρ_2 , which means our linear approximation predicts the cost reduction well, then we may choose to enlarge the trust region in Step 3, i.e., we put more faith in our approximation. Otherwise, we may keep the trust region unchanged, or contract its radius if needed. The most unwanted situation is when r^k is negative, or close to zero. The current step will be rejected in this case, and one has to contract the trust region and re-optimize at (x^k, u^k) .

III. CONVERGENCE ANALYSIS

In this section, we present a convergence analysis in the continuous-time setting. Hence we re-formulate the continuous-time optimal control problem as an infinite dimensional optimization problem in Banach space to simplify notations. To do this, we treat both the states and the control as independent variables, so that in Problem 1 and Problem 2, (2b) and (4) are considered as equality

Algorithm 1 Successive Convexification (SCvx)

Input Select initial state x^1 and control u^1 s.t. $x^1 \in X$ and $u^1 \in U$. Initialize trust region radius with positive Δ^1 and lower bound Δ_l . Select positive penalty weight λ , and parameters $0 < \rho_0 < \rho_1 < \rho_2 < 1$ and $\alpha > 1$.

Step 1 At each succession k , solve Problem 2 at (x^k, u^k, Δ^k) to get an optimal solution d^k .

Step 2 Compute the "actual" change in the penalized cost (6):

$$\Delta J(x^k, u^k) := \Delta J^k = J(x^k, u^k) - J(x^k + d^k, u^k + w^k), \quad (8)$$

and the "predicted" change by linear approximation:

$$\Delta L(d^k, w^k) := \Delta L^k = J(x^k, u^k) - L(d^k, w^k). \quad (9)$$

if $\Delta L^k = 0$ **then**

Stop, and **return** (x^k, u^k) ;

else

Compute the ratio $r^k = \Delta J^k / \Delta L^k$.

end if

Step 3

if $r^k < \rho_0$ **then**

Reject this step, contract the trust region radius, i.e. $\Delta_k \leftarrow \Delta_k / \alpha$ and go back to Step 1;

else

Accept this step, i.e. $x^{k+1} \leftarrow x^k + d^k$, $u^{k+1} \leftarrow u^k + w^k$, and update Δ^{k+1} by

$$\Delta^{k+1} = \begin{cases} \Delta^k / \alpha, & \text{if } r^k < \rho_1; \\ \Delta^k, & \text{if } \rho_1 \leq r^k < \rho_2; \\ \alpha \Delta^k, & \text{if } \rho_2 \leq r^k. \end{cases}$$

end if

$\Delta^{k+1} \leftarrow \max\{\Delta^{k+1}, \Delta_l\}$, $k \leftarrow k + 1$, and go to Step 1.

constraints. For simplicity, we denote our variable pair (x, u) as x in the sequel. Thus the space X will be re-defined as $X := W_{1,\infty}[0, T]^n \times L_\infty[0, T]^m$. Following this definition, the original dynamics will be represented by a set of algebraic equations:

$$g_i(x) = 0, \quad i = 1, 2, \dots, n, \quad (10)$$

and the original state and control constraints will be represented by a set of inequalities:

$$h_j(x) \leq 0, \quad j = 1, 2, \dots, n + m. \quad (11)$$

Denote the set of x satisfying (11) as $F = \{x \mid h_j(x) \leq 0, \quad j = 1, 2, \dots, n + m\}$. For objective functional, the original one (2a) becomes

$$\Psi(x) := \varphi(x(T), T) + \int_0^T L(x(t), t) dt. \quad (12)$$

Note that $\Psi(\cdot)$ is Fréchet differentiable, since $\varphi(\cdot)$ and $L(\cdot)$ are both differentiable. Now Problem 1 is ready to be reformulated as:

Problem 3. Determine $x \in X$, which minimizes $\Psi(x)$ defined in (12), subject to constraints (10) and (11).

If we rewrite (10) in a more concise form, $g(x) = 0$, then the penalized cost defined in (6) will be

$$J(x, \lambda) := \Psi(x) + \lambda \gamma(g(x)). \quad (13)$$

Then the associated penalty problem can be established in a similar manner:

Problem 4. Determine $x \in X$, which minimizes $J(x, \lambda)$ defined in (13), subject to constraints (11).

A. *Exactness of Penalty Function* $\gamma(g(x))$

We say a penalty function is *exact*, if there exists a finite λ such that problems before and after penalizing are equivalent in the sense of optimality conditions. A geometric justification of exactness for $\gamma(g(x))$ with different norms can be found in [22]. Alternatively, we can also show it via analysis. The detailed analysis is given in the full version paper we put onto arXiv, [25]. Here we only present the final results. Works we referred to in the following are [19], [21], [26], [27], [28].

Lemma 1. If \bar{x} is a local optimum of Problem 4, then there exist multipliers $\sigma_i \geq 0$ such that

$$0 \in \partial J(\bar{x}) + \left\{ y \mid y = \sum_{i \in I(\bar{x})} \sigma_i \nabla h_i(\bar{x}) \right\} \triangleq D(\bar{x}),$$

where the "+" sign stands for the Minkowski sum.

Define the set of constrained stationary points of Problem 4 as

$$S := \{x \mid x \in F \text{ and } 0 \in D(x)\}.$$

Lemma 1 states that if \bar{x} solves Problem 4, then $\bar{x} \in S$. Next theorem gives the main result in this subsection, that is, the exactness of the penalty problem.

Theorem 1. If \bar{x} is a stationary point of Problem 3 with multipliers $\bar{\mu}_i$, and $\bar{\sigma}_i$, and if penalty weight λ satisfies

$$\lambda \geq |\bar{\mu}_i|, \quad \forall i = 1, 2, \dots, n,$$

then \bar{x} is a constrained stationary point of Problem 4, that is, $\bar{x} \in S$. Conversely, if $\bar{x} \in S$ is feasible for Problem 3, then it is a stationary point of Problem 3.

Although Theorem 1 does not suggest a constructive way to find such a λ , it still has important theoretical values. In our current implementations, we select a "sufficiently" large λ and keep it fixed for the whole process. Numerical results show that this works well for most applications.

B. *Convergence Analysis*

Two major convergence results will be stated in this subsection. The first one deals with the case of finite convergence, while the second will handle the situation where an infinite sequence $\{x^k\}$ is generated by the SCvx algorithm. For the finite case, we have

Theorem 2. The "predicted" changes ΔL^k defined in (9) are nonnegative for any k . Furthermore, $\Delta L^k = 0$ implies that x^k is a stationary point of Problem 4, i.e. $x^k \in S$.

Now the real challenge is when $\{x^k\}$ is an infinite sequence. In this case, the non-differentiable penalty function $\gamma(\cdot)$ poses difficulties to our analysis. To facilitate further results, we first note that for any x ,

$$J(x+d) = L(d) + o(\|d\|)$$

and $o(\|d\|)$ is independent of x . This can be verified by simply writing out the Taylor expansion of $J(x)$ and using the fact that $\gamma(\cdot)$ is a linear operator.

Next lemma is the major preliminary result, and its proof also provides some geometric insights of the SCvx algorithm we described. See [25] for the full proof.

Lemma 2. *Let $\bar{x} \in F$ but $\bar{x} \notin S$, i.e., \bar{x} is feasible but not a stationary point of the penalty problem, and any scalar $c \leq 1$. Then there exist positive $\bar{\Delta}$ and $\bar{\varepsilon}$ such that for all $x \in N(\bar{x}, \bar{\varepsilon}) \cap F$ and $0 < \Delta \leq \bar{\Delta}$, any optimal solution \bar{d} of the convex subproblem, Problem 2 solved at x with trust region radius Δ satisfies*

$$r(x, \Delta) = \frac{J(x) - J(x + \bar{d})}{J(x) - L(\bar{d})} \geq c,$$

where $N(\bar{x}, \bar{\varepsilon})$ represents an open neighborhood of \bar{x} with radius $\bar{\varepsilon}$.

Remark. A frustrating situation happens when the algorithm keeps rejecting our trial steps. Fortunately, Lemma 2 provides some assurance that the SCvx algorithm won't reject the trial steps forever. At some point, usually after reducing Δ^k several times, the ratio metric r^k will be greater than any prescribed c , hence $r^k \geq \rho_0$ is guaranteed.

We can now present our main result in this section. The proof is by contradiction. See [25] for the full proof.

Theorem 3. *If the SCvx algorithm generates an infinite sequence $\{x^k\}$, then $\{x^k\}$ has limit points, and any limit point \bar{x} is a constrained stationary point of Problem 4, i.e. $\bar{x} \in S$.*

IV. NUMERICAL SIMULATIONS

This section presents a numerical example to demonstrate the proposed algorithm's capabilities. The example consists of a mass subject to "double-integrator" dynamics with nonlinear aerodynamic drag. The mass is controlled using a thrust input, \mathbf{T} , that is limited to a maximum magnitude of T_{max} . The problem formulation is summarized below, and its parameters are given in Table I.

$$\begin{aligned} & \text{minimize} \int_0^{t_f} \Gamma(\tau) d\tau \quad \text{s.t.} \\ & \mathbf{x}(0) = \mathbf{x}_i \quad \mathbf{v}(0) = \mathbf{v}_i \\ & \mathbf{x}(t_f) = \mathbf{x}_f \quad \mathbf{v}(t_f) = \mathbf{v}_f \\ & \dot{\mathbf{x}}(t) = \mathbf{v}(t) \quad \dot{\mathbf{v}}(t) = \mathbf{a}(t) \\ & \mathbf{a}(t) = \frac{1}{m} (\mathbf{T}(t) - k_d \|\mathbf{v}(t)\| \mathbf{v}(t)) \\ & \|\mathbf{T}(t)\| \leq \Gamma(t) \leq T_{max}. \end{aligned}$$

Here, we linearize the first succession about a constant velocity straight-line trajectory from the initial to the final

TABLE I
SIMULATION PARAMETERS

Parameter	Value
t_f	10
m	1.0
k_d	0.25
T_{max}	2.0
\mathbf{x}_i	$[0 \ 0]^T$
\mathbf{x}_f	$[10 \ 10]^T$
\mathbf{v}_i	$[5 \ 0]^T$
\mathbf{v}_f	$[5 \ 0]^T$
Δ_i	0.0
ρ_0	0.0
ρ_1	0.25
ρ_2	0.9
α	2.0

position. The convergence history is presented in Fig. 2, where the first 10 accepted successions are shown, after which a desirable convergence is achieved. Fig. 3 shows

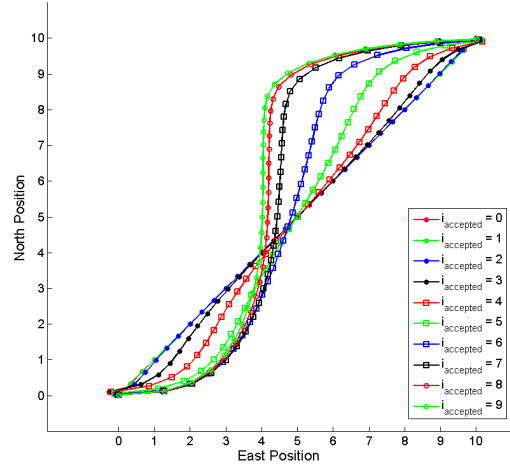


Fig. 2. Convergence history of two-dimensional trajectory. The dots along the trajectory indicate time-discretization points.

the converged thrust magnitude profile. The plot also shows a no-drag thrust magnitude profile for the same scenario. Note that the no-drag case renders the problem convex, thus recovering the classical "bang-bang" solution. In contrast, the case with aerodynamic drag takes advantage of drag at the beginning by decreasing its thrust magnitude, and must sustain a non-zero thrust in the middle of the trajectory to counter the energy loss induced by drag. Note that the area under the thrust curve in the case with drag is less than that of the case without drag. Thus, we see evidence of a situation where the optimal cost of the problem is reduced due to the inclusion of nonlinear dynamics.

Fig. 4 show the process the algorithm undergoes in obtaining the solution. The top plot shows how r^k varies as a function of iteration number, and how the trial step is rejected when r^k drops below ρ_0 . Note that the trust region radius increases whenever r^k is greater than ρ_2 , stays the same when $\rho_1 \leq r^k \leq \rho_2$ or $r^k \leq \rho_0$, and decreases when $\rho_0 \leq r^k \leq \rho_1$. More importantly, we can see how the process of rejecting the step results in decreasing the trust region radius, Δ^k , thus increasing r^k back above ρ_0 .

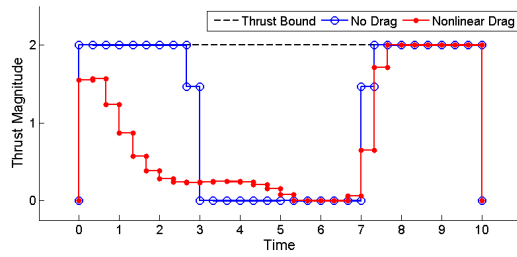


Fig. 3. Convergence history of solution and thrust profile: The thrust magnitude profile of the converged solution (red) and of the corresponding no-drag problem (blue). The markers indicate time-discretization points.

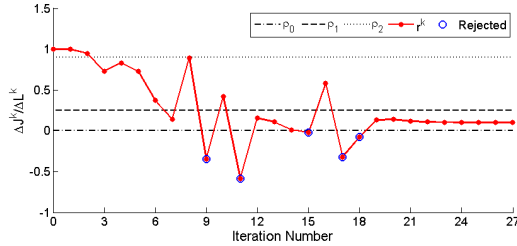


Fig. 4. Algorithm behavior during the solution process. The plot shows the time history of r^k relative to the thresholds p_0 , p_1 , and p_2 , and indicates the rejected steps.

V. CONCLUDING REMARKS

In this paper, we have proposed the SCvx algorithms. It successively convexifies nonlinear dynamics, and enables us to solve non-convex optimal control problems in real-time by actually solving a sequence of convex subproblem. We have given a relatively thorough description and analysis of the proposed algorithm, with the aid of a simple, yet illustrative numerical example. Convergence has been demonstrated in theoretical proofs, as well as illustrative figures and plots. The main result is: for the infinite convergent case, we have proved that every limit point will satisfied the optimality condition for the original non-convex optimal control problems.

A key next step is to incorporate this algorithm with other convexification techniques to construct a comprehensive algorithmic framework to deal with a large class of non-convexities.

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