5.11

① Suppose $V$ is a monoidal category: consisting of a category $V$, a bifunctor $-\otimes-: V\times V\to V$ called the monoidal product, and a unit object $*\in V$ with natural isomorphisms

$$u\otimes(v\otimes w) \underset{\alpha}{\cong} (u\otimes v)\otimes w \qquad\qquad *\otimes v \underset{\lambda}{\cong} v \underset{\rho}{\cong} v\otimes *$$

satisfying certain coherence properties.

i.e. $V$ is a category with a bifunctor $\otimes: V\times V\to V$ that is associative up to coherent natural isomorphism (i.e. there is a well-defined $n$-ary functor $V^{\times n} \xrightarrow{\otimes^n} V$ built from $\otimes$) and a unit object $*\in V$ satisfying the natural isomorphisms $\lambda$, $\rho$.

Suppose $V$ has all finite coproducts and $\otimes$ preserves them in each variable; that is,

$$(v\sqcup v')\otimes(w\sqcup w') \cong v\otimes w \sqcup v'\otimes w \sqcup v\otimes w' \sqcup v'\otimes w'$$

Show that $T(X)=\coprod_{n\geq 0} X^{\otimes n}$ defines a monad on $V$ by defining the unit $\eta: 1_V\Rightarrow T$ and $\mu: T^2\Rightarrow T$ satisfying the commutativity of the diagrams

$$\begin{tikzcd}
T^3 \arrow[r, Rightarrow, "T\mu"] \arrow[d, Rightarrow, "\mu T"'] & T^2 \arrow[d, Rightarrow, "\mu"] \\
T^2 \arrow[r, Rightarrow, "\mu"'] & T
\end{tikzcd}
\qquad
\begin{tikzcd}
T \arrow[r, Rightarrow, "\eta T"] \arrow[dr, equal, "1_T"'] & T^2 \arrow[d, Rightarrow, "\mu"] & T \arrow[l, Rightarrow, "T\eta"'] \arrow[dl, equal, "1_T"] \\
 & T &
\end{tikzcd}$$

PROOF. First of all, the functor $\otimes^n: V^{\times n}\to V$ is well-defined by assumption, and its action is the freely-associative application of an object $X\in V$ $n$ times. For example, $\otimes^4(Y)=(Y\otimes Y)\otimes(Y\otimes Y)$, with varying associativities being identical up to isomorphism.

Indeed the functor $T: V\to V$ defined by $T(X)=\coprod_{n\geq 0} X^{\otimes n}$ is also well-defined, as $V$ admits all finite colimits, and the action on morphisms by $T$ is the evident one sending morphisms from objects in $V$ to unique factorizations through a colimit cocone under a particular diagram, as admitted by the universal property of colimits.

We define the unit $\eta$ as follows. For $X\in V$, $\eta_X: X\to \coprod_{n\geq 0} X^{\otimes n} = \iota_X$, i.e. the inclusion of the unary product on $X$ into the coproduct. By uniqueness of factorization through coproducts, this definition of $\eta$ is natural (a morphism $X\to Y$ taken by $T$ yields a unique factorization under a particular diagram's same construction as in 4.6.iii).

$$\begin{tikzcd}
X \arrow[r, hook, "\iota_X"'] \arrow[d, "f"'] & \coprod_{n\geq 0} X^{\otimes n} \arrow[d, dashed, "\exists!"] & X\otimes X \arrow[l, hook', "\iota_{X^{\otimes 2}}"] \arrow[d, "\otimes^2 f"] \\
fX \arrow[r, hook, "\iota_{fX}"'] & \coprod_{n\geq 0} fX^{\otimes n} & fX\otimes fX \arrow[l, hook', "\iota_{fX^{\otimes 2}}"]
\end{tikzcd}$$

Now we need to define the multiplication $\mu : T^2 \Rightarrow T$. The monad laws also require, for one, the commutativity

$$\begin{tikzcd} T \arrow[r, Rightarrow, "\eta T"] \arrow[dr, Rightarrow, "1_T"'] & T^2 \arrow[d, Rightarrow, "\mu"] & T \arrow[l, Rightarrow, "T\eta"'] \arrow[dl, Rightarrow, "1_T"] \\ & T & \end{tikzcd}$$

On the left, this means we need to find an arrow $\mu_A : T^2A \to TA$ s.t.

$$\begin{tikzcd} \coprod_{n\geq 0} A^{\otimes n} \arrow[r, "\eta_{TA}"] \arrow[dr, "1_{TA}"'] & \coprod_{m\geq 0}\left(\coprod_{n\geq 0} A^{\otimes n}\right)^{\otimes m} \arrow[d, dashed, "\mu_A"] & \left(\coprod_{n\geq 0} A^{\otimes n}\right)\otimes\left(\coprod_{n\geq 0} A^{\otimes n}\right) \arrow[l, "\eta_{TA}^{\otimes 2}"'] \cong \coprod_{n,m\geq 0} A^{\otimes m}\otimes A^{\otimes n} \arrow[dl, "\phi"] \\ & \coprod_{n\geq 0}(A^{\otimes n}) & \end{tikzcd}$$

$$\underbrace{\hspace{4em}}_{\text{mandatory}} \qquad \underbrace{\hspace{4em}}_{\text{possible, to get universal arrow}}$$

commutes. On the right, the arrow $\mu_A : T^2A \to TA$ must be s.t.

$$\begin{tikzcd} \coprod_{n\geq 0} A^{\otimes n} \arrow[r, "T\eta_A"', "= T\eta_A^{\otimes 2} = \eta_{TA}"] \arrow[dr, "1_{TA}"'] & \coprod_{m\geq 0}\left(\coprod_{n\geq 0} A^{\otimes n}\right)^{\otimes m} \arrow[d, dashed, "\mu_A"] & \coprod_{n,m\geq 0} A^{\otimes m}\otimes A^{\otimes n} \arrow[l, "\eta_{TA}^{\otimes 2}"'] \arrow[dl, "\phi"] \\ & \coprod_{n\geq 0}(A^{\otimes n}) & \end{tikzcd}$$

$$\begin{tikzcd} X\otimes X \arrow[r, "\iota_{X^{\otimes 2}}"] \arrow[d, "\eta_A^{\otimes 2} = \iota_{X^{\otimes 2}}"'] & \coprod X^{\otimes n} \arrow[d, "T\eta_A^{\otimes 2}"] \\ \coprod X^{\otimes n} \arrow[r, "\iota_{(\coprod X^{\otimes n})}"'] & \coprod\left(\coprod X^{\otimes n}\right)^{\otimes m} \end{tikzcd}$$

$$\underbrace{\hspace{6em}}_{\eta T = T\eta}$$

commutes.

How to get $\phi : \coprod_{n,m\geq 0} A^{\otimes m}\otimes A^{\otimes n} \to \coprod_{n\geq 0} A^{\otimes n}$ ? There is a particular diagram...

$$\begin{tikzcd} \cdots\ A^{\otimes 2}\otimes A^{\otimes 3} \cong A^{\otimes 5} \cong A^{\otimes 3}\otimes A^{\otimes 2}\ \cdots \arrow[d] \arrow[dd, bend right] \arrow[dd, bend left] \\ \coprod_{n,m\geq 0} A^{\otimes m}\otimes A^{\otimes n} \arrow[d, dashed, "\exists!\phi"] \\ \coprod_{n\geq 0} A^{\otimes n} \end{tikzcd}$$

$\phi$ over $\coprod_{n\geq 0} A^{\otimes n} \to \coprod_{n\geq 0} A^{\otimes}$ must be $1_{\coprod A^{\otimes n}}$ ... defining this way we would get a unique $\mu_A : T^2A \to TA$ factoring (some commuting of $TA$ and $\phi$) through $T^2A$. Still have to show

$$\begin{tikzcd} T^3 \arrow[r, Rightarrow, "T\mu"] \arrow[d, Rightarrow, "\mu T"'] & T^2 \arrow[d, Rightarrow, "\mu"] \\ T^2 \arrow[r, Rightarrow, "\mu"'] & T \end{tikzcd}$$

(and that $\mu$ is natural). For the second monad law, we have that

$$\begin{tikzcd} T^2A \arrow[r, "\eta_{TA}^2 = T\iota_A^2"] \arrow[d, "\mu_A"'] & T^3A \arrow[d, "T\mu_A \text{ and } T^2\iota_A"] \\ TA \arrow[r, "\eta_{TA} = T\iota_A"'] & T^2A \end{tikzcd} \qquad \begin{tikzcd} T^2A \arrow[r, "\mu_A"] \arrow[d, "T\iota_A = T\eta_A = \eta_{TA} = \iota_{TA}"'] & TA \arrow[d] \\ T^3A \arrow[r, "\mu_{TA}"'] & T^2A \end{tikzcd}$$

commute (if $\mu$ natural). Then $\mu T \circ \iota_{TA}^2 = \eta T \circ \mu = T\mu \circ \iota_{T^2A}$. And $T\mu_A \circ \iota_{T^2A} = T\mu_A \circ T^2\eta_A = T(\mu_A \circ T\eta_A) = 1_{TA}$ ...

③ The adjunction associated to a reflective subcategory of $\mathcal{C}$ induces an idempotent monad on $\mathcal{C}$.

Prove that the following three characterizations of an idempotent monad $(T, \eta, \mu)$ are equivalent:

(i) The multiplication $\mu : T^2 \Rightarrow T$ is a natural isomorphism (hence the terminology "idempotent")

(ii) The natural transformations $\eta T, T\eta : T \Rightarrow T^2$ are equal

(iii) Each component of $\mu$ is a monomorphism.

[(i) ⟺ (ii)] If $\mu$ is a natural isomorphism, then by the commutativity of the monadic identity

$$\begin{tikzcd} T \arrow[r, Rightarrow, "\eta T"] \arrow[dr, Rightarrow, "1_T"'] & T^2 \arrow[d, Rightarrow, "\mu"] & T \arrow[l, Rightarrow, "T\eta"'] \arrow[dl, equal, "1_T"] \\ & T & \end{tikzcd}$$

we have that $\mu \circ \eta T = \mu \circ T\eta \Rightarrow \eta T = T\eta$.

Conversely, suppose $\eta T, T\eta$. By the naturality of $\eta$ and $\mu$ we have that

$$\begin{tikzcd} T^2 \arrow[r, Rightarrow, "\eta TT"] \arrow[d, Rightarrow, "\mu"'] & T^3 \arrow[d, Rightarrow, "T\mu"] \\ T \arrow[r, Rightarrow, "\eta T"'] & T^2 \end{tikzcd}$$

commutes. We have that $T\mu \circ \eta TT = T\mu \circ T\eta T = T(\mu \circ \eta T) = T(1_T) = 1_{T^2}$, and so $\eta T \circ \mu = 1_{T^2}$, hence $\eta T, \mu$ are inverse isomorphisms.

[(i) ⟺ (iii)] If $\mu$ is a natural isomorphism, each component of $\mu$ is monic by definition.

Conversely, supposing each component of $\mu$ is monic, we have that $\mu \circ T\eta = \mu \circ \eta T \Rightarrow \eta T = T\eta$, and application of (i) ⟺ (ii) yields that $\mu$ is a natural isomorphism. ∎

5.2.1 Pick your favorite monad in Set induced from a free ⊣ forgetful adjunction, identify its algebras and show that algebra morphisms are precisely the homomorphisms in the appropriate sense.

① Let us examine the maybe monad, induced by the adjunction
$$\begin{tikzcd} Set \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & Set^* \arrow[l, bend left, "U"] \end{tikzcd}.$$

By 5.1.3, this adjunction induces a monad with the endofunctor $(-)_+ = UF : Set \to Set$, adding a free point to a set. The unit of the monad is the unit of the adjunction $U \dashv F$, denoted $\eta$, whose components are inclusions $\eta_A : A \hookrightarrow A_+$. The multiplication $\mu : (-)_+^2 \Rightarrow (-)_+$ is given by "collapsing" free points in $(A_+)_+$ to the free point in $A_+$ and acting as the identity for $a \in A$, at all components $A \in Set$.

The algebras of $(-)_+$ are objects in the Eilenberg-Moore category $Set^{(-)_+}$, given by pairs $(A \in Set, a : A_+ \to A)$ s.t.

$$\begin{tikzcd} A \arrow[r, "\eta_A"] \arrow[dr, "1_A"'] & A_+ \arrow[d, "a"] \\ & A \end{tikzcd} \qquad \begin{tikzcd} (A_+)_+ \arrow[r, "\mu_A"] \arrow[d, "a_+"'] & A_+ \arrow[d, "a"] \\ A_+ \arrow[r, "a"'] & A \end{tikzcd}$$

both commute. The triangle forces $a : A_+ \to A$ to act as the identity on element in $A$, and both diagrams force the free element of $A_+$ to be sent to some a priori "picked-out" $a \in A$.

Hence the algebras of $Set^{(-)_+}$ are pairs $(A, a : A_+ \to A)$ where $A$ is a pointed set with some basepoint $a' \in A$, and $a : A_+ \to A$ is a morphism between pointed sets acting as the identity on $A$ and sending the free point on $A_+$ to $a'$.

The $(-)_+$-algebra homomorphisms are maps $f : A \to B \in Set$ yielding a morphism $(A,a) \to (B,b)$ s.t.

$$\begin{tikzcd} A_+ \arrow[r, "f_+"] \arrow[d, "a"'] & B_+ \arrow[d, "b"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

commutes in Set. That is, $f$ is just a homomorphism between pointed sets $(A, a') \to (B, b')$ in the category $Set^*$. ⊠

Note: the "homomorphisms" are defined in the "codomain" of the adjunction, not visible from the monad!

(2) Fill in the remaining details in the proof of 5.2.8 to show that the free and forgetful functors relating a cat $C$ w/ monad $T$ to the cat $C^T$ of $T$-algebras are adjoints inducing the given monad $T$.

As in 5.2.8, the adjunction
\begin{tikzcd} C \arrow[r, shift left, "F^T"] \arrow[r, phantom, "\perp"] & C^T \arrow[l, shift left, "U^T"] \end{tikzcd}
, where $C$ has monad $(T, \eta, \mu)$, is given by

- $F^T$: $F^T A = (TA, \mu_A : T^2 A \to TA)$
$F^T(f : A \to B) = Tf : (TA, \mu_A) \to (TB, \mu_B)$

Indeed for $A \in C$, $F^T A = (TA, \mu_A)$ satisfies the $T$-algebra identities

$$\begin{tikzcd} TA \arrow[r, "\eta_{TA}"] \arrow[dr, "1_{TA}"'] & TTA \arrow[d, "\mu_A"] \\ & TA \end{tikzcd} \qquad \begin{tikzcd} T^3 A \arrow[r, "\mu_{TA}"] \arrow[d, "T\mu_A"'] & T^2 A \arrow[d, "\mu_A"] \\ T^2 A \arrow[r, "\mu_A"'] & TA \end{tikzcd}$$

by virtue of the monadic identities, and for $f : A \to B$,

$$\begin{tikzcd} TTA \arrow[r, "TTf"] \arrow[d, "\mu_A"'] & TTB \arrow[d, "\mu_B"] \\ TA \arrow[r, "Tf"'] & TB \end{tikzcd}$$

$Tf : TA \to TB$ is a morphism between algebras by the naturality of $\mu$.

- $U^T$: is the evident forgetful functor.

Now, $U^T F^T$ takes objects $A$ to $TA$ and morphisms $f$ to $Tf$, so $U^T F^T = T$; hence if we can show $U^T \dashv F^T$, this adjunction induces the monad $T$.

The unit of $U^T \dashv F^T$, $\eta : 1_C \Rightarrow U^T F^T = T$ is given by $\eta$ inherited from the monad $T$.

The counit $\epsilon : F^T U^T \Rightarrow 1_{C^T}$ is given by $\epsilon_{(A,a)} := (TA, \mu_A) \xrightarrow{a} (A, a)$, the algebra map $a : TA \to A$.

Indeed this def is valid b/c
$$\begin{tikzcd} T^2 A \arrow[r, "Ta"] \arrow[d, "\mu_A"'] & TA \arrow[d, "a"] \\ TA \arrow[r, "a"'] & A \end{tikzcd}$$
commutes by the definition of $T$-algebra.

Now to show the $\Delta$-identities hold. First, WTS $1_{F^T} = \epsilon F^T \circ F^T \eta$. Taking $A \in C$,
$$(\epsilon F^T \circ F^T \eta)_A (F^T A) = (\epsilon F^T \circ F^T \eta)_A (TA, \mu_A) = (\epsilon F^T)_A (TTA, \mu_{TA}) = \epsilon_{(TA, \mu_A)} (TTA, \mu_{TA}) = (TA, \mu_A) = F^T A.$$

Next, to show $1_{U^T} = U^T \epsilon \circ \eta U^T$, let $(A, a) \in C^T$. Then
$$(U^T \epsilon \circ \eta U^T)_{(A,a)} (U^T (A,a)) = (U^T \epsilon)_{(A,a)} \circ \eta_A (A) = U^T \epsilon_{(A,a)} (TA) = A = U^T (A, a).$$ $\boxtimes$

③ Dualize 5.2.4 and 5.2.8 to define the category of coalgebras for a comonad together w/ its forgetful-cofree adjunction.

Recall: A comonad $(K, \epsilon, \delta)$ over a category $C$ is a monad over $C^{op}$. That is, it consists of $K: C \to C$, $\epsilon: K \Rightarrow 1_C$, $\delta: K \Rightarrow K^2$ such that

$$\begin{tikzcd} K \arrow[r, Rightarrow, "\delta"] \arrow[d, Rightarrow, "\delta"'] & K^2 \arrow[d, Rightarrow, "K\delta"] \\ K^2 \arrow[r, Rightarrow, "\delta K"'] & K^3 \end{tikzcd} \qquad \begin{tikzcd} K & K^2 \arrow[l, Rightarrow, "\epsilon K"'] \arrow[r, Rightarrow, "K\epsilon"] & K \\ & K \arrow[ul, Rightarrow, "1_K"] \arrow[u, Rightarrow, "\delta"] \arrow[ur, Rightarrow, "1_K"'] & \end{tikzcd}$$

commute in $C^C$. The category of $K$-coalgebras is the category $C^K$ where

- objects: are pairs $(A \in C, a: A \to KA)$ so that

$$\begin{tikzcd} A & KA \arrow[l, "\epsilon_A"'] \\ & A \arrow[ul, "1_A"] \arrow[u, "a"'] \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "a"] \arrow[d, "a"'] & KA \arrow[d, "\delta_A"] \\ KA \arrow[r, "Ka"'] & K^2A \end{tikzcd}$$

commute.

- morphisms: $f: (A,a) \to (B,b)$ is given by a map $f: A \to B$ s.t.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "a"'] & B \arrow[d, "b"] \\ KA \arrow[r, "Kf"'] & KB \end{tikzcd}$$

commutes, with composition and identities as in $C$.

There is an adjunction $\begin{tikzcd} C \arrow[r, shift left, "F^K"] \arrow[r, phantom, "\top"] & C^K \arrow[l, shift left, "U^K"] \end{tikzcd}$ given by the evident forgetful functor $U^K: C^K \to C$ and the cofree functor $F^K: C \to C^K$ carrying an object $A \in C$ to the cofree $K$-coalgebra

$F^K A = (KA, \delta_A: KA \to K^2A)$; indeed

$$\begin{tikzcd} KA \arrow[r, "\delta_A"] \arrow[dr, "1_{KA}"'] & K^2A \arrow[d, "\epsilon_{KA}"] \\ & KA \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} KA \arrow[r, "\delta_A"] \arrow[d, "\delta_A"'] & K^2A \arrow[d, "\delta_{KA}"] \\ K^2A \arrow[r, "K\delta_A"'] & K^3A \end{tikzcd}$$

commute.

$F^K$ carries a morphism $f: A \to B$ to the cofree $K$-coalgebra morphism

$F^K f = (KA, \delta_A) \xrightarrow{Kf} (KB, \delta_B)$; indeed $\begin{tikzcd} KA \arrow[r, "Kf"] \arrow[d, "\delta_A"'] & KB \arrow[d, "\delta_B"] \\ KKA \arrow[r, "KKf"'] & KKB \end{tikzcd}$ commutes by nat. of $\delta$.

Notice $U^K F^K(A) = KA$ and $U^K F^K(f: A \to B) = Kf$, so $U^K F^K = K$. The counit $\epsilon$ of $U^K \dashv F^K$ is given by $\epsilon: K \Rightarrow 1_C = U^K F^K \Rightarrow 1_C$ inherited from the comonad $(K, \epsilon, \delta)$, and the unit $\eta: 1_{C^K} \Rightarrow F^K U^K$ is given by $\eta_{(A,a)} = (A,a) \xrightarrow{a} (KA, \delta_A)$; indeed, this satisfies the commutativity of $\begin{tikzcd} A \arrow[r, "a"] \arrow[d, "a"'] & KA \arrow[d, "\delta_A"] \\ KA \arrow[r, "Ka"'] & K^2A \end{tikzcd}$ simply by definition of coalgebra. Verification that $\eta, \epsilon$ satisfy the $\Delta$-identities is as in ①.

(4) Verify that the Kleisli category is indeed a category by checking the composition operator is associative and unital.

For a cat. $C$ with monad $(T, \eta, \mu)$, the Kleisli cat. $C_T$ is given by

- objects: objects of $C$
- morphisms: $A \rightsquigarrow B \in C_T$ is a morphism $A \to TB \in C$.
- identities: $1_A : A \rightsquigarrow A \in C_T$ is given by $\eta_A : A \to TA$.
- composition: for $f : A \rightsquigarrow B$, $g : B \rightsquigarrow C \in C_T$, $g \circ f$ is defined by $A \rightsquigarrow C = A \to TC$

$$A \xrightarrow{f} TB \xrightarrow{Tg} T^2C \xrightarrow{\mu_C} TC$$

[Unitality] Let $f : A \rightsquigarrow B \in C_T$ be given. WTS $1_B \circ f = f = f \circ 1_A$.

$1_B \circ f$ is given by the composite

$$A \xrightarrow{f} TB \xrightarrow{T\eta_B} TTB \xrightarrow{\mu_B} TB$$

which is equivalently $f : A \rightsquigarrow B$ because $\mu_B \circ T\eta_B = 1_B$.

$f \circ 1_A$ is given by the composite

$$A \xrightarrow{\eta_A} TA \xrightarrow{Tf} TTB \xrightarrow{\mu_B} TB$$

$$\begin{tikzcd} A \arrow[r, "\eta_A"] \arrow[d, "f"'] & TA \arrow[d, "Tf"] \\ TB \arrow[r, "\eta_{TB}"'] & TTB \end{tikzcd}$$

which, by the naturality of $\eta$, is equivalently the composite

$$A \xrightarrow{f} TB \xrightarrow{T\eta_B} T^2B \xrightarrow{\mu_B} TB$$

which equals $1_B \circ f$; hence $f = 1_B \circ f = f \circ 1_A$.

[Associativity] Let $f : A \rightsquigarrow B$, $g : B \rightsquigarrow D$, $h : D \rightsquigarrow E \in C_T$ be given, and WTS $h(gf) = (hg)f$.

We have that $h(gf)$ is the composite

$$A \xrightarrow{f} TB \xrightarrow{Tg} T^2D \xrightarrow{\mu_D} TD \xrightarrow{Th} T^2E \xrightarrow{\mu_E} TE$$

$$\begin{tikzcd} T^2D \arrow[r, "\mu_D"] \arrow[d, "T^2h"'] & TD \arrow[d, "Th"] \\ T^3E \arrow[r, "\mu_{TE}"'] & T^2E \end{tikzcd}$$

which by naturality of $\mu$ is equivalent to the composite

$$A \xrightarrow{f} TB \xrightarrow{Tg} T^2D \xrightarrow{T^2h} T^3E \xrightarrow{\mu_{TE}} T^2E \xrightarrow{\mu_E} TE$$

which by monadic identity is equivalent to

$$A \xrightarrow{f} TB \xrightarrow{Tg} T^2D \xrightarrow{T^2h} T^3E \xrightarrow{T\mu_E} T^2E \xrightarrow{\mu_E} TE$$

which is the composite $(hg)f$. $\boxtimes$

(5) Verify that $F_T, U_T$ in 5.2.11 are functorial.

Given a monad $(T, \eta, \mu)$ on a cat. $C$, the functors $C \underset{U_T}{\overset{F_T}{\rightleftarrows}} C_T$ relaying in and out of the Kleisli cat $C_T$ are given by

[$F_T$] - on objects, $F_T$ is the identity. On a morphism $f: A \to B$,

$$F_T f = A \xrightarrow{f} B \xrightarrow{\eta_B} TB$$

For $1_A : A \to A \in C$, $F_T 1_A = \eta_A \circ 1_A = \eta_A$, precisely the identity morphism $A \rightsquigarrow A \in C_T$.

For $f: A \to B$, $g: B \to D \in C$, $F_T g \circ F_T f$ is the composite

$$A \xrightarrow{f} B \xrightarrow{\eta_B} TB \xrightarrow{Tg} TD \xrightarrow{T\eta_D} T^2 D \xrightarrow{\epsilon_D} TD$$

as given by composites in $C_T$. By the monadic identity $\epsilon \circ T\eta = 1_T$, we have that this is equivalent to the composite

$$A \xrightarrow{f} B \xrightarrow{\eta_B} TB \xrightarrow{Tg} TD$$

which by naturality of $\eta$ is equivalent to

$$A \xrightarrow{\eta_A} TA \xrightarrow{Tf} TB \xrightarrow{Tg} TD$$

which by functoriality is equivalent to

$$A \xrightarrow{\eta_A} TA \xrightarrow{T(g\circ f)} TD$$

which again by naturality of $\eta$ is equivalent to

$$A \xrightarrow{g\circ f} D \xrightarrow{\eta_D} TD$$

which is the definition of $F_T(g\circ f)$. Hence $F_T g \circ F_T f = F_T(g\circ f)$.

[$U_T$] - for $A \in C_T$, $U_T A = TA \in C$. For $f: A \rightsquigarrow B \in C_T$, $U_T f = TA \xrightarrow{Tf} T^2 B \xrightarrow{\mu_B} TB$.

Given $1_A = \eta_A : A \rightsquigarrow A \in C_T$, $U_T \eta_A = \mu_A \circ T\eta_A = 1_{TA}$.

Given $f: A \rightsquigarrow B$, $g: B \rightsquigarrow D \in C_T$, $U_T(g\circ f)$ is the composite

$$TA \xrightarrow{Tf} T^2 B \xrightarrow{T^2 g} T^3 D \xrightarrow{T\mu_D} T^2 D \xrightarrow{\mu_D} TD$$

$$= TA \xrightarrow{Tf} T^2 B \xrightarrow{T^2 g} T^3 D \xrightarrow{\mu_{TD}} T^2 D \xrightarrow{\mu_D} TD \quad \text{(by monadic identity)}$$

$$= TA \xrightarrow{Tf} T^2 B \xrightarrow{\mu_B} TB \xrightarrow{Tg} T^2 D \xrightarrow{\mu_D} TD \quad \text{(by naturality of } \mu)$$

$$= U_T g \circ U_T f.$$

∎

7) From the canonical functor $K: \mathcal{C}_T \to \mathcal{C}^T$ of Lemma 5.2.13 that embeds the Kleisli category into the Eilenberg-Moore category for a monad $T$ on $\mathcal{C}$, we can define a "restricted Yoneda" functor

$$\mathcal{C}^T \xrightarrow{\mathcal{C}^T(K-,-)} \mathrm{Set}^{\mathcal{C}_T^{op}}$$

$$(A,a) \mapsto \mathcal{C}^T(K-,(A,a))$$

that yields a presheaf on the Kleisli category for each $T$-algebra $(A,a)$. Show that the presheaves in the image of this functor restrict along $F_T: \mathcal{C} \to \mathcal{C}_T$ to representable functors in $\mathrm{Set}^{\mathcal{C}^{op}}$. In fact, this functor defines an isomorphism b/w the Eilenberg-Moore category and the full subcategory of presheaves on the Kleisli category that restrict along $F_T$ to representable functors.

First, let us recall the action of $K$. $K$ is a morphism in the category of adjunctions inducing $T$, specifically that which defines a functor $\mathcal{C}_T \to \mathcal{C}^T$ such that the diagram

$$\begin{tikzcd}
\mathcal{C}_T \arrow[rr, dashed, "K"] \arrow[dr, shift left, "U_T"] & & \mathcal{C}^T \arrow[dl, shift left, "U^T"] \\
& \mathcal{C} \arrow[ul, shift left, "F_T"] \arrow[ur, shift left, "F^T"] \arrow[ul, phantom, "\perp"] \arrow[ur, phantom, "\perp"] &
\end{tikzcd}$$

commutes. In particular, as in 5.2.12, for objects $c \in \mathcal{C}_T$ (equiv. objects in $\mathcal{C}$), $Kc = F^T c$.

For morphisms $f: c \rightsquigarrow c' \in \mathcal{C}_T$, $Kf = F^T c \xrightarrow{F^T f} F^T U^T F^T c' \xrightarrow{\epsilon F^T c'} F^T c'$

$$= (Tc, \mu_c) \xrightarrow{Tf} (TTc', \mu_{Tc'}) \xrightarrow{\epsilon_{(Tc',\mu_{c'})}} (Tc', \mu_{c'})$$

$$= (Tc, \mu_c) \xrightarrow{Tf} (TTc', \mu_{Tc'}) \xrightarrow{\mu_{c'}} (Tc', \mu_{c'})$$

where $\mu$ is the multiplication of the monad $(T, \eta, \mu)$ on $\mathcal{C}$.

Now a presheave in the image of $\mathcal{C}^T(K-,-)$ restricted on $F_T$ is a functor $\mathcal{C}^T(KF_T-,(A,a)): \mathcal{C}^{op} \to \mathrm{Set}$.

On objects, $KF_T(A \in \mathcal{C}) = K(A \in \mathcal{C}_T) = F^T A$.

On morphisms, $KF_T(f \in \mathcal{C}(A,B)) = K(\eta_B \circ f) = \mu_B \circ T\eta_B \circ Tf = Tf = F^T f$.

Hence $\mathcal{C}^T(KF_T-,(A,a)) = \mathcal{C}^T(F^T-,(A,a)): \mathcal{C}^{op} \to \mathrm{Set}$. But by the adjunction $F^T \dashv U^T$, we have that $\mathcal{C}^T(F^T-,(A,a)) \cong \mathcal{C}(-,A)$, and hence we see that $\mathcal{C}^T(KF_T-,(A,a))$ is represented by the object $A \in \mathcal{C}$. $\blacksquare$

5.31

① Prove that the category of algebras for an idempotent monad on $C$ defines a reflective subcategory of $C$. Show further that the Kleisli category for an idempotent monad is also reflective.

Let $(T, \eta, \mu)$ be an idempotent monad on $C$; in particular we have that $\mu$ is a natural iso., and equivalent (by 5.1.iii), $\eta T = T\eta$, and each component of $\mu$ is monic.

The category of $T$-algebras consists of objects that are pairs $(A \in C,\ a: TA \to A)$ satisfying the commutativity of

$$\begin{tikzcd} A \arrow[r, "\eta_A"] \arrow[dr, "1_A"'] & TA \arrow[d, "a"] \\ & A \end{tikzcd} \qquad \begin{tikzcd} T^2A \arrow[r, "\mu_A"] \arrow[d, "Ta"'] & TA \arrow[d, "a"] \\ TA \arrow[r, "a"'] & A \end{tikzcd}$$

Hence, for any $T$-algebra $(A,a)$ we have $a \circ \eta_A = 1_A$, and also

$$\eta_A \circ a = Ta \circ \eta_{TA} = Ta \circ T\eta_A = T(a \circ \eta_A) = 1_{TA}.$$

Thus, $a, \eta_A$ are inverse isomorphisms for any $T$-algebra $(A,a)$. With this in mind, we can also see that for any pair of $T$-algebras $(A,a)$, $(B,b)$, any morphism $f: A \to B$ yields an algebra homomorphism. In particular, the condition of commutativity of

$$\begin{tikzcd} TA \arrow[r, "Tf"] \arrow[d, "\eta_A \cong a"'] & TB \arrow[d, "b \cong \eta_B"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

is satisfied by the naturality of $\eta$ and the fact that $\eta_A, \eta_B$ are isomorphisms with inverses $a, b$ respectively (c.f. Lemma 1.5.10).

Indeed this tells us that the subcategory of $C$ included by the forgetful functor on the described category of $T$-algebras is a full subcategory of $C$. Furthermore, the free $T$-algebra functor $F^T: C \to C^T$ is a left adjoint to the inclusion $U^T$.

Hence the category of $T$-algebras on $C$ is indeed a reflective subcategory of $C$, when $T$ is idempotent.

Furthermore, regarding the Kleisli category over $T$ - by 5.2.13, we know the canonical functor $C_T \ni A \xrightarrow{K} KA = (TA, \mu_A) \in C^T$ is fully faithful. For any $T$-algebra $(A,a) \in C^T$, we have that $a$ induces an algebra homomorphism $(TA, \mu_A) \xrightarrow{a} (A,a)$ by the commutativity of

$$\begin{tikzcd} TTA \arrow[r, "Ta"] \arrow[d, "\mu_A"'] & TA \arrow[d, "a"] \\ TA \arrow[r, "a"'] & A \end{tikzcd}$$

given by the definition of a $T$-algebra. But $a$ is an isomorphism, so $(A,a) \cong (TA, \mu_A)$ $\forall (A,a) \in C^T$. This means $K$ is essentially surjective, so $C_T \simeq C^T$ in the presence of an idempotent monad, and hence $C_T$ is also a reflective subcategory of $C$. ∎