4.6.31

($\Rightarrow$) Suppose C is locally small with coproducts, and suppose $F: C \to Set$ is represented by $c \in C$; in particular $C(c,-) \cong F$. We want to show F admits a left adjoint $L: Set \to C$.

L takes $A \in Set$ to $\coprod_{c \in A} c \in C$, and $f: A \to B \in Set$ to the unique morphism factoring a cone under A with nadir LB through LA. Uniqueness of this map ensures functoriality, as presented last time.

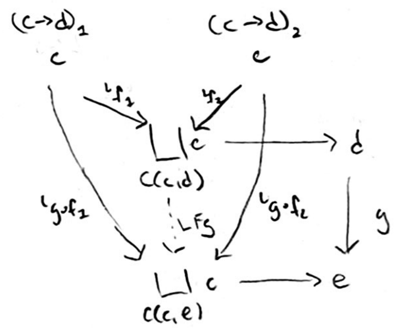
To show $L \dashv F$, we will define the counit of the adjunction and demonstrate it yields a certain universal property defining the isomorphism $(\epsilon_d)_* F: C(c, Gd) \cong D(Fc, d)$.

The d'th component of ϵ is a map $\epsilon_d: LFd \to d = L\,C(c,d) \to d = \coprod_{C(c,d)} c \to d$.

We will define this map to be the unique map factoring the cone under $C(c,d)$ with nadir d whose components are given by the fully-indexed set $C(c,d)$, through $\coprod_{C(c,d)} c$.

In particular, we can define $\epsilon_d(\iota_{f_i: c \to d}\, c) = f_i c$.

$$\begin{tikzcd}
\coprod_{C(c,d)} c \arrow[r, "\epsilon_d"] \arrow[d, "LFg"'] & d \arrow[d, "g"] \\
\coprod_{C(c,e)} e \arrow[r, "\epsilon_e"'] & e
\end{tikzcd}$$



To observe the naturality of ϵ, notice that $g \circ \epsilon_d(\iota_{f_i: c \to d}\, c) = g \circ f_i c$, and $\epsilon_e \circ LFg(\iota_{f_i: c \to d}\, c) = \epsilon_e(\iota_{g \circ f_i}\, c) = g \circ f_i c$, $\forall\, g: d \to e \in C$.

Now we wish to demonstrate the composite

$$Set(A, Fd) \xrightarrow{L} C(LA, LFd) \xrightarrow{(\epsilon_d)_*} C(LA, d)$$

or equivalently,

$$Set(A, C(c,d)) \xrightarrow{L} C\left(\coprod_A c, \coprod_{C(c,d)} c\right) \xrightarrow{(\epsilon_d)_*} C\left(\coprod_A c, d\right)$$

is a bijection $\forall\, A \in Set$, $d \in C$. Taking distinct $f, g: A \rightrightarrows C(c,d)$, Lf and Lg are unique factorizations of distinct cocones under A, and by the universal property of $\coprod_A c$ are thus also distinct.

Furthermore, if Lf, Lg are distinct, then $\exists\, a \in A$ s.t. $\iota_{f_j: c \to d}\, c = Lf\, \iota_a c \neq Lg\, \iota_a c = \iota_{f_k: c \to d}\, c$, $f_j \neq f_k$. Then $\epsilon_d(\iota_{f_j} c) = f_j c \neq f_k c = \epsilon_d(\iota_{f_k} c)$, demonstrating the injectivity of this map.

To see surjectivity, let $f: \coprod_A c \to d$ be given. By the representability of $\mathrm{Cone}(A,-)$, f corresponds to a unique cocone under A with nadir d. The components of this cocone are maps $\{\mu_a : c \to d\}_{a \in A}$ s.t. $\mu_a = f \circ (\iota_a : c \to \coprod_A c)$. But now we can construct a cocone under A with nadir $\coprod_{C(c,d)} c$, whose components are $\{\mu'_a = \iota_{\mu_a} = \iota_{f \circ \iota_a}\}_{a \in A}$. Indeed this is well-defined as for any map $c \to d$ there is an inclusion of the image of the diagram $C(c,d) \to c$ into $\coprod_{C(c,d)} c$. Furthermore $\epsilon_d(\mu'_a(c)) = f \circ \iota_a(c)$, so $(\epsilon_d)_* \mu'$ yields the cocone μ. Again applying the representability of $\mathrm{Cone}(A,-)$, we have that μ' corresponds to a unique map $f' : \coprod_A c \to \coprod_{C(c,d)} c$. So far, we have shown that there is a map $f' \in C(\coprod_A c, \coprod_{C(c,d)} c)$ s.t. $\epsilon_d \circ f' = f \in C(\coprod_A c, d)$, for arbitrary f. It remains to be seen that we can lift f' to $\mathrm{Set}(A, C(c,d))$ by way of L. Defining $f'' : A \to C(c,d)$ by $f''(a) = f \circ (\iota_a : c \to \coprod_A c)$, we have Lf'' is the unique morphism factoring the cocone under A given by components $\{\iota_{f \circ \iota_a} : c \to \coprod_{C(c,d)} c\}_{a \in A}$ through $\coprod_A c$. But this cocone is exactly μ', and uniqueness of the factorization forces $Lf'' = f'$. Hence $(\epsilon_d)_* L$ is injective and surjective, admitting a bijection $C(LA, d) \cong \mathrm{Set}(A, Fd)$, and so $L \dashv F$. $\boxtimes$

② Use the monadicity theorem to prove that another one of the categories listed in 5.5.3 is monadic over sets.

We will show that Set_* is monadic over Set via the forgetful right adjoint $U: Set_* \to Set$. By the monadicity theorem, we must show that U creates coequalizers of U-split pairs.

Let $f, g : (A, a) \rightrightarrows (B, b) \in Set_*$ s.t.

$$\begin{tikzcd} A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \arrow[r, "h"] \arrow[l, bend left=60, "t"] & C \arrow[l, bend left, "s"] \end{tikzcd}$$

is a U-splitting, admitting a split coequalizer in Set. We want to show C lifts to a pointed set (C, c) and h lifts to a map of pointed sets coequalizing f, g. In particular we will show C lifts to (C, hb). Indeed $U(C, hb) = C$ and h is a pointed set morphism for it is a map $B \to C$ preserving the pointed elements.

We claim also that h coequalizes f, g in Set_*. To show this, let $k : (B, b) \to (D, d)$ any other map s.t. $kf = kg$, forming a cone under a certain diagram and WTS $\exists ! \, j : (C, hb) \to (D, d)$ yielding the factorization

$$\begin{tikzcd} (A,a) \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & (B,b) \arrow[r, "h"] \arrow[dr, "k"'] & (C, hb) \arrow[d, dashed, "j"] \\ & & (D, d) \end{tikzcd}$$

But in particular k is a map $B \to D$ with the property $k(b) = d$, which under the coequalizer in C yields a unique map $j : C \to D$, with furthermore the property that $d = k(b) = j(h(b))$. Hence j is also the unique map factoring k through h. ☒

③ Prove 5.5.P(RTT): If $U: D \to C$ has a left adjoint and if

(i) D has coequalizers of reflexive pairs (i.e. for which $f, g : A \rightrightarrows B$ admit a common section $s : B \to A$)

(ii) U preserves coequalizers of reflexive pairs

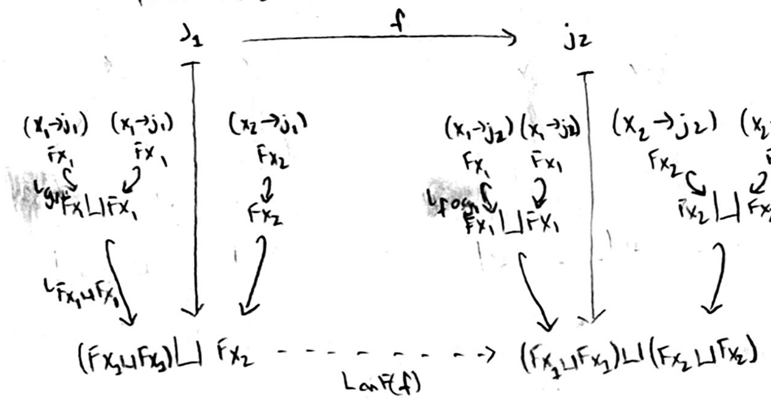
(iii) U reflects isomorphisms

Then U is monadic.

(5) For any small cat. J and any cocomplete category C, the forgetful functor $U: C^J \to C^{\mathrm{ob}J}$ admits a left adjoint $\mathrm{Lan}: C^{\mathrm{ob}J} \to C^J$ that sends a functor $F \in C^{\mathrm{ob}J}$ to the functor $\mathrm{Lan}F \in C^J$ defined by

$$\mathrm{Lan}F(j) = \coprod_{x \in J} \coprod_{J(x,j)} Fx .$$

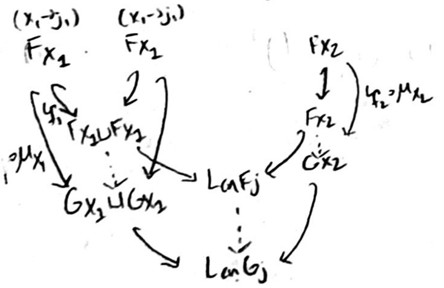
(i) Define $\mathrm{Lan}F$ on morphisms in J.



Given $f: j \to k \in J$, $\mathrm{Lan}F(f): \coprod_{x \in J} \coprod_{J(x,j)} Fx \to \coprod_{x \in J} \coprod_{J(x,k)} Fx$ is the unique morphism mapping the domain coproduct in the following way. $\mathrm{Lan}F(f)\left(\iota_{\coprod_{J(x_n,j)} Fx_n}\left(\iota_{g_m: x_n \to j} Fx_n\right)\right) = \iota_{\coprod_{J(x_n,k)} Fx_n}\left(\iota_{f \circ g_m: x_n \to k} Fx_n\right)$.
In particular this defines the action on inclusions of coproducts factored by a unique morphism under a certain diagram.

(ii) Define Lan on morphisms in $C^{\mathrm{ob}J}$.

Let $\mu: F \Rightarrow G$ be a nat. transformation b/w functors $F, G: \mathrm{ob}J \to C$. $\mathrm{Lan}\mu: \mathrm{Lan}F \Rightarrow \mathrm{Lan}G$ is defined by components $(\mathrm{Lan}\mu)_j: \mathrm{Lan}F_j \to \mathrm{Lan}G_j$, each of which is the unique map factoring a certain cone under $\mathrm{ob}J$ with nadir $\mathrm{Lan}G_j$ through $\mathrm{Lan}F_j$. We can enumerate the factorization as follows. For a coproduct $\coprod_{J(x_n,j)} Fx_n$ included into $\mathrm{Lan}F_j$ by $\iota_{x_n}: \coprod_{J(x_n,j)} Fx_n \to \mathrm{Lan}F_j$, there is a morphism $(\iota_{x_n}: \coprod_{J(x_n,j)} Gx_n \to \mathrm{Lan}(G_j)) \circ \varphi_n$, where φ_n is a unique morphism $\coprod_{J(x_n,j)} Fx_n \to \coprod_{J(x_n,j)} Gx_n$, taking $\coprod_{J(x_n,j)} Fx_n$ into $\mathrm{Lan}G_j$. If we can construct the map φ_n, the set $\{\iota_{x_n} \circ \varphi_n\}_{x_n \in \mathrm{ob}J}$ yields a cocone with nadir $\mathrm{Lan}G_j$ under a diagram whose colimit is $\mathrm{Lan}F_j$, and we will define $(\mathrm{Lan}\mu)_j$ to be the unique factorization of this cocone through $\mathrm{Lan}F_j$. Now, the collection of arrows $\{(\iota_{f_n}: Gx_n \to \coprod_{J(x_n,j)} Gx_n) \circ \mu_{x_n} : Fx_n \to Gx_n \to \coprod_{J(x_n,j)} Gx\}_{x_n \in \mathrm{ob}J}$ yield a cocone under a diagram whose colimit is $\coprod_{J(x_n,j)} Fx_n$, and so there is a unique map $\varphi_n: \coprod_{\overline{J}(x_n,j)} Fx_n \to \coprod_{J(x_n,j)} Gx_n$ factoring this cocone through the colimit cone. This completes the construction of $\mathrm{Lan}\mu$.



$$\begin{tikzcd} \mathrm{Lan}\,Fj \arrow[r, "\mathrm{Lan}\,\mu_j"] \arrow[d, "\mathrm{Lan}\,Ff"'] & \mathrm{Lan}\,Gj \arrow[d, "\mathrm{Lan}\,Gf"] \\ \mathrm{Lan}\,Fk \arrow[r, "\mathrm{Lan}\,\mu_k"'] & \mathrm{Lan}\,Gk \end{tikzcd}$$

A diagram chase reveals the naturality of μ. Taking $f: j \to k \in J$, we have that

$$\mathrm{Lan}\,Gf \circ \mathrm{Lan}\,\mu_j (\iota_{x_n}(\iota_{g_m} Fx_n)) = \mathrm{Lan}\,Gf(\iota_{x_n} \circ \Psi_n(\iota_{g_m} Fx_n))$$
$$= \mathrm{Lan}\,Gf(\iota_{x_n}(\iota_{g_m} \circ \mu_{x_n} Fx_n)) = \iota_{x_n}(\iota_{f\circ g_m} \circ \mu_{x_n} Fx_n)$$

and $\mathrm{Lan}\,\mu_k \circ \mathrm{Lan}\,Ff(\iota_{x_n}(\iota_{g_m} Fx_n)) = \mathrm{Lan}\,\mu_k(\iota_{x_n}(\iota_{f\circ g_m} Fx_n)) = \iota_{x_n} \circ \Psi_n(\iota_{f\circ g_m} Fx_n)$

$$= \iota_{x_n}(\iota_{f\circ g_m} \circ \mu_{x_n} Fx_n) = \mathrm{Lan}\,Gf \circ \mathrm{Lan}\,\mu_j.$$

(iii) Use the Yoneda Lemma to show that Lan is left adjoint to the forgetful (restriction) functor $U: C^J \to C^{\mathrm{ob}J}$.

We wish to show there is an isomorphism $C^J(\mathrm{Lan}\,G, H) \cong C^{\mathrm{ob}J}(G, UH)$ natural in G and H.

$\mathrm{Hom}(C^J(-, \mathrm{Lan}\,G), C^J(-,H))$? $\mathrm{Hom}(C^{\mathrm{ob}J}(-,G), C^{\mathrm{ob}J}(-,UH))$ Uniqueness of adjoints up to $\cong$?

$\cong$ $\cong$

$C^J(\mathrm{Lan}\,G, H) \cong C^{\mathrm{ob}J}(G, UH)$

$\cong$ $\cong$

$\mathrm{Hom}(C^J(H,-), C^J(\mathrm{Lan}\,G,-))$? $\mathrm{Hom}(C^{\mathrm{ob}J}(UH,-), C^{\mathrm{ob}J}(G,-))$

A direct pf could be as follows. We first define the forward isomorphism $\phi: C^J(\mathrm{Lan}\,G, H) \to C^{\mathrm{ob}J}(G, UH)$.

$$\begin{tikzcd} \coprod\limits_{J(x,j)}\coprod Gx \arrow[r] \arrow[d] & Hj \arrow[d] & \arrow[r, phantom, "\overset{\phi}{\mapsto}"] & {} & Gj \arrow[r] & UHj \\ \coprod\limits_{J(x,k)}\coprod Gx \arrow[r] & Hk & & & Gk \arrow[r] & UHk \end{tikzcd}$$

$\sim$ $Gj \hookrightarrow \coprod\limits_{J(j,j)} Gj \hookrightarrow \coprod\limits_{J(x,j)}\coprod Gx \to Hj$

Given a natural transformation $\mu: \mathrm{Lan}\,G \Rightarrow H$, $\phi(\mu): G \Rightarrow UH$ is given by $\mu_j \circ \iota_{\coprod_{J(j,j)} G_j} \circ \iota_{G_j}: Gj \to Hj$.

Naturality of μ is trivial as $\mathrm{ob}J$ has no morphisms.

As for $\phi^{-1}: C^{\mathrm{ob}J}(G, UH) \to C^J(\mathrm{Lan}\,G, H)$, we need to take a nat. transformation $\lambda: G \Rightarrow UH$ to a nat. transformation $\phi^{-1}(\lambda): \mathrm{Lan}\,G \Rightarrow H$ s.t. $\phi(\phi^{-1}(\lambda)) = \lambda$ and $\phi^{-1}(\phi(\mu: \mathrm{Lan}\,G \Rightarrow H)) = \mu$.

$$\begin{tikzcd} Gj \arrow[r, "\lambda"] & UHj & \arrow[r, phantom, "\overset{\phi^{-1}}{\mapsto}"] & {} & \coprod\limits_{J(x,j)}\coprod Gj \arrow[r, "\phi^{-1}(\lambda)"] \arrow[d] & Hj \arrow[d] \\ Gk \arrow[r] & UHk & & & \coprod\limits_{J(x,k)}\coprod Gk \arrow[r] & Hk \end{tikzcd}$$

$$\begin{tikzcd} Gj \arrow[d, "\lambda_j"'] \arrow[dr] & & Gj \arrow[dl] \arrow[d, "\lambda_j"] & & Gk \arrow[dl] \arrow[d, "\lambda_k"] \\ Hj \arrow[ddrr, bend right, "Hf_{j_i}"'] & \coprod\limits_{J(j,j)} Gj \arrow[dr] & Hj \arrow[dd, "Hf_{j_i}"] & \coprod\limits_{J(k,j)} Gk \arrow[dl] & Hk \arrow[ddll, bend left, "Hf_{k_i}"] \\ & & \coprod\limits_{x\in J}\coprod\limits_{J(x,j)} Gx \arrow[d, "\phi^{-1}(\lambda)_j"] & & \\ & & Hj & & \end{tikzcd}$$

$f_{x_i} \in J(x,j)$

Given $\lambda: G \Rightarrow UH$, we can define a cocone under J with components $\{Hf_{x_i} \circ \lambda_x\}_{x\in J, f_{x_i} \in J(x,j)}$. By the universal property of $\mathrm{Lan}\,Gj$, $\exists!$ $\mathrm{Lan}\,Gj \to Hj$ factoring this cocone through $\mathrm{Lan}\,Gj$; we will define $\phi^{-1}(\lambda)_j$ to be this arrow.

To see that ϕ, ϕ^{-1} are inverses first let $\mu : \mathrm{Lan}\, G \Rightarrow H$ be given. $\phi^{-1}(\phi(\mu)) = \phi^{-1}(\mu \circ \iota \circ \iota)$, but inspection tells us this yields a diagram of the form

$$\begin{tikzcd}
G_j \arrow[dr, "\iota"] & & G_j \arrow[dl, "\iota"'] \\
& \coprod_{J(j,j)} G_j \arrow[dr, "\iota"] \arrow[ddr, bend right, "\mu_j"'] & \\
& & \coprod_{x \in J} \coprod_{J(x,j)} G_x \arrow[d, dashed] \\
& & H_j
\end{tikzcd}$$

and so the unique arrow factoring the cocone with nadir H_j through $\mathrm{Lan}\, G_j$ must be exactly μ_j.

OTOH let $\lambda : G \Rightarrow UH$ be given. Then $\phi^{-1}(\lambda)_j$ is the unique map s.t.

$$\phi^{-1}(\lambda)_j \circ \iota_{\coprod_{J(j,j)} G_j} \circ \iota_{1_j} = H(1_j) \circ \lambda_j$$

Hence $\phi(\phi^{-1}(\lambda_j)) = \phi^{-1}(\lambda_j) \circ \iota \circ \iota = H(1_j) \circ \lambda_j = 1_{H_j} \circ \lambda_j = \lambda_j$.

(iv) Prove that this adjunction is monadic by appealing to the monadicity theorem.

Our attempt will be to show that $U: C^J \to C^{\mathrm{ob}J}$ strictly creates all colimits, from whence the proposition that U strictly creates coequalizers of U-split pairs follows immediately.

$C \sim \mathrm{Set}$, $J \sim \begin{smallmatrix} a \\ \downarrow \\ b \end{smallmatrix}$, indexed by $\bullet \quad \bullet \to \bullet$

$$\begin{pmatrix} a \to 1 \\ b \to 2 \end{pmatrix} \quad \begin{pmatrix} a \to 2 \\ b \to 1 \end{pmatrix} \Rightarrow \begin{pmatrix} a \to 3 \\ b \to 2 \end{pmatrix}$$

$$\Downarrow$$

$$\begin{pmatrix} a \to \mathrm{colim}(1 \quad 2 \to 3) \\ b \to \mathrm{colim}(2 \quad 1 \to 2) \end{pmatrix}$$

$$\Downarrow$$

$$\begin{pmatrix} a \to 4 \\ b \to 10 \end{pmatrix}$$

$\mapsto$

$$\begin{pmatrix} a & \to & 1 \\ \downarrow & & \downarrow \\ b & \to & 2 \end{pmatrix} \quad \begin{pmatrix} a & \to & 2 \\ \downarrow & & \downarrow \\ b & \to & 1 \end{pmatrix} \Rightarrow \begin{pmatrix} a & \to & 3 \\ \downarrow & & \downarrow \\ b & \to & 2 \end{pmatrix}$$

$$\Downarrow$$

$$\begin{pmatrix} a & \to & \mathrm{colim}(1 \quad 2 \to 3) \\ \downarrow & & \downarrow \\ b & \to & \mathrm{colim}(2 \quad 1 \to 2) \end{pmatrix}$$

$$\Downarrow$$

$$\begin{pmatrix} a & \to & 4 \\ \downarrow & & \downarrow \\ b & \to & 10 \end{pmatrix}$$

Let $K: D \to C^J$ be a small diagram and suppose $UK: D \to C^{\mathrm{ob}J}$ admits a colimit cone $\lambda: UK \Rightarrow \mathrm{colim}\, UK$ in $C^{\mathrm{ob}J}$. We want to show that λ uniquely lifts to a colimit cone $\lambda': K \Rightarrow \mathrm{colim}K$ in C^J, such that this cone is also preserved by U.

Firstly, we know that $\mathrm{colim}\, UK$ is defined "component-wise"; in particular, $\mathrm{colim}\, UK: \mathrm{ob}J \to C$ is such that $\mathrm{colim}\, UK(j \in J)$ is the colimit of the diagram $(UK)_j: D \to C$ given by $(UK)_j(d) = UK(d)(j)$.

In order for U to preserve the lifted colimit, it must be that $\mathrm{colim}K: J \to C$ is such that $\mathrm{colim}K(j \in J) = \mathrm{colim}\,(UK)_j$. Furthermore the action of $\mathrm{colim}K$ on morphisms is forced: An inclusion natural transformation $\mu_d: (\Phi: J \to C) \to \mathrm{colim}K$ and morphism $f: j \to j' \in J$, the naturality of μ_d

$$\begin{array}{ccc} \Phi_j & \xrightarrow{\mu_{dj}} & \mathrm{colim}K(j) = \mathrm{colim}\,(UK)_j \\ \Phi f \downarrow & & \downarrow \mathrm{colim}K(f) \\ \Phi_{j'} & \xrightarrow[\mu_{dj'}]{} & \mathrm{colim}K(j') = \mathrm{colim}\,(UK)_{j'} \end{array}$$

$$\begin{pmatrix} j & \to & x \\ \downarrow & & \downarrow f \\ j' & \to & a \end{pmatrix} \quad \begin{pmatrix} j & \to & y \\ \downarrow & & \downarrow f \\ j' & \to & b \end{pmatrix} \to \begin{pmatrix} j & \to & z \\ \downarrow & & \downarrow f \\ j' & \to & c \end{pmatrix}$$

$$\begin{pmatrix} j & \to & \mathrm{colim}(x \quad y \to z) \\ \downarrow & & \downarrow f \\ j' & \to & \mathrm{colim}(a \quad b \to c) \end{pmatrix}$$

induces a cocone under $(UK)_j$ with nadir $\mathrm{colim}\,(UK)_{j'} = \mathrm{colim}K(j')$, hence there is a unique morphism $\mathrm{colim}K(f): \mathrm{colim}K(j) \to \mathrm{colim}K(j')$ for each $f \in J$, that satisfies the naturality of μ_d. This shows $\mathrm{colim}K$, if lifted, is unique. It remains to show it really is a colimit, with the evidently lifted colimit cone. First of all, by our preceding construction, maps of the colimit cone λ have been lifted appropriately in λ'. Now taking any other colimit cone $\gamma: K \Rightarrow (A: J \to C)$, we need to show it factors uniquely through $\mathrm{colim}K$ via a natural transformation $\phi: \mathrm{colim}K \Rightarrow A$. But such a natural transformation is given uniquely by the colimit cone λ, since the data of ϕ are just maps $\mathrm{colim}K(j) \to A_j$ $\forall j \in J$, and $\mathrm{colim}K(j) = \mathrm{colim}\, UK(j)$ and $A(j) = UA(j)$. It remains to show that ϕ is natural in J.

$$\begin{array}{ccccc} \Phi_j & \xrightarrow{\beta_j} & \mathrm{colim}K_j & \xrightarrow{\phi_j} & A_j \\ \downarrow & & \downarrow & & \downarrow \\ \Phi_{j'} & \longrightarrow & \mathrm{colim}K_{j'} & \longrightarrow & A_{j'} \end{array}$$

(2) Consider the Kleisli category Set_T for a monad T acting on Set and choose a skeleton $N \xrightarrow{\simeq} Fin \hookrightarrow Set$ for the full subcategory of finite sets. Let L be the opposite of the full subcategory of the Kleisli cat. spanned by $0, 1, 2, \ldots \in N$, so there is an identity-on-objects functor

$$\begin{tikzcd}
N^{op} \arrow[r, dashed, "I"] \arrow[d, tail, "\simeq"'] & & L \arrow[d, tail] \\
Fin^{op} \arrow[r, tail] & Set^{op} \arrow[r, "F_T"] & Set_T^{op}
\end{tikzcd}$$

(i) Show that the categories N^{op} and L have strictly associative finite products that are preserved by the functor $I : N^{op} \to L$.

Finite products in N^{op} are equivalently finite coproducts in N. Indeed N admits all finite coproducts, as ordinal sums. For a finite diagram whose image is $\{n_i\}_{i=1,\ldots,M}$, the coproduct of this diagram is given by $\sum_{i=1}^{M} n_i \in N$. This corresponds exactly to a disjoint union in finite sets, and furthermore we can see that this coproduct is strictly associative because $n_1 + \sum_{i=2}^{M} n_2 = \sum_{i=1}^{M} n_i = (\sum_{i=1}^{M-1} n_i) + n_M$ thanks to the skeletal nature of N.

Let us enumerate what the category L is. Its objects are $0, 1, \ldots \in N$, and its morphisms $B \to A$ are opposite those in Set_T, and hence are arrows $TB \to A$ in Set. To see that L has all strictly associative finite products is to see that L^{op} has all strictly associative finite coproducts.

First let us see that Set_T has all finite coproducts. Taking $\{A_i\}_{i=1,\ldots,M} \subseteq ob\, Set = ob\, Set_T$, we claim the coproduct $\coprod_{i=1}^{M} A_i$ and cocone $\{\eta_{\coprod_{i=1}^{M} A_i} \circ \iota_{A_i}\}_{i=1}^{M}$ form a coproduct and colimit cone in Set_T. Taking any other cocone $\{A_i\}_{i=1}^{M} \Rightarrow D$ in Set_T, we have that this is equivalently a cocone $\{A_i\}_{i=1}^{M} \Rightarrow TD$

$$\begin{tikzcd}
A \arrow[dr, "\iota_A"'] \arrow[ddr, bend right] & & B \arrow[dl, "\iota_B"] \arrow[ddl, bend left] \\
& A \sqcup B \arrow[d, "\phi"] & \\
& TD &
\end{tikzcd}
\iff
\begin{tikzcd}
A \arrow[dr, "\eta_{\sqcup} \circ \iota_A"'] \arrow[ddr, bend right] & & B \arrow[dl, "\eta_{\sqcup} \circ \iota_B"] \arrow[ddl, bend left] \\
& A \sqcup B \arrow[d, "\phi"] & \\
& D &
\end{tikzcd}$$

in Set. By the universal property of $\coprod_{i=1}^{M} A_i$, $\exists! \phi : \coprod_{i=1}^{M} A_i \to TD$ factoring the cocone through the colimit cone in Set. But also $\phi \circ (\eta_{\coprod_{i=1}^{M} A_i} \circ \iota_{A_j})$ in Set_T is a composite $\mu_D \circ T\phi \circ \eta_{\sqcup A_i} \circ \iota_{A_j}$ $= \mu_D \circ \eta_{TD} \circ \phi \circ \iota_{A_j}$ (by the naturality of η) $= \phi \circ \iota_{A_j}$, so ϕ also factors the cocone in Set_T through the presented colimit cone, hence the described coproduct + colimit cone indeed are admitted as a coproduct definition under the diagram A in Set_T.

Now, taking $\{n_i\}_{i=1}^{M} \subseteq ob\, L^{op} = ob\, N$, we have firstly that $\sum_{i=1}^{M} n_i \in N$ is coproduct of this diagram in L^{op} as described above, and since L^{op} is a full subcategory spanned by $ob\, N$, $\{\eta_{\sum n_i} \circ \iota_{n_i}\}_{i=1}^{M}$ forms is admitted as a colimit cone in L^{op}. Hence L^{op} has all finite coproducts, and since these coproducts are objects of N they are strictly associative as argued earlier.

Dually, L has all finite products, which are strictly associative.

It remains to define the functor $I : N^{op} \to L$ and show that it preserves all finite products in N. By hypothesis, I is the identity on objects. On morphisms, the constraint that I preserve finite products mandates that morphisms $f : A \to B \in N$ be taken to $\eta_B \circ f : A \to TB \in L^{op}$, as done in the construction of colimit over in L^{op}. But this definition is precisely the action of the free functor into the kleisli category Set_T, and so we see that $I = F_T|N^{op}$ (F_T restricted on N^{op}).

When the monad T is finitary, $I : N^{op} \to L$ defines its associated <u>Lawvere Theory</u>.
Objects in L are iterated finite products of 1, so the data of L is the set $L(n,1)$ of n-ary ops.
From this data, the category of T-algebras can be recovered. A <u>model</u> of the Lawvere Theory in Set is a finite product preserving functor $L \to Set$. A morphism b/w models is a nat. transform.
(Bartosz ~ Lawvere Theory)

(iii) Define a functor from the category of T-algebras to the category of models for the Lawvere theory $I : N^{op} \to L$ (Hint: see 5.2.viii).

We want to define a functor from the Eilenberg-Moore category Set^T to the category spanned by models $L \to Set$, finite-product preserving functors.
Considering the functor $Set^T(K -, -) : Set^T \to Set^{Set_T^{op}}$ from 5.2.viii, we can see that the restriction of this functor on the inclusion $y : L \to Set_T^{op}$ defines a functor $Set^T(Ky -, -) : Set^T \to Set^L$ from the category of T-algebras over sets to the category of models of L, where K is the canonical functor from the kleisli category to the category of T-algebras.

In particular, this functor takes an algebra (A, a) to model $Set^T(Ky -, (A,a)) : L \to Set$ that yields the set of algebra homomorphisms from Tn, $n \in N$ and A. For example, for $1 \in L$ we are yielded the set of homomorphisms $Set^T((T1, \mu_1), (A, a))$, and for any other $n \in N$ the set $Set^T((Tn, \mu_n), (A, a))$. For a map $f : n \to 1 \in L$, this model takes f to a map $Set^T((T1, \mu_1), (A,a)) \to Set^T((Tn, \mu_n), (A,a))$ that in some sense seems to characterize the operation in the L-model to an algebra homomorphism.

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(1) 3.3.4 suggests an alternate form of the monadicity theorem: A functor $U: D \to C$ is monadic if

- U has a left adjoint
- U reflects isomorphisms
- D has and U preserves U-split coequalizers

Use this, along with a direct proof that any cont. bijection b/w compact Hausdorff spaces is a homeomorphism, to re-prove 5.5.6 by showing cHaus has U-split coequalizers, constructed as in Top.

(Hint: see that the coequalizer of maps $f, g: X \rightrightarrows Y$ of compact spaces is compact. To prove a U-split coequalizer is Hausdorff, use the split coequalizer diagram in Set to prove that the kernel pair $E \hookrightarrow Y \times Y$ of the quotient map $Y \twoheadrightarrow Z$ is closed.)

(5.5.6) The underlying set functor $U: \text{cHaus} \to \text{Set}$ from the category of compact Hausdorff spaces is monadic.

First of all U admits a left adjoint $F: \text{Set} \to \text{cHaus}$; this is given by 5.1.4(v).

To see that U reflects isomorphisms, a continuous function that is an isomorphism in Set is a continuous bijection in cHaus, and all cont. bijections from compact to Hausdorff spaces are homeomorphisms.

Now let $f, g: X \rightrightarrows Y$ be parallel maps in cHaus admitting a split coequalizer in Set. Since Top is cocomplete, these maps admit a coequalizer $h: Y \twoheadrightarrow Y/\sim$ in Top. $Y/\sim$ is defined to be the quotient of Y under the equivalence relation $f(x) \sim g(x)$ $\forall x \in X$ with the quotient topology, s.t. $U \subseteq Y/\sim$ is open when its preimage under the quotient map $h: Y \twoheadrightarrow Y/\sim$ is open in Y.

U preserves this coequalizer, for the coequalizer of $f, g: UX \rightrightarrows UY$ in Set is exactly the quotient of Y under $f(x) \sim g(x)$, with the quotient map $h: Y \twoheadrightarrow Y/\sim$.

It remains to show that the coequalizer $Y/\sim$ is really a coequalizer in cHaus, not just Top.

Since $h: Y \twoheadrightarrow Y/\sim$ is a continuous surjection and Y is compact, $Y/\sim$ is compact.

Now, per the hint, we will use the U-split coequalizer diagram in Set to prove that the kernel pair $E \hookrightarrow Y \times Y$ of the quotient map $Y \twoheadrightarrow Y/\sim$ is closed.

RECALL: the kernel pair of the arrow h is the pullback of h along itself, yielding a monomorphism $(s, t): E \rightarrowtail Y \times Y$

Per 3.5,