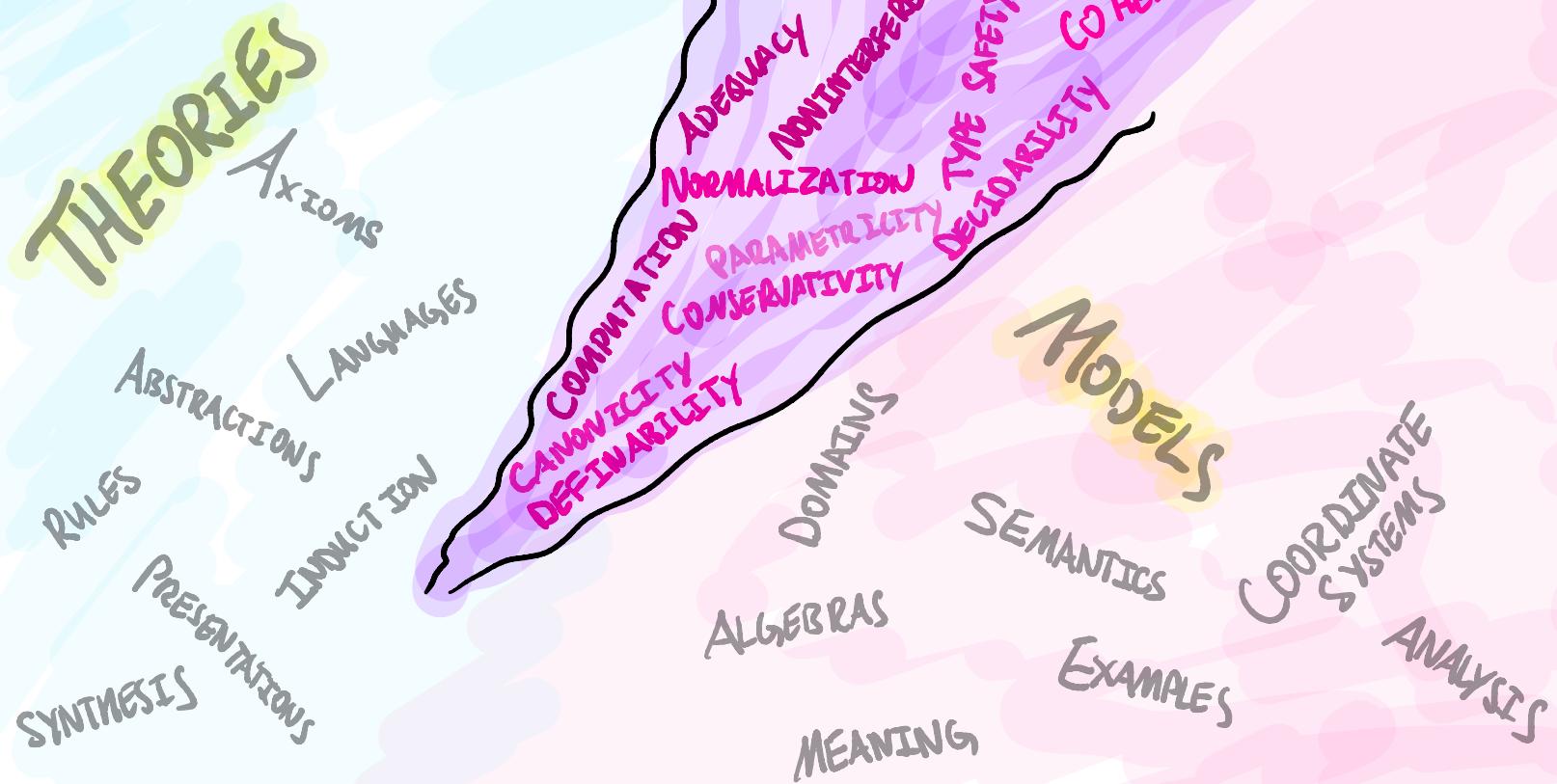


# Towards a Geometry for syntax.

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“Σχῆμα εστι τὸ ἔνδο τοῦ ἣ τίνες ἐγενέρησαν.”  
— Euclid

Syntactic metatheorists study:



There are three kinds of facts that can be known about a theory:

### Derivability

Statements that make sense inside the doctrine of a theory.

e.g. " $x : A \vdash \psi_x$  type"

Can always be proved using the rules of the theory

### Non-derivability

e.g. consistency.

Goes outside the theory.

Proved by means of

~ countermodels ~

(denotational semantics)

Geometrical / topological models

### Admissibility

Positive statements about derivability that go outside the theory.

e.g. normalization, decidability, computational adequacy

Proved by means of

(Kripke) logical relations.

Non-derivability is studied geometrically

To show that  $\vdash 0=1:\text{nat}$  in PCF, we interpret PCF into a category of geometric spaces, e.g. pointed sets.

$$[\![\text{nat}]\!] = \mathbb{N}_\perp$$

$$[0] = 0$$

$$[1] = 1$$

Suppose that  $\vdash 0=1:\text{nat}$ . Then  $[0]=[1] \in \mathbb{N}_\perp$ , a contradiction.  $\square$

But derivative can also be studied geometrically..

## Example ( $\text{ST}\lambda\mathcal{C}$ )

- \* The typed  $\lambda$ -calculus has two sorts: types and terms.  
(judgments)
- \* We have a 'category of contexts'  $\mathcal{C}$ , a free cartesian closed category.
- \* The category of presheaves  $\text{Pr}(\mathcal{C}) = [\mathcal{C}^{\text{op}}, \text{Set}]$  allows us to treat each type of  $\text{ST}\lambda\mathcal{C}$  as an étale space/sheaf over a topos (=generalized space)  $\hat{\mathcal{C}}$ . [write  $\text{Set}_{\hat{\mathcal{C}}} = \text{Pr}(\mathcal{C})$ ].

Derivability is captured by morphisms of ētale spaces:

$$\text{bool} \xrightarrow{\text{not}} \text{bool}$$

$$\begin{array}{c} \searrow \\ \hat{\mathcal{C}} \end{array}$$

$$\in \text{Set}_{\hat{\mathcal{C}}} \subseteq \text{Topos}/\hat{\mathcal{C}}$$

$$\in \text{Topos}$$

But  $\text{Set}_{\hat{\mathcal{C}}}$  contains many useful spaces beyond those that come from STLC.

For instance, there is a "generic type"  $(\text{tm} \rightarrow \text{tp})$  in  $\text{Set}_{\hat{\mathcal{C}}}$  from which every type of STLC arises by pullback:

[[univalence theorem, Awodey]]

"local universe"

$$\begin{array}{ccc} \text{Nat} & \xrightarrow{\quad} & \text{tm} \\ \downarrow & & \downarrow \\ \mathbb{I} & \xrightarrow{c_{\text{Nat}}} & \text{tp} \end{array}$$

$$\begin{array}{ccc} \text{tm}^2 & \xrightarrow{\text{pair}} & \text{tm} \\ \downarrow & & \downarrow \\ \text{tp}^2 & \xrightarrow{(x)} & \text{tp} \end{array}$$

The generalized derivability embodied in  $\hat{\mathcal{C}}$  allows us to express many useful concepts, such as schematic polymorphism:

e.g.  $\prod_{\alpha: \text{tp}} \text{tm}[\alpha \Rightarrow \alpha]$

Interesting "noninterference" properties are reflected in the internal language of  $\text{Set}_{\hat{\mathcal{C}}}$ . For instance:

$$\hat{\mathcal{C}} \models \forall A: \text{bool} \Rightarrow \text{tp}. \quad 'A \text{ is constant}'.$$

(Because STLC doesn't have dependent types!)

Other "generalized" derivabilities provide useful reduction techniques  
— See A. Kock's observation that the generic local ring is a field!

What about ADMISSIBILITY?

(What even is admissibility?)

## Admissibility

In a formal system, the rule  $\frac{\Gamma \vdash J}{\Gamma' \vdash J'}$  is "admissible" when the derivability of  $(\Gamma \vdash J)$  implies the derivability of  $(\Gamma' \vdash J')$ .

Very imprecise!

What actually are  $\Gamma, \Gamma', J, J'$ ? Do they have "parameters"?

What kind of transformation  $(\Gamma \vdash J) \rightsquigarrow (\Gamma' \vdash J')$  is allowed? Should it commute with replacement of "parameters"?

A more rational definition is needed. START FROM THE GEOMETRY!

Example: injectivity of type constructors.

Injectivity of  $\Rightarrow$

$$\frac{\vdash A \rightarrow B = A' \rightarrow B' : \mathcal{U}}{\vdash A = A' : \mathcal{U} \quad \vdash B = B' : \mathcal{U}} \quad (*)$$

Over the syntactic space  $\hat{\mathcal{C}}$ , we first note that the following statement is not true, corresponding to the non-derivability of (\*):

$$\hat{\mathcal{C}} \not\models \Gamma(\Rightarrow) : \mathcal{U}^2 \rightarrow \mathcal{U} \text{ is injective?} \quad (\dagger)$$

Easily refuted using a countermodel. But what were the syntactical obstructions? Roughly that the "proof" of injectivity is not natural/does not commute with substitution.

New plan:

Find a way to "enlarge"  $\hat{C}$  to make (7) true,  
without losing the connection to syntax.

Let  $\mathcal{A}$  be the cartesian category of contexts and renamings of variables (not all substitutions!).

We have :

$$\mathcal{A} \xrightarrow{\alpha} \mathcal{C} : \text{CAT}$$

hence:

$$\hat{\mathcal{A}} \xrightarrow{\hat{\alpha}} \hat{\mathcal{C}} : \text{Topos}$$

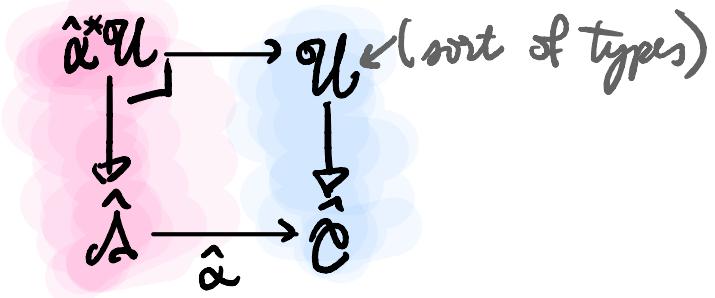
given by:

$$\hat{\alpha}^*: (E: \text{Set}_{\hat{\mathcal{C}}}) \longmapsto (\lambda a: \mathcal{A}^{\text{op}}. E(\alpha(a)))$$

$\perp$   
 $\hat{\alpha}_x$ 
↑  
precomposition

Idea: restrict the sorts of our language along  $\hat{\alpha}:\hat{A}\rightarrow\hat{C}$ :

e.g.



Theorem. We have:

$$\hat{A} \models [\hat{\alpha}^*(\Rightarrow) : \hat{\alpha}^*U^2 \rightarrow \hat{\alpha}^*U \text{ is injective}]$$

Theorem. We have:

$$\hat{\lambda} \models [\hat{\alpha}^*(\Rightarrow) : \hat{\alpha}^*\mathcal{U}^2 \rightarrow \hat{\alpha}^*\mathcal{U} \text{ is injective}]$$



Suggests a notion of " $\alpha$ -admissibility" for each  $\alpha : A \rightarrow C$ .

But how do we prove such admissibilities?

'Kripke Logical Predicates'

In essence, we need to do an 'inductive proof' over  $\mathcal{C}$ , where the motive of induction is valued in  $\text{Set}_{\mathcal{A}}$ . That's what Kripke logical relations do!

1) To each  $C : \mathcal{C}$ , assign a subobject of

$$\begin{array}{c} [c] \\ \downarrow \\ \hat{\alpha}^* \gamma_C^c \end{array} : \text{Set}_{\mathcal{A}}$$

2) To each  $c \xrightarrow{f} d : \mathcal{C}$ , assign a diagram:

$$\begin{array}{ccc} [c] & \xrightarrow{[f]} & [d] \\ \downarrow & & \downarrow \\ \hat{\alpha}^* \gamma_C^c & \xrightarrow{\hat{\alpha}^* f} & \hat{\alpha}^* \gamma_C^d \end{array}$$

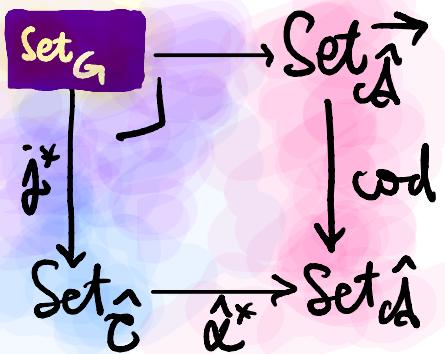
(Funitrivially)

But this is almost a comma category / katin gluing!

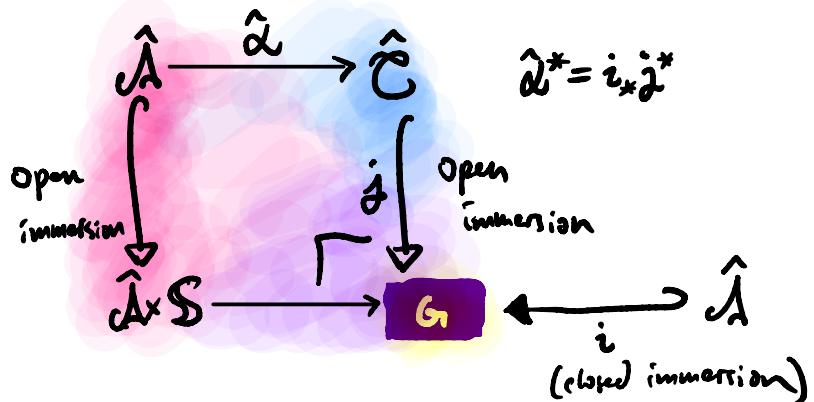
$$\begin{array}{ccc} \text{KripRel}_{\mathcal{A}} & \longrightarrow & \text{Sub}(\text{Set}_{\mathcal{A}}) \\ \downarrow & & \downarrow \\ \mathcal{C} & \xrightarrow{\gamma_0} & \text{Set}_{\mathcal{B}} \xrightarrow{\hat{\alpha}} \text{Set}_{\mathcal{A}} \end{array}$$

We may simplify things by relaxing the requirement that  $\mathcal{L}^{\text{op}} \downarrow \mathcal{C}$  be a mono.

We have a category (topos!) of generalized, proof-relevant Kripke Logical Predicates:

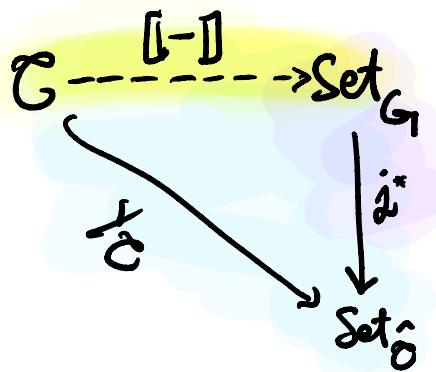


$\uparrow$   
(in category theory)



$\uparrow$   
(in topos geometry)

To make a Kripke logical predicate model, "just" define a structure preserving functor:



| IDEA:

Just as the internal language of  $\text{Set}_G$  has provided a rational/objective basis to study derivability,

we may study  $\alpha$ -admissibility by grasping the internal language of  $\text{Set}_{\hat{G}}$ .

I characterize the internal language of  $\text{Set}_G$  by a simple set of postulates called:

**SYNTHETIC TAIT COMPUTABILITY**

# Euclid's Postulates for syntactic metatheory?

We abstract the following properties of  $\text{Set}_{\mathcal{G}_7}$ :

- 1) Ext. Martin-Löf type theory with universes  $\mathcal{U}_1, \mathcal{U}_2$  and QITs.
- 2) There exists a proposition  $\varphi : \mathcal{U}_2$ .

Def.: A type  $E$  is  $\varphi$ -modal when  $E \rightarrow E^\varphi$  is iso  
" "  $\varphi$ -connected "  $\text{Ex}\varphi \rightarrow \varphi$  is iso

In  $\text{Set}_{G_1}$ ,  $\psi$  is the subterminal

$$\left( \begin{array}{c} \emptyset \rightarrow || \\ \downarrow \\ \hat{\alpha}^* 11 = \hat{\alpha}^* x|| \end{array} \right)$$

Then we reconstruct  $\text{Set}_{\tilde{G}}$  as  $\text{Set}_{G_1}/\psi$ . [Artin, Grothendieck, Ex. 1962]

$$\begin{array}{ccccc}
 & \text{Set}_{\tilde{G}} & \xrightarrow{\sim} & \text{Set}_{G_1}/\psi & \\
 \uparrow E & \begin{array}{c} \uparrow -1 \\ \text{Set}_{\tilde{C}} \\ \downarrow E \\ \text{Set}_{G_1} \end{array} & & \begin{array}{c} \uparrow \psi \times E \\ \text{Set}_{G_1}/\psi \\ \downarrow -1 \\ \text{Set}_{G_1} \end{array} & \downarrow \psi \times E \\
 & \left( \begin{array}{c} \uparrow -1 \\ \text{Set}_{\tilde{C}} \\ \downarrow E \\ \text{Set}_{G_1} \end{array} \right) & & \left( \begin{array}{c} \uparrow -1 \\ \text{Set}_{G_1}/\psi \\ \downarrow E \\ \text{Set}_{G_1} \end{array} \right) & \downarrow \Pi_{\psi E} \\
 & & \xrightarrow{=} & &
 \end{array}$$

3) Given a  $\varphi$ -modal type  $E$  and a family  $x:E \vdash F(x)$  of  $\varphi$ -connected types, there exists a type  $E \times F$  and an iso.

$$\sum_E F \xrightarrow{\text{glue}} E \times F \quad \text{s.t.} \quad -\vdash \varphi \vdash E \times F = E \text{ strictly and}$$

$$-\vdash \varphi \vdash \text{glue} = \pi_1 : \sum_E F \rightarrow E.$$


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These axioms are called **Synthetic Tait Computability**, after Bill Tait's introduction of the method of computability / logical predicates in the late 1960s.

Two modalities:

[Rijkhuizen, Shulman, Spitters 2017]

$$\textcircled{O} A := \varphi \Rightarrow A$$

$$\bullet A := \varphi \sqcup_{\varphi \times A} A$$

(open modality)

(closed modality)

Facts:  $\textcircled{O} \bullet A = \perp$ .

$$\begin{array}{ccc} A & \xrightarrow{\eta_0} & \bullet A \\ \downarrow \eta_0^A & \nearrow & \downarrow \eta_A \\ \textcircled{O} A & \longrightarrow & \bullet \textcircled{O} A \\ & \eta_{\textcircled{O} A} & \end{array}$$

Example:

$$[\![\text{bool}]\!] = \sum_{b: \text{Bool}} \bullet \left\{ i: 2 \mid \begin{cases} i=0 \rightarrow b=\text{true} \\ i=1 \rightarrow b=\text{false} \end{cases} \right\}$$

$$[\![\text{true}]\!] = (\text{true}, \eta_0 \circ \textcircled{O})$$

$$[\![\text{false}]\!] = (\text{false}, \eta_0^{-1})$$

With just (\*) these axioms, we can reconstruct most Kripke Logical Predicate arguments in a simpler, less error-prone way:

- \* Canonicity and normalization for MLTT (S.)
- \* Parametricity and data abstraction for ML modules (S., Harper)
- \* Computational adequacy for denotational semantics à la Plotkin (S.)

But we have also proved new theorems that were previously out of reach:

\* Normalization & decidability of type-checking for cubical type theory (S., Angiuli)

\* Normalization & decidability of type checking for multi-modal type theory (Gratzer)

# Prospects for language design

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STC is not only a tool to make hard theorems.

The STC modalities express a form of non-interference or phase distinction, e.g. between syntax & semantics:

Any map  $\bullet A \rightarrow \circ B$  is constant.

But syntax/semantics is not the only useful phase distinction!

# Phase Distinctions & Noninterference for language design

- 1) Static/dynamic phase separation in ML Modules (Harper, Mitchell, Moggi 199)
- reconstructed by S. E. Harper in our 2021 J.ACM paper.
- 2) Behavior / complexity : Niu, S., Grodin, Harper (POPL 2022)
- 3) Public vs. private : STC for security typing / IFC.  
Termination-insensitive noninterference for security modalities.

Thanks!

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See my website [jonmsterling.com](http://jonmsterling.com) for links to papers,  
including my PhD thesis:

First Steps in Synthetic Tait Computability:  
the Objectin Metatheory of Cubical Type Theory.