Patch Locale of a Spectral Locale in Univalent Type Theory

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Goal

Implement the patch locale in univalent type theory predicatively i.e. without using resizing axioms.

What is a locale?

Notion of space characterised solely by its frame of opens.

What is the patch locale?

What is a spectral locale?

A locale in which the compact opens form a basis closed under finite meets.

What is a Stone locale?

A compact locale in which the clopens form a basis.

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Stone $\Rightarrow$ Spectral

Every Stone locale is spectral as the clopens coincide with the compact opens in Stone locales.

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Patch transforms spectral locales into Stone ones.
It is the universal such transformation.

Patch as a coreflector

A spectral map is a map of locales reflecting compact opens.

Spec is the category of spectral locales together with spectral maps.

The category of Stone locales forms a full subcategory of **Spec** that is denoted **Stone**.

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Stone **Spec**

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$$\mathbf{Stone} \overset{i}{\hookrightarrow} \mathbf{Spec}$$

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\end{tikzcd}
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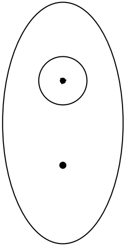
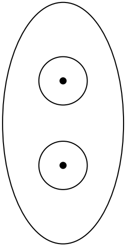
$$\begin{tikzcd} \mathbf{Stone} \arrow[r, hook, bend left, "i"] \arrow[r, phantom, "\perp"] & \mathbf{Spec} \arrow[l, dashed, bend left, "Patch"] \end{tikzcd}$$

Some examples of patch

Spectral locale in consideration	**Its patch**
Sierpiński space (Ω)	Booleans ($\mathbf{2}$)
	
Scott topology of a (Scott) domain	Lawson topology
$\mathcal{P}(\mathbb{N}) \simeq \Omega^{\mathbb{N}}$	Cantor space ($\mathbf{2}^{\mathbb{N}}$)
Scott topology of domain $\mathbb{N}_\perp$	$\mathbb{N}_\infty$

Frames in type theory

Write $\mathsf{Fam}_{\mathcal{W}}(A) :\equiv \Sigma_{I:\mathcal{W}} I \to A$.

Definition (Frame)

A $(\mathcal{U}, \mathcal{V}, \mathcal{W})$-*frame* consists of

- a type $A : \mathcal{U}$,
- a partial order $- \leq - : A \to A \to \mathsf{hProp}_{\mathcal{V}}$,
- a top element $\top : A$,
- a binary meet operation $- \wedge - : A \to A \to A$,
- a join operation $\bigvee \_ : \mathsf{Fam}_{\mathcal{W}}(A) \to A$,
- satisfying distributivity i.e. $x \wedge \bigvee_{i:I} y_i = \bigvee_{i:I} x \wedge y_i$ for every $x : A$ and family $\{y_i\}_{i:I}$ in A.

The carrier type does not have to be explicitly required to be a set since this follows from the existence of a partial order on it.

Some notation

A frame homomorphism is a function preserving finite meets and arbitrary joins.

The category of frames and their homomorphisms is denoted **Frm**; its opposite is denoted **Loc**.

- Morphisms of **Loc** are called continuous maps.

The frame corresponding to a locale X is denoted $\mathcal{O}(X)$.

We pretend *as though* locales were spaces and use the letters

- $X, Y, Z, \ldots$ for them;
- $f, g : X \to Y$ for their continuous maps; and
- $U, V : \mathcal{O}(X)$ for their opens.

$f^* : \mathcal{O}(Y) \to \mathcal{O}(X)$ denotes the frame homomorphism corresponding to a continuous map $f : X \to Y$ of locales.

Patch as the frame of Scott-continuous nuclei

A nucleus on frame L is an endofunction $j : |L| \to |L|$ that is inflationary, idempotent, and preserves binary meets.

A nucleus is called Scott-continuous if it preserves joins of directed families.

Patch of L is the frame of Scott-continuous nuclei on L.

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This description of Patch was used by Escardó [1] to give a constructive, yet *impredicative*, treatment of the patch frame.

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- This question turns out to be nontrivial.

Bases for frames

Consider a $(\mathcal{U}, \mathcal{V}, \mathcal{W})$-locale X.

Defn. (Basis)

A $\mathcal{W}$-family $\{B_i\}_{i:I}$ over a $(\mathcal{U}, \mathcal{V}, \mathcal{W})$-locale X is said to form a basis for X if

for any $U : \mathcal{O}(X)$, there is a *subfamily* $\{B_l\}_{l \in L}$ of $\{B_i\}_{i:I}$ such that
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In our work, we are primarily interested in frames with bases of the form $(\mathcal{U}^+, \mathcal{U}, \mathcal{U})$ i.e.

large and locally small frames with small bases.

Spectrality revisited (1)

Recall the impredicative definition of a spectral locale as:

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- B_i is compact for each $i : I$, and
- $\{B_i\}_{i:I}$ is closed under finite meets i.e. there exists some $t : I$ such that $\top = B_t$ and for any two $j, k : I$, there exists some $l : I$ such that $B_l = B_j \wedge B_k$.

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We use the same idea for Stone-ness.

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Proposition

Let X be a spectral locale with basis $\{B_i\}_{i:I}$. Given any compact $K : \mathcal{O}(X)$, there is some $k : I$ such that $K = B_k$.

Ordering on nuclei – *size matters* (1)

Let

- X be a large and locally small spectral locale with basis $\{B_i\}_{i:I}$, and
- j and k be two Scott-continuous nuclei on X.

Define $j \preceq k :\equiv \prod_{U:\mathcal{O}(X)} j(U) \leq k(U)$.

Problem: $j \preceq k$ lives in universe $\mathcal{U}^+$.

This means Patch(X) is a $(\mathcal{U}^+, \mathcal{U}^+, \mathcal{U})$-locale i.e. it is *not* locally small.

Solution: define a small version of the ordering.

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Ordering on nuclei – *size matters* (2)

Proposition

$j \preceq k$ iff $j \preceq_S k$.

Proof

- The nontrivial direction is $j \preceq_S k \to j \preceq k$.
- Let $U = \bigvee_{l \in L} B_l$ be an open of locale X.
- $j\left(\bigvee_{l \in L} B_l\right) = \bigvee_{l \in L} j(B_l) \leq \bigvee_{l \in L} k(B_l) = k\left(\bigvee_{l \in L} B_l\right)$.

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Notice the use of Scott-continuity!
It is crucial to the local smallness of Patch.

Closed and open nuclei

Let X be a spectral locale and $U : \mathcal{O}(X)$ an open.

We embed the opens of X into Patch(X) using the closed and open nuclei.

Closed nucleus of U: $\text{‘}U\text{’} :\equiv V \mapsto U \vee V$.
Open nucleus of U: $\neg\text{‘}U\text{’} :\equiv V \mapsto U \Rightarrow V$.

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- The usual definition of Heyting implication (e.g. via the Adjoint Functor Theorem) is impredicative.
- We use (a version of the) Adjoint Functor Theorem for locally small frames with small bases.

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Formalised in modules `AdjointFunctorTheoremForFrames`, `GaloisConnection`, `HeytingImplication` of Escardó's `TypeTopology` [2] Agda development.

Patch is Stone

Theorem

Given a spectral $(\mathcal{U}^+, \mathcal{U}, \mathcal{U})$-locale X with small basis $\{B_i\}_{i:I}$, Patch(X) is a Stone locale.

Proof idea

The family

$$\{‘B_k’ \wedge \neg ‘B_l’ \mid k, l : I\}$$

forms a basis for Patch(X) and the covering subfamily for a given Scott-continuous nucleus $j : \mathcal{O}(X) \to \mathcal{O}(X)$ is

$$\{‘B_k’ \wedge \neg ‘B_l’ \mid B_k \leq j(B_l), k, l : I\}$$

Summary

We set out to implement a rather important construction of pointfree topology in univalent type theory, without using resizing.

Doing this predicatively turned out to involve surprising challenges.

We had to reformulate quite a few things in the theory itself to obtain a **type-theoretic understanding** of the construction in consideration.

Our work has been almost completely formalised in the Agda proof assistant, as part of Escardó's `TypeTopology` [2] library.

References I

[1] Escardó, Martín H. "On the Compact-regular Coreflection of a Stably Compact Locale". In: *Electronic Notes in Theoretical Computer Science* 20 (1999), pp. 213–228. ISSN: 15710661. DOI: `10.1016/S1571-0661(04)80076-8`.

[2] Escardó, Martín H. and contributors. *TypeTopology*. Agda development. URL: `https://github.com/martinescardo/TypeTopology`.