# A Construction of The Patch Frame in Univalent Type Theory

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**Abstract**

Lorem ipsum.

## 1 Preliminaries

**Definition 1** (Family). A "$\mathcal{U}$-family on type $A$" is[1] simply a pair $(I, f)$ where $I : \mathcal{U}$ and $f : I \to A$. We denote the type of $\mathcal{U}$-families by $\mathsf{Fam}$ i.e. $\mathsf{Fam}_{\mathcal{U}}(A) :\equiv \Sigma_{(I:\mathcal{U})} I \to A$.

**Definition 2** (Directed family). A family $(I, f)$ on some type $A$ is called *directed* iff (1) $I$ is inhabited, and (2) for every $i, j : I$, there exists some $k : I$ such that $f(k)$ is the upper bound of $\{f(i), f(j)\}$.

Our definition of a frame is parameterised by three universes: one for the carrier set, one for the order, and one for the index type of families on which the join operation is defined. We adopt the convention of using universe variables of $\mathcal{U}$, $\mathcal{V}$, and $\mathcal{W}$ for these respectively.

**Definition 3** (Frame). A $(\mathcal{U}, \mathcal{V}, \mathcal{W})$-frame consists of:

- an h-set $|F| : \mathcal{U}$,

- a partial order $\_ \leq \_ : |F| \to \Omega_{\mathcal{V}}$,

- a top element $\top : |F|$,

- a meet operation i.e. an operation $\_ \wedge \_ : |F| \to |F| \to |F|$ such that $x \wedge y$ is the GLB of $x$ and $y$ for every $x, y : |F|$, and

- a join operation i.e. an operation $\bigvee \_ : \mathsf{Fam}_{\mathcal{W}}(|F|) \to |F|$ such that given a family $\{x_i\}_{i \in I}$, $\bigvee_i x_i$ is the LUB of $\{x_i\}_{i \in I}$,

such that binary meets distribute over arbitrary joins:

$$x \wedge \bigvee_{i:I} y_i = \bigvee_{i:I} x \wedge y_i$$

for every $x : |F|$, family $\{y\} : \mathsf{Fam}_{\mathcal{W}}(|F|)$.

A nucleus on a frame $F$ is an equivalent characterisation of a regular monomorphism on the locale corresponding to $F$.

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[1] It would be reasonable to require $I$ to be an h-set and $f$ to be injective but we are providing a direct translation of the definition in our formalisation. This definition might have to be updated with these requirements.

**Definition 4** (Nucleus)**.** A nucleus on a $(\mathcal{U}, \mathcal{V}, \mathcal{W})$-frame $F$ is an endofunction $j : |F| \to |F|$ satisfying:

- (meet-preservation) $j(x \wedge y) = j(x) \wedge j(y)$;

- (inflation) $x \leq j(x)$; and

- (idempotence) $j(j(x)) \leq j(x)$.

for every $x, y : |F|$.

If an endofunction satisfies meet-preservation and inflation but not necessarily idempotence, it is called a *prenucleus.*

We write $\mathsf{Nucleus}(F)$ to denote the type of nuclei on a given frame $F$.

**Proposition 1** (Prenuclei are monotonic)**.** Every prenucleus is monotonic.

*Proof.* Let $j : |F| \to |F|$ be a nucleus on some frame $F$ and let $x, y : |F|$ s.t. $x \leq y$. We have:

$$j(x) = j(x \wedge y) = j(x) \wedge j(y) \leq j(y),$$

using meet-preservation and the fact that $x \leq y \leftrightarrow x \wedge y = x$. $\qquad\square$

**Proposition 2.** Given any two prenuclei $j$ and $k$ on some frame $F$, $j \circ k$ is an upper bound of $\{j, k\}$.

*Proof.* Given any $x : |F|$, we have that $k(x) \leq j(k(x))$ by the inflation of $j$ and that $j(x) \leq j(k(x))$ by the monotonicity of $j$, since $x \leq k(x)$ by the inflation of $k$. $\qquad\square$

**Proposition 3.** Given any nucleus $F$, $\mathsf{Nucleus}(F)$ is an h-set.

**Definition 5** (Scott-continuity)**.** A function $f : |F| \to |G|$ from a $(\mathcal{U}, \mathcal{V}, \mathcal{W})$-frame $F$ to a $(\mathcal{U}, \mathcal{V}, \mathcal{W})$-frame $G$ is called *Scott-continuous* iff given any *directed* $\{x_i\} : \mathsf{Fam}_{\mathcal{W}}(|F|)$,

$$f\left(\bigvee_{i:I}^{F} x_i\right) = \bigvee_{i:I}^{G} f(x_i).$$

## 2 Meet-semilattice of nuclei

Nuclei form a meet-semilattice when ordered pointwise.

**Definition 6.** The type of nuclei on a given frame $F$ forms a meet-semilattice as follows:

- order: given nuclei $j$ and $k$, $j \leq k :\equiv \forall x \in |F|.\ j(x) \leq k(x)$;

- top nucleus: $\_ \mapsto \top_F$; and

- meet of two nucleus: $j \wedge k :\equiv x \mapsto j(x) \wedge_F k(x)$.

The fact that $j \wedge k$ satisfies the nucleus laws is given in Proposition 4.

We denote this $\mathcal{N}(F)$.

**Proposition 4.** Given two nuclei $j$ and $k$ on some frame $F$, the function $x \mapsto j(x) \wedge k(x)$ is a nucleus.

*Proof.* Inflation is direct from the inflation of $j$ and $k$ and the fact that $j(x) \wedge k(x)$ is the GLB of $j(x)$ and $k(x)$. To see that meet-preservation holds, let $x, y : |F|$. We have:

$$
\begin{aligned}
j(x \wedge y) \wedge k(x \wedge y) &= j(x) \wedge j(y) \wedge k(x) \wedge k(y) \\
&= (j(x) \wedge k(x)) \wedge (j(y) \wedge k(y)).
\end{aligned}
$$

For idempotence, let $x : |F|$. We have:

$$
\begin{aligned}
j(j(x) \wedge k(x)) \wedge k(j(x) \wedge k(x)) &\leq j(j(x)) \wedge j(k(x)) \wedge k(j(x)) \wedge k(k(x)) \\
&\leq j(j(x)) \wedge k(k(x)) \\
&\leq j(x) \wedge k(x).
\end{aligned}
$$

$\square$

**Definition 7.** The type of Scott-continuous nuclei on a given $(\mathcal{U}, \mathcal{V}, \mathcal{W})$-frame $F$ forms a meet-semilattice in the same way as in Definition 6. Its top element is $\top_{\mathcal{N}(F)}$ which is trivially Scott-continuous. It remains only to be shown that the meet of two Scott-continuous nuclei is a Scott-continuous nucleus. Consider two Scott-continuous nuclei $j$ and $k$ on $F$ and a directed family $\{x_i\}_{i:I} : \mathsf{Fam}_{\mathcal{W}}(|F|)$. This follows as:

$$
\begin{aligned}
(j \wedge k)\left(\bigvee_{i:I} x_i\right) &\equiv j\left(\bigvee_{i:I} x_i\right) \wedge k\left(\bigvee_{i:I} x_i\right) \\
&= \left(\bigvee_{i:I} j(x_i)\right) \wedge \left(\bigvee_{i:I} k(x_i)\right) && \text{[Scott-continuity of } j \text{ and } k] \\
&= \bigvee_{(i,j):I \times I} j(x_i) \wedge k(x_i) && \text{[distributivity]} \\
&= \bigvee_{i:I} j(x_i) \wedge k(x_i) && [\dagger]
\end{aligned}
$$

where, for the $\dagger$ step, we use antisymmetry. The backwards direction is immediate whereas the forwards direction follows, essentially, from the monotonicity of nuclei (Proposition 1). We omit the details.

## 3 Joins

The nontrivial part of the patch frame construction is the join of a family $\{k_i\}_{i:I}$ of Scott-continuous nuclei as defining it pointwise,

$$
\bigvee_{i:I} k_i \quad :\equiv \quad x \mapsto \bigvee_{i:I} k_i(x),
$$

does not work. The problem is that this is not idempotent and hence not a nucleus. Our construction (that follows Escardó [1]) is based on the idea that, if we start with a family $\{k_i\}_{i:I}$ of nuclei, we can consider the family

$$
\{k_{i_0} \circ \cdots \circ k_{i_n}\}_{(i_0, \cdots, i_n):\mathsf{List}(I)},
$$

which will always be directed. However, in a predicative setting, this works only on Scott-continuous nuclei. Let us write this a bit more precisely as follows:

**Definition 8.** Let $K :\equiv \{k_i\}_{i \in I}$ be a family of nuclei on a given $(\mathcal{U}, \mathcal{V}, \mathcal{W})$-frame $F$. We define $K^*$ as the family $(\mathsf{List}(I), \mathfrak{d})$ where $\mathfrak{d}$ is defined as:

$$
\begin{aligned}
\mathfrak{d}([\,]) &:\equiv \mathsf{id}; \\
\mathfrak{d}(i :: is) &:\equiv \mathfrak{d}(is) \circ k_i.
\end{aligned}
$$

**Proposition 5.** Let $K :\equiv \{k_i\}_{i:I}$ be a family of prenuclei on a frame $F$. Given any $is, js : \mathsf{List}(I)$, $\mathfrak{d}(is \mathbin{++} js) = \mathfrak{d}(is) \circ \mathfrak{d}(js)$.

*Proof.* Straightforward induction, using function extensionality. $\square$

**Proposition 6.** Given a family $K :\equiv \{k_i\}_{i:I}$ of nuclei on a $(\mathcal{U}, \mathcal{V}, \mathcal{W})$-frame $F$, every $\alpha \in K^*$ is a prenucleus, that is, for every $is : \mathsf{List}(I)$, $\mathfrak{d}(is)$ is a prenucleus

*Proof.* If $is = [\,]$, we are done as it is immediate that the identity function $\mathsf{id}$ is a prenucleus. If $is = i :: is'$, we need to show that $\mathfrak{d}(is') \circ k_i$ is a prenucleus. For meet-preservation, consider some $x, y : |F|$. We have that:

$$
\begin{aligned}
(\mathfrak{d}(is') \circ k_i)(x \wedge y) &\equiv \mathfrak{d}(is')(k_i(x \wedge y)) \\
&= \mathfrak{d}(is')(k_i(x) \wedge k_i(y)) && [k_i \text{ is a nucleus}] \\
&= \mathfrak{d}(is')(k_i(x)) \wedge \mathfrak{d}(is')(k_i(y)) && [\text{IH}] \\
&\equiv (\mathfrak{d}(is') \circ k_i)(x) \wedge (\mathfrak{d}(is') \circ k_i)(y).
\end{aligned}
$$

For inflation, consider some $x : |F|$. We have that

$$
x \leq k_i(x) \leq \mathfrak{d}(is')(k_i(x)),
$$

by the inflation $k_i$ and the IH. $\square$

**Proposition 7.** Given a family $\{k_i\}_{i:I}$ of Scott-continuous nuclei, every prenucleus $\alpha \in K^*$, is Scott-continuous

*Proof.* In the base case of $is = [\,]$, it is trivial that the identity prenucleus is Scott-continuous. For the case of $is = i :: is'$, let $\{x_i\}_{i:I}$ be a directed family on $F$. The result follows equationally as follows:

$$
\begin{aligned}
(\mathfrak{d}(is') \circ k_i)\left(\bigvee_i x_i\right) &\equiv \mathfrak{d}(is')\left(k_i\left(\bigvee_i x_i\right)\right) \\
&= \mathfrak{d}(is')\left(\bigvee_{i:I}(k_i(x_i))\right) && [\text{Scott-continuity of } k_i] \\
&= \bigvee_{i:I} \mathfrak{d}(is')(k_i(x_i)) && [\text{IH}].
\end{aligned}
$$

Note that to be able to appeal to the IH, it must be shown that $\{k_i(x_i)\}_{i:I}$ is a directed family which follows from **TODO**. $\square$

**Proposition 8.** Given a family $K :\equiv \{k_i\}_{i:I}$ of nuclei (on some frame $F$), $K^*$ is a directed family.

*Proof.* $K^*$ is always inhabited by $\mathfrak{d}([\,])$. Upwards-closure also holds since, given $is, js : \mathsf{List}(I)$, $\mathfrak{d}(is \mathbin{++} js)$ is the upper bound of $\{\mathfrak{d}(is), \mathfrak{d}(js)\}$: $\mathfrak{d}(is \mathbin{++} js) = \mathfrak{d}(is) \circ \mathfrak{d}(js)$ (by Proposition 5) which is the upper bound of $\{\mathfrak{d}(is), \mathfrak{d}(js)\}$ by Proposition 2. $\square$

To be able to define the patch frame on $F$, we will also need some lemmas about nuclei in general.

**Proposition 9.** Given a nucleus $j$ on some frame $F$ and a family $K :\equiv \{k_i\}_{i:I}$ of nuclei on $F$, if we denote by $L$ the set $\{j \wedge k \mid k \in K\}$, we have that $L_{is}^*$ is a lower bound of $\{K_{is}^*, j\}$.

*Proof.* Let $is : \mathsf{List}(I)$ and $x : |F|$. In the base case of $is \equiv \texttt{[]}$, we are done as both sides of the inequality reduce to the application of $\mathsf{id}(x)$. If we have $is \equiv i \mathbin{+\!\!+} is'$, we have that

$$
\begin{aligned}
L_{i \,::\, is'}^*(x) \quad &\equiv \quad L_{is'}^*(L_i(x)) \quad \equiv \quad L_{is'}^*(j(x) \wedge k_i(x)) & \\
&\leq \quad K_{is'}^*(j(x) \wedge k_i(x)) & [\text{IH}] \\
&\leq \quad K_{is'}^*(j(x)) \wedge K_{is'}^*(k_i(x)) & [K_{is'}^* \text{ is a prenucleus}] \\
&\leq \quad K_{is'}^*(k_i(x)), &
\end{aligned}
$$

and that

$$
\begin{aligned}
L_{i \,::\, is'}^*(x) \quad &\equiv \quad L_{is'}^*(L_i(x)) \quad \equiv \quad L_{is'}^*(j(x) \wedge k_i(x)) & \\
&\leq \quad j(j(x) \wedge k_i(x)) & [\text{IH}] \\
&\leq \quad j(j(x)) \wedge j(k_i(x)) & [j \text{ preserves meets}] \\
&\leq \quad j(j(x)) & \\
&= \quad j(x), & [j \text{ is idempotent}]
\end{aligned}
$$

which is to say $L_{i \,::\, is'}^*(x)$ is lower than both of $K_{i \,::\, is'}^*(x)$ and $j(x)$. $\qquad\square$

**Definition 9** (Patch frame over a frame $F$)**.** Let $F$ be a $(\mathcal{U}, \mathcal{V}, \mathcal{W})$-frame. We define the patch frame on $F$ as follows:

- the underlying meet-semilattice is as given in Definition 6;

- given a family $K :\equiv \{k_i\}_{i \in I}$ of Scott-continuous nuclei, its join is defined as:

$$
\bigvee K \quad :\equiv \quad x \mapsto \{\alpha(x) \mid \alpha \in K^*\}.
$$

The proof of the distributivity law is not straightforward and we omit it.

# References

[1] Martín Hötzel Escardó. "Properly injective spaces and function spaces". In: *Topology and its Applications*. Domain Theory 89.1 (Nov. 20, 1998), pp. 75–120.