A Construction of The Patch Frame in Univalent Type Theory

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Abstract

To-do: write an abstract.

1 Preliminaries

Definition 1 (Family)**.** A "$\mathcal{U}$-family on type A" is[1] simply a pair (I, f) where $I : \mathcal{U}$ and $f : I \to A$. We denote the type of $\mathcal{U}$-families by Fam i.e. $\mathsf{Fam}_{\mathcal{U}}(A) :\equiv \Sigma_{(I:\mathcal{U})} I \to A$.

Definition 2 (Directed family)**.** A family (I, f) on some type A is called *directed* iff (1) I is inhabited, and (2) for every $i, j : I$, there exists some $k : I$ such that $f(k)$ is the upper bound of $\{f(i), f(j)\}$.

Our definition of a frame is parameterised by three universes: one for the carrier set, one for the order, and one for the index type of families on which the join operation is defined. We adopt the convention of using universe variables of $\mathcal{U}$, $\mathcal{V}$, and $\mathcal{W}$ for these respectively.

Definition 3 (Frame)**.** A $(\mathcal{U}, \mathcal{V}, \mathcal{W})$-frame consists of:

- an h-set $|F| : \mathcal{U}$,
- a partial order $\_ \leq \_ \ : \ |F| \to \Omega_{\mathcal{V}}$,
- a top element $\top : |F|$,
- a meet operation i.e. an operation $\_ \wedge \_ \ : \ |F| \to |F| \to |F|$ such that $x \wedge y$ is the GLB of x and y for every $x, y : |F|$, and
- a join operation i.e. an operation $\bigvee \_ : \mathsf{Fam}_{\mathcal{W}}(|F|) \to |F|$ such that given a family $\{x_i\}_{i \in I}$, $\bigvee_i x_i$ is the LUB of $\{x_i\}_{i \in I}$,

such that binary meets distribute over arbitrary joins:

$$x \wedge \bigvee_{i:I} y_i = \bigvee_{i:I} x \wedge y_i$$

for every $x : |F|$, family $\{y\} : \mathsf{Fam}_{\mathcal{W}}(|F|)$.

A nucleus on a frame F is an equivalent characterisation of a regular monomorphism on the locale corresponding to F.

[1] It would be reasonable to require I to be an h-set and f to be injective but we are providing a direct translation of the definition in our formalisation. This definition might have to be updated with these requirements.

Definition 4 (Nucleus)**.** A nucleus on a $(\mathcal{U}, \mathcal{V}, \mathcal{W})$-frame F is an endofunction $j : |F| \to |F|$ satisfying:

- (meet-preservation) $j(x \wedge y) = j(x) \wedge j(y)$;
- (inflation) $x \leq j(x)$; and
- (idempotence) $j(j(x)) \leq j(x)$.

for every $x, y : |F|$.

If an endofunction satisfies meet-preservation and inflation but not necessarily idempotence, it is called a *prenucleus*.

We write $\mathsf{Nucleus}(F)$ to denote the type of nuclei on a given frame F.

Proposition 1 (Prenuclei are monotonic)**.** Every prenucleus is monotonic.

Proof. Let $j : |F| \to |F|$ be a nucleus on some frame F and let $x, y : |F|$ s.t. $x \leq y$. We have:

$$j(x) = j(x \wedge y) = j(x) \wedge j(y) \leq j(y),$$

using meet-preservation and the fact that $x \leq y \leftrightarrow x \wedge y = x$. $\square$

Proposition 2. Given any two prenuclei j and k on some frame F, $j \circ k$ is an upper bound of $\{j, k\}$.

Proof. Given any $x : |F|$, we have that $k(x) \leq j(k(x))$ by the inflation of j and that $j(x) \leq j(k(x))$ by the monotonicity of j, since $x \leq k(x)$ by the inflation of k. $\square$

Proposition 3. Given any nucleus F, $\mathsf{Nucleus}(F)$ is an h-set.

Definition 5 (Scott-continuity)**.** A function $f : |F| \to |G|$ from a $(\mathcal{U}, \mathcal{V}, \mathcal{W})$-frame F to a $(\mathcal{U}, \mathcal{V}, \mathcal{W})$-frame G is called *Scott-continuous* iff given any *directed* $\{x_i\} : \mathsf{Fam}_{\mathcal{W}}(|F|)$,

$$f\left(\bigvee_{i:I}^{F} x_i\right) = \bigvee_{i:I}^{G} f(x_i).$$

2 Meet-semilattice of nuclei

Nuclei form a meet-semilattice when ordered pointwise.

Definition 6. The type of nuclei on a given frame F forms a meet-semilattice as follows:

- order: given nuclei j and k, $j \leq k :\equiv \forall x \in |F|.\ j(x) \leq k(x)$;
- top nucleus: $\_ \mapsto \top_F$; and
- meet of two nucleus: $j \wedge k :\equiv x \mapsto j(x) \wedge_F k(x)$.

The fact that $j \wedge k$ satisfies the nucleus laws is given in Proposition 4.

We denote this $\mathcal{N}(F)$.

Proposition 4. Given two nuclei j and k on some frame F, the function $x \mapsto j(x) \wedge k(x)$ is a nucleus.

Proof. Inflation is direct from the inflation of j and k and the fact that $j(x) \wedge k(x)$ is the GLB of $j(x)$ and $k(x)$. To see that meet-preservation holds, let $x, y : |F|$. We have:

$$\begin{aligned} j(x \wedge y) \wedge k(x \wedge y) &= j(x) \wedge j(y) \wedge k(x) \wedge k(y) \\ &= (j(x) \wedge k(x)) \wedge (j(y) \wedge k(y)). \end{aligned}$$

For idempotence, let $x : |F|$. We have:

$$\begin{aligned} j(j(x) \wedge k(x)) \wedge k(j(x) \wedge k(x)) &\leq j(j(x)) \wedge j(k(x)) \wedge k(j(x)) \wedge k(k(x)) \\ &\leq j(j(x)) \wedge k(k(x)) \\ &\leq j(x) \wedge k(x). \end{aligned}$$

□

Definition 7. The type of Scott-continuous nuclei on a given $(\mathcal{U}, \mathcal{V}, \mathcal{W})$-frame F forms a meet-semilattice in the same way as in Definition 6. Its top element is $\top_{\mathcal{N}(F)}$ which is trivially Scott-continuous. It remains only to be shown that the meet of two Scott-continuous nuclei is a Scott-continuous nucleus. Consider two Scott-continuous nuclei j and k on F and a directed family $\{x_i\}_{i:I} : \mathsf{Fam}_{\mathcal{W}}(|F|)$. This follows as:

$$\begin{aligned} (j \wedge k)\left(\bigvee_{i:I} x_i\right) &\equiv j\left(\bigvee_{i:I} x_i\right) \wedge k\left(\bigvee_{i:I} x_i\right) \\ &= \left(\bigvee_{i:I} j(x_i)\right) \wedge \left(\bigvee_{i:I} k(x_i)\right) && \text{[Scott-continuity of } j \text{ and } k\text{]} \\ &= \bigvee_{(i,j):I \times I} j(x_i) \wedge k(x_i) && \text{[distributivity]} \\ &= \bigvee_{i:I} j(x_i) \wedge k(x_i) && [\dagger] \end{aligned}$$

where, for the † step, we use antisymmetry. The backwards direction is immediate whereas the forwards direction follows, essentially, from the monotonicity of nuclei (Proposition 1). We omit the details.

3 Joins

The nontrivial part of the patch frame construction is the join of a family $\{k_i\}_{i:I}$ of Scott-continuous nuclei as defining it pointwise,

$$\bigvee_{i:I} k_i \quad :\equiv \quad x \mapsto \bigvee_{i:I} k_i(x),$$

does not work. The problem is that this is not idempotent and hence not a nucleus. Our construction (that follows Escardó [1]) is based on the idea that, if we start with a family $\{k_i\}_{i:I}$ of nuclei, we can consider the family

$$\{k_{i_0} \circ \cdots \circ k_{i_n}\}_{(i_0, \cdots, i_n) : \mathsf{List}(I)},$$

which will always be directed. However, in a predicative setting, this works only on Scott-continuous nuclei. Let us write this a bit more precisely as follows:

Definition 8. Let $K :\equiv \{k_i\}_{i\in I}$ be a family of nuclei on a given $(\mathcal{U},\mathcal{V},\mathcal{W})$-frame F. We define K^* as the family $(\mathsf{List}(I), \mathfrak{d})$ where $\mathfrak{d}$ is defined as:

$$\begin{aligned} \mathfrak{d}([\,]) &:\equiv \mathsf{id}; \\ \mathfrak{d}(i :: is) &:\equiv \mathfrak{d}(is) \circ k_i. \end{aligned}$$

Proposition 5. Let $K :\equiv \{k_i\}_{i:I}$ be a family of prenuclei on a frame F. Given any $is, js : \mathsf{List}(I)$, $\mathfrak{d}(is \mathbin{+\!\!+} js) = \mathfrak{d}(is) \circ \mathfrak{d}(js)$.

Proof. Straightforward induction, using function extensionality. □

Proposition 6. Given a family $K :\equiv \{k_i\}_{i:I}$ of nuclei on a $(\mathcal{U},\mathcal{V},\mathcal{W})$-frame F, every $\alpha \in K^*$ is a prenucleus, that is, for every $is : \mathsf{List}(I)$, $\mathfrak{d}(is)$ is a prenucleus

Proof. If $is = [\,]$, we are done as it is immediate that the identity function id is a prenucleus. If $is = i :: is'$, we need to show that $\mathfrak{d}(is') \circ k_i$ is a prenucleus. For meet-preservation, consider some $x, y : |F|$. We have that:

$$\begin{aligned} (\mathfrak{d}(is') \circ k_i)(x \wedge y) &\equiv \mathfrak{d}(is')(k_i(x \wedge y)) \\ &= \mathfrak{d}(is')(k_i(x) \wedge k_i(y)) && [k_i \text{ is a nucleus}] \\ &= \mathfrak{d}(is')(k_i(x)) \wedge \mathfrak{d}(is')(k_i(y)) && [\text{IH}] \\ &\equiv (\mathfrak{d}(is') \circ k_i)(x) \wedge (\mathfrak{d}(is') \circ k_i)(y). \end{aligned}$$

For inflation, consider some $x : |F|$. We have that

$$x \leq k_i(x) \leq \mathfrak{d}(is')(k_i(x)),$$

by the inflation k_i and the IH. □

Proposition 7. Given a nucleus j and a family $K :\equiv \{k_i\}_{i:I}$ of nuclei on some frame F, if j is an upper bound of K then j is an upper bound of K^*.

Proof. Let $is : \mathsf{List}(I)$. If $is = \texttt{[]}$, we have that $\mathsf{id}(x) \equiv x \leq j(x)$. If $is = i :: is'$, then we have

$$\begin{aligned} K^*_{is'}(K_i(x)) &\leq K^*_{is'}(j(x)) && [\text{monotonicity of } K^*_{is'} \text{ (Prop. 6 and Prop. 1)}] \\ &\leq j(j(x)) && [\text{IH}] \\ &\leq j(x). && [\text{idempotence of } j] \end{aligned}$$

□

Proposition 8. Given a family $\{k_i\}_{i:I}$ of Scott-continuous nuclei, every prenucleus $\alpha \in K^*$, is Scott-continuous

Proof. In the base case of $is = [\,]$, it is trivial that the identity prenucleus is Scott-continuous. For the case of $is = i :: is'$, let $\{x_i\}_{i:I}$ be a directed family on F. The result follows equationally as follows:

$$\begin{aligned} (\mathfrak{d}(is') \circ k_i)\left(\bigvee_i x_i\right) &\equiv \mathfrak{d}(is')\left(k_i\left(\bigvee_i x_i\right)\right) \\ &= \mathfrak{d}(is')\left(\bigvee_{i:I} (k_i(x_i))\right) && [\text{Scott-continuity of } k_i] \\ &= \bigvee_{i:I} \mathfrak{d}(is')(k_i(x_i)) && [\text{IH}]. \end{aligned}$$

Note that to be able to appeal to the IH, it must be shown that $\{k_i(x_i)\}_{i:I}$ is a directed family which follows from ... To-do: complete. □

Proposition 9. Given a family $K :\equiv \{k_i\}_{i:I}$ of nuclei (on some frame F), K^* is a directed family.

Proof. K^* is always inhabited by $\mathfrak{d}([\,])$. Upwards-closure also holds since, given $is, js : \mathsf{List}(I)$, $\mathfrak{d}(is \mathbin{+\!+} js)$ is the upper bound of $\{\mathfrak{d}(is), \mathfrak{d}(js)\}$: $\mathfrak{d}(is \mathbin{+\!+} js) = \mathfrak{d}(is) \circ \mathfrak{d}(js)$ (by Proposition 5) which is the upper bound of $\{\mathfrak{d}(is), \mathfrak{d}(js)\}$ by Proposition 2. ◻

To be able to define the patch frame on F, we will also need some lemmas about nuclei in general.

Proposition 10. Given a nucleus j on some frame F and a family $K :\equiv \{k_i\}_{i:I}$ of nuclei on F, if we denote by L the set $\{j \wedge k \mid k \in K\}$, we have that L^*_{is} is a lower bound of $\{K^*_{is}, j\}$.

Proof. Let $is : \mathsf{List}(I)$ and $x : |F|$. In the base case of $is \equiv \texttt{[]}$, we are done as both sides of the inequality reduce to the application of $\mathsf{id}(x)$. If we have $is \equiv i \mathbin{+\!+} is'$, we have that

$$
\begin{array}{rclcll}
L^*_{i\,::\,is'}(x) & \equiv & L^*_{is'}(L_i(x)) & \equiv & L^*_{is'}(j(x) \wedge k_i(x)) & \\
 & & & \leq & K^*_{is'}(j(x) \wedge k_i(x)) & [\text{IH}] \\
 & & & \leq & K^*_{is'}(j(x)) \wedge K^*_{is'}(k_i(x)) & [K^*_{is'} \text{ is a prenucleus}] \\
 & & & \leq & K^*_{is'}(k_i(x)), &
\end{array}
$$

and that

$$
\begin{array}{rclcll}
L^*_{i\,::\,is'}(x) & \equiv & L^*_{is'}(L_i(x)) & \equiv & L^*_{is'}(j(x) \wedge k_i(x)) & \\
 & & & \leq & j(j(x) \wedge k_i(x)) & [\text{IH}] \\
 & & & \leq & j(j(x)) \wedge j(k_i(x)) & [j \text{ preserves meets}] \\
 & & & \leq & j(j(x)) & \\
 & & & = & j(x), & [j \text{ is idempotent}]
\end{array}
$$

which is to say $L^*_{i\,::\,is'}(x)$ is lower than both of $K^*_{i\,::\,is'}(x)$ and $j(x)$. ◻

Definition 9 (Join of a family of Scott-continuous nuclei)**.** Let $K :\equiv \{k_i\}_{i:I}$ be a family of Scott-continuous nuclei. Its join is given by the operation $\bigvee^N$ defined as:

$$\bigvee_i^N k_i \quad :\equiv \quad x \mapsto \bigvee^F \{\alpha(x) \mid \alpha \in K^*\}\,.$$

We need to show that $\bigvee_i^N k_i$ is (a) a nucleus, (b) Scott-continuous, and (c) the LUB of K.

(a). Inflation is direct. For meet-preservation, consider some $x, y : |F|$. We have:

$$
\begin{array}{rcll}
\left(\bigvee_i^N k_i\right)(x) & \equiv & \bigvee^F \{\alpha(x \wedge y) \mid \alpha \in K^*\} & \\
 & = & \bigvee^F \{\alpha(x) \wedge \alpha(y) \mid \alpha \in K^*\} & [\alpha \text{ is a prenucleus}] \\
 & = & \bigvee^F \{\alpha(x) \wedge \beta(y) \mid \alpha, \beta \in K^*\} & [\text{To-do: complete.}] \\
 & = & \left(\bigvee^F \{\alpha(x) \mid \alpha \in K^*\}\right) \wedge \left(\bigvee^F \{\beta(y) \mid \beta \in K^*\}\right) & [\text{distributivity}] \\
 & \equiv & \left(\bigvee_i^N k_i\right)(x) \wedge \left(\bigvee_i^N k_i\right)(y). &
\end{array}
$$

For idempotence, let $x : |F|$. We have that:

$$\begin{aligned}
\left(\bigvee_i^N k_i\right)\left(\left(\bigvee_i^N k_i\right)(x)\right) &\equiv \bigvee^F \left\{\bigvee^F \{\alpha(\beta(x)) \mid \beta \in K^*\} \mid \alpha \in K^*\right\} \\
&\leq \bigvee^F \{\alpha(\beta(x)) \mid \alpha, \beta \in K^*\} && \text{[To-do: explain.]} \\
&\leq \bigvee^F \{\alpha(x) \mid \alpha \in K^*\} && \text{[To-do: explain.]} \\
&\equiv \left(\bigvee_i^N k_i\right)(x).
\end{aligned}$$

(b). Let $U :\equiv \{x_j\}_{j:J}$ be a directed family over $|F|$. The result follows as:

$$\begin{aligned}
\bigvee^F \left\{\alpha\left(\bigvee_j^F x_j\right) \mid \alpha \in K^*\right\} &= \bigvee^F \left\{\bigvee^F \{\alpha(x_j) \mid x_j \in U\} \mid \alpha \in K^*\right\} && \text{[Scott-continuity of } \alpha\text{]} \\
&= \bigvee^F \left\{\bigvee^F \{\alpha(x_j) \mid \alpha : K^*\} \mid x_j \in U\right\}. && \text{[joins commute]}
\end{aligned}$$

(c). The fact that $\bigvee_i^N k_i$ is an upper bound of K is easy to verify: given some k_i and $x : |F|$, $k_i(x) \in \{\alpha(x) \mid \alpha \in K^*\}$ since $k_i \in K^*$. To see that it is *the least* upper bound, consider a Scott-continuous nucleus j that is an upper bound of K. Let $x : |F|$. We need to show that $\left(\bigvee_i^N k_i\right)(x) \leq j(x)$. We know by Proposition 7 that j is an upper bound of K^*, since it is an upper bound of K, which is to say $K_{is}^*(x) \leq j(x)$ for every $is : \mathsf{List}(I)$ i.e. $j(x)$ is an upper bound of the family $\{\alpha(x) \mid \alpha \in K^*\}$. Since $\left(\bigvee_i^N k_i\right)(x)$ is the least upper bound of this family by definition, it follows that it is below $j(x)$.

Proposition 11 (Distributivity)**.** Given a nucleus j and a family of nuclei $K :\equiv \{k_i\}_{i:I}$ on some frame F, we have that $j \wedge (\bigvee_{i:I} k_i) = \bigvee_{i:I} j \wedge k_i$.

Proof. To-do: complete the proof. □

References

[1] Martín Hötzel Escardó. "Properly injective spaces and function spaces". In: *Topology and its Applications*. Domain Theory 89.1 (Nov. 20, 1998), pp. 75–120.