A Construction of The Patch Frame in Univalent Type Theory

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Abstract

Lorem ipsum.

1 Preliminaries

Definition 1 (Family)**.** A "$\mathcal{U}$-family on type A" is[1] simply a pair (I, f) where $I : \mathcal{U}$ and $f : I \to A$. We denote the type of $\mathcal{U}$-families by Fam i.e. $\mathsf{Fam}_{\mathcal{U}}(A) :\equiv \Sigma_{(I:\mathcal{U})} I \to A$.

Definition 2 (Directed family)**.** A family (I, f) on some type A is called *directed* iff (1) I is inhabited, and (2) for every $i, j : I$, there exists some $k : I$ such that $f(k)$ is the upper bound of $\{f(i), f(j)\}$.

Our definition of a frame is parameterised by three universes: one for the carrier set, one for the order, and one for the index type of families on which the join operation is defined. We adopt the convention of using universe variables of $\mathcal{U}$, $\mathcal{V}$, and $\mathcal{W}$ for these respectively.

Definition 3 (Frame)**.** A $(\mathcal{U}, \mathcal{V}, \mathcal{W})$-frame consists of:

- an h-set $|F| : \mathcal{U}$,
- a partial order $\_ \leq \_ \ : \ |F| \to \Omega_{\mathcal{V}}$,
- a top element $\top : |F|$,
- a meet operation i.e. an operation $\_ \wedge \_ \ : \ |F| \to |F| \to |F|$ such that $x \wedge y$ is the GLB of x and y for every $x, y : |F|$, and
- a join operation i.e. an operation $\bigvee \_ : \mathsf{Fam}_{\mathcal{W}}(|F|) \to |F|$ such that given a family $\{x_i\}_{i \in I}$, $\bigvee_i x_i$ is the LUB of $\{x_i\}_{i \in I}$,

such that binary meets distribute over arbitrary joins:

$$x \wedge \bigvee_{i:I} y_i = \bigvee_{i:I} x \wedge y_i$$

for every $x : |F|$, family $\{y\} : \mathsf{Fam}_{\mathcal{W}}(|F|)$.

A nucleus on a frame F is an equivalent characterisation of a regular monomorphism on the locale corresponding to F.

[1] It would be reasonable to require I to be an h-set and f to be injective but we are providing a direct translation of the definition in our formalisation. This definition might have to be updated with these requirements.

Definition 4 (Nucleus)**.** A nucleus on a $(\mathcal{U}, \mathcal{V}, \mathcal{W})$-frame F is an endofunction $j : |F| \to |F|$ satisfying:

- (meet-preservation) $j(x \wedge y) = j(x) \wedge j(y)$;
- (inflation) $x \leq j(x)$; and
- (idempotence) $j(j(x)) \leq j(x)$.

for every $x, y : |F|$.

If an endofunction satisfies meet-preservation and inflation but not idempotence, it is called a *prenucleus*.

We write $\mathsf{Nucleus}(F)$ to denote the type of nuclei on a given frame F.

Proposition 1 (Nuclei are monotonic)**.** Every nucleus is monotonic.

Proof. Let $j : |F| \to |F|$ be a nucleus on some frame F and let $x, y : |F|$ s.t. $x \leq y$. We have:

$$j(x) = x \wedge j(x) \leq x \leq y \leq j(y).$$

$\square$

Proposition 2. Given any nucleus F, $\mathsf{Nucleus}(F)$ is an h-set.

Definition 5 (Scott-continuity)**.** A function $f : |F| \to |G|$ from a $(\mathcal{U}, \mathcal{V}, \mathcal{W})$-frame F to a $(\mathcal{U}, \mathcal{V}, \mathcal{W})$-frame G is called *Scott-continuous* iff given any *directed* $\{x_i\} : \mathsf{Fam}_{\mathcal{W}}(|F|)$,

$$f\left(\bigvee_{i:I}^{F} x_i\right) = \bigvee_{i:I}^{G} f(x_i).$$

2 Meet-semilattice of nuclei

Nuclei form a meet-semilattice when ordered pointwise.

Definition 6. The type of nuclei on a given frame F forms a meet-semilattice as follows:

- order: given nuclei j and k, $j \leq k :\equiv \forall x \in |F|.\ j(x) \leq k(x)$;
- top nucleus: $\_ \mapsto \top_F$; and
- meet of two nucleus: $j \wedge k :\equiv x \mapsto j(x) \wedge_F k(x)$.

The fact that $j \wedge k$ satisfies the nucleus laws is given in Proposition 3.

We denote this $\mathcal{N}(F)$.

Proposition 3. Given two nuclei j and k on some frame F, the function $x \mapsto j(x) \wedge k(x)$ is a nucleus.

Proof. Inflation is direct from the inflation of j and k and the fact that $j(x) \wedge k(x)$ is the GLB of $j(x)$ and $k(x)$. To see that meet-preservation holds, let $x, y : |F|$. We have:

$$\begin{aligned} j(x \wedge y) \wedge k(x \wedge y) &= j(x) \wedge j(y) \wedge k(x) \wedge k(y) \\ &= (j(x) \wedge k(x)) \wedge (j(y) \wedge k(y)). \end{aligned}$$

For idempotence, let $x : |F|$. We have:

$$\begin{aligned} j(j(x) \wedge k(x)) \wedge k(j(x) \wedge k(x)) \quad &\leq \quad j(j(x)) \wedge j(k(x)) \wedge k(j(x)) \wedge k(k(x)) \\ &\leq \quad j(j(x)) \wedge k(k(x)) \\ &\leq \quad j(x) \wedge k(x). \end{aligned}$$

□

Definition 7. The type of Scott-continuous nuclei on a given $(\mathcal{U}, \mathcal{V}, \mathcal{W})$-frame F forms a meet-semilattice in the same way as in Definition 6. Its top element is $\top_{\mathcal{N}(F)}$ which is trivially Scott-continuous. It remains only to be shown that the meet of two Scott-continuous nuclei is a Scott-continuous nucleus. Consider two Scott-continuous nuclei j and k on F and a directed family $\{x_i\}_{i:I} : \mathsf{Fam}_{\mathcal{W}}(|F|)$. This follows as:

$$\begin{aligned} (j \wedge k)\left(\bigvee_{i:I} x_i\right) &\equiv j\left(\bigvee_{i:I} x_i\right) \wedge k\left(\bigvee_{i:I} x_i\right) \\ &= \left(\bigvee_{i:I} j(x_i)\right) \wedge \left(\bigvee_{i:I} k(x_i)\right) && [\text{Scott-continuity of } j \text{ and } k] \\ &= \bigvee_{(i,j):I \times I} j(x_i) \wedge k(x_i) && [\text{distributivity}] \\ &= \bigvee_{i:I} j(x_i) \wedge k(x_i) && [\dagger] \end{aligned}$$

where, for the † step, we use antisymmetry. The backwards direction is immediate whereas the forwards direction follows, essentially, from the monotonicity of nuclei (Proposition 1). We omit the details.

3 Joins

The nontrivial part of the patch frame construction is the join of a family $\{k_i\}_{i:I}$ of Scott-continuous nuclei as defining it pointwise,

$$\bigvee_{i:I} k_i \quad :\equiv \quad x \mapsto \bigvee_{i:I} k_i(x),$$

does not work. The problem is that this is not idempotent and hence not a nucleus. Our construction (that follows Escardó [1]) is based on the idea that, if we start with a family $\{k_i\}_{i:I}$ of nuclei, we can consider the family

$$\{k_{i_0} \circ \cdots \circ k_{i_n}\}_{(i_0, \cdots, i_n):\mathsf{List}(I)},$$

which will always be directed. However, in a predicative setting, this works only on Scott-continuous nuclei. Let us write this a bit more precisely as follows:

Definition 8. Let $K :\equiv \{k_i\}_{i \in I}$ be a family of nuclei on a given $(\mathcal{U}, \mathcal{V}, \mathcal{W})$-frame F. We define K^* as the family $(\mathsf{List}(I), \mathfrak{d})$ where $\mathfrak{d}$ is defined as:

$$\begin{aligned} \mathfrak{d}([\,]) \quad &:\equiv \quad \mathsf{id}; \\ \mathfrak{d}(i :: is) \quad &:\equiv \quad \mathfrak{d}(is) \circ k_i. \end{aligned}$$

Proposition 4. Given a family $K :\equiv \{k_i\}_{i:I}$ of nuclei on a $(\mathcal{U}, \mathcal{V}, \mathcal{W})$-frame F, every $\alpha \in K^*$ is a prenucleus, that is, for every $is : \mathsf{List}(I)$, $\mathfrak{d}(is)$ is a prenucleus

Proof. If $is = [\,]$, we are done as it is immediate that the identity function id is a prenucleus. If $is = i :: is'$, we need to show that $\mathfrak{d}(is') \circ k_i$ is a prenucleus. For meet-preservation, consider some $x, y : |F|$.

$$\begin{aligned}
(\mathfrak{d}(is') \circ k_i)(x \wedge y) &\equiv \mathfrak{d}(is')(k_i(x \wedge y)) \\
&= \mathfrak{d}(is')(k_i(x) \wedge k_i(y)) && [k_i \text{ is a nucleus}] \\
&= \mathfrak{d}(is')(k_i(x)) \wedge \mathfrak{d}(is')(k_i(y)) && [\text{IH}] \\
&\equiv (\mathfrak{d}(is') \circ k_i)(x) \wedge (\mathfrak{d}(is') \circ k_i)(y).
\end{aligned}$$

□

We use the following proposition to construct the patch frame.

Proposition 5. Given a family K of Scott-continuous nuclei, K^* is always a directed family of Scott-continuous prenuclei (i.e. a prenucleus is an endofunction satisfying all nuclei laws except idempotence).

Definition 9 (Patch frame over a frame F)**.** Let F be a $(\mathcal{U}, \mathcal{V}, \mathcal{W})$-frame. We define the patch frame on F as follows:

- the underlying meet-semilattice is as given in Defn. 6;
- given a family $K :\equiv \{k_i\}_{i \in I}$ of Scott-continuous nuclei, its join is defined as:

$$\bigvee K \quad :\equiv \quad x \mapsto \{\alpha(x) \mid \alpha \in K^*\}.$$

The proof of the distributivity law is not straightforward and we omit it.

References

[1] Martín Hötzel Escardó. “Properly injective spaces and function spaces”. In: *Topology and its Applications*. Domain Theory 89.1 (Nov. 20, 1998), pp. 75–120.