CENG 384 - Signals and Systems for Computer Engineers
Spring 2022
Homework 2

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1. (a) Following the graph, we can find out that:

$$x(t) - \frac{2dx(t)}{dt} + 3y(t) - 2\int_{-\infty}^{t} y(t)dt = \frac{dy(t)}{dt}$$

(b) To ease our job, we can differentiate both sides and get rid of the integral. Then, we get the following equation:

$$\frac{dx(t)}{dt} - \frac{2d^2x(t)}{dt^2} + 3\frac{dy(t)}{dt} - 2y(t) = \frac{d^2y(t)}{dt^2}$$

$$\frac{2d^2y(t)}{dt^2} - 3\frac{dy(t)}{dt} + 2y(t) = -\frac{2d^2x(t)}{dt^2} + \frac{dx(t)}{dt}$$

Now, we can start solving by using the linearity of the system:

$$y_g(t) = y_h(t) + y_p(t)$$

$$x(t) = x_1(t) + x_2(t) \ where \ x_1(t) = e^{-t}u(t) \ and \ x_2(t) = e^{-2t}u(t)$$

1) Finding the partial solution:
We know that $y_p(t) \ and \ x(t)$ is in the form:

$$y_p(t) = H(\lambda)e^{\lambda t} \ , \ x(t) = e^{\lambda t}$$

Taking the derivatives:

$$y_p'(t) = \lambda H(\lambda)e^{\lambda t} \ , \ y_p''(t) = \lambda^2 H(\lambda)e^{\lambda t} \ , \ x' = \lambda e^{\lambda t} \ , \ x'' = \lambda^2 e^{\lambda t}$$

Putting them into the equation:

$$\lambda^2 H(\lambda)e^{\lambda t} - 3\lambda H(\lambda)e^{\lambda t} + 2H(\lambda)e^{\lambda t} = \lambda e^{\lambda t} - 2\lambda^2 e^{\lambda t}$$

$$H(\lambda) = \frac{\lambda - 2\lambda^2}{\lambda^2 - 3\lambda + 2} \ (transfer \ function)$$

$$y_{p1} = H(-1)e^{-t}u(t) = -\frac{1}{2}e^{-t}u(t)$$

$$y_{p2} = H(-2)e^{-2t}u(t) = -\frac{5}{6}e^{-2t}u(t)$$

$$y_p = -\frac{1}{2}e^{-t}u(t) - \frac{5}{6}e^{-2t}u(t)$$

2) Finding the homogeneous solution:

We know that y_h is in the form:

$$y_h(t) = Ce^{\alpha t}$$

Taking the derivatives:

$$y_h'(t) = \alpha Ce^{\alpha t} \ , \ y_h''(t) = \alpha^2 Ce^{\alpha t}$$

Putting it into the equation $\frac{2d^2y(t)}{dt} - 3\frac{dy(t)}{dt} + 2y(t) = 0$ we get;

$$\alpha^2 Ce^{\alpha t} - 3\alpha Ce^{\alpha t} + 2Ce^{\alpha t} = 0$$

$$Ce^{\alpha t}(\alpha^2 - 3\alpha + 2) = 0$$

Since C is a nonzero constant and $e^{\alpha t}$ cannot be 0 for any t,

$$\alpha^2 - 3\alpha + 2 = 0$$

$$\alpha_1 = 1 \ , \ \alpha_2 = 2$$

$$y_h(t) = C_1 e^t + C_2 e^{2t}$$

$$y_g(t) = C_1 e^t + C_2 e^{2t} - \frac{1}{2} e^{-t} u(t) - \frac{5}{6} e^{-2t} u(t)$$

Since the system is initially at rest, y(0)=0 and y'(0)=0:

$$y_g(0) = C_1 + C_2 - \frac{1}{2} - \frac{5}{6} = 0 \ , \ C_1 + C_2 = \frac{4}{3}$$

$$y_g'(0) = C_1 + 2C_2 + \frac{1}{2} + \frac{10}{6} = 0 \ , \ C_1 + 2C_2 = -\frac{13}{6}$$

$$C_1 = \frac{29}{6} \ , \ C_2 = -\frac{7}{2}$$

$$y_g(t) = \frac{29}{6} e^t - \frac{7}{2} e^{2t} - \frac{1}{2} e^{-t} u(t) - \frac{5}{6} e^{-2t} u(t)$$

2. (a) We know that $x[n] * \delta[n-t] = x[n-t]$. By using this convolution result and distributive property of convolution:

$$y[n] = x[n] * h[n]$$

$$y[n] = (\delta[n-1] + 3\delta[n+2]) * (2\delta[n+2] - \delta[n+1])$$

$$y[n] = (\delta[n-1] * 2\delta[n+2]) - (\delta[n-1] * \delta[n+1]) + (3\delta[n+2] * 2\delta[n+2]) - (3\delta[n+2] * \delta[n+1])$$

$$y[n] = 2\delta[n+1] - \delta[n] + 6\delta[n+4] - 3\delta[n+3]$$

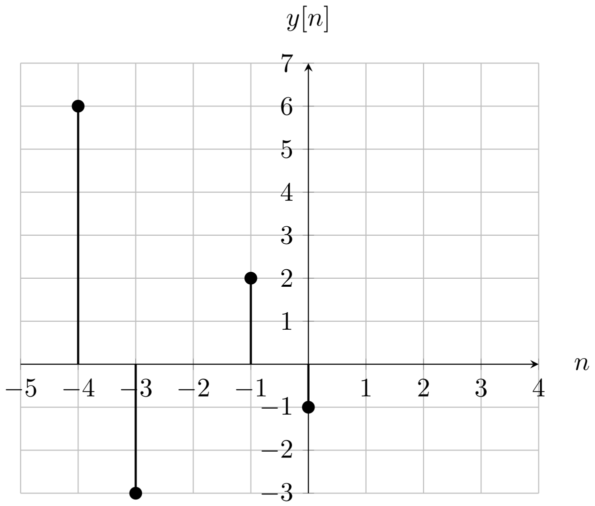

Figure 1: n vs. $y[n]$.

(b) By using the property of $u[n] - u[n-1] = \delta[n]$, we can write $x[n]$ and $h[n]$ as in the following way:

$$x[n] = \delta[n+1] + \delta[n] + \delta[n-1]$$

$$h[n] = \delta[n-4] + \delta[n-5]$$

By using the same properties in the part a:

$$y[n] = (\delta[n+1] + \delta[n] + \delta[n-1]) * (\delta[n-4] + \delta[n-5])$$

$$y[n] = (\delta[n+1]*\delta[n-4]) + (\delta[n+1]*\delta[n-5]) + (\delta[n]*\delta[n-4]) + (\delta[n]*\delta[n-5]) + (\delta[n-1]*\delta[n-4]) + (\delta[n-1]*\delta[n-5])$$

$$y[n] = \delta[n-3] + 2\delta[n-4] + 2\delta[n-5] + \delta[n-6]$$

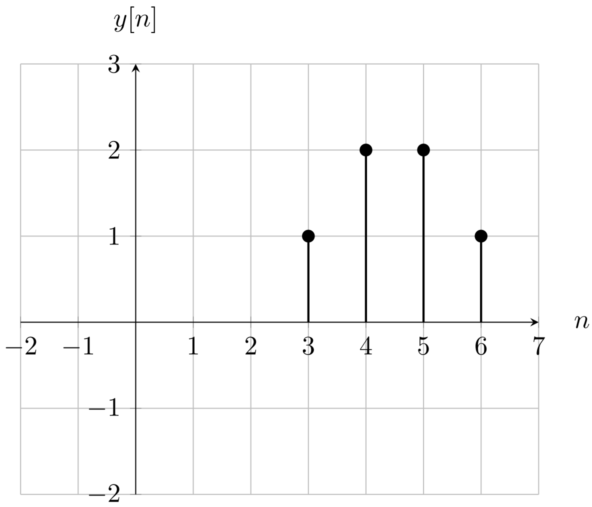

Figure 2: n vs. $y[n]$.

3. (a)

$$h(t) = \begin{cases} e^{-\frac{t}{2}} & t \geq 0 \\ 0 & t < 0 \end{cases}$$

$$x(t) = \begin{cases} e^{-t} & t \geq 0 \\ 0 & t < 0 \end{cases}$$

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

$$x(\tau) = e^{-\tau}u(\tau)\ ,\ h(t-\tau) = e^{-\frac{1}{2}(t-\tau)}u(t-\tau)$$

$$y(t) = \int_{-\infty}^{\infty} e^{-\tau}u(\tau)e^{-\frac{1}{2}(t-\tau)}u(t-\tau)d\tau$$

This integral takes nonzero values only when t$\geq \tau \geq 0$ since $u(\tau) \neq 0\ when\ \tau \geq 0\ and\ u(t-\tau) \neq 0\ when\ t-\tau \geq 0$ Hence,

$$y(t) = \int_{0}^{t} e^{-\tau}e^{-\frac{1}{2}(t-\tau)}d\tau$$

$$y(t) = \int_{0}^{t} e^{-\tau}e^{\frac{1}{2}\tau}e^{-\frac{1}{2}t}d\tau$$

$$y(t) = \int_{0}^{t} e^{-\frac{1}{2}\tau}e^{-\frac{1}{2}t}d\tau$$

Since $e^{-\frac{1}{2}t}$ does not depend on τ, it is a constant.

$$y(t) = e^{-\frac{1}{2}t}\int_{0}^{t} e^{-\frac{1}{2}\tau}d\tau$$

Taking the integral:

$$y(t) = e^{-\frac{1}{2}t}\Big[-2e^{-\frac{1}{2}\tau}\Big|_{0}^{t}\Big]$$

$$y(t) = (2 - 2e^{-\frac{1}{2}t})e^{-\frac{1}{2}t}$$

$$y(t) = 2e^{-\frac{t}{2}} - 2e^{-t}$$

As this only holds for the values $t \geq 0$

$$y(t) = (2e^{-\frac{t}{2}} - 2e^{-t})u(t)$$

(b)

$$h(t) = \begin{cases} e^{-3t} & t \geq 0 \\ 0 & t < 0 \end{cases}$$

$$x(t) = \begin{cases} 0 & t > 4 \\ 1 & 0 \leq t \leq 4 \\ 0 & t < 0 \end{cases}$$

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

$$x(\tau) = u(\tau) - u(\tau - 4)\ ,\ h(t-\tau) = e^{-3(t-\tau)}u(t-\tau)$$
$$y(t) = \int_{-\infty}^{\infty} (u(\tau) - u(\tau-4))e^{-3(t-\tau)}u(t-\tau)d\tau$$
$$y(t) = \int_{-\infty}^{\infty} (u(\tau)e^{-3(t-\tau)}u(t-\tau)d\tau \ - \ \int_{-\infty}^{\infty} u(\tau-4)e^{-3(t-\tau)}u(t-\tau)d\tau$$

First part of the integral takes nonzero values only for $t \geq \tau \geq 0$ and the second part of the integral takes nonzero values only for $t \geq \tau \geq 4$
Hence,

$$y(t) = \int_{0}^{t} e^{-3(t-\tau)}d\tau \ - \ \int_{4}^{t} e^{-3(t-\tau)}d\tau$$

Since e^{-3t} does not depend on τ, it is a constant.

$$y(t) = e^{-3t}\int_{0}^{t} e^{3\tau}d\tau \ - \ e^{-3t}\int_{4}^{t} e^{3\tau}d\tau$$

Taking the integral:

$$y(t) = e^{-3t}\left[\frac{1}{3}e^{3\tau}\Big|_{0}^{t}\right]u(t) \ - \ e^{-3t}\left[\frac{1}{3}e^{3\tau}\Big|_{4}^{t}\right]u(t-4)$$
$$y(t) = e^{-3t}\left[\frac{1}{3}e^{3t} - \frac{1}{3}\right]u(t) - e^{-3t}\left[\frac{1}{3}e^{3t} - \frac{1}{3}e^{12}\right]u(t-4)$$
$$y(t) = (\frac{1}{3} - \frac{1}{3}e^{-3t})u(t) + (\frac{1}{3}e^{-3t+12} - \frac{1}{3})u(t-4)$$

4. (a) We know that:

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$
$$x(t) = x(t) * \delta(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau$$

and also,

$$x(t) = \delta(t) * x(t) = \int_{-\infty}^{\infty} x(t-\tau)\delta(\tau)d\tau$$

Hence we can directly write $\delta(\tau)$ in place of $x(\tau)$ in the first equation to obtain h(t) and it would give us:

$$\int_{-\infty}^{\infty} \delta(\tau)h(t-\tau)d\tau = \delta(t) * h(t) = h(t)$$

y(t) is given as:

$$y(t) = \int_{-\infty}^{t} e^{-(t-\tau)}x(\tau-3)d\tau$$

in the question. We can write it as:

$$h(t) = \int_{-\infty}^{t} e^{-(t-\tau)}\delta(\tau-3)d\tau$$
$$h(t) = e^{-(t-3)} \ for \ t \geq 3$$
$$h(t) = e^{-(t-3)}u(t-3)$$

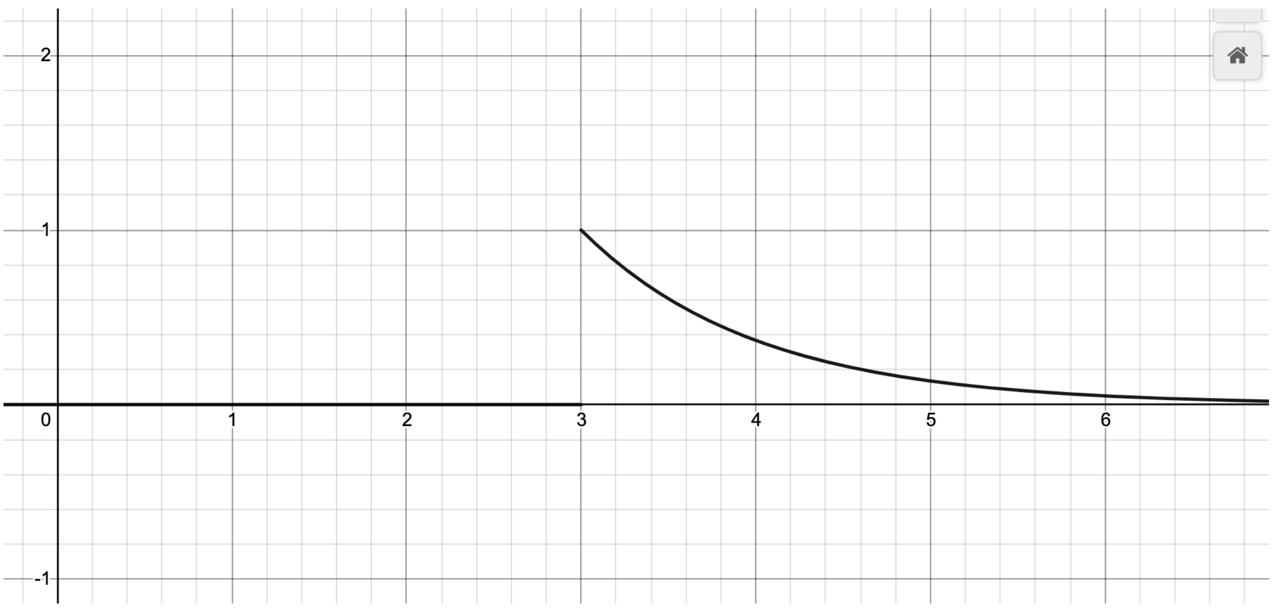

Figure 3: Graph of h(t)

(b)

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = \int_{-\infty}^{\infty} x(t-\tau)h(\tau)d\tau$$
$$x(t) = u(t+2) - u(t-1)$$
$$y(t) = \int_{-\infty}^{\infty} e^{-(t-\tau-3)}u(t-\tau-3)(u(\tau+2) - u(\tau-1))d\tau$$
$$y(t) = \int_{-2}^{t-3} e^{-(t-\tau-3)}d\tau - \int_{1}^{t-3} e^{-(t-\tau-3)}d\tau$$
$$y(t) = e^{3-t}\int_{-2}^{t-3} e^{\tau}d\tau - e^{3-t}\int_{1}^{t-3} e^{\tau}d\tau$$
$$y(t) = e^{3-t}(e^{t-3} - e^{-2})u(t-1) - e^{3-t}(e^{t-3} - e)u(t-4)$$
$$y(t) = (1 - e^{1-t})u(t-1) - (1 - e^{4-t})u(t-4)$$

5. (a) We know that convolution of a function with inverse of itself is $\delta[n]$. By using this fact:

$$h_1[n] * h_1^{-1} = \delta[n]$$
$$h_1[n] * ((\frac{1}{2})^n u[n]) = \delta[n]$$
$$h_1[n] = \delta[n] - \frac{1}{2}\delta[n-1]$$

We can now apply convolution operation to function itself:

$$h_1[n] * h_1[n] = (\delta[n] - \frac{1}{2}\delta[n-1]) * (\delta[n] - \frac{1}{2}\delta[n-1])$$

We can use distribution property of convolution as we used in the second question. After all the operations we can find:

$$h_1[n] * h_1[n] = \delta[n] - \delta[n-1] + \frac{1}{4}\delta[n-2]$$

(b) We know that $h[n] = h_0[n] * h_1[n] * h_1[n]$. To find $h_0[n]$, we can convolute both side with $h_1^{-1}[n] * h_1^{-1}[n]$. Then, we get the following:

$$h[n] * h_1^{-1}[n] * h_1^{-1}[n] = h_0[n]$$

Using the commutative property of convolution, we can first calculate:

$$h_1^{-1}[n] * h_1^{-1}[n] = (\frac{1}{2})^n u[n] * (\frac{1}{2})^n u[n]$$
$$h_1^{-1}[n] * h_1^{-1}[n] = \sum_{k=-\infty}^{\infty} \frac{1}{2}^k u[k] \frac{1}{2}^{n-k} u[n-k]$$
$$h_1^{-1}[n] * h_1^{-1}[n] = \sum_{k=-\infty}^{\infty} \frac{1}{2}^n u[k]u[n-k]$$
$$h_1^{-1}[n] * h_1^{-1}[n] = \sum_{k=0}^{n} \left(\frac{1}{2}\right)^n$$
$$h_1^{-1}[n] * h_1^{-1}[n] = (n+1)\left(\frac{1}{2}\right)^n u[n]$$

We know that $h[n] = 4\delta[n] + \delta[n-2] - 3\delta[n-3] + \delta[n-4]$ from the graph of $h[n]$. Now, we can convolute $h[n]$ with the result we found:

$$h[n] * ((n+1)\left(\frac{1}{2}\right)^n u[n]) = h_0[n]$$
$$\Big(4\delta[n] + \delta[n-2] - 3\delta[n-3] + \delta[n-4]\Big) * \left((n+1)\left(\frac{1}{2}\right)^n u[n]\right)$$

Again, using the distributive property of convolutions:

$$h_0[n] = 4(n+1)\left(\frac{1}{2}\right)^n u[n] + (n-1)\left(\frac{1}{2}\right)^{n-2} u[n-2] - 3(n-2)\left(\frac{1}{2}\right)^{n-3} u[n-3] + (n-3)\left(\frac{1}{2}\right)^{n-4} u[n-4]$$

(c) From the block diagram, we can understand that $y[n] = x[n] * h_0[n]$. Thus:

$$y[n] = \left(4(n+1)\left(\frac{1}{2}\right)^{n} u[n] + (n-1)\left(\frac{1}{2}\right)^{n-2} u[n-2] - 3(n-2)\left(\frac{1}{2}\right)^{n-3} u[n-3] + (n-3)\left(\frac{1}{2}\right)^{n-4} u[n-4]\right) * \left(\delta[n] + \delta[n-2]\right)$$

We will divide convolution into $y_1[n] = h_0[n] * \delta[n]$ and $y_2[n] = h_0[n] * \delta[n-2]$:

$$y[n] = y_1[n] + y_2[n]$$

$$y_1[n] = 4(n+1)\left(\frac{1}{2}\right)^{n} u[n] + (n-1)\left(\frac{1}{2}\right)^{n-2} u[n-2] - 3(n-2)\left(\frac{1}{2}\right)^{n-3} u[n-3] + (n-3)\left(\frac{1}{2}\right)^{n-4} u[n-4]$$

$$y_2[n] = 4(n-1)\left(\frac{1}{2}\right)^{n-2} u[n-2] + (n-3)\left(\frac{1}{2}\right)^{n-4} u[n-4] - 3(n-4)\left(\frac{1}{2}\right)^{n-5} u[n-5] + (n-5)\left(\frac{1}{2}\right)^{n-6} u[n-6]$$