CENG 384 - Signals and Systems for Computer Engineers
Spring 2022
Homework 3

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1. (a) We know that;

$$cos(wt) = \frac{1}{2}(e^{jwt} + e^{-jwt}) \quad (a)$$

$$sin(wt) = \frac{1}{2j}(e^{jwt} - e^{-jwt}) \quad (b)$$

So, we can write x(t) as;

$$x(t) = \frac{1}{2}e^{j\frac{\pi}{4}t} + \frac{1}{2}e^{-j\frac{\pi}{4}t} + \frac{1}{2j}e^{j\frac{\pi}{5}t} - \frac{1}{2j}e^{-j\frac{\pi}{5}t} \quad (1)$$

To be able to use the synthesis equation:

$$x(t) = \Sigma_{k=-\infty}^{\infty} a_k e^{jkw_0t}$$

We need to determine w_0. Since $w = \frac{2\pi}{T}$, T1=8 and T2=10.
Therefore, the fundamental period T_0 is 40 and w_0 is $\frac{\pi}{20}$.

$$x(t) = \Sigma_{k=-\infty}^{\infty} a_k e^{jk\frac{\pi}{20}t} \quad (2)$$

Nonzero terms in the equation 2 should correspond to the terms in equation 1.
Hence, the coefficient $a_k's$ from $-\infty$ to ∞ except a_4, a_{-4}, a_5, a_{-5} are 0, and

$$a_4 = a_{-4} = \frac{1}{2j}$$

$$a_5 = a_{-5} = \frac{1}{2}$$

(b) The spectral coefficients for discrete signals are periodic with the fundamental period N_0.
To determine N_0 we need to find the period of each term in x[n].

$$w = \frac{2\pi k}{N}$$

Period for $sin(4\pi n)$, $N1 = \frac{k}{2}$; period for $cos(2\pi n)$, N2=m and the period for $e^{j\pi n}$, N3=2.
So, for k=4 and m=2, $N_0 = 2$
As $N_0 = 2$ and the coefficients are periodic, to find $a_k's$ it is enough to calculate a_0 and a_1.
Using the analysis equation with $N_0 = 2$:

$$a_k = \frac{1}{2}\Sigma_{n=0}^{1} x[n]e^{-jk\pi n}$$

$$x[0] = \frac{5}{2}$$

$$x[1] = \frac{3}{2} + e^{j\pi}$$

$$a_k = \frac{1}{2}(\frac{5}{2} + \frac{3}{2}e^{-jk\pi} + e^{j\pi(1-k)})$$

Using the formula we obtained for a_k:

$$a_0 = \frac{3}{2}\ ,\ a_1 = 1\ and\ a_k = a_{k+2}\ \ for\ k > 0$$

2.

$$w_0 = \frac{2\pi}{N} = \frac{2\pi}{7}$$

Since we know the nonzero terms, by using the formula:

$$x[n] = \Sigma_{k \subset N} a_k e^{-jw_0 n}$$

We can write x(t) as:

$$x[n] = 2je^{j\frac{2\pi}{7}n} - 2je^{-j\frac{2\pi}{7}n} + 2je^{j\frac{4\pi}{7}n} + 2je^{-j\frac{2\pi}{7}n} + 2je^{j\frac{6\pi}{7}n} - 2je^{-j\frac{6\pi}{7}n}$$

By using the equations a and b to write x(t) in the form:

$$x(n) = 4j^2 sin(\frac{2\pi n}{7}) + 4cos(\frac{4\pi n}{7}) + 4j^2 sin(\frac{6\pi n}{7})$$

$$x(n) = -4sin(\frac{2\pi n}{7}) + 4cos(\frac{4\pi n}{7}) - 4sin(\frac{6\pi n}{7})$$

$$x(n) = -4sin(\frac{2\pi n}{7}) - 4sin(\frac{4\pi n}{7} - \frac{\pi}{2}) - 4sin(\frac{6\pi n}{7})$$

where A_0=0.

3. (a) We can write sin(wt) as the following form using Euler's Formula:

$$sin(wt) = \frac{1}{2j}(e^{jwt} - e^{-jwt})$$

Then, $w = \frac{\pi}{8}$

$$sin(\frac{\pi}{8}t) = \frac{1}{2j}(e^{j\frac{\pi}{8}t} - e^{-j\frac{\pi}{8}t})$$

$$sin(\frac{\pi}{8}t) = \frac{1}{2j}e^{j\frac{\pi}{8}t} - \frac{1}{2j}e^{-j\frac{\pi}{8}t}$$

we know that $x(t)$ is of the form:

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jkw_0 t}$$

When me make them equal, we found out that:

$$a_{-1} = -\frac{1}{2j} \qquad a_1 = \frac{1}{2j}$$

Otherwise $a_k = 0$.

(b) We can write cos(wt) as the following form using Euler's Formula:

$$cos(wt) = \frac{1}{2}(e^{jwt} + e^{-jwt})$$

Then, $w = \frac{\pi}{8}$

$$cos(\frac{\pi}{8}t) = \frac{1}{2}(e^{j\frac{\pi}{8}t} + e^{-j\frac{\pi}{8}t})$$

$$cos(\frac{\pi}{8}t) = \frac{1}{2}e^{j\frac{\pi}{8}t} + \frac{1}{2}e^{-j\frac{\pi}{8}t}$$

we know that $x(t)$ is of the form:

$$x(t) = \sum_{k=-\infty}^{\infty} b_k e^{jkw_0 t}$$

When me make them equal, we found out that:

$$b_{-1} = \frac{1}{2} \qquad b_1 = \frac{1}{2}$$

Otherwise $b_k = 0$.

(c) Given that $x(t) \overset{\text{FS}}{\longleftrightarrow} a_k$ and $y(t) \overset{\text{FS}}{\longleftrightarrow} b_k$ then,

$$x(t)y(t) = z(t) \overset{\text{FS}}{\longleftrightarrow} c_k = \sum_{l=-\infty}^{\infty} a_l b_{k-l}$$

Since a_l is nonzero only for $l = -1, 1$, we can write the result of summation as:

$$c_k = a_{-1}b_{k+1} + a_1 b_{k-1}$$

$$c_{-2} = a_{-1}b_{-1} + a_1 b_{-3} = -\frac{1}{4j}$$

$$c_0 = a_{-1}b_1 + a_1 b_{-1} = 0$$

$$c_2 = a_{-1}b_3 + a_1 b_1 = \frac{1}{4j}$$

Otherwise $c_k = 0$.

4. From the given conditions, we can infer followings:

- Since signal is odd, there is no DC component. Thus, $a_0 = 0$.
- For odd signals, we know that $a_k = -a_{-k}$. Thus, given that $a_2 = 3j$, $a_{-2} = -3j$.
- Given that

$$\frac{1}{4}\int_0^4 |x(t)|^2 dt = \sum_{k=-2}^{2} |a_k|^2 = 18$$

from the Parseval's Equality. We know that $|a_{-2}| = |a_2| = 3$ and $|a_0| = 0$. Using the Parseval's Equality, we find out that both a_{-1} and a_1 is 0. Then, we know that:

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jkw_0 t}$$

We are given that $T = 4$ then we can infer w_0 from the equality of $w_0 = \frac{2\pi}{T} = \frac{\pi}{2}$. Using the a_k's we can write signal as:

$$x(t) = -3je^{-j2\frac{\pi}{2}t} + 3je^{j2\frac{\pi}{2}t}$$

$$x(t) = 3j(e^{j\pi t} - e^{-j\pi t})$$

Using the Euler's Formula:

$$x(t) = -6sin(\pi t)$$

5. We can find the spectral coefficients from the equation:

$$a_k = \frac{1}{N}\sum_{n=-\infty}^{\infty} f[n]e^{-jkw_0 n}$$

where $w_0 = \frac{2\pi}{N} = \frac{2\pi}{9}$ for both cases.

(a)

$$a_k = \frac{1}{9}\sum_{n=0}^{8} x[n]e^{-jk\frac{2\pi}{9}n}$$

$$a_k = \frac{1}{9}\sum_{n=0}^{4} e^{-jk\frac{2\pi}{9}n}$$

$$a_k = \frac{1}{9}(1 + e^{-jk\frac{2\pi}{9}} + e^{-jk\frac{4\pi}{9}} + e^{-jk\frac{6\pi}{9}} + e^{-jk\frac{8\pi}{9}})$$

(b)

$$b_k = \frac{1}{9}\sum_{n=0}^{8} y[n]e^{-jk\frac{2\pi}{9}n}$$

$$b_k = \frac{1}{9}\sum_{n=0}^{3} e^{-jk\frac{2\pi}{9}n}$$

$$b_k = \frac{1}{9}(1 + e^{-jk\frac{2\pi}{9}} + e^{-jk\frac{4\pi}{9}} + e^{-jk\frac{6\pi}{9}})$$

(c) Given that $x[n]$ and $H(e^{jw_0})$, we know that $y[n] = H(e^{jw_0}x[n])$. Also, we know that relationship between the spectral coeffients of $y[n]$ and $x[n]$ is written as below:

$$b_k = H(e^{jkw_0})a_k$$

Then, we can write eigenvalue $H(e^{jkw_0})$ as:

$$H(e^{jkw_0}) = \frac{b_k}{a_k}$$

$$H(e^{jkw_0}) = \frac{1 + e^{-jkw_0} + e^{-2jkw_0} + e^{-3jkw_0}}{1 + e^{-jkw_0} + e^{-2jkw_0} + e^{-3jkw_0} + e^{-4jkw_0}}$$

To get frequency response of the system from eigenvalue, we need to make $kw_0 \to w$ transformation. Then,

$$H(e^{jw}) = \frac{1 + e^{-jkw} + e^{-2jkw} + e^{-3jkw}}{1 + e^{-jkw} + e^{-2jkw} + e^{-3jkw} + e^{-4jkw}}$$