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Answer 1

$a_n = a_{n-1} + 2^n,\ n \geq 1,\ a_0 = 1$
$(a_0, a_1, a_2, ...a_n) = (1, 3, 7....)$

To find the general rule for a_n
$\sum_{n=1}^{\infty} a_n \cdot x^n = \sum_{n=1}^{\infty}(a_{n-1} + 2^n) \cdot x^n$

where $A(x) = \sum_{n=0}^{\infty} a_n \cdot x^n$;
$\sum_{n=1}^{\infty} a_n \cdot x^n = A(x) - a_0$
$A(x) - a_0 = \sum_{n=1}^{\infty} a_{n-1} \cdot x^n + \sum_{n=1}^{\infty} 2^n \cdot x^n$
$A(x) - a_0 = x \cdot \sum_{n=1}^{\infty} a_{n-1} \cdot x^{n-1} + \sum_{n=1}^{\infty} 2^n \cdot x^n$

Since $\sum_{n=1}^{\infty} a_{n-1} \cdot x^{n-1} = \sum_{n=0}^{\infty} a_n = A(x)$;
$A(x) - a_0 = x \cdot A(x) + \sum_{n=1}^{\infty} 2^n \cdot x^n$
$A(x) - a_0 = x \cdot A(x) + \sum_{n=1}^{\infty}(2x)^n$
$\sum_{n=1}^{\infty}(2x)^n = \sum_{n=0}^{\infty}(2x)^n - 1\ as\ (2x)^n\ where\ n = 0\ is\ 1.$
$\sum_{n=0}^{\infty}(2x)^n = 1/(1 - 2x)$

$A(x) - 1 = x \cdot A(x) + 1/(1 - 2x) - 1$
$A(x) = x \cdot A(x) + 1/(1 - 2x)$
$A(x) - x \cdot A(x) = 1/(1 - 2x)$
$A(x) \cdot (1 - x) = 1/(1 - 2x)$
$A(x) = 1/((1 - 2x) \cdot (1 - x))$

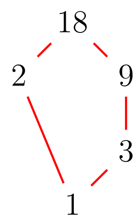
$A(x) = A/(1 - 2x) + B/(1 - x) = 1/((1 - 2x) \cdot (1 - x))$ where A and B are constants.
$A \cdot (1 - x) + B \cdot (1 - 2x) = 1$
$A - A \cdot x + B - 2B \cdot x = 1$
$x \cdot (-A - 2B) + A + B = 1$
$-A - 2B = 0\ and\ A + B = 1$
A=2 , B=-1

$A(x) = 2/(1 - 2x) + -1/(1 - x)$
$-1/(1 - x) \iff (-1, -1, -1, -1..... - 1^n)$
$2/(1 - 2x) \iff (2, 4, 8, 16......2^{n+1})$
$A(x) = 2/(1 - 2x) + -1/(1 - x) \iff (1, 3, 7, 15....2^{n+1} - 1)$
$a_n = 2^{n+1} - 1$

Answer 2

Given the relation R $= \{(a, b) \mid a \textit{ divides } b\} \textit{ on } A = \{1, 2, 3, 9, 18\};$

a) Hasse Diagram of R:



b) Matrix representation for R:

$$M_R = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

c)Yes it is a lattice. A lattice is a poset in which every pair of elements has both a least upper bound and a greatest lower bound.
(A,R) is a lattice since every pair of elements in our poset has both a least upper and greatest lower bound.
Pairs with their LUBs and GLBs:
(1,2) LUB: 2 , GLB: 1
(1,3) LUB: 3 , GLB: 1
(1,9) LUB: 9 , GLB: 1
(1,18) LUB: 18 , GLB: 1
(2,3) LUB: 18, GLB: 1
(2,9) LUB: 18 , GLB: 1
(2,18) LUB: 18 , GLB: 2
(3,9) LUB: 9 , GLB: 3
(3,18) LUB: 18 , GLB: 3
(9,18) LUB: 18, GLB: 9
....

d) $R_s = R \cup R^{-1}$ where $R^{-1} = \{(b, a) \mid (a, b) \in R\}$

$$M_{R^{-1}} = M_R^T = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

Matrix representation for $R_s = M_s = M_R \vee M_{R^{-1}}$

$$\text{Ms=} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \vee \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

e) For this binary relation, two integers a and b are comparable if $a \mid b \ or \ b \mid a$.
Since $2 \not| 9$ and $9 \not| 2$ and neither divides the other, the integers 2 and 9 are uncomparable in (A,R). However; even if $18 \not| 3$, since $3|18$, the integers 3 and 18 are comparable in (A,R).

Answer 3

a) $2^n \cdot 3^{n \cdot (n-1)/2}$

Let's think of a matrix representation of a binary relation on a set A with n elements.
Any anti-symmetric relation has no restriction on the reflexivity. Therefore, the main diagonal elements of the matrix can be either 0 or 1. It means we have 2 options for each element on the main diagonal. By the product rule, we have 2^n ways to construct the diagonal. Then, if we choose an element a_{ij} from the upper right triangle, its symmetric element a_{ji} on the left lower triangle must be related to a_{ij}.
We have 3 alternatives that satisfies the anti-symmetric properties:
$a_{ij} = 0, a_{ji} = 1$
$a_{ij} = 1, a_{ji} = 0$
$a_{ij} = 0, a_{ji} = 0$
The number of elements we have on the upper right triangle is total number of elements - number of main diagonal elements divided by 2;
$(n^2 - n)/2 = n \cdot (n-1)/2$
This is equal to number of pairs we have like (a_{ij}, a_{ji}) and for each possible pair, we have 3 options. By the product rule, we have $3^{n \cdot (n-1)/2}$ ways to construct the matrix except the diagonal.
Thus, in total we have $2^n \cdot 3^{n \cdot (n-1)/2}$ ways for the whole matrix and this gives us the number of different binary relations on A that are anti-symmetric.

Matrix example for one of the anti-symmetric binary relations on A $\begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \end{bmatrix}$

b) $3^{n \cdot (n-1)/2}$

As we did in part a, we can again think of the matrix representation.
This time, we have a restriction on reflexivity and reflexive relations are represented by a matrix that has 1 on the main diagonal since every element must have a relation to itself. Therefore, we have only 1 way to construct the diagonal.
For anti-symmetry, we have the same 3 alternatives from part a for each pair like (a_{ij}, a_{ji}) and this again gives us $3^{n \cdot (n-1)/2}$ ways to construct the rest of the matrix. Hence, in total we have, $3^{n \cdot (n-1)/2}$ ways for the whole matrix and this gives us the number of different binary relations on A that are both reflexive and anti-symmetric.

Matrix example for one of the reflexive and anti-symmetric binary relations on A $\begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \end{bmatrix}$