Student Information

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Answer 1

a) If X and Y are independent, the joint density function f(x,y) must be equal to f(x)f(y). We found f(x) and f(y) in part b.

$$f_X(x)f_Y(y) = \frac{2\sqrt{1-x^2}}{\pi}\frac{2\sqrt{1-y^2}}{\pi} \neq \frac{1}{\pi}\ ,\ (x^2+y^2 \leq 1)$$

Since they are not equal, X and Y are not independent.

b) We can calculate the marginal pdfs from:

$$f_X(x) = \int f_{X,Y}(x,y)dy$$

$$f_Y(y) = \int f_{X,Y}(x,y)dx$$

$f_{X,Y}(x,y) = 0$ when x and y does not satisfy the equation $x^2+y^2 \leq 1$. Therefore, integrals are zero at those points and we do not have to calculate it.
At other points, we can use the equation $x^2+y^2 = 1$ to find the lower and upper limit of the integral for x and y.

$$y_1 = \sqrt{1-x^2}\ ,\ y_2 = -\sqrt{1-x^2}$$

$$x_1 = \sqrt{1-y^2}\ ,\ x_2 = -\sqrt{1-y^2}$$

$$f_X(x) = \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{1}{\pi}dy = \frac{2\sqrt{1-x^2}}{\pi}$$

$$f_Y(y) = \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \frac{1}{\pi}dx = \frac{2\sqrt{1-y^2}}{\pi}$$

c) We know that:

$$E(X) = \int xf(x)dx$$

Also we know f(x) from part b, so we can calculate E(X) as:

$$E(X) = \int_{-1}^{1} \frac{x2\sqrt{1-x^2}}{\pi}dx$$

By change of variable $u = 1 - x^2$, du=-2xdx and both the lower and upper limit of the integral becomes 0.

$$E(X) = \int_0^0 \frac{-\sqrt{u}}{\pi} du$$

Hence,

$$E(X) = 0$$

d) We know that:

$$Var(X) = \int x^2 f(x) dx - \mu^2$$

Taking the integral by again using the f(x) we found from part b:

$$Var(X) = \int_{-1}^{1} \frac{x^2 2\sqrt{1-x^2}}{\pi} dx = 0.25$$

And we already know the value of μ from part c as it is equal to E(X).

$$Var(X) = 0.25 - 0^2 = 0.25$$

Answer 2

a) Joint density function for a uniform distribution is

$$f_x(x) = \frac{1}{b-a}$$

For T_A and T_B b=100 and a=0;

$$f_{T_A}(t_a) = \frac{1}{100}\ ,\ f_{T_B}(t_b) = \frac{1}{100}\ ,\ \ 0 < t_a, t_b < 100$$

Since they are independent,

$$f_{T_A,T_B}(t_a, t_b) = f_{T_A}(t_a) f_{T_B}(t_b) = \frac{1}{100}\frac{1}{100} = \frac{1}{10000}\ ,\ \ 0 < t_a, t_b < 100$$

Joint CDF can be found from the density functions as:

$$F_x(x) = \int f_x(x) dx$$

$$F_{T_A}(t_a) = \int_0^{t_a} f(t_a) dt_a = \frac{t_a}{100}\ ,\ F_{T_B}(t_b) = \int_0^{t_b} f(t_b) dt_b = \frac{t_b}{100}\ ,\ \ 0 < t_a, t_b < 100$$

Since they are independent,

$$F_{T_A,T_B}(t_a, t_b) = F_{T_A}(t_a) F_{T_B}(t_b) = \frac{t_a}{100}\frac{t_b}{100} = \frac{t_a t_b}{10000}\ ,\ \ 0 < t_a, t_b < 100$$

b) We can find the probability that subject A pushes the button in the first 10 seconds and subject B pushes the button in the last 10 seconds bu using the Joint CDF's we found in part a:

$$P\{T_A \leq 10\} = F_{t_a}(10) = \int_0^{10} \frac{1}{100} dt_a = \frac{1}{10}$$

$$P\{T_B \geq 90\} = 1 - F_{t_b}(90) = 1 - \int_0^{10} \frac{1}{100} dt_b = 1 - \frac{9}{10} = \frac{1}{10}$$

Since these events are independent,

$$P\{T_A \leq 10, T_B \geq 90\} = P\{T_A \leq 10\} P\{T_B \geq 90\} = \frac{1}{100}$$

c) Let t_b be the second B pushes the button and t_a be the second A pushes the button. Since subject A can push the button at most 20 seconds after subject B, unless B pushes the button at $t_b > 80$, t_a can only get values between 0 and $t_b + 20$. When $t_b > 80$, t_a can get the values from 0 to 100 as the difference between t_a and t_b is always less than 20.

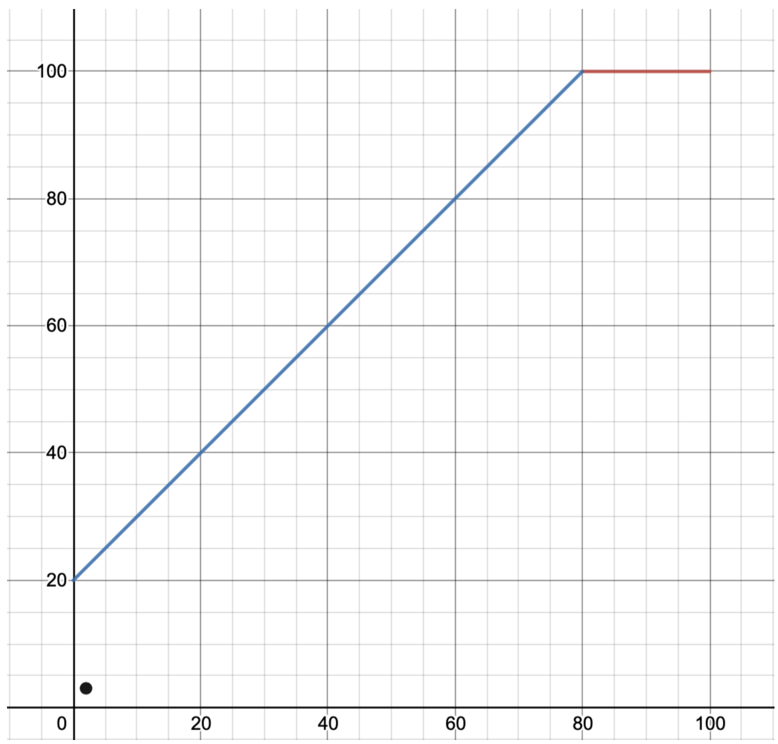

Figure 1: Graph of t_b vs t_a

The ratio of the area under the curve to the total area which is 100x100 defines the probability that subject A pushes the button at most 20 seconds after subject B.

Taking integral to find the probability:

$$\frac{1}{10000}\int_0^{80}(t_b+20)dt_b \;+\; \frac{1}{10000}\int_{80}^{100}(100)dt_b$$

$$0.48+0.20=0.68$$

d) Let t_b be the second B pushes the button and t_a be the second A pushes the button. Since the elapsed time of subject B and A cannot differ by more than 30 seconds, when B pushes the button at $0<t_b<30$ t_a can take values between 0 to t_b+30. When $30<t_b<70$, t_a can take values between t_b-30 to t_b+30. When $70<t_b<100$, t_a can take values between t_b-30 to 100.

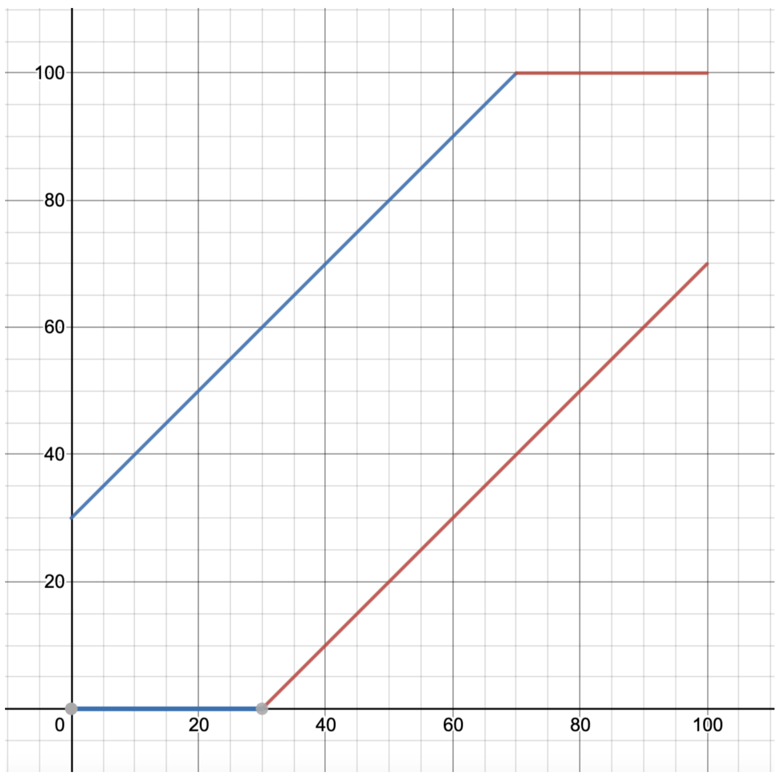

Figure 2: Graph of t_b vs t_a

The ratio of the area between the two curves to the total area which is 100x100 defines the probability that they pass the test.

$$\frac{1}{10000}\int_0^{30}(t_b+30)dt_b \;+\; \frac{1}{10000}\int_{30}^{70}(t_b+30)-(t_b-30)dt_b \;+\; \frac{1}{10000}\int_{70}^{100}100-(t_b-30)dt_b$$
$$=0.135+0.240+0.135=0.51$$

Answer 3

a)

$$F_{X_i} = P(X_i \geq x) = e^{-\lambda_i x_i}$$

To find the $F_{X_n} = P(T \leq t)$:

$$P(T \leq t) = 1 - P(T \geq t)$$

$$= 1 - P(min\{X_1, X_2, X_3....X_n\} \geq t)$$

Since T is the minimum which means it is less than all X_i's and greater than t, all X_i's will be greater than t.

$$= 1 - P(X_1 \geq t, X_2 \geq t, X_3 \geq t, ...X_n \geq t)$$

Since those are independent,

$$= 1 - P(X_1 \geq t)P(X_2 \geq t)P(X_3 \geq t).....P(X_n \geq t)$$

$$= 1 - e^{-\lambda_1 t}e^{-\lambda_2 t}e^{-\lambda_3 t}.....e^{-\lambda_n t}$$

$$= 1 - e^{-\lambda_1 t - \lambda_2 t - \lambda_3 t.....-\lambda_n t}$$

$$= 1 - e^{-\Sigma_{i=1}^{n} \lambda_i t}$$

b) μ is given as $\frac{10}{n}$ in the question.
Since $\mu = E(X) = \frac{1}{\lambda}$, we know that:

$$\lambda_n = \frac{n}{10}$$

$$\lambda_1 = \frac{1}{10}\ ,\ \lambda_1 = \frac{1}{10}\ ,\ \lambda_2 = \frac{2}{10}......\lambda_1 0 = \frac{10}{10}$$

$$\Sigma_{n=0}^{n=10} \lambda_n = \frac{55}{10}$$

Exponential distribution has the density function:

$$f(t) = \lambda e^{-\lambda t}$$

$$f(t) = \frac{55}{10} e^{-\frac{55}{10} t}$$

Finding the expected time:

$$E(X) = \int_0^{\propto} t f(t) dt$$

$$E(X) = \int_0^{\propto} t f(t) dt$$

$$E(X) = \int_0^{\propto} t \frac{55}{10} e^{-\frac{55}{10} t} dt$$

$$= 0.\overline{18}$$

Answer 4

We are gonna apply the central limit theorem and use continuity correction.

$$\mu = np \ , \ \sigma = \sqrt{np(1-p)}$$

a)

Probability of being an undergraduate student is 0.74, p=0.74 The number of students participating in a poll is 100, n=100

$$\mu = np = 74 \ , \ \sigma = \sqrt{np(1-p)} = \sqrt{74(1-0.74)} = \sqrt{19.24}$$

Let U denote the random variable of participants being an undergraduate students.
Instead of calculating the probability that at least %70 of the participants being undergraduate students, we can calculate the probability that at most %70 (without including 70) of the participants being undergraduate students and substract it from 1.

$$P(U < 70) = P(U < 69.5) = P\Big\{\frac{U-\mu}{\sigma} < \frac{69.5-\mu}{\sigma}\Big\}$$

$$\approx P\Big\{\frac{69.5-74}{\sqrt{19.24}}\Big\} = \Phi(-1.026)$$

Hence,

$$P(U \geq 70) \approx 1 - \Phi(-1.026) = 0.8485$$

b)

Probability of pursuing a doctoral degree is 0.1, p=0.1 The number of students participating in a poll is 100, n=100

$$\mu = np = 10 \ , \ \sigma = \sqrt{np(1-p)} = \sqrt{10(1-0.1)} = \sqrt{9} = 3$$

Let D denote the random variable of participants pursuing a doctoral degree.

$$P(D \leq 5) = P(D < 5.5) = P\Big\{\frac{D-\mu}{\sigma} < \frac{5.5-\mu}{\sigma}\Big\}$$

$$\approx P\Big\{\frac{5.5-10}{3}\Big\} = \Phi(-1.5)$$

Hence,

$$P(D \leq 5) \approx \Phi(-1.5) = 0.066807$$