CENG 384 - Signals and Systems for Computer Engineers
Spring 2022
Homework 4

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July 15, 2022

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1. (a) From the block diagram:

$$\frac{dx(t)}{dt} - \int x(t) + x(t) - \int y(t) - 2y(t) = \frac{dy(t)}{dt}$$

Taking derivative with respect to t to get rid of the integrals:

$$\frac{d^2x(t)}{dt^2} - x(t) + \frac{dx(t)}{dt} - y(t) - 2\frac{dy(t)}{dt} = \frac{d^2y(t)}{dt^2}$$

$$y^{''} + 2y^{'} + y = x^{''} + x^{'} - x$$

(b)

$$(jw)^2Y(jw) + 2jwY(jw) + Y(jw) = (jw)^2X(jw) + jwX(jw) - X(jw)$$

$$Y(jw)((jw)^2 + 2jw + 1) = X(jw)((jw)^2 + jw - 1)$$

We know that:

$$H(jw) = \frac{Y(jw)}{X(jw)}$$

Hence,

$$H(jw) = \frac{(jw)^2 + jw - 1}{(jw)^2 + 2jw + 1} = 1 + \frac{-X - 2}{(X+1)^2}$$

To get the fractions:

$$\frac{A}{X+1} + \frac{B}{(X+1)^2} = \frac{-X-2}{(X+1)^2}$$

$$AX + A + B = -X - 2$$

$$A = -1\ ,\ B = -1$$

So,

$$H(jw) = 1 - \frac{1}{jw+1} - \frac{1}{(jw+1)^2}$$

(c) Taking the inverse fourier transform to find h(t) from H(jw) that we found in part b:

$$H(jw) \xleftrightarrow{\text{F.T}} = h(t)$$

$$1 \xleftrightarrow{\text{F.T}} = \delta(t)$$

$$\frac{1}{(jw+1)^2} \xleftrightarrow{\text{F.T}} = te^{-t}u(t)$$

$$\frac{1}{jw+1} \xleftrightarrow{\text{F.T}} = e^{-t}u(t)$$

$$h(t) = \delta(t) - (t+1)e^{-t}u(t)$$

(d) We know that:

$$Y(jw) = X(jw)H(jw)$$

From the previous part:

$$H(jw) = 1 - \frac{1}{jw+1} - \frac{1}{(jw+1)^2}$$

Taking fourier transform to find $X(jw)$ from x(t):

$$x(t) \xleftrightarrow{\text{F.T}}= X(jw)$$

$$e^{-t}u(t) \xleftrightarrow{\text{F.T}}= \frac{1}{1+jw}$$

$$Y(jw) = \frac{1}{1+jw}(1 - \frac{1}{jw+1} - \frac{1}{(jw+1)^2})$$
$$Y(jw) = \frac{1}{1+jw} - \frac{1}{(jw+1)^2} - \frac{1}{(jw+1)^3}$$

Taking the inverse fourier transform to find y(t) from $Y(jw)$:

$$Y(jw) \xleftrightarrow{\text{F.T}}= y(t)$$

$$y(t) = (1 - t - \frac{t^2}{2})e^{-t}u(t)$$

2. (a) Let us find first the frequency response of the system. Assume that:

$$y(t) \xleftrightarrow{\mathcal{F}} Y(jw) \qquad x(t) \xleftrightarrow{\mathcal{F}} X(jw)$$

From differentiation property:

$$\frac{dy(t)}{dt} \xleftrightarrow{\mathcal{F}} jwY(jw)$$

And from time shift property:

$$x(t-t_0) \xleftrightarrow{\mathcal{F}} e^{-jwt_0}X(jw)$$

Then,

$$\frac{dy(t)}{dt} = x(t+1) - x(t-1) \xleftrightarrow{\mathcal{F}} jwY(jw) = X(jw)(e^{jw} - e^{-jw})$$

We find that frequency response is:

$$H(jw) = \frac{Y(jw)}{X(jw)} = \frac{e^{jw} - e^{-jw}}{jw}$$

(b) We can find impulse response of this system by applying inverse transform to frequency response we found in part a. Apply euler's formula first and we get:

$$H(jw) = \frac{2sinw}{w}$$

From the table of fourier transform, we get:

$$h(t) = u(t+1) - u(t-1)$$

3. (a)

$$x[n] * h_1[n] * h_2[n] = y[n]$$
$$x[n] * h[n] = y[n]$$
$$h[n] = h_1[n] * h_2[n] \quad (1)$$

Since convolution in the time domain is equivalent to multiplication in the frequency domain, we can use fourier transform with $h_1[n]$ and $h_2[n]$ to directly obtain $H(e^{jw})$

$$h_1[n] * h_2[n] = H_1(e^{jw})H_2(e^{jw}) \quad (2)$$

We know that;

$$a^n u[n] \xleftrightarrow{\text{F.T}}= \frac{1}{1 - ae^{-jw}}$$

$$h_1[n] \xleftrightarrow{\text{F.T}} H_1(e^{jw}) = \frac{1}{1 - \frac{1}{2}e^{-jw}}$$

$$h_2[n] \xleftrightarrow{\text{F.T}} H_2(e^{jw}) = \frac{1}{1 - \frac{1}{2}e^{-jw}}$$

Using eq 1 and 2:

$$h[n] \xleftrightarrow{\text{F.T}} H(e^{jw}) = H_1(e^{jw})H_2(e^{jw})$$

$$H(e^{jw}) = (\frac{1}{1 - \frac{1}{2}e^{-jw}})^2$$

(b) To find Fourier transform of $x[n]$, let us first find $\mathcal{F}\{sin(\frac{\pi n}{3})\}$ and shift it by $\frac{\pi}{4}$.

$$\mathcal{F}\{sin(\frac{\pi n}{3})\} = \frac{\pi}{j}\sum_{k=-\infty}^{\infty}\delta(w-\frac{\pi}{3}-2\pi k)-\delta(w+\frac{\pi}{3}-2\pi k)$$

$$\mathcal{F}\{sin(\frac{\pi n}{3}+\frac{\pi}{4})\} = e^{jw\frac{\pi}{4}}\frac{\pi}{j}\sum_{k=-\infty}^{\infty}\delta(w-\frac{\pi}{3}-2\pi k)-\delta(w+\frac{\pi}{3}-2\pi k)$$

(c) We know that:

$$y[n] = x[n]*h[n] \overset{\mathcal{F}}{\longleftrightarrow} H(e^{jw})Y(e^{jw})$$

Then, we get:

$$Y(e^{jw}) = \frac{1}{(1-\frac{e^{-jw}}{2})^2}e^{jw\frac{\pi}{4}}\frac{\pi}{j}\sum_{k=-\infty}^{\infty}\delta(w-\frac{\pi}{3}-2\pi k)-\delta(w+\frac{\pi}{3}-2\pi k)$$

4. (a) Assume $h[n] = g_1[n] + g_2[n]$ where:

$$g_1[n] = 2\delta[n] \qquad g_2[n] = 2^{-n}u[n]$$

Using the linearity of discrete Fourier Transformation:

$$H(e^{jw}) = G_1(e^{jw}) + G_2(e^{jw})$$

Now, find fourier transformations of G_1 and G_2 respectively. We know that $\delta[n] \overset{\text{F}}{\longleftrightarrow} 1$ and $a^n u[n] \overset{\text{F}}{\longleftrightarrow} \frac{1}{1-ae^{-jw}}$ for $|a| < 1$ from the transform table. Then,

$$G_1(e^{jw}) = 2$$

$$G_2(e^{jw}) = \frac{1}{1-\frac{1}{2}e^{-jw}}$$

$$H(e^{jw}) = 2 + \frac{1}{1-\frac{1}{2}e^{-jw}}$$

(b) We can use the following formula to find the difference equation describing this system:

$$Y(e^{jw}) = H(e^{jw})X(e^{jw})$$

Let's arrange frequency response a bit:

$$H(e^{jw}) = 2 + \frac{1}{1-\frac{1}{2}e^{-jw}} = 2 + \frac{2}{2-e^{-jw}}$$

Put it into the equation above:

$$Y(e^{jw}) = (2 + \frac{2}{2-e^{-jw}})X(e^{jw})$$

$$Y(e^{jw}) = 2X(e^{jw}) + \frac{2X(e^{jw})}{2-e^{-jw}}$$

$$(2-e^{-jw})Y(e^{jw}) = (4-2e^{-jw})X(e^{jw}) + 2X(e^{jw})$$

Then use inverse fourier transformation and lookup table. We get:

$$2y[n] - y[n-1] = 6x[n] - 2x[n-1]$$

$$y[n] = \frac{y[n-1] + 6x[n] - 2x[n-1]}{2}$$

(c) We can write $x[n] = e^{j\pi n}$ instead of $(-1)^n$ where $w_0 = \pi$. To get $Y(e^{jw})$, we can use the equation below:

$$Y(e^{jw}) = H(e^{jw})X(e^{jw})$$

We know $H(e^{jw}) = 2 + \frac{2}{2-e^{-jw}}$. Let's find $X(e^{jw})$. From the lookup table:

$$X(e^{jw}) = 2\pi\sum_{k=-\infty}^{\infty}\delta(w-\pi-2\pi k)$$

Put $X(e^{jw})$ into equation.

$$Y(e^{jw}) = \frac{6-2e^{-jw}}{2-e^{-jw}}2\pi\sum_{k=-\infty}^{\infty}\delta(w-\pi-2\pi k)$$