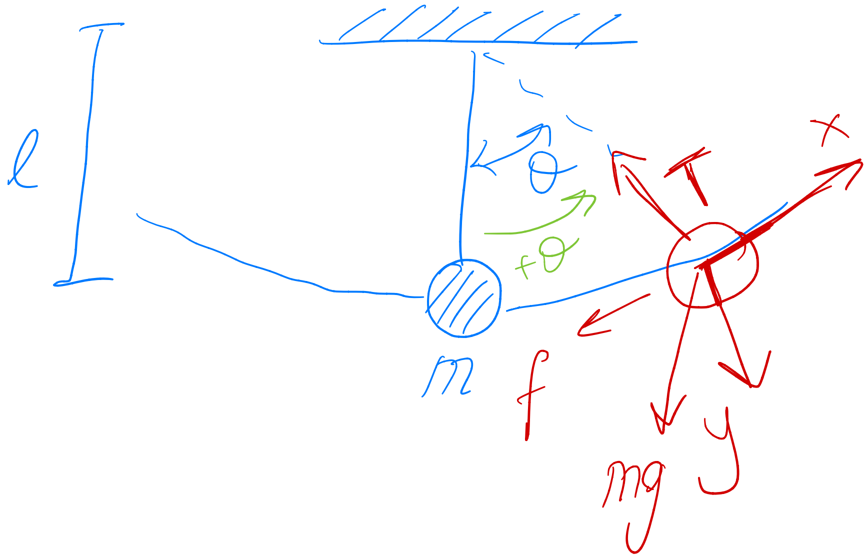
FLIGHT DYNAMICS

#1

11/07/2024

Introduction to System Dynamics:

K: friction coef.



$\dot{\theta} = \frac{d\theta}{dt}$ Angular velo.

Analyze the "motion"

"dynamics"

$\sum \vec{F} = m\vec{a}$ Newton

y-direction:

$$-T + mg\cos\theta = m \cdot a_y$$

$$\vec{a} = \frac{d\vec{V}}{dt}$$

change in velo.

<u>vector</u>

(it can be change in velocity or direction)

$$m\frac{v^2}{R}$$

$v = \omega r$ angular velo.

$= \dot{\theta} \ell$

$$= -T + mg\cos\theta = m \cdot \frac{\dot{\theta}^2 \cancel{\ell^2}}{\cancel{\ell}}$$

inside?

$v = \dot{\theta}\ell$

centrifugal force

$$= \boxed{-T + mg\cos\theta = -m\dot{\theta}^2\ell}$$

motion dynamics in y-direc.

Velocity and acceleration change wrt to the reference frames. $\Rightarrow$ Reference Frames

I am siting outside and looking otha coordinate system

$$\boxed{\sum \vec{F} = m\vec{a}}$$

then writing this

Observer

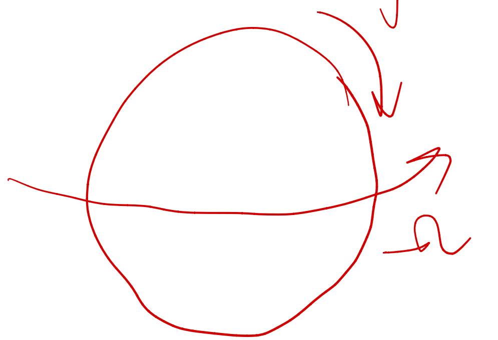
Who is the Observer then?

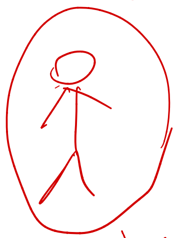
$\Downarrow$

α Earth is not rotating

and

α Earth is flat





Observer sits outside and fixed

he watches the moving object from fixed point

Coordinate system is not an observer
it is just a measurement device

x-direction: $\sum \vec{F} = ma$

x -direction: $\boxed{-mg\sin\theta - k\dot{\theta} = m\ddot{\theta}l}$

$V = \omega r$

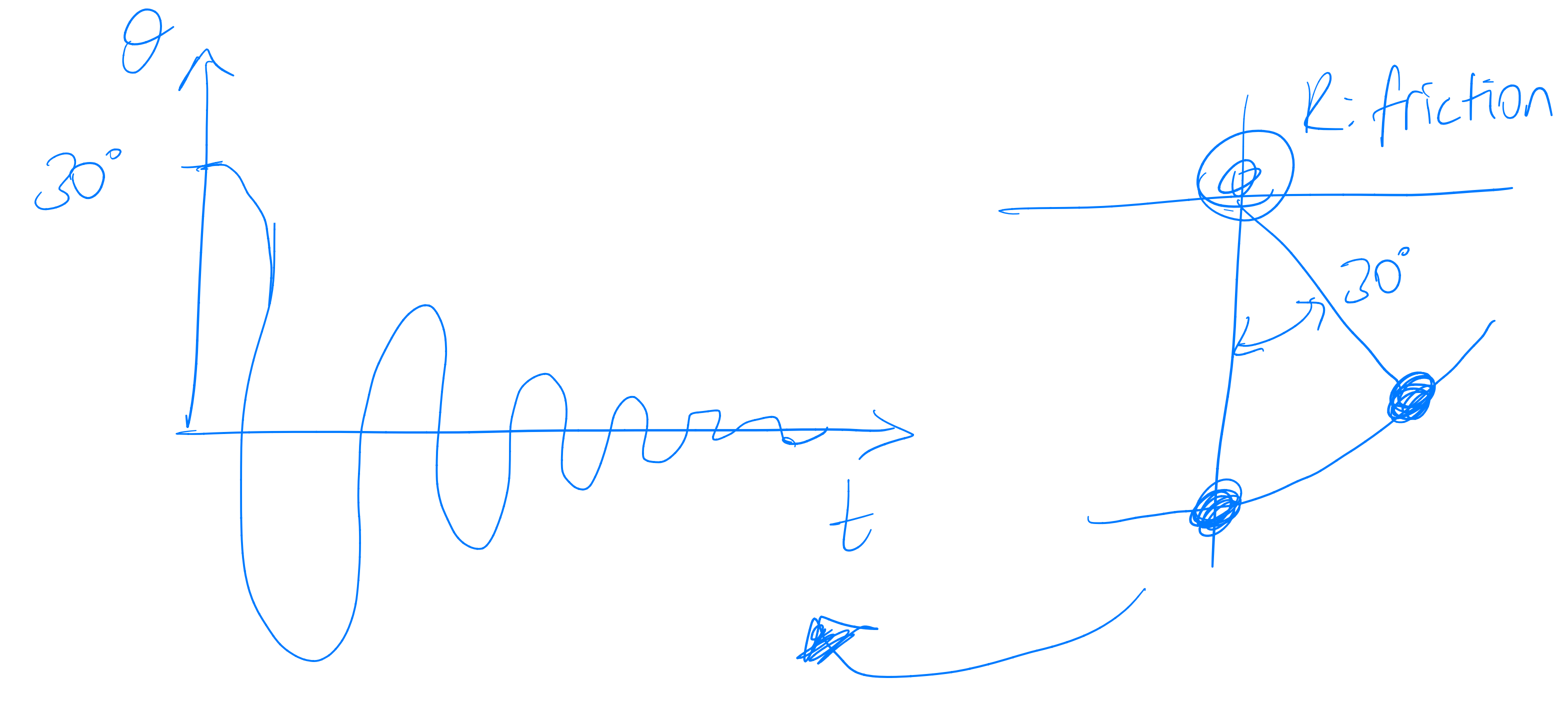
$V_x = \dot{\theta} l$

$\frac{dV_x}{dt} = \ddot{\theta} l \qquad \dot{\theta} = \frac{d\theta}{dt} \qquad \ddot{\theta} = \frac{d^2\theta}{dt^2}$

Equations of motion:

x-direction: $-mg\sin\theta - k\dot{\theta} = m\ddot{\theta}\,l$ → 2nd order Diff.

y-direction: $-T + mg\cos\theta = -m\dot{\theta}^2 l$ → 1st order Diff.



You need this eqns. for an airplane ! for example
what will be the behaviour when a gust hits or turbulence comes

The movement of pendelum let's look at equation in the x-direction:

$$-mg\sin\theta - k\dot{\theta} = m\ddot{\theta}l$$

$\theta(t)$, θ_0, $\dot{\theta}_0$

2nd order diff. eqn.

We divide 2nd order eqns into 2, 1 order dif. eqns.

$\theta_1 = \theta$ $\qquad$ $\theta_2 = \dot{\theta}$

$$\ddot{\theta} = -\frac{g}{l}\sin\theta - \frac{k}{ml}\dot{\theta}$$

$$\boxed{\begin{aligned} \dot{\theta}_2 &= -\frac{g}{l}\sin\theta_1 - \frac{k}{ml}\theta_2 \\ \dot{\theta}_1 &= \theta_2 \end{aligned}}$$

2, 1st order diff. eqn.

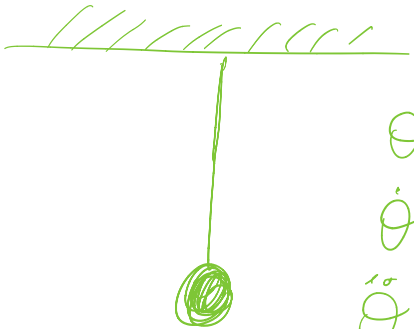
θ_1, θ_2 state variables

* θ_1, θ_2 are enough to describe the dynamics of the system. > The variable define the state.

* How many state variable do I need to fully define the dynamic behaviour of the pendulum? $\Rightarrow 2$

* How many state variables do I need to define the dynamic movement of an airplane?

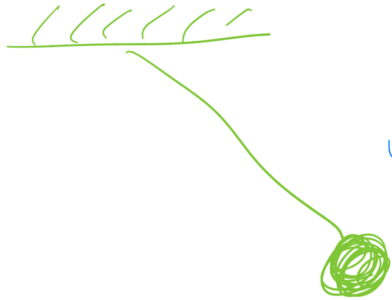
State variables > The variables that I need to describe the motion dynamics of a system



$\theta = 0$

$\dot{\theta} = 0$

$\ddot{\theta} = 0$



$\underbrace{\theta = 30^\circ}_{\theta_1}$, $\underbrace{\dot{\theta} = 10/s}_{\theta_2}$ $\underbrace{\ddot{\theta} = 0}_{\dot{\theta}_2}$

Equilibrium Point

If you dont touch it will stay forever

Equilibrium point of a dynamic system denotes a steady state, one which all states variables do not change (stay constant) in time (with no input)

$$\dot{\theta}_2 = 0 = -\frac{g}{l}\sin\theta_1 - \frac{K}{ml}\theta_2$$

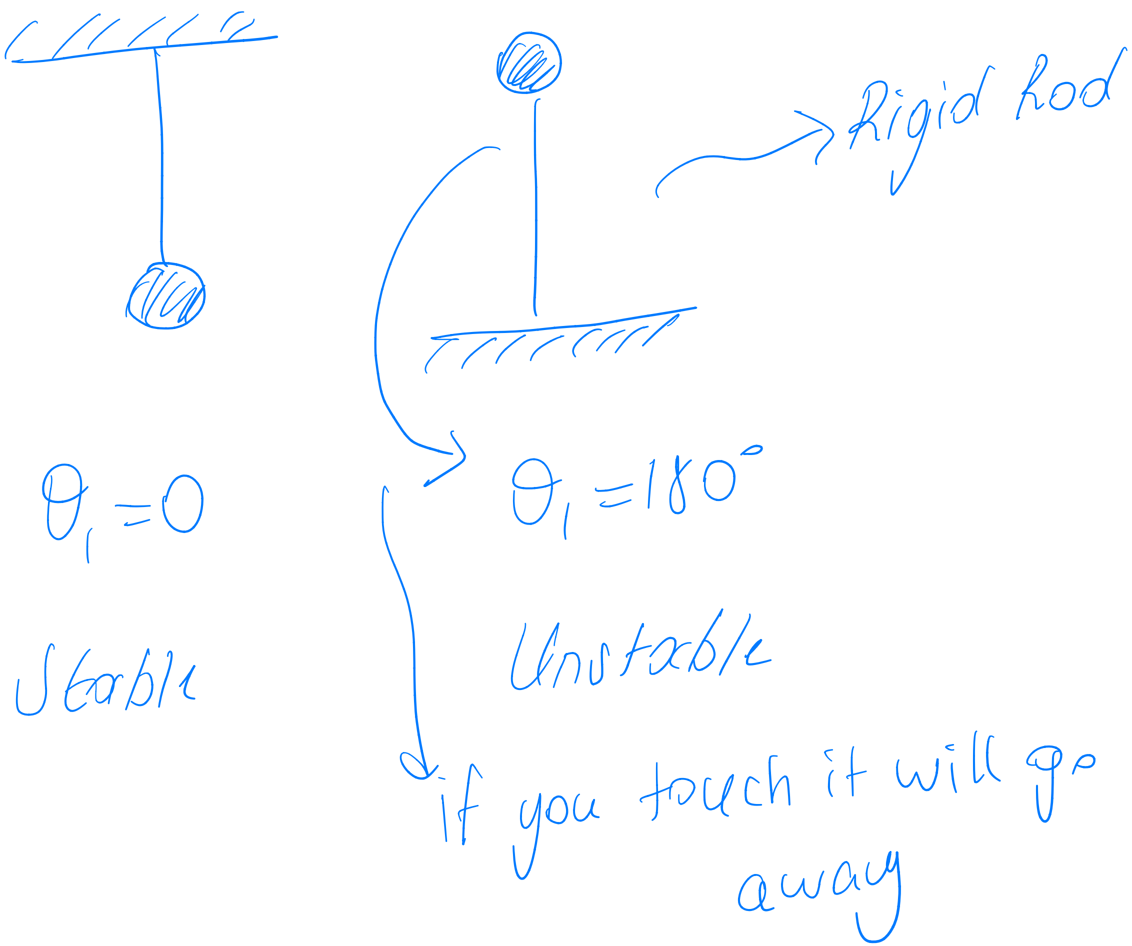
$$\dot{\theta}_1 = 0 = \theta_2$$

$$\boxed{\theta_2 = 0}$$

it does not always means that in eq. point velo is zero

$$0 = -\frac{g}{l}\sin\theta_1 - \frac{K}{ml}\cancel{0}$$

$$\boxed{\sin\theta_1 = 0} \quad \boxed{\theta_1 = 0^\circ} \quad \boxed{\theta_1 = 180^\circ}$$



If the pendalum stable? $\Rightarrow$ No answer to this question.

The pendelum has two equilibrium points 1 is stable, 1 is unstable.

~~F16 is unstable~~ ⇒ Wrong Statement

F16 has equilibrium points that are unstable, if fact some of the equilibrium points may be stable

Stability is the property of equilibrium points

Static Stability, is the initial tendency of the system to return to its eq. pt. after it has been perturbed from it.

☆ How long does it take? — mins / — hours } Dynamic Stability

☆

$$\dot{\theta}_2 = -\frac{g}{\ell}\sin\theta_1 - \frac{K}{m\ell}\theta_2$$

$$\dot{\theta}_1 = \theta_2$$

How can I analyze if this system is statically stable or unstable?

add a little force / perturbation and see what happens in time

Characteristic eqn.

denominator of TF

Laplace

This is a nonlinear eqn so I cannot find Laplace

because of the sinus

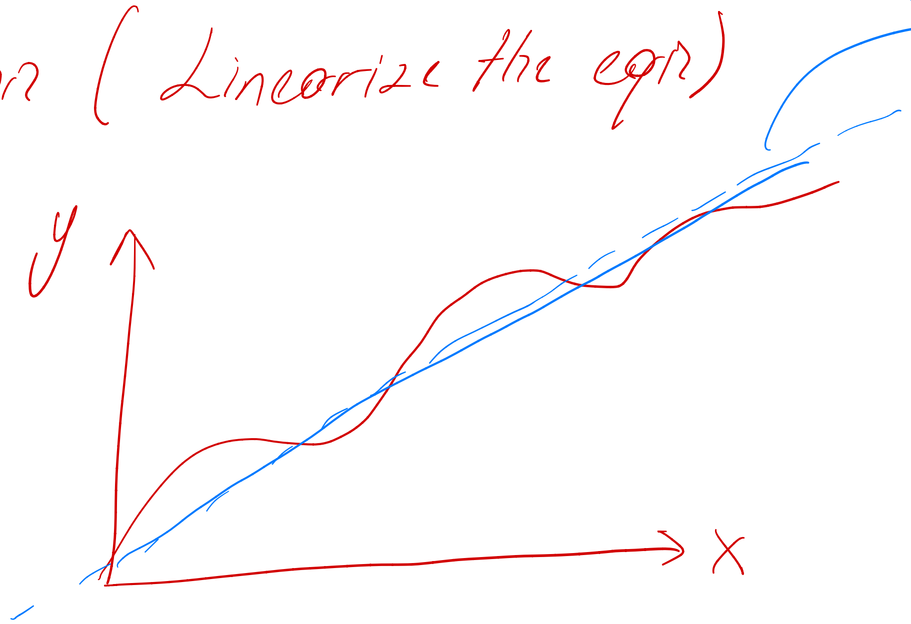
Transfer functions are defined for linear systems

We have to linearize the equations

Taylor Series Expansion (Linearize the eqn)

$y = f(x)$

Linearizes the function around a point



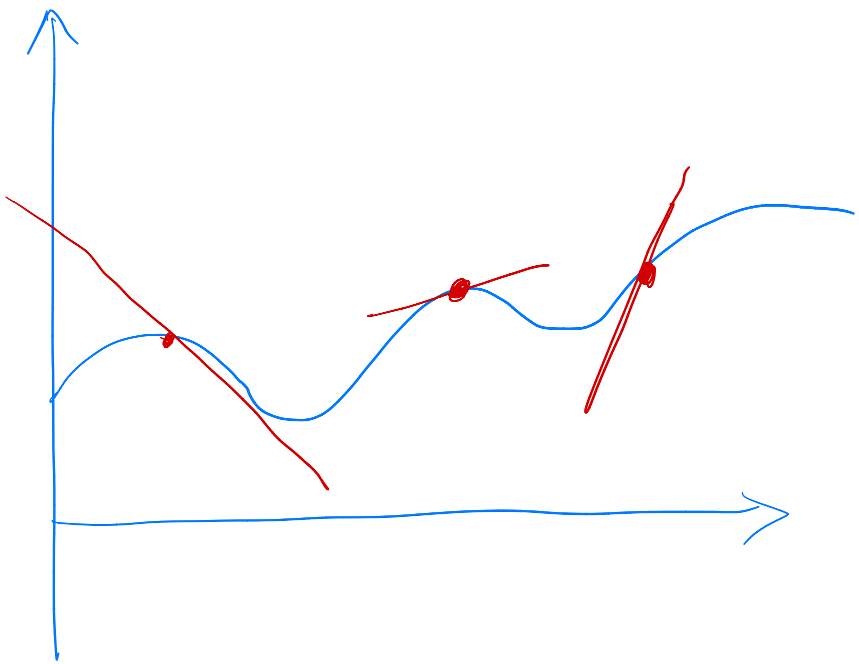
Linearization By using Taylor Ser.

Will I get the blue line?

No

This is a curve fit, approximation of the function

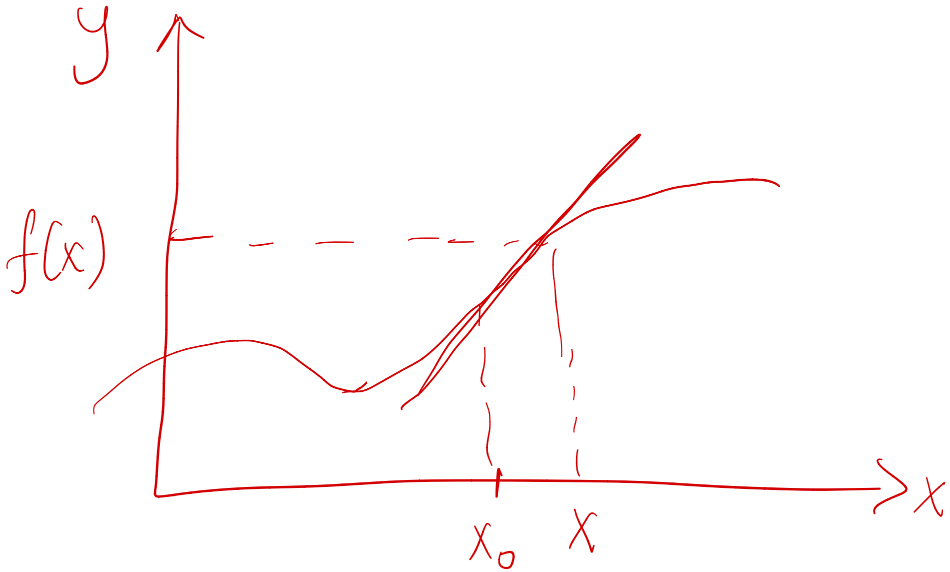
Taylor series will choose a point, then Taylor series will linearize the function around that point



Here 3 different points are chosen and 3 different lines are obtained

$$y = f(x_0)$$

linearize wrt x_0 in Taylor series



$$y = f(x_0) + \frac{\partial f}{\partial x}\bigg|_{x=x_0}(x - x_0) + HOT$$

HOT → higher order terms

⇓

If you want to linearize use this

Taylor Series

12 July 2024

2nd Lecture

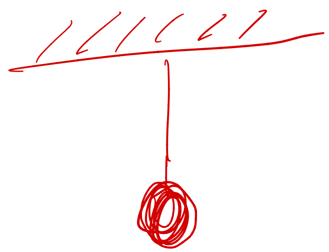
$$\boxed{\begin{aligned}\dot{\theta}_2 &= -\frac{g}{l}\sin\theta_1 - \frac{k}{ml}\theta_2 \\ \dot{\theta}_1 &= \theta_2\end{aligned}}$$

$$\dot{\theta}_2 = f(\theta_1, \theta_2)$$

$$\dot{\theta}_1 = \theta_2$$

Linearize around eq. pt.

$$\theta_{10} = 0 \qquad \theta_{20} = 0$$



$$\dot{\theta}_2 = f(0,0) + \left.\frac{\partial f}{\partial \theta_1}\right|_{\substack{\theta_1=0\\ \theta_2=0}} (\theta_1 - \theta_{10}) + \left.\frac{\partial f}{\partial \theta_2}\right|_{\substack{\theta_1=0\\ \theta_2=0}} (\theta_2 - \theta_{20})$$

$$= 0 + \left(-\frac{g}{l}\cos\theta_1\right)\Big|_{\substack{\theta_1=0\\ \theta_2=0}} \theta_1 + \left(-\frac{k}{ml}\right)\Big|_{\substack{\theta_1=0\\ \theta_2=0}} \theta_2$$

$$\boxed{\dot{\theta}_2 = -\frac{g}{l}\theta_1 - \frac{k}{ml}\theta_2}$$

linearized around

$\theta_1 = 0 \quad \theta_2 = 0$

$$\boxed{\dot{\theta}_1 = \theta_2}$$

$$\begin{bmatrix} \dot\theta_1 \\ \dot\theta_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -g/l & -k/ml \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix}$$

linear state-space representation of the dynamic system

$$x = \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix}$$ state vector

$$\boxed{\dot x = Ax}$$

$$0 = -\frac{g}{\ell}\theta_1 - \frac{\ell}{m\ell}\theta_2$$

$$0 = \theta_2$$

$$\boxed{\begin{aligned}\theta_2 &= 0\\ \theta_1 &= 0\end{aligned}}$$

Equilibrium points

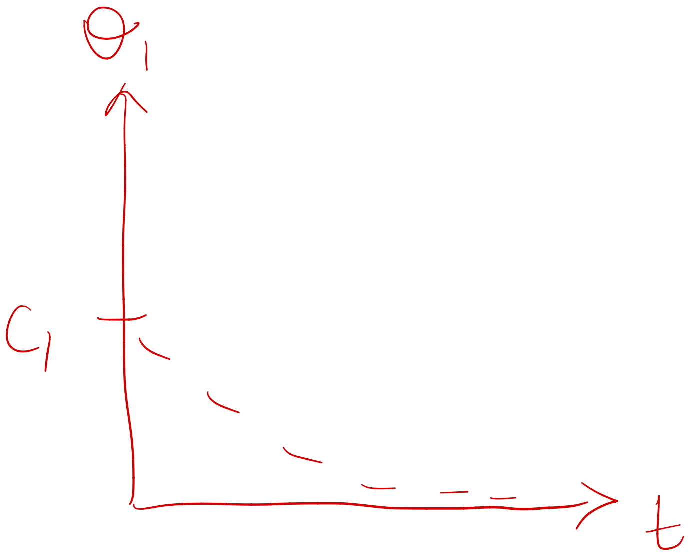
Let's start the system some where else from the equilibrium point

$\Rightarrow$ Solve the linear system $\dot{x} = Ax$ starting from $x = x_0$ where x_0 is not the eq. point numerical example : $g/\ell = 2$

$\ell/m\ell = 3$

$$\begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix}$$

with $\theta_1 = c_1$
$\theta_2 = c_2$
@ $t = 0$



$\theta_1(t) = ?$

$\theta_2(t) = ?$

2D $\dot{x} = Ax$

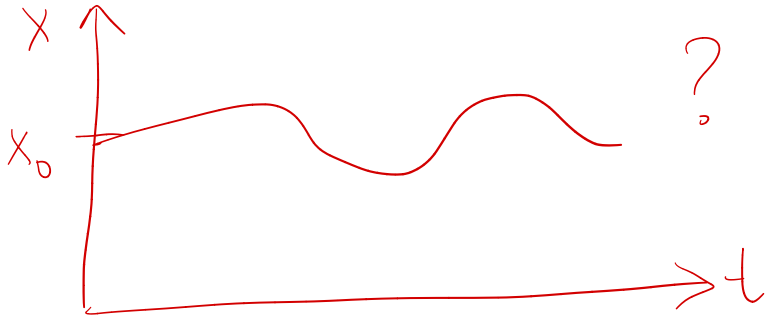
NEXT

1-D problem

$\dot{x} = ax$ a: scalar

$$\boxed{\frac{dx}{dt} = ax}$$

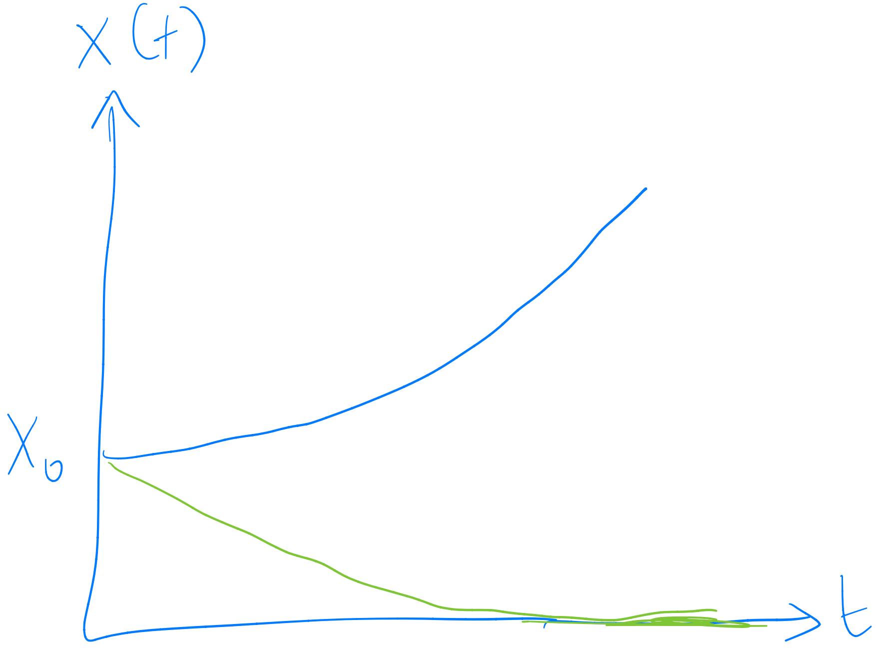
$x = x_0$ @ $t = 0$



I want to know the time history of $x(t) = ?$

$x_0 a e^{at} = a x_0 e^{at}$ ✓

$x(t) = x_0 e^{at}$



$a > 0$

$a < 0$ → converge to zero

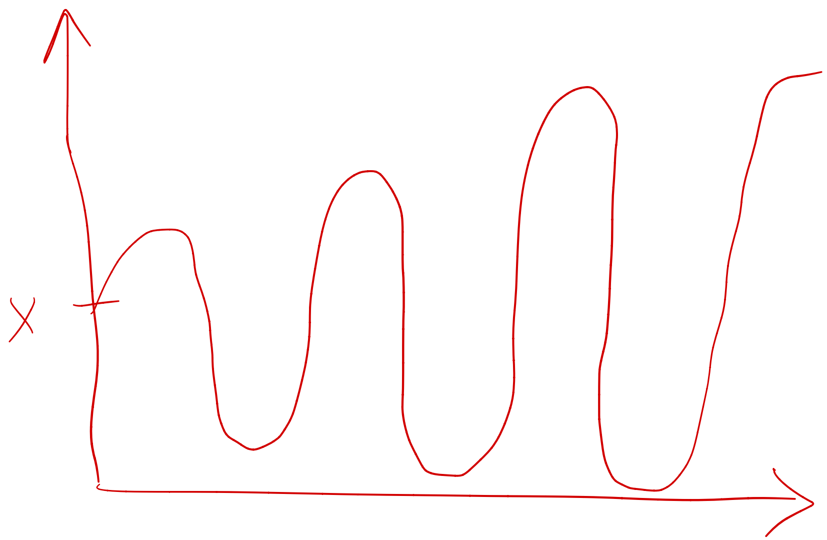
What if "a" is a complex number?

$a = a_1 + b_1 i$

$x(t) = x_0 e^{(a_1 + b_1 i)t} = x_0 e^{a_1 t} e^{b_1 i t}$

Oscillation coming from here

$e^{b_1 i t} = \underbrace{\cos b_1 t + i \sin b_1 t}_{\text{Oscillation}}$

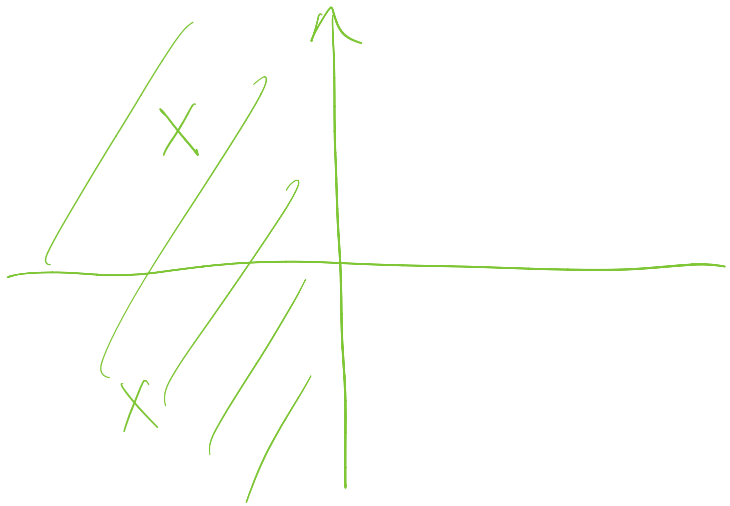


$a_1 > 0$ growing oscillation

$a_1 < 0$ Oscillation that decay

(At last coming back to zero)

$\underbrace{a_1 \text{ is deciding stable or unstable}}_{\text{The real part}}$



If the real part is negative the oscillation will decay.

Very similiar to this $\dot{x} = ax$ we can look at

$\dot{x} = Ax$ and say something about stability.

2-D Problem

$\dot{x} = Ax$, $x = x_0$ @ $t=0$

$$\begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix}$$

@ $t=0$

$\theta_1 = c_1$

$\theta_2 = c_2$

Assume a soln, $\theta_1(t) = e^{\lambda t} y$

$\theta_2(t) = e^{\lambda t} z$

y, z, λ are constant

$$\dot{\theta}_1 = \lambda \cancel{e^{\lambda t}} y = \cancel{e^{\lambda t}} z$$

$$\dot{\theta}_2 = \lambda \cancel{e^{\lambda t}} z = -2 \cancel{e^{\lambda t}} y - 3 \cancel{e^{\lambda t}} z$$

$$\left.\begin{aligned} \lambda y &= z \\ \lambda z &= -2y - 3z \end{aligned}\right\} \lambda \underbrace{\begin{bmatrix} y \\ z \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} y \\ z \end{bmatrix}}_{X}$$

2 eqns

3 unknowns

it look like I cannot find a soln. But look

$$\lambda X = AX$$

$$\boxed{Ax = \lambda x}$$

Eigenvalue problem

$$eig(A) = \det \begin{bmatrix} 0-\lambda & 1 \\ -2 & -3-\lambda \end{bmatrix} = 0$$

$$-\lambda(-3-\lambda) + 2 = 0$$

$$\lambda^2 + 3\lambda + 2 = 0$$

$$\lambda_1 = -2 \qquad \lambda_2 = -1$$

$\lambda_1 = -2$

Eigenvalues

$\lambda_2 = -1$

$x_1 = ?$

$x_2 = ?$

$$\begin{bmatrix} -2y \\ -2z \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix}$$

$$x_2 = \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

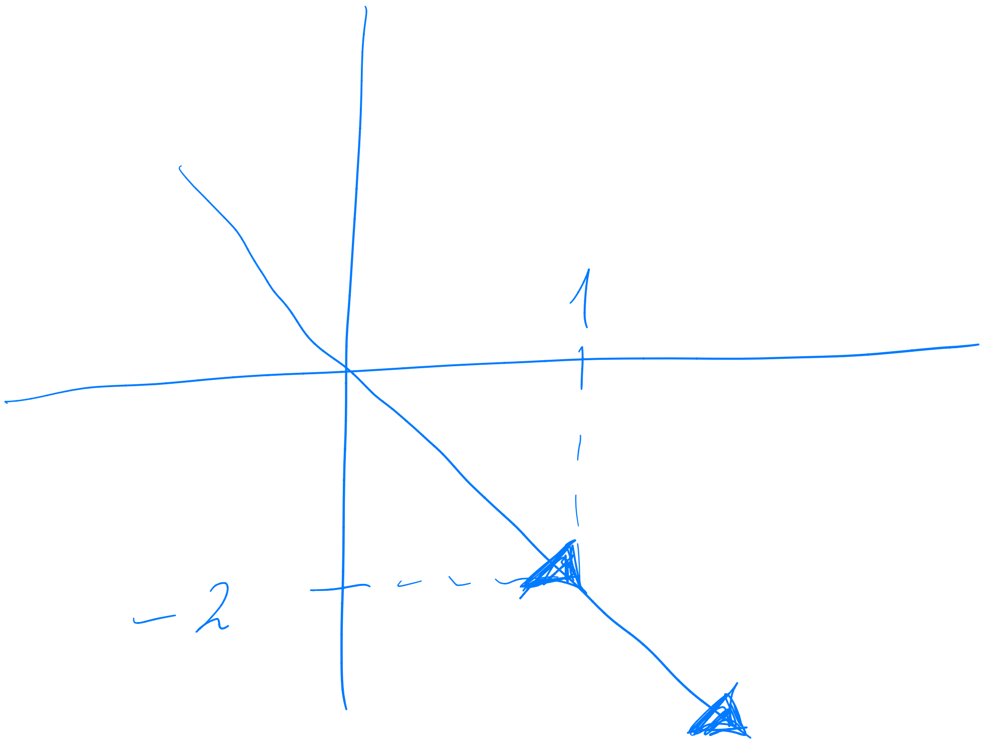
$$-2y = z$$
$$-2z = -2y - 3z$$

$$x_1 = \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

Eigenvectors

any multiples is a soln.

$$x_1 = \begin{bmatrix} 2 \\ -4 \end{bmatrix}, \quad x_1 = \begin{bmatrix} 4 \\ -8 \end{bmatrix}$$



$\lambda_2 = -1 \qquad x_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\lambda_1 = -2 \qquad x_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

$\theta_1 = e^{-t}, \quad \theta_2 = -e^{-t}$

$\theta_1 = e^{-2t} \qquad \theta_2 = -2e^{-2t}$

$e^{\lambda t} y \qquad e^{\lambda t} z$

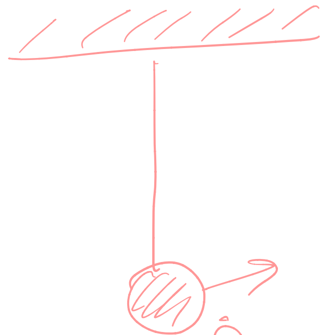
What is the soln?

$$\theta_1(t) = c_1 e^{-t} + c_2 e^{-2t}$$
$$\theta_2(t) = c_1(-e^{-t}) + c_2(-2)e^{-2t}$$
} General Solution

Actual soln. is the linear combination of these two

EX Find c_1, c_2 initial condition

numeric example:



$\theta_1 = \theta = 0$ rad

$\theta_2 = \dot{\theta} = 0.1$ rad/s

$\dot{\theta} = 0.1$ rad/s

pendulum is not in equilibrium point since it has a velocity

when $t=0$

$$\left.\begin{aligned} 0 &= c_1 + c_2 \\ 0.1 &= -c_1 + (-2)c_2 \end{aligned}\right\}$$

$$\begin{bmatrix} 0 \\ 0.1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

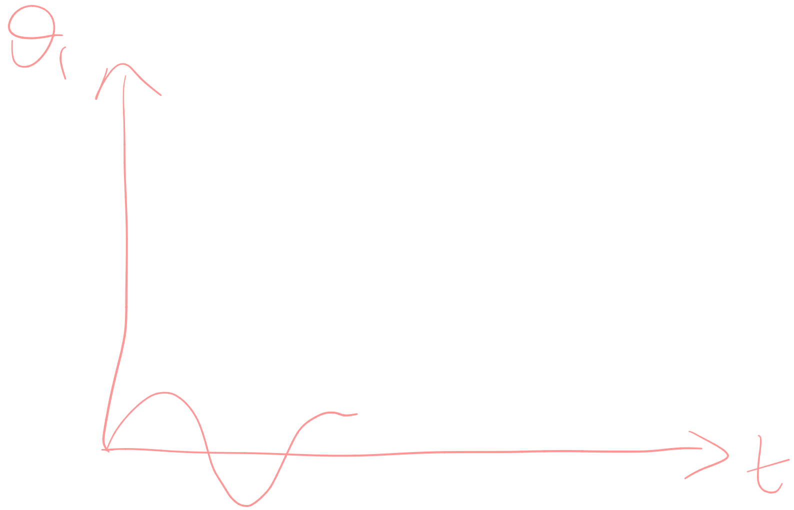
$c_1 = 0.1$

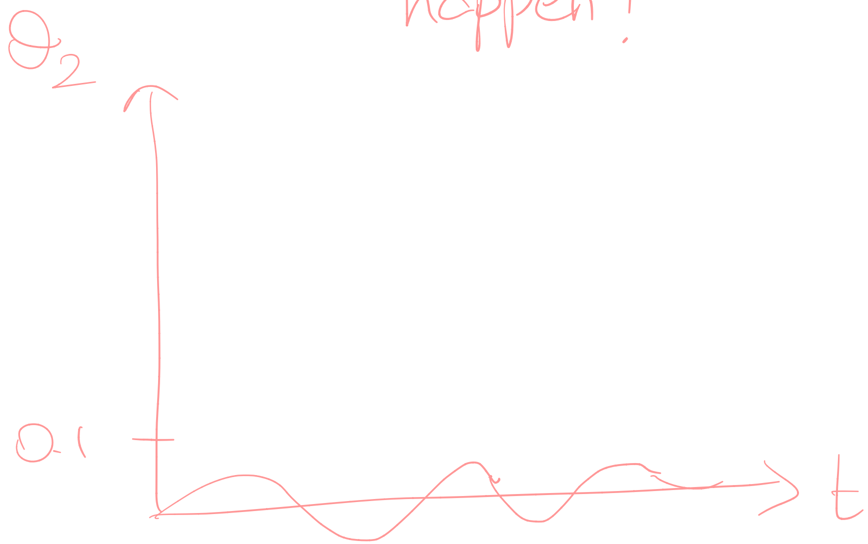
$c_2 = -0.1$

$$\boxed{\begin{aligned}\theta_1(t) &= 0.1e^{-t} - 0.1e^{-2t}\\ \theta_2(t) &= -0.1e^{-t} + 0.2e^{-2t}\end{aligned}}$$

Eigenvalues of the A matrix

if I start from the initial condition this is what's going to happen!





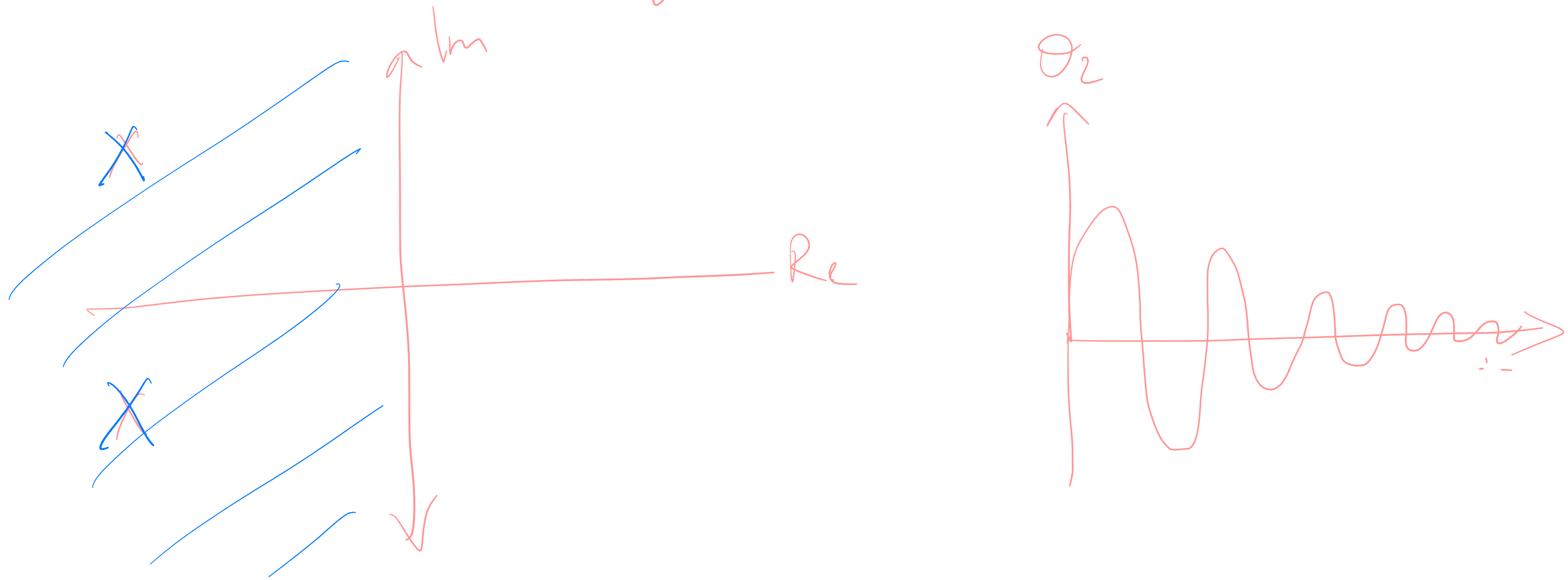
as $t \to \infty$ $\quad \underbrace{\theta_1(t) = 0 \quad \theta_2(t) = 0}_{\text{equilibrium point}}$

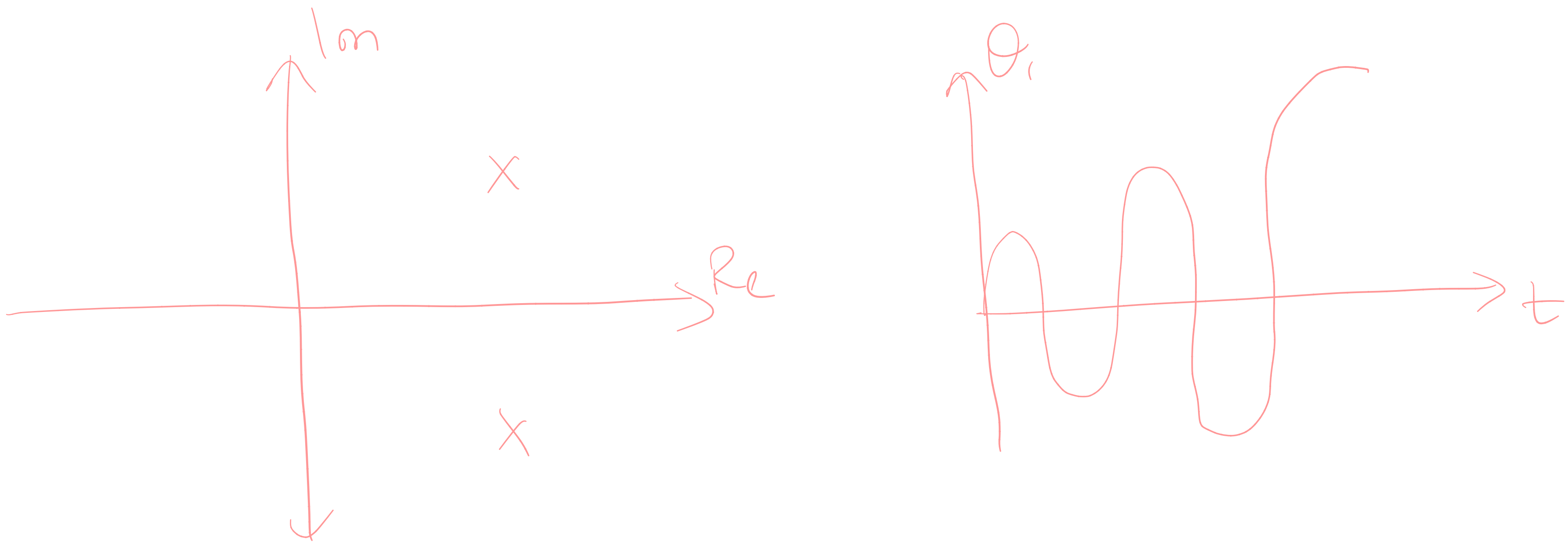
A matrix was find by linearizing the nonlinear eqn. around a point

As long as the eigenvalues of the system is negative θ_1, θ_2 will go to zero. I do not need to solve the problem all the time. If eigenvalues of A is negative, states will go to zero.

All Eigen values of A, must be negative. If one of the eigenvalue is positive, it will go to infinity.

If we have complex eigenvalue which is $\lambda = a + bi$





$$\theta_1(t) = 0.1e^{-t} - 0.1e^{-2t}$$

if its bigger it will diverge very fast

$$\theta_2(t) = -0.1e^{-t} + 0.2e^{-2t}$$

Large Eigenvalues (Abs)

system will converge or diverge very quickly

Take Laplace → find TF → Look at Characteristic eqn.

↓

You will find again the eigenvalues ← Find the roots of the Charac eqn.

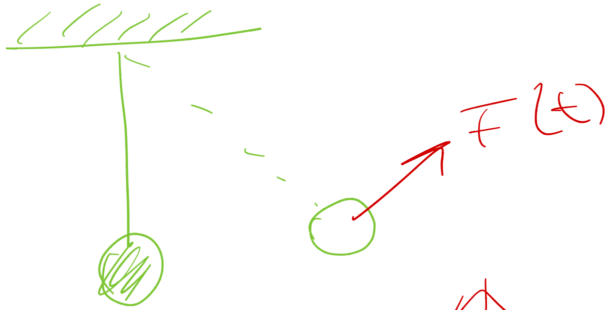
if A matrix was 10x10, you would have 10 eigenvalues

Eigenvalues of A matrix = Root of the characteristic eqn.

Doing the same thing using Laplace Transform:

$$\dot{\theta}_1 = \theta_2$$

$$\dot{\theta}_2 = -\frac{g}{\ell}\theta_1 - \frac{k}{m\ell}\theta_2$$



→ Find input versus output

There is no input so I have to invent an input

I added a force

$$\dot{\theta}_1 = \theta_2$$

$$\dot{\theta}_2 = -\frac{g}{\ell}\theta_1 - \frac{k}{m\ell}\theta_2 + \underbrace{F(t)}_{\text{input}}$$

Laplace

$$s\Theta_1(s) = \Theta_2(s)$$

$$s\Theta_2(s) = -\frac{g}{\ell}\Theta_1(s) - \frac{k}{m\ell}\Theta_2(s) + F(s)$$

$$s^2\Theta_1(s) = -\frac{g}{\ell}\Theta_1(s) - \frac{k}{m\ell}s\Theta_1(s) + F(s)$$

$$\left(s^2 + s\frac{k}{ml} + \frac{g}{l}\right)\theta_1(s) = F(s)$$

$$G(s) = \frac{\theta_1(s)}{F(s)} = \frac{1}{s^2 + s\frac{k}{ml} + \frac{g}{l}}$$

$$\frac{k}{ml} \rightarrow 3 \qquad \frac{g}{l} \rightarrow 2$$

$$G(s) = \frac{1}{s^2 + 3s + 2}$$

→ characteristic eqn

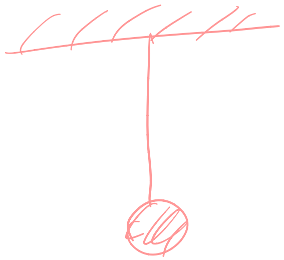
$$\boxed{s^2 + 3s + 2 = 0}$$

$$s_1 = -1 \quad , \quad s_2 = -2$$

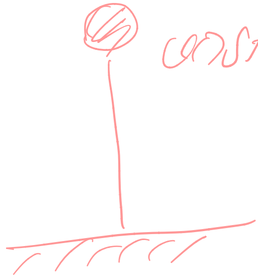
★ If the linearized system (around the eq. pt.) is unstable (not all eigenvalues are negative), then the eq. pt. of the non-linear system is unstable.

★ If the eq. pt. of the linear system is stable then the eq. pt. of the non-linear system (around which it was linearized) is also stable.

An airplane might have hundreds of equilibrium points. Every altitude, velocity is an equilibrium point. You can linearize the system around these equilibrium points and investigate the stability.



stable



unstable

$$\dot{x} = Ax$$

$x = 0$ equilibrium

$$\left.\begin{array}{l}\theta_1 = 0 \\ \theta_2 = 0\end{array}\right\} \text{Non-linear}$$

$$\left.\begin{array}{l}\theta_1 = 180^\circ \\ \theta_2 = 0^\circ\end{array}\right\} \text{Nonlinear}$$

$$\text{linear} \left\{\begin{array}{l}\theta_1 = 0 \\ \theta_2 = 0\end{array}\right.$$

$$\left.\begin{array}{l}\theta_1 = 0^\circ \\ \theta_2 = 0^\circ\end{array}\right\} \text{linear}$$

??? Find it! HmW!

Reference Frame (Frame of Reference)

any rigid body can be a reference frame

Reference frame is a rigid body or a set of rigidly connected points that can be used to define (establish) velocities, acceleration etc.

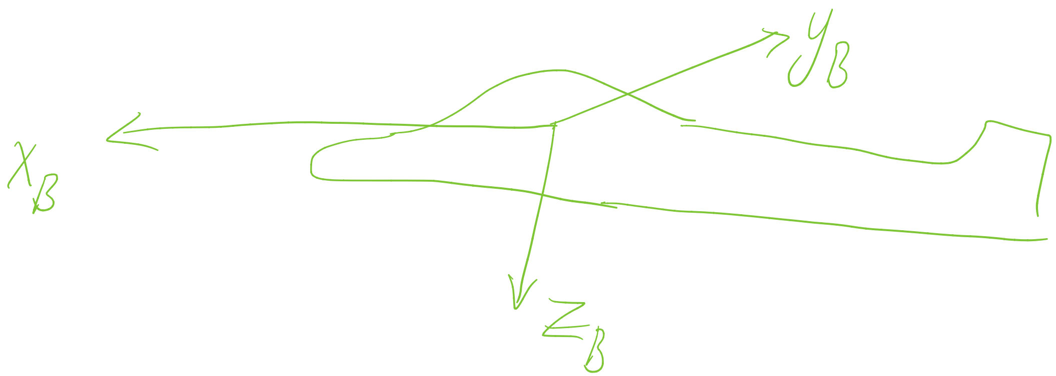
Coordinate system: is a measurement "device" usually attached to a reference frame.

When you say the velocity $\vec{V} = \begin{bmatrix} 3 \\ 5 \\ 6 \end{bmatrix}$ m/s you have to mention about the reference frame and the coordinate system.

Reference Frames and coordinate System used in Flight Mechanics

1) ~~Body Fixed Reference Frame~~

Body-Fixed coordinate system (F_B)



c.g. : center of gravity

x_B : pointing towards the nose of the aircraft

y_B, z_B: right-handed coordinate system

z_B: symmetry axis of the aircraft

origin of coordinate system is at the cg

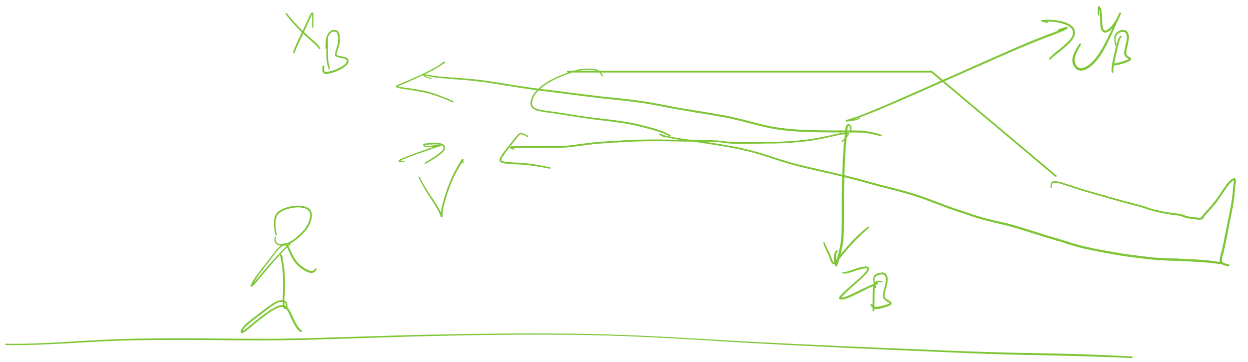
The coordinate system is attached to the airplane. Moves, rotates with the aircraft

Right hand Rule   rotate your fingers from x to y your thumb will show you the z direction.

What is the velocity of aircraft wrt to the body fixed coordinate frame? $\longrightarrow$ zero

(You sit in the airplane and look that the aircraft is not moving)

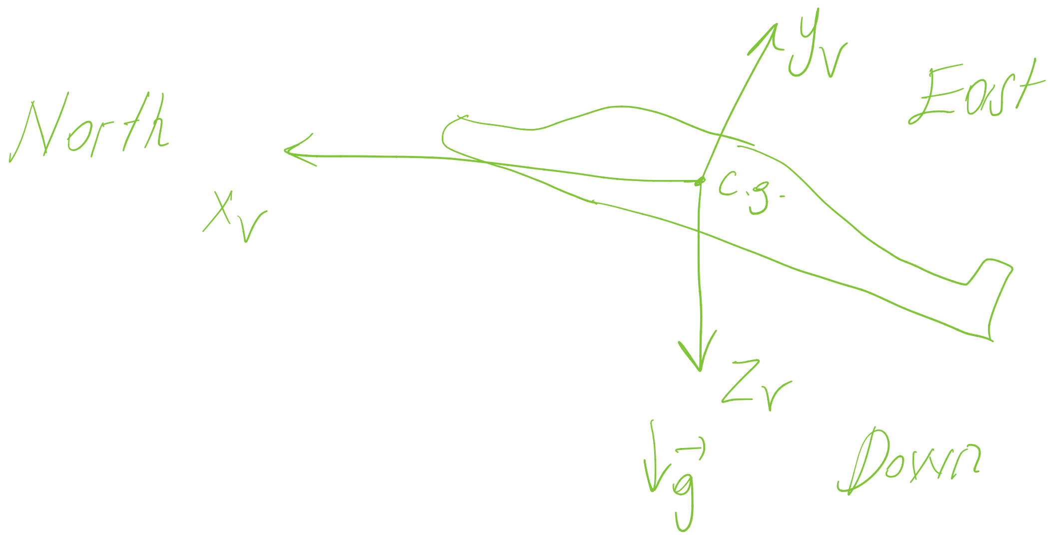
What is the velo of Earth wrt the body fixed coor. frame?

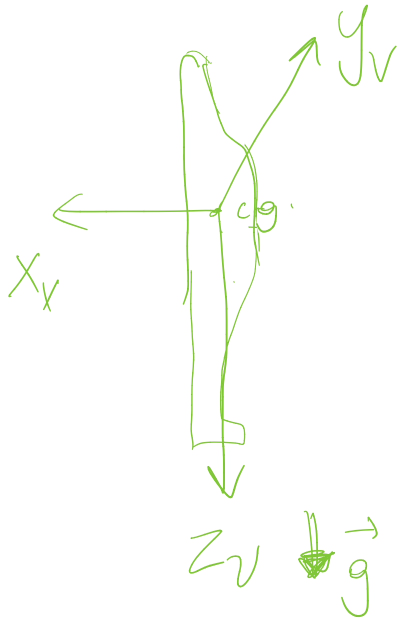


$$\vec{V} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

2) Vehicle Carried Reference Frame

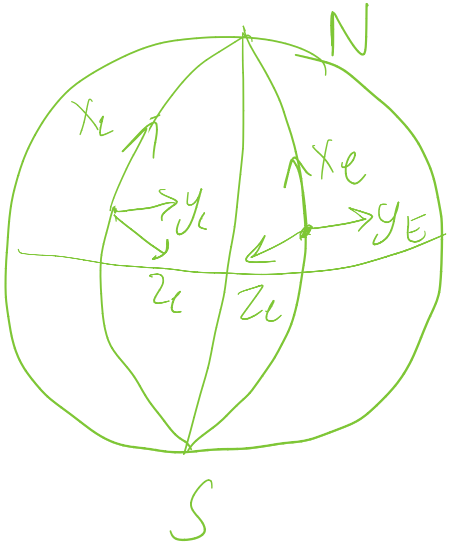
Vehicle Carried Coordinate Sys.





3) Earth Fixed Reference Frame

" Coordinate Sys. (F_E)

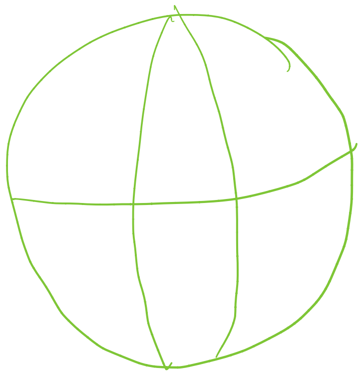


Z_E – center of the Earth

X_E – North pole

Y_E – East

4) Earth Centered Coordinate System?

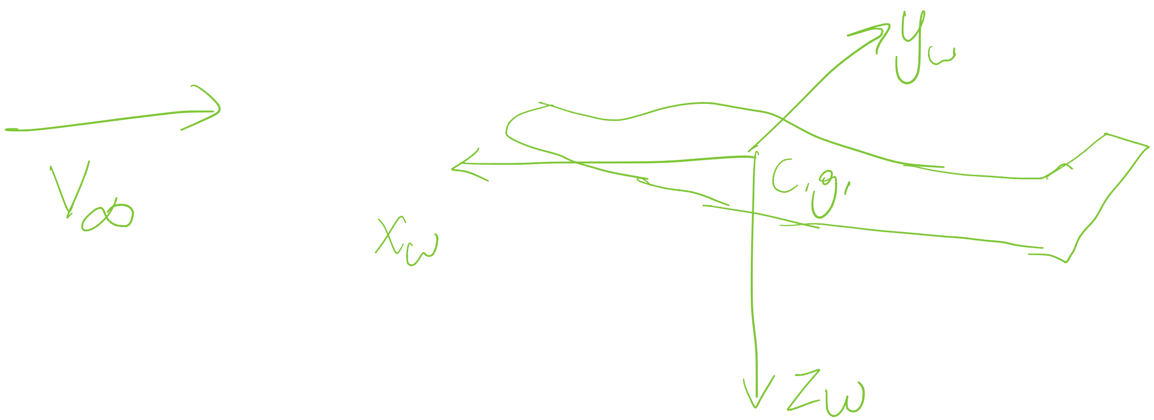


Z_{EC} – North

X_{EC}, Y_{EC} – Plane of equator

5) Air Trajectory Reference Frame

Wind - Fixed Coordinate System (F_W)



x_w = towards the local air velocity vector

z_w = symmetry plane of A/C

y_w = right handed coordinate system

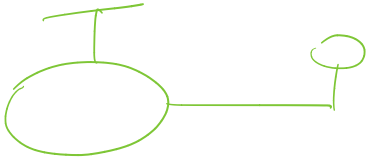
most used in aerodynamics

since

Drag is in ~~same~~ oppo direc. with V_∞

V_∞ $\longrightarrow$

In forward flight



In Hover there is no V_∞

$\longleftarrow$ V_∞

In backward flight

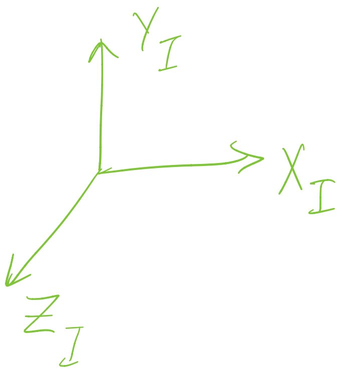
} Helicopter Example

For H/C in Hover V_∞ in changing direction, so it does not make sense to use wind fixed Coor. sys.

For A/C is useful while defining Lift and Drag

6) Inertial Frame (F_I)

Inertial Fixed Coordinate System



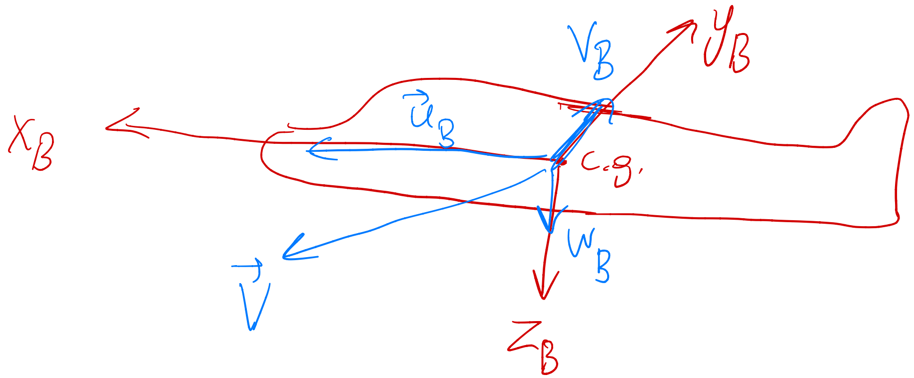
$O_{X_I Y_I Z_I}$ is fixed wrt $\underbrace{\text{distant stars}}_{\text{not to be moving}}$

$$\sum \vec{F} = m\vec{a}$$

7) Stability Frame of Reference (Later)

Some Important Symbology and Definitions in the Body Fixed Coordinate System

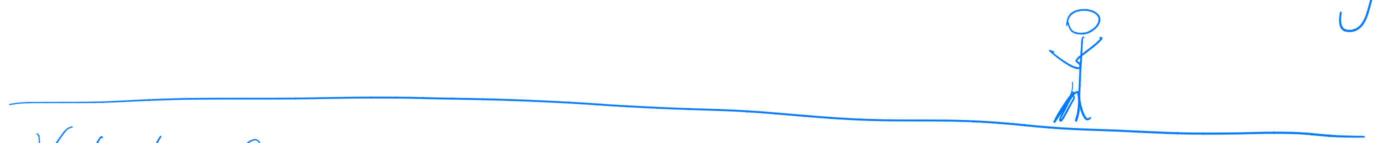
I



velocity wrt Earth

$$\vec{V}^E_B = \begin{bmatrix} u_B \\ v_B \\ w_B \end{bmatrix}$$

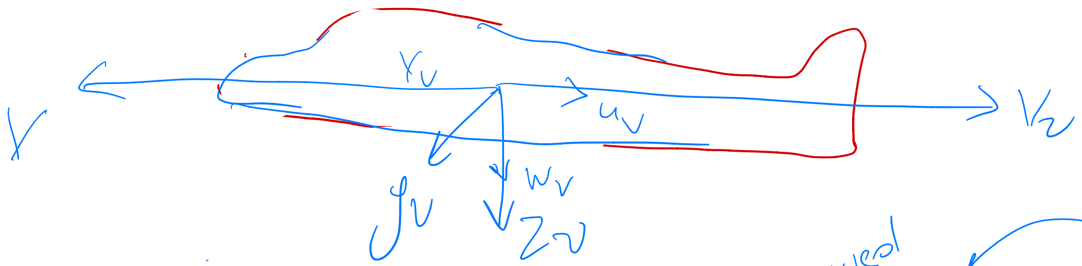
vector is decomposed in body axes (written in BFCS)



Earth

Vehicle Carr. Coord. Sys.

II



North

$$\vec{V}^E_V = \begin{bmatrix} u_V \\ v_V \\ w_V \end{bmatrix}$$

observed from Earth

Earth

$$|\vec{V}_B^{\,E}| = |\vec{V}_V^{\,E}|$$

$\vec{V}_B^{\,I}$

observed from Inertial frame
but written in body axis

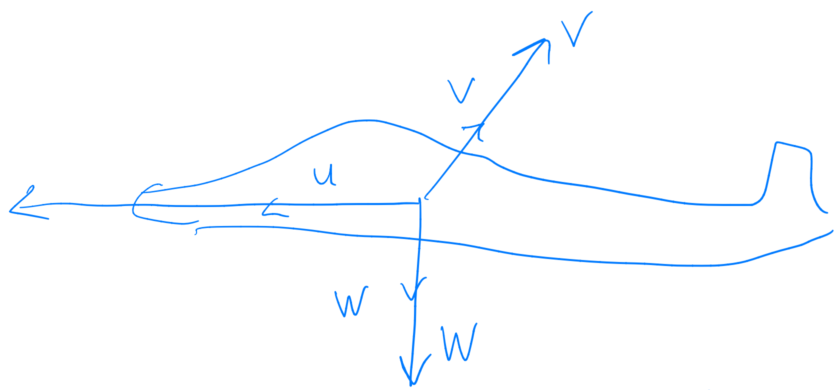
$$\vec{V}_B^{\,B} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\vec{V}_B^{\,\circ} \rightarrow$ Inertial Frame

$\rightsquigarrow \vec{V}_B$

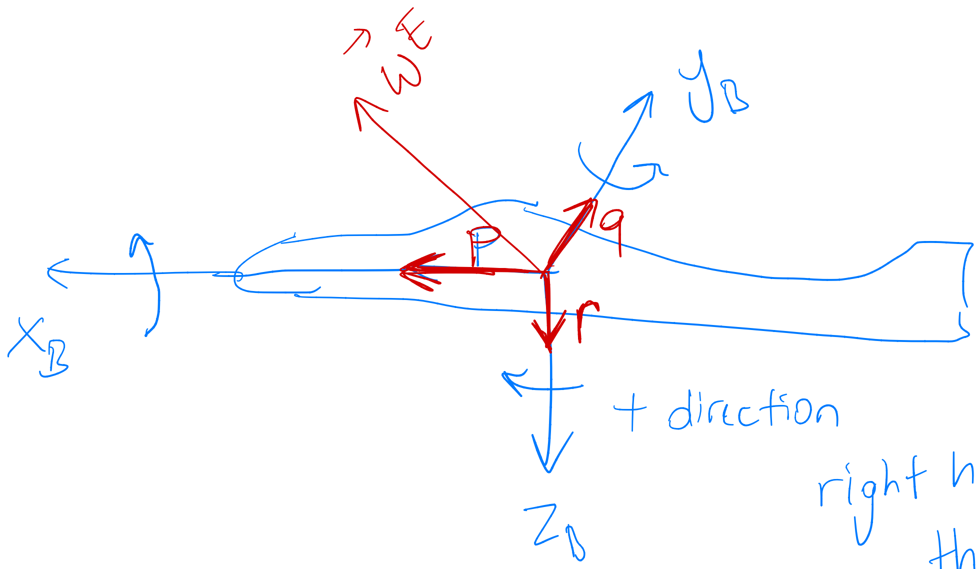
$$\begin{array}{ll} & \\ \vec{V}_B = \begin{bmatrix} u \\ v \\ w \end{bmatrix} & \begin{array}{l} \rightarrow x \text{ axis} \\ \rightarrow y_B \\ \rightarrow z_B \end{array} \end{array} \Bigg\} \text{Inertial Frame}$$

$$\vec{V}_B = \begin{bmatrix} 200 \\ 0 \\ 10 \end{bmatrix} \text{km/s}$$



u, v, w velocity vector components along Ox_B, y_B, z_B

wrt Inertial Ref. Frame



+ direction

right hand rule
thumb

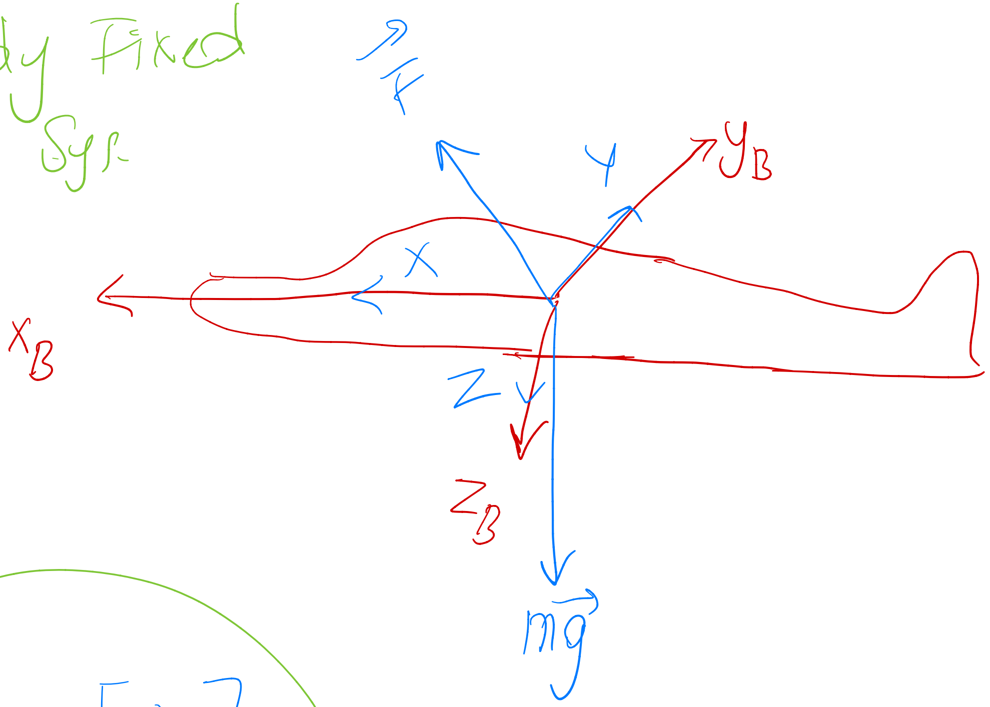
P vector denotes a rotation

$$\vec{W}_B^E = \begin{array}{cl} \begin{bmatrix} p_B \\ q_B \\ r_B \end{bmatrix} & \begin{array}{l} \text{roll} \\ \text{pitch} \\ \text{yaw} \end{array} \end{array}$$

$$\underbrace{\vec{W}_B = \begin{bmatrix} p \\ q \\ r \end{bmatrix}}_{\text{Rotation Vector}}$$

wrt inertial frame written in the body fixed coordinate system

Ex Body Fixed Coor. Sys.



$$\text{gravity} \; \Big\} \; mg$$

$$\left.\begin{array}{l} \text{Lift} \\ \text{Drag} \\ \text{Thrust} \end{array}\right\} F$$

+

TOTAL

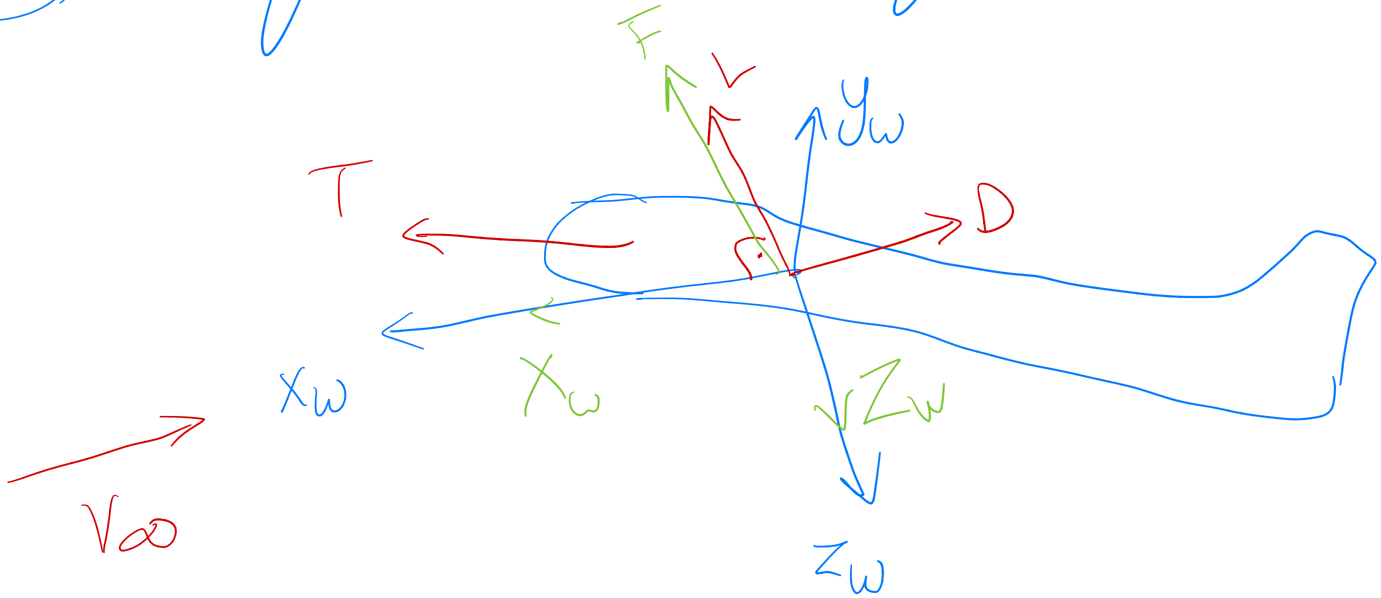
$$\vec{F}_B = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$$

The force that does NOT include the gravitational force written in body fixed coordinate system.

"Aero-Propulsive Forces"

Ex

Wing Fixed Coordinate system



— Drag parallel to x_w

Lift perpendicular to x_w

$$\vec{F}_w = \begin{bmatrix} X_w \\ Y_w \\ Z_w \end{bmatrix}$$

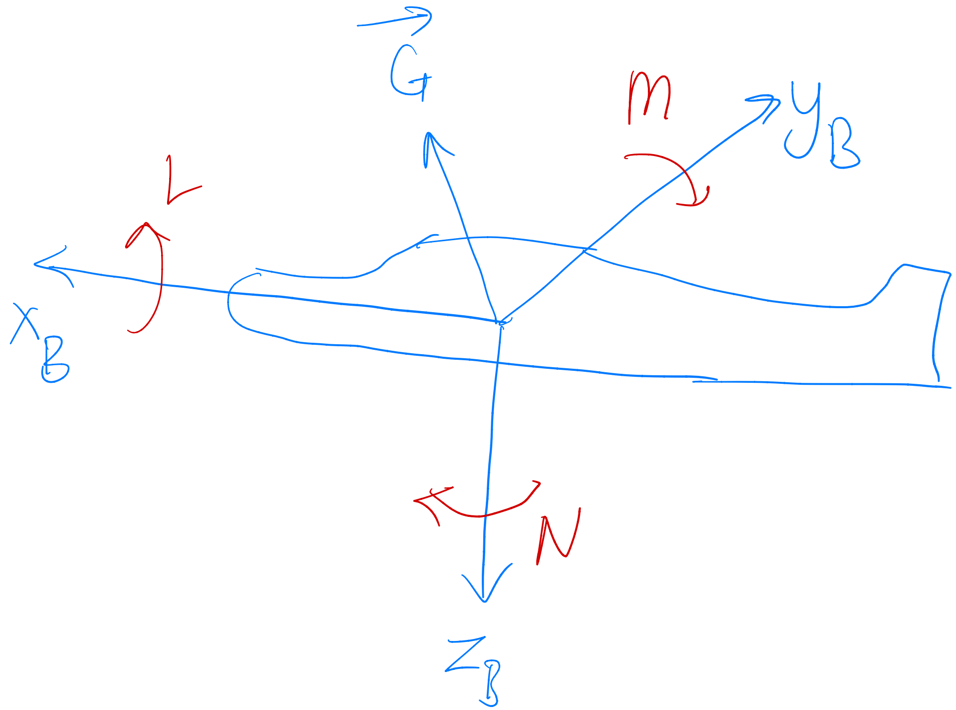
F includes

Lift, Drag, Thrust

X, Y, Z are the aeropropulsive forces that act along

x_B, y_B, z_B

Moments



$\vec{G}$: External moments that act on the A/C.

$$\vec{G}_B = \begin{bmatrix} L \\ M \\ N \end{bmatrix} \begin{array}{l} \rightarrow \text{roll moment} \\ \rightarrow \text{pitch moment} \\ \rightarrow \text{yaw moment} \end{array}$$

$$\overbrace{\underbrace{u, v, w, p, q, r}_{\text{Reference Frame (Inertial)}}, \underbrace{X, Y, Z, L, M, N}_{\text{Do not defined in Reference Frame}}}^{\text{Defined in body-fixed coordinate systems}}$$

$u, v, w \rightarrow$ velocities

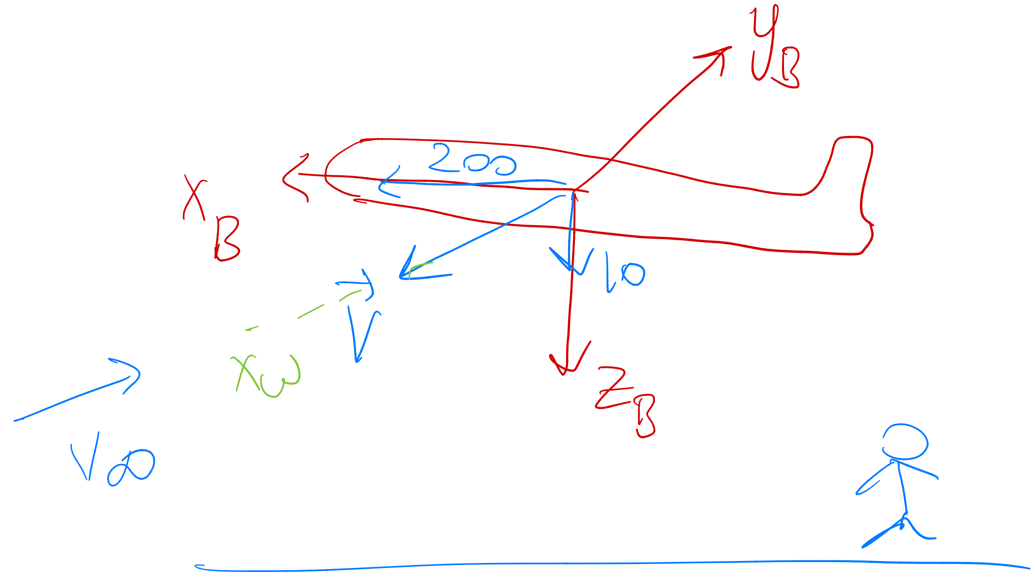
$p, q, r \rightarrow$ angular velo.

$X, Y, Z \rightarrow$ aero propulsive Force

$L, M, N \rightarrow$ Moment

$$\vec{V}_B = \begin{bmatrix} 200 \\ 0 \\ 10 \end{bmatrix} \text{km/h}$$

$$\vec{V}_w = \begin{bmatrix} \sqrt{200^2 + 10^2} \\ 0 \\ 0 \end{bmatrix} \text{km/h}$$



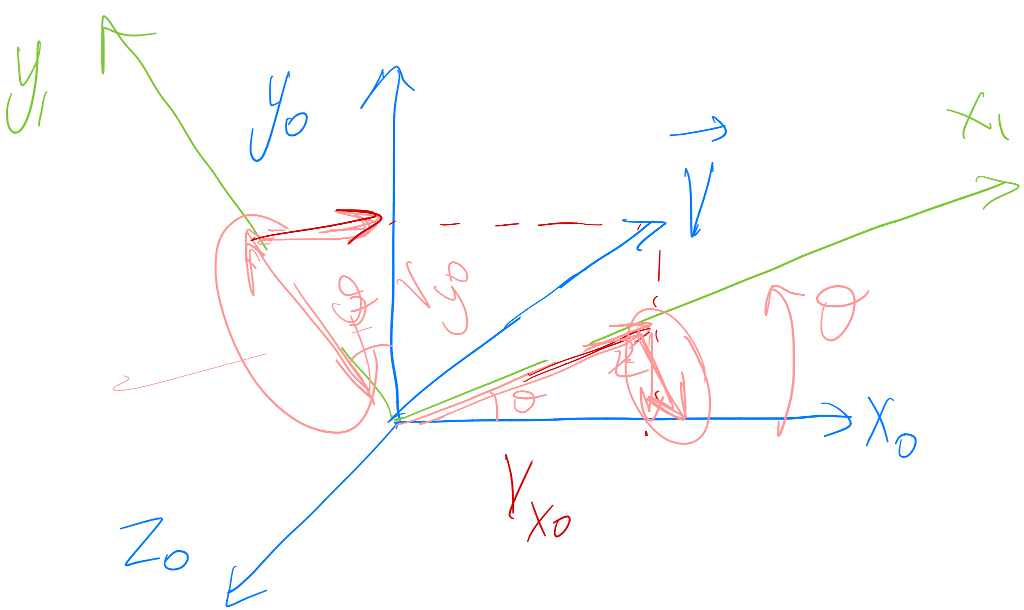
Earth is flat, non-rotating

How to relat $\vec{V_B}$ to $\vec{V_W}$?

How do we find the new components?

Vector Transformations (Coordinate Transformations)

(Re-writing a vector in a different coordinate system)

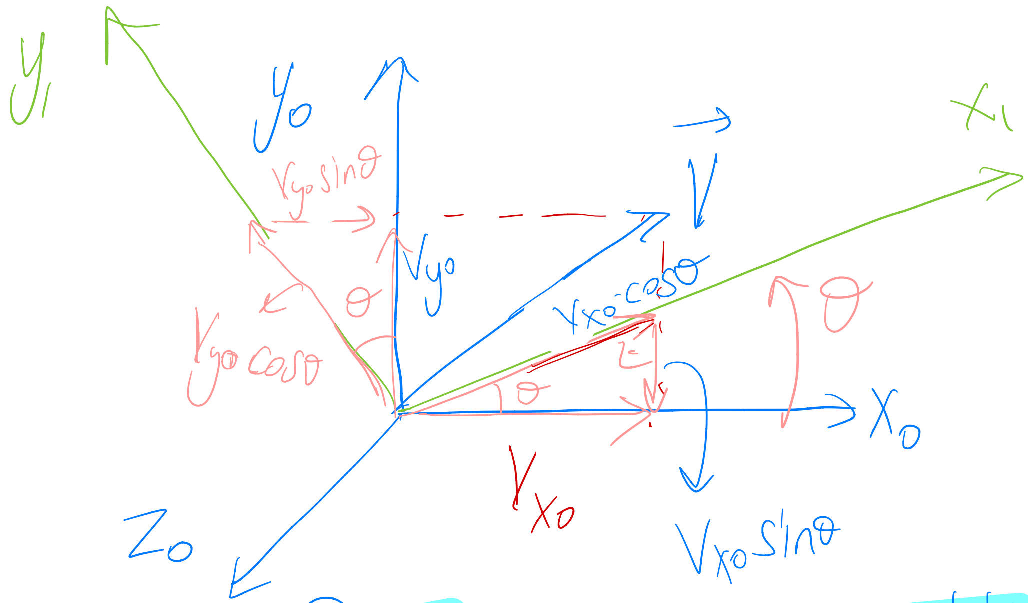


$$\vec{V_0} = \begin{bmatrix} V_{x_0} \\ V_{y_0} \end{bmatrix}$$

$$V_{x_1} = V_{x_0}\cos\theta + V_{y_0}\sin\theta$$

$$V_{y_1} = -V_{x_0}\sin\theta + V_{y_0}\cos\theta$$

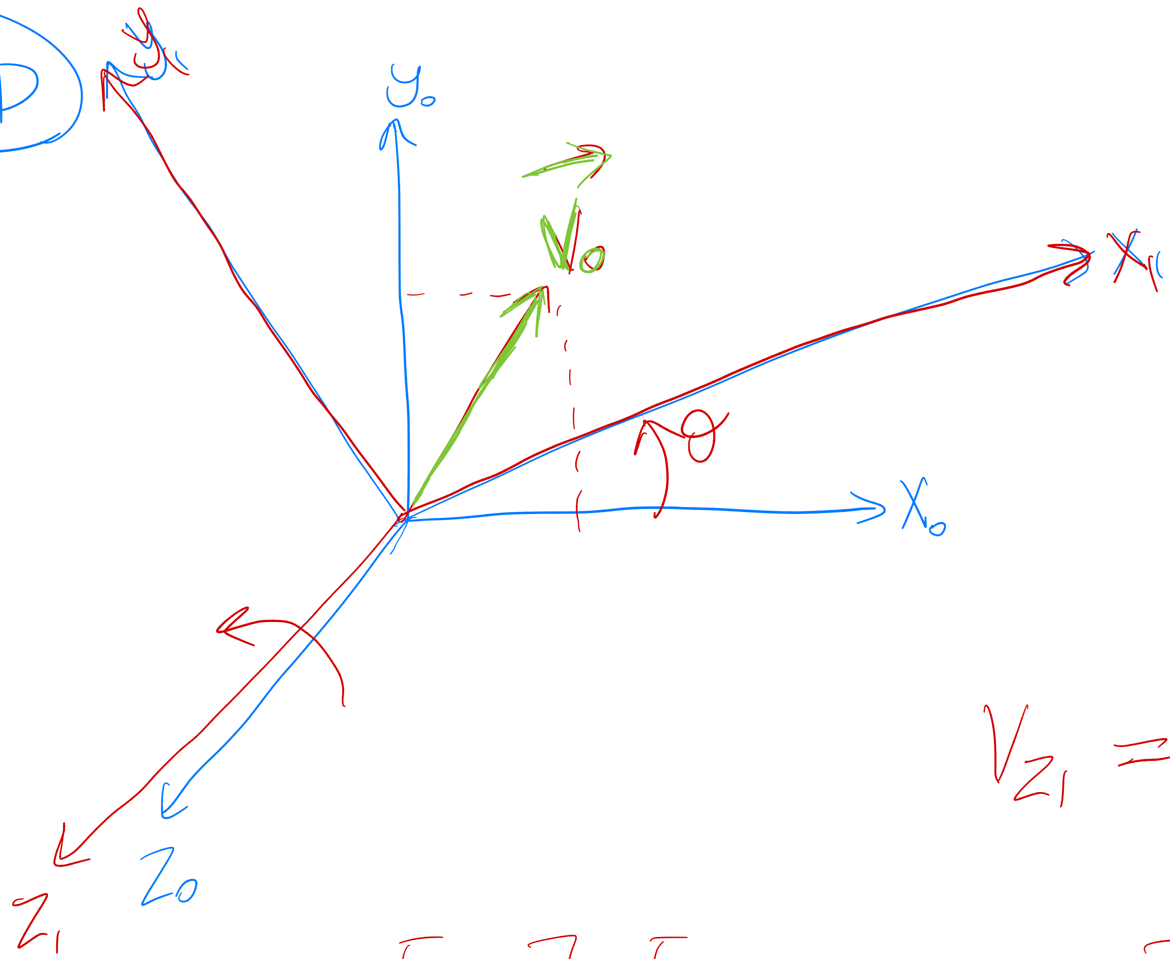
$$\vec{V}_1 = \begin{bmatrix} V_{x_1} \\ V_{y_1} \end{bmatrix}$$



$$\overbrace{\begin{bmatrix} V_{x_1} \\ V_{y_1} \end{bmatrix}}^{O_{x_1y_1}} = \overbrace{\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}}^{\text{Transformation matrix}} \overbrace{\begin{bmatrix} V_{x_0} \\ V_{y_0} \end{bmatrix}}^{O_{x_0y_0}}$$

Same vector written in different coordinate system

3-D



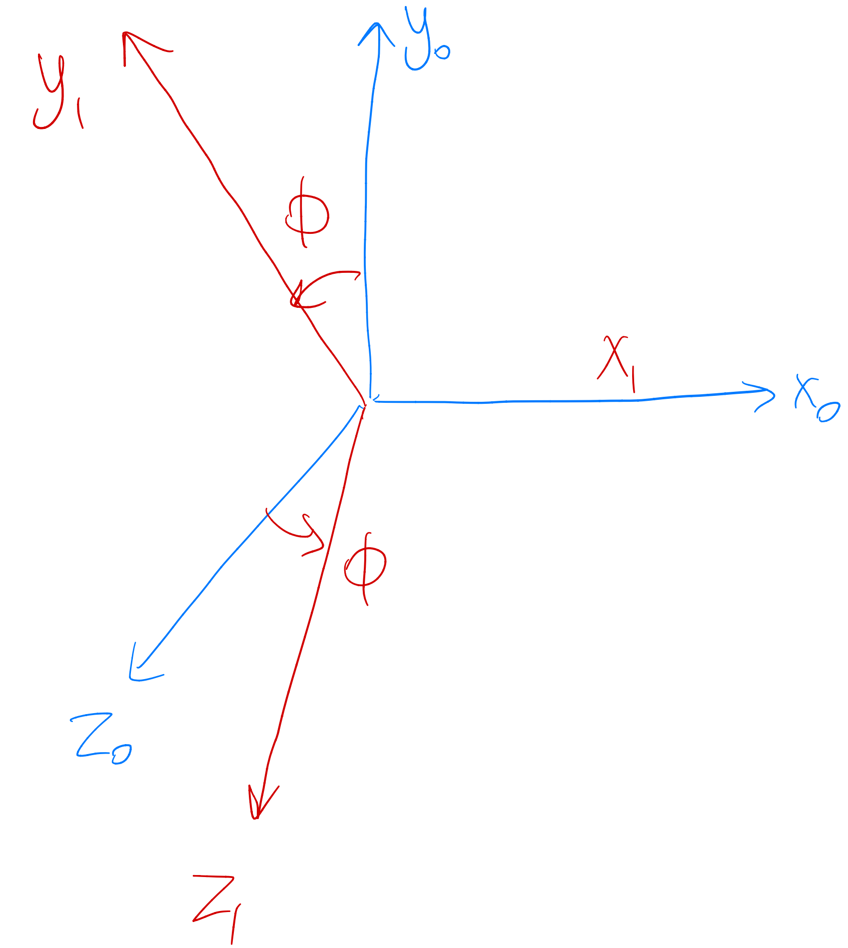
Pozitif rotation since
z axis rotation (+)

Rotation
Around
z-axis
by θ

$$V_{z_1} = V_{z_0}$$

$$\begin{bmatrix} V_{x_1} \\ V_{y_1} \\ V_{z_1} \end{bmatrix} = \underbrace{\begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\text{Transformation matrix}} \begin{bmatrix} V_{x_0} \\ V_{y_0} \\ V_{z_0} \end{bmatrix}$$

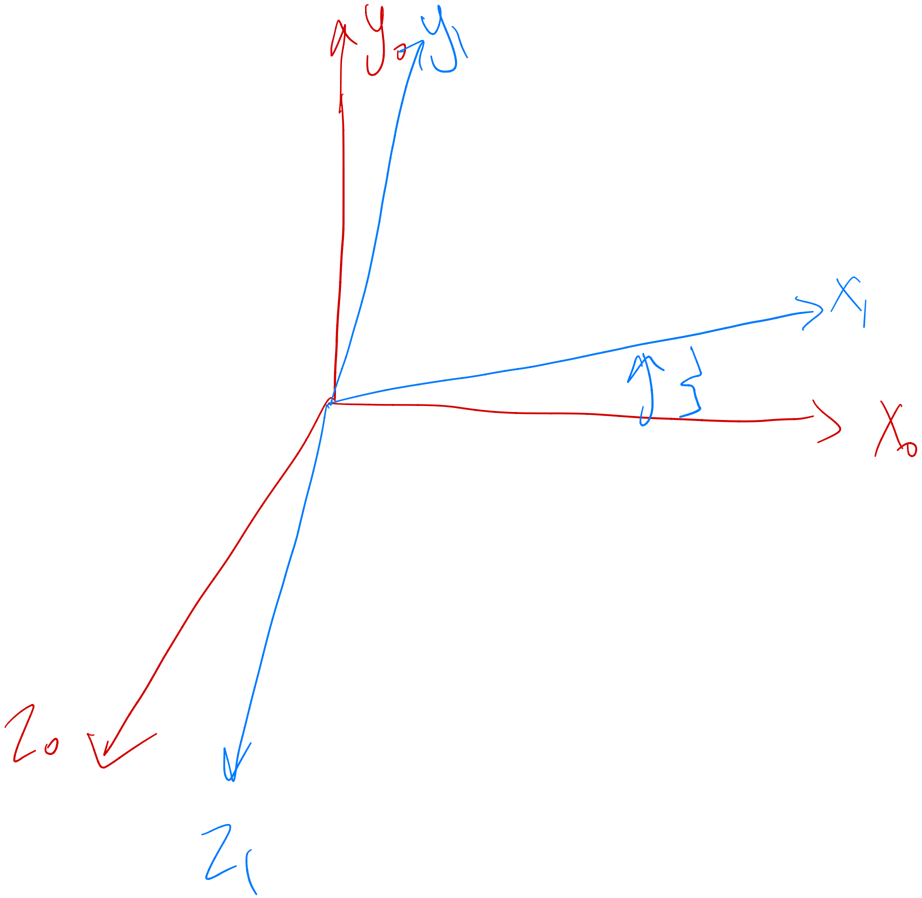
Rotate a coordinate system around its x-axis by ϕ



$$\begin{bmatrix} V_{x_1} \\ V_{y_1} \\ V_{z_1} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} V_{x_0} \\ V_{y_0} \\ V_{z_0} \end{bmatrix}$$

some vector

Rotate the coordinate system around its y-axis by ξ



$$\begin{bmatrix} V_{x_1} \\ V_{y_1} \\ V_{z_1} \end{bmatrix} = \underbrace{\begin{bmatrix} \cos\xi & 0 & -\sin\xi \\ 0 & 1 & 0 \\ \sin\xi & 0 & \cos\xi \end{bmatrix}}_{\text{Transformation matrix}} \begin{bmatrix} V_{x_0} \\ V_{y_0} \\ V_{z_0} \end{bmatrix}$$

$$\underbrace{\text{Rotate around All axes at once , not just one axis}}_{\text{We want this?}}$$

Mathematical Fact: By rotating the original coordinate system $(O x_0 y_0 z_0)$ 1-by-1 around each axis, we can find (re-present) a vector in any new coordinate system $(O x_1, y_1, z_1)$

Take a coordinate system Oxyz. Represent a vector

$\vec{V} = \begin{bmatrix} V_x \\ V_y \\ V_z \end{bmatrix}$ in an arbitrary new coordinate system.

Vehicle carried coordinate system

$$\begin{bmatrix} V_x \\ V_y \\ V_z \end{bmatrix} \xrightarrow[\text{around its z-axis by } \psi]{\text{Rotate the coordinate system}} \begin{bmatrix} V_{x_1} \\ V_{y_1} \\ V_{z_1} \end{bmatrix} \xrightarrow[\text{around its } y_1\text{-axis by } \theta]{\text{Rotate the coordinate } x_1, y_1, z_1} \begin{bmatrix} V_{x_2} \\ V_{y_2} \\ V_{z_2} \end{bmatrix} \longrightarrow$$

$$\xrightarrow[x_2\text{-axis by } \phi]{\text{Rotate } x_2, y_2, z_2 \text{ around}} \begin{bmatrix} V_{x_3} \\ V_{y_3} \\ V_{z_3} \end{bmatrix}$$

→ Body Fixed Coordinate System

In Flight Mechanics we use this Order!

$$\begin{bmatrix} V_{x_1} \\ V_{y_1} \\ V_{z_1} \end{bmatrix} = \underbrace{\begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{L_1(\psi)} \begin{bmatrix} V_x \\ V_y \\ V_z \end{bmatrix} \quad ?$$

$$\begin{bmatrix} V_{x_2} \\ V_{y_2} \\ V_{z_2} \end{bmatrix} = \underbrace{\begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}}_{L_2(\theta)} \begin{bmatrix} V_{x_1} \\ V_{y_1} \\ V_{z_1} \end{bmatrix}$$

Give me 3 angles at any instant of time, I can tell you the angular position relative to north-east-down !

Help me describe the angular position in space $\Leftarrow$ $\left.\theta, \phi, \psi\right\}$ Euler Angles

$$\begin{bmatrix} V_{x_3} \\ V_{y_3} \\ V_{z_3} \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix}}_{L_3(\phi)} \begin{bmatrix} V_{x_2} \\ V_{y_2} \\ V_{z_2} \end{bmatrix}$$

?

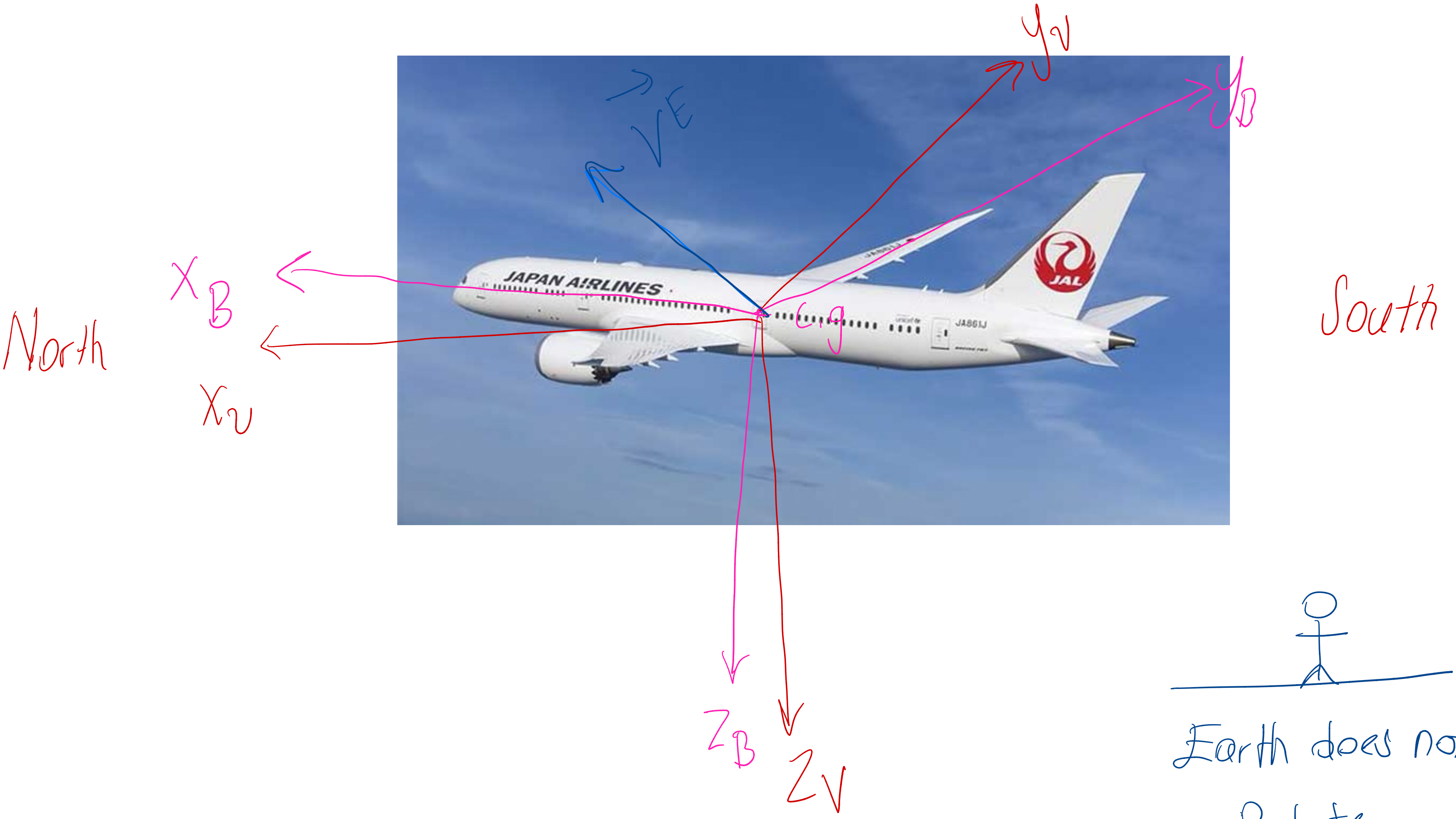
$$\begin{bmatrix} V_{x_3} \\ V_{y_3} \\ V_{z_3} \end{bmatrix} = \underbrace{L_3(\phi)\, L_2(\theta)\, L_1(\psi)}_{\text{Transformation matrix}} \begin{bmatrix} V_x \\ V_y \\ V_z \end{bmatrix}$$

Final Coordinate system

Original coordinate

$$
\begin{bmatrix} V_{x_3} \\ V_{y_3} \\ V_{z_3} \end{bmatrix}
=
\underbrace{\begin{bmatrix}
c\theta c\psi & c\theta s\psi & -s\theta \\
s\phi s\theta c\psi - c\phi s\psi & s\phi s\theta s\psi + c\phi c\psi & s\phi c\theta \\
c\phi s\theta s\psi + s\phi s\psi & c\phi s\theta s\psi - s\phi c\psi & c\phi c\theta
\end{bmatrix}}_{L_3(\phi)\,L_2(\theta)\,L_1(\psi)}
\begin{bmatrix} V_x \\ V_y \\ V_z \end{bmatrix}
$$

3x3



Earth does not Rotate

observed someone from Earth

$$\vec{V}_B^{E} = \begin{bmatrix} u \\ v \\ w \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} u \\ v \\ w \end{bmatrix}}_{\text{Body Fixed Coordinate System}} = \underbrace{L_3(\phi)\,L_2(\theta)\,L_1(\psi)}_{L_{BV}} \underbrace{\begin{bmatrix} u_V \\ v_V \\ w_V \end{bmatrix}}_{\text{Vehicle Carried Coordinate Sys.}}$$

$\leftarrow$

Transformation matrix transforming to Vehicle to Body Fixed Coordinet system

$$\boxed{\vec{V}_B^{E} = L_{BV}\,\vec{V}_V^{E}}$$

It could be an angular velocity, position vector. Not just <u>velocity</u> ?

If I say θ, everyone understands θ angle

take north-east-down, rotate it around ψ, and then

rotate that coord. sys. by θ

θ, ϕ, ψ are called Euler Angles. In Flight Mechanics

they define the rotations from the Vehicle carried coordinate

system to the body fixed coordinate system.

θ – Pitch angle
ϕ – Roll angle
ψ – Yaw angle

"Attitude of an Aircraft"

Angular Orientation of the aircraft with respect to Vehicle Carried Coordinate system (North-East-Down)

$$\vec{V}_B = L_{BV}\vec{V}_V = \underbrace{L_3(\phi)\,L_2(\theta)\,L(\psi)}_{L_{BV}}\vec{V}_V$$

$$\vec{V}_V = L_{VB}\vec{V}_B = \underbrace{L_1(-\psi)\,L_2(-\theta)\,L_3(-\phi)}_{L_{VB}}\vec{V}_B$$

$$\vec{V}_B = L_{BV}\vec{V}_V \longrightarrow \vec{V}_V = L_{BV}^{-1}\vec{V}_B$$

$$\vec{V}_V = L_{BV}\vec{V}_B$$

$$\boxed{L_{BV}^{-1} = L_{VB}}$$

$$L_{VB}^{-1} = L_{BV}$$

$\vec{V}_B \quad , \quad \vec{V}_V$

$|\vec{V}_B| = |\vec{V}_V|$ → since they are the same vector just in different coordinates

$\vec{V}_B^T \vec{V}_B = \vec{V}_V^T \vec{V}_V$

$V = \begin{bmatrix} x \\ y \end{bmatrix} \quad V^T = [x \quad y]$

$V^T V = x^2 + y^2$

$|V| = \sqrt{x^2 + y^2}$

$|V|^2 = x^2 + y^2$

$\vec{V}_V^T \vec{V}_V = \underbrace{(L_{VB} \vec{V}_B)^T}_{V_V^T} \underbrace{(L_{VB} \vec{V}_B)}_{V_V}$

$= \vec{V}_B^T L_{VB}^T \underbrace{L_{VB} \vec{V}_B}_{V_V}$

Identity

$= \vec{V}_B^{\,T} \vec{V}_B$

$$\vec{V}_V = L_{VB} \vec{V}_B$$

$$\vec{V}_V = L_{BV}^{-1} \vec{V}_B$$

$$\vec{V}_V = L_{BV}^{T} \vec{V}_B$$

$$\vec{V}_B = L_{BV} \vec{V}_V$$

$$\vec{V}_B = L_{VB}^{-1} \vec{V}_V$$

$$\vec{V}_B = L_{VB}^{T} \vec{V}_V$$

$$L_{VB}^{T} L_{VB} = I$$

$$L_{VB}^{T} = L_{VB}^{-1}$$

A property of all transformation matrices

Example

$$\vec{V}_B^{E} = \begin{bmatrix} 900 \\ 0 \\ 0 \end{bmatrix} km/h \qquad \begin{array}{l} \theta = 0^\circ \\ \phi = 0^\circ \\ \psi = 90^\circ \rightarrow \text{Flying East} \end{array}$$

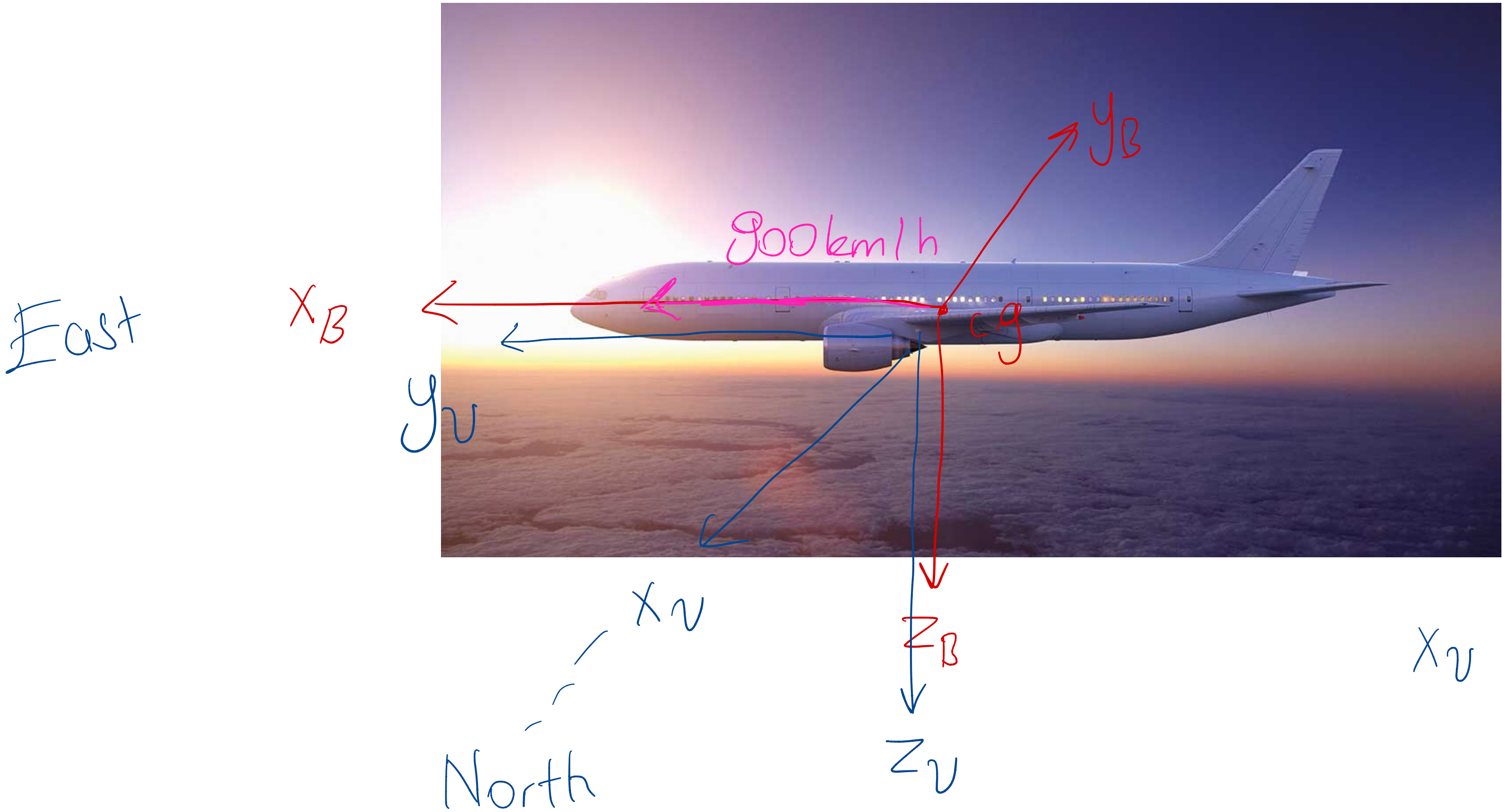
Write the velocity vector of the aircraft in the vehicle-carried coordinate system.

$$\vec{V}_V = L_{VB} \vec{V}_B = L_1(-\psi) L(-\theta) L(-\phi) \vec{V}_B$$

$$= \underbrace{\left[L_3(\phi) L_2(\theta) L_1(\psi) \right]}_{L_{BV}}{}^T V_B$$

$$= \begin{bmatrix} \cos 90 & \sin 90 & 0 \\ -\sin 90 & \cos 90 & 0 \\ 0 & 0 & 1 \end{bmatrix}^T \begin{bmatrix} 900 \\ 0 \\ 0 \end{bmatrix}$$

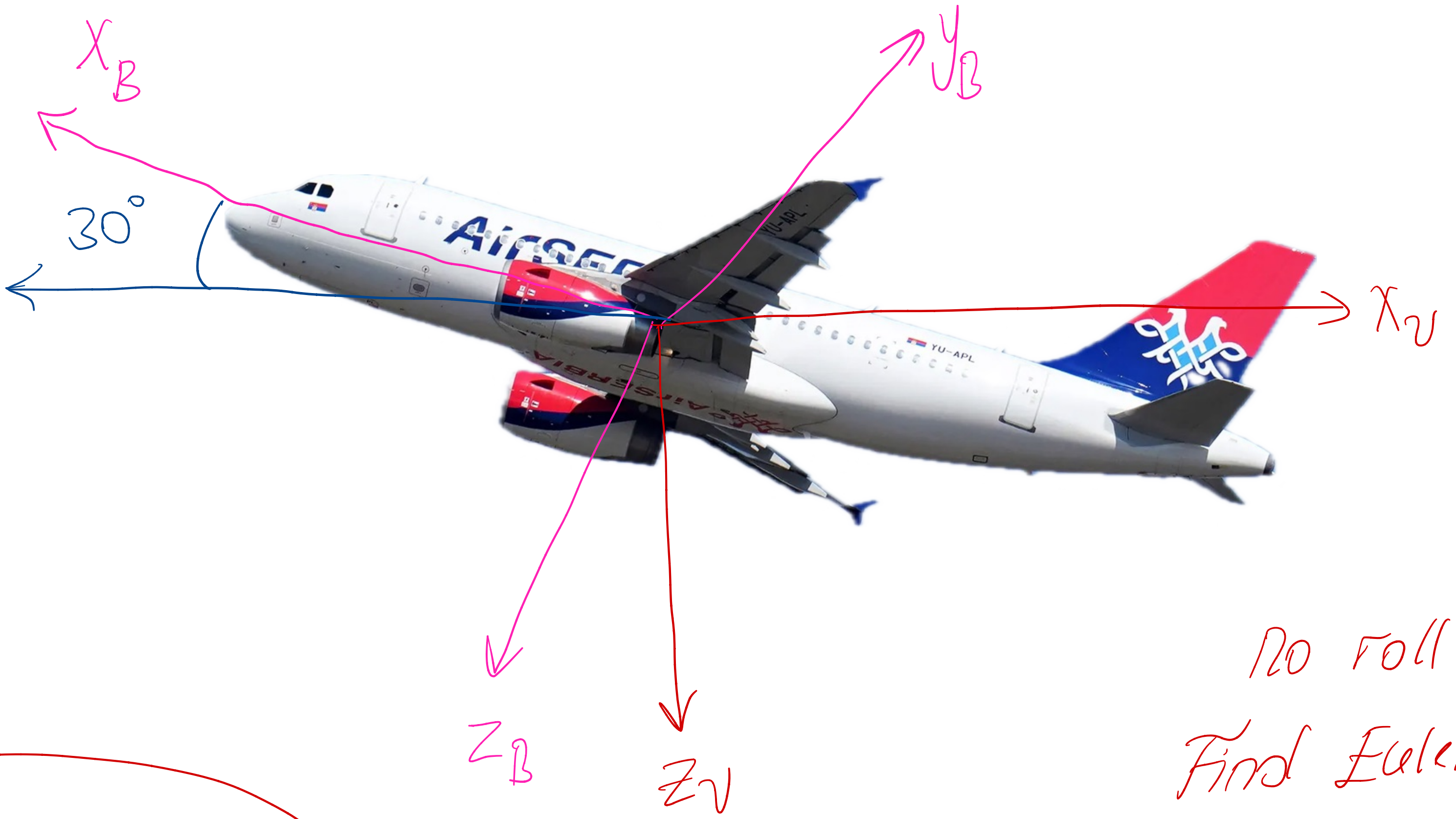
$$\vec{V}_V = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 900 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 900 \\ 0 \end{bmatrix} \text{km/h}$$



x_V is pointing towards us.

EX

South



North

no roll

Find Euler Angles?

$$-\pi \leq \psi < \pi$$
$$-\pi/2 \leq \theta < \pi/2$$
$$-\pi \leq \phi < \pi$$

$$\phi = 0$$
$$\psi = -180^\circ$$
$$\theta = 30^\circ$$