CS419 Homework 1

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Question 1

Convolution Commutative

We want to prove that convolution operation is commutative such that $f * g = g * f$. The definition of convolution is given by:

$$(f * g)(x) = \int_{-\infty}^{+\infty} f(y)g(x-y)\,dy$$

Now, let's change the integration variable by letting $y' = x - y$, hence $dy' = -dy$. Substituting this into the integral, we get:

$$(f * g)(x) = \int_{+\infty}^{-\infty} g(y')f(x-y')(-1)\,dy'$$

Simplifying, we have:

$$(f * g)(x) = \int_{-\infty}^{+\infty} g(y')f(x-y')\,dy'$$

Thus, we conclude that $(f * g)(x) = (g * f)(x)$, and convolution is commutative.

Cross-correlation Commutative

We want to prove that cross-correlation operation is not commutative such that $f * g \neq g * f$. The definition of cross-correlation is given by:

$$(f * g)(x) = \int_{-\infty}^{+\infty} f(y)g(x+y)\,dy$$

$$(g * f)(x) = \int_{-\infty}^{+\infty} g(y)f(x+y)\,dy$$

Define u= x+y, then y=u-x , dy=du , then we have the following:

$$(g * f)(x) = \int_{-\infty}^{+\infty} g(u-x)f(u)\,du$$

If we make the same substitution for (f * g)(x),then we have the following:

$$(f * g)(x) = \int_{-\infty}^{+\infty} f(u-x)g(u)\,du \neq \int_{-\infty}^{+\infty} g(u-x)f(u)\,du$$

Doing the same changes/substitutions to both side showed us that these two expressions are not same because if they were when we did the same changes to both side the end result would be the same so $f * g \neq g * f$

Convolution Associative

We want to prove that convolution operation has associative property.

$$(f * g) * h = f * (g * h)$$

Extend the expression according to definition of Convolution operation:

$$(f * g) * h = \int_{-\infty}^{+\infty} (f * g)(y) h(x - y)\, dy$$

Starting with the left-hand side:

$$\int_{-\infty}^{+\infty} (f * g)(y) h(x - y)\, dy$$

Extend the convolution inside the expression (f * g) (y)

$$\int_{-\infty}^{+\infty} \left(\int_{-\infty}^{+\infty} f(u) g(y - u)\, du \right) h(x - y)\, dy$$

Combine the two and write as double integral

$$\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(u) g(y - u) h(x - y)\, du\, dy$$

$$= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(u) g(y - u) h(x - y)\, dy\, du$$

Now separate the integrals to show the inner integral

$$\int_{-\infty}^{+\infty} f(u) \left(\int_{-\infty}^{+\infty} g(y - u) h(x - y)\, dy \right) du$$

For inner integral define z=y-u then replace all z related expressions according to new one.

$$\int_{-\infty}^{+\infty} g(z) h(x - u - z)\, dz$$

As we can see if we look at it from different way we can detect a convolution operation inside the existing expression.

$$\int_{-\infty}^{+\infty} g(z) h((x - u) - z)\, dz = (g * h)(x - u)$$

Now that we found a new expression for the integral put the inner integral back to where it was. After that again we can see a new convolution operation.

$$\int_{-\infty}^{+\infty} f(u) (g * h)(x - u)\, du = f * (g * h)(x)$$

Therefore, we have shown that $(f * g) * h(x) = f * (g * h)(x)$, and convolution is associative.

Question 2

In a 2D Cartesian coordinate system, when you perform a rotation in the xy-plane, the transformation equations for the new coordinates (x', y') in terms of the original coordinates (x, y) are:

$$x' = x\cos(\theta) - y\sin(\theta)$$
$$y' = x\sin(\theta) + y\cos(\theta)$$

Now, let's consider the Laplacian operator $\nabla^2 f'$ applied to a scalar field $f'(x', y')$ after this rotation:

$$\nabla^2 f' = \frac{\partial^2 f}{\partial x'^2} + \frac{\partial^2 f}{\partial y'^2}$$

To prove that Laplacian operator is rotation invariant we must show that

$$\nabla^2 f = \nabla^2 f'$$

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 f}{\partial x'^2} + \frac{\partial^2 f}{\partial y'^2}$$

Let's break down each component starting with $\frac{\partial^2 f}{\partial y^2}$

$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial x'}\frac{\partial x'}{\partial y} + \frac{\partial f}{\partial y'}\frac{\partial y'}{\partial y} = -\frac{\partial f}{\partial x'}\sin\theta + \frac{\partial f}{\partial y'}\cos\theta$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}\left(-\frac{\partial f}{\partial x'}\sin\theta + \frac{\partial f}{\partial y'}\cos\theta\right) = -\frac{\partial}{\partial y}\frac{\partial f}{\partial x'}\sin\theta + \frac{\partial}{\partial y}\frac{\partial f}{\partial y'}\cos\theta$$

Now the problem is to compute $\frac{\partial}{\partial y}\frac{\partial f}{\partial x'}$ and $\frac{\partial}{\partial y}\frac{\partial f}{\partial y'}$. We can think $\frac{\partial}{\partial y}\frac{\partial f}{\partial x'}$ as $\frac{\partial}{\partial x'}\frac{\partial f}{\partial y}$ and similarly for $\frac{\partial}{\partial y}\frac{\partial f}{\partial y'}$ as $\frac{\partial}{\partial y'}\frac{\partial f}{\partial y}$

$$\frac{\partial}{\partial x'}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x'}\left(-\frac{\partial f}{\partial x'}\sin\theta + \frac{\partial f}{\partial y'}\cos\theta\right) = -\frac{\partial^2 f}{\partial x'^2}\sin\theta + \frac{\partial^2 f}{\partial x'\partial y'}\cos\theta$$

$$\frac{\partial}{\partial y'}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y'}\left(-\frac{\partial f}{\partial x'}\sin\theta + \frac{\partial f}{\partial y'}\cos\theta\right) = -\frac{\partial^2 f}{\partial y'\partial x'}\sin\theta + \frac{\partial^2 f}{\partial y'^2}\cos\theta$$

Now plug this back in to $\frac{\partial^2 f}{\partial y^2}$ equation.

$$\frac{\partial^2 f}{\partial y^2} = -\left(-\frac{\partial^2 f}{\partial x'^2}\sin\theta + \frac{\partial^2 f}{\partial x'\partial y'}\cos\theta\right)\sin\theta + \left(-\frac{\partial^2 f}{\partial y'\partial x'}\sin\theta + \frac{\partial^2 f}{\partial y'^2}\cos\theta\right)\cos\theta$$

Expand the expression and do elimination.

$$= \frac{\partial^2 f}{\partial x'^2}\sin^2\theta - \frac{\partial^2 f}{\partial x'\partial y'}\cos\theta\sin\theta - \frac{\partial^2 f}{\partial y'\partial x'}\sin\theta\cos\theta + \frac{\partial^2 f}{\partial y'^2}\cos^2\theta$$

Now let's break done next component which is $\frac{\partial^2 f}{\partial x^2}$

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial x'}\frac{\partial x'}{\partial x} + \frac{\partial f}{\partial y'}\frac{\partial y'}{\partial x} = \frac{\partial f}{\partial x'}\cos\theta + \frac{\partial f}{\partial y'}\sin\theta$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x'}\cos\theta + \frac{\partial f}{\partial y'}\sin\theta\right) = \frac{\partial}{\partial x}\frac{\partial f}{\partial x'}\cos\theta + \frac{\partial}{\partial x}\frac{\partial f}{\partial y'}\sin\theta$$

Now the problem is to compute $\frac{\partial}{\partial x}\frac{\partial f}{\partial x'}$ and $\frac{\partial}{\partial x}\frac{\partial f}{\partial y'}$. We can think $\frac{\partial}{\partial x}\frac{\partial f}{\partial x'}$ as $\frac{\partial}{\partial x'}\frac{\partial f}{\partial x}$ and similarly for $\frac{\partial}{\partial x}\frac{\partial f}{\partial y'}$ as $\frac{\partial}{\partial y'}\frac{\partial f}{\partial x}$

$$\frac{\partial}{\partial x'}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x'}\left(\frac{\partial f}{\partial x'}\cos\theta + \frac{\partial f}{\partial y'}\sin\theta\right) = \frac{\partial^2 f}{\partial x'^2}\cos\theta + \frac{\partial^2 f}{\partial x'\partial y'}\sin\theta$$

$$\frac{\partial}{\partial y'}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y'}\left(\frac{\partial f}{\partial x'}\cos\theta + \frac{\partial f}{\partial y'}\sin\theta\right) = \frac{\partial^2 f}{\partial y'\partial x'}\cos\theta + \frac{\partial^2 f}{\partial y'^2}\sin\theta$$

Now plug this back in to $\frac{\partial^2 f}{\partial x^2}$ equation.

$$\frac{\partial^2 f}{\partial x^2} = \left(\frac{\partial^2 f}{\partial x'^2}\cos\theta + \frac{\partial^2 f}{\partial x'\partial y'}\sin\theta\right)\cos\theta + \left(\frac{\partial^2 f}{\partial y'\partial x'}\cos\theta + \frac{\partial^2 f}{\partial y'^2}\sin\theta\right)\sin\theta$$

Expand the expression and do elimination.

$$= \frac{\partial^2 f}{\partial x'^2}\cos^2\theta + \frac{\partial^2 f}{\partial x'\partial y'}\sin\theta\cos\theta + \frac{\partial^2 f}{\partial y'\partial x'}\cos\theta\sin\theta + \frac{\partial^2 f}{\partial y'^2}\sin^2\theta$$

Now to find the final result of $\nabla^2 f$ let's add the results of $\frac{\partial^2 f}{\partial x^2}$ and $\frac{\partial^2 f}{\partial y^2}$

$$\begin{aligned} =&\frac{\partial^2 f}{\partial x'^2}\cos^2\theta + \frac{\partial^2 f}{\partial x'\partial y'}\sin\theta\cos\theta \\ &+ \frac{\partial^2 f}{\partial y'\partial x'}\cos\theta\sin\theta + \frac{\partial^2 f}{\partial y'^2}\sin^2\theta \\ &+ \frac{\partial^2 f}{\partial x'^2}\sin^2\theta - \frac{\partial^2 f}{\partial x'\partial y'}\cos\theta\sin\theta \\ &- \frac{\partial^2 f}{\partial y'\partial x'}\sin\theta\cos\theta + \frac{\partial^2 f}{\partial y'^2}\cos^2\theta \end{aligned}$$

$$= \frac{\partial^2 f}{\partial x'^2}\cos^2\theta + \frac{\partial^2 f}{\partial y'^2}\sin^2\theta + \frac{\partial^2 f}{\partial x'^2}\sin^2\theta + \frac{\partial^2 f}{\partial y'^2}\cos^2\theta$$

In the end because $\sin^2\theta + \cos^2\theta = 1$ the final result will be the same as $\nabla^2 f'$

$$= \frac{\partial^2 f}{\partial x'^2} + \frac{\partial^2 f}{\partial y'^2} = \nabla^2 f'$$

With this we proved that the Laplacian operator is rotation invariant meaning $\nabla^2 f = \nabla^2 f'$

Question 3

The given filter function is:

$$h(x,y) = 3f(x,y) + 2f(x-1,y) + 2f(x+1,y) - 17f(x,y-1) + 99f(x,y+1)$$

linear filter

Let's consider two images a,b that have points (x_1, y_1) and (x_2, y_2).A filter is linear if it satisfies both additivity and homogeneity rules for the filter h on the a,b images. A filter is additive if

$$h(a(x_1,y_1)+, b(x_2,y_2)) = h(a(x_1,y_1)) + h(b(x_2,y_2))$$

and it is homogeneous if

$$h(c \cdot a(x_1,y_1)) = c \cdot h(a(x_1,y_1))$$

Additivity Check

Let's consider two images a,b that have points (x_1, y_1) and (x_2, y_2). The sum of h at these points for these two images should be equal to h at the point $a(x_1,y_1)+, b(x_2,y_2)$ if the filter is additive.

$$\begin{aligned}
h(a(x_1,y_1) + b(x_2,y_2)) = \\
& 3(a(x_1,y_1) + b(x_2,y_2)) \\
& \quad + 2(a(x_1-1,y_1) + b(x_2-1,y_2)) \\
& \quad + 2(a(x_1+1,y_1) + b(x_2+1,y_2)) \\
& \quad - 17(a(x_1,y_1-1) + b(x_2,y_2-1)) \\
& \quad + 99f(a(x_1,y_1+1) + b(x_2,y_2+1)) \\
& = 3a(x_1,y_1) + 2a(x_1-1,y_1) + 2a(x_1+1,y_1) \\
& \quad - 17a(x_1,y_1-1) + 99a(x_1,y_1+1) \\
& \quad + 3b(x_2,y_2) + 2b(x_2-1,y_2) + 2b(x_2+1,y_2) \\
& \quad - 17b(x_2,y_2-1) + 99b(x_2,y_2+1)
\end{aligned}$$

$$= h(a(x_1,y_1)) + h(b(x_2,y_2))$$

As we can see the filter h satisfies additivity.

Homogeneity Check

A filter is homogeneous if $h(c \cdot a(x_1,y_1)) = c \cdot h(a(x_1,y_1))$ for any constant c.

Let's check this property:

$$h(c{\cdot}a(x_1,y_1)) = 3(c{\cdot}a(x_1,y_1)) + 2(c{\cdot}a(x_1-1,y_1)) + 2(c{\cdot}a(x_1+1,y_1)) - 17(c{\cdot}a(x_1,y_1-1)) + 99(c{\cdot}a(x_1,y_1+1))$$

$$= c\cdot(3a(x_1,y_1)+2a(x_1-1,y_1)+2a(x_1+1,y_1)-17a(x_1,y_1-1)+99a(x_1,y_1+1))$$

$$= c \cdot h(a(x_1, y_1))$$

If h is homogeneous, these two expressions $h(c \cdot a(x_1, y_1)) = c \cdot h(a(x_1, y_1))$ should be equal for any constant c and as we can see they are so this filter does satisfy homogeneity as well.

Conculusion

With both properties being satisfied by the filter h we can say that filter h is linear.

Convolution Mask

The correct convolution mask for the filter is:

$$h_{\text{mask}} = \begin{bmatrix} 0 & 99 & 0 \\ 2 & 3 & 2 \\ 0 & -17 & 0 \end{bmatrix}$$

Here's the breakdown:

- The central element (3) corresponds to the coefficient of $f(x, y)$.
- The right and left elements of the central element (both 2) correspond to the coefficients of $f(x \pm 1, y)$.
- The bottom center element (-17) corresponds to the coefficient of $f(x, y - 1)$.
- The top-center element (99) corresponds to the coefficient of $f(x, y + 1)$.