Inverse of a matrix

Michael Friendly

April 04, 2018

The inverse of a matrix plays the same roles in matrix algebra as the reciprocal of a number and division does in ordinary arithmetic: Just as we can solve a simple equation like $4x = 8$ for x by multiplying both sides by the reciprocal

$$4x = 8 \Rightarrow 4^{-1}4x = 4^{-1}8 \Rightarrow x = 8/4 = 2$$

we can solve a matrix equation like $\mathbf{Ax} = \mathbf{b}$ for the vector $\mathbf{x}$ by multiplying both sides by the inverse of the matrix $\mathbf{A}$,

$$\mathbf{Ax} = \mathbf{b} \Rightarrow \mathbf{A}^{-1}\mathbf{Ax} = \mathbf{A}^{-1}\mathbf{b} \Rightarrow \mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$$

The following examples illustrate the basic properties of the inverse of a matrix.

Load the `matlib` package

This defines: `inv()`, `Inverse()`; the standard R function for matrix inverse is `solve()`

```
library(matlib)
```

Create a 3 x 3 matrix

The ordinary inverse is defined only for square matrices.

```
 A <- matrix( c(5, 1, 0,
               3,-1, 2,
               4, 0,-1), nrow=3, byrow=TRUE)
det(A)
```

```
## [1] 16
```

Basic properties

1. `det(A) != 0`, so inverse exists

Only non-singular matrices have an inverse.

```
(AI  <- inv(A))
```

```
##        [,1]    [,2]   [,3]
## [1,] 0.0625  0.0625  0.125
## [2,] 0.6875 -0.3125 -0.625
## [3,] 0.2500  0.2500 -0.500
```

2. Definition of the inverse: $A^{-1}A = AA^{-1} = I$ or `AI * A = diag(nrow(A))`

The inverse of a matrix A is defined as the matrix A^{-1} which multiplies A to give the identity matrix, just as, for a scalar a, $aa^{-1} = a/a = 1$.

NB: Sometimes you will get very tiny off-diagonal values (like `1.341e-13`). The function `zapsmall()` will round those to 0.

```
AI %*% A
```

```
##      [,1] [,2] [,3]
## [1,]    1    0    0
## [2,]    0    1    0
## [3,]    0    0    1
```

3. Inverse is reflexive: `inv(inv(A)) = A`

Taking the inverse twice gets you back to where you started.

```
inv(AI)
```

```
##      [,1] [,2] [,3]
## [1,]    5    1    0
## [2,]    3   -1    2
## [3,]    4    0   -1
```

4. `inv(A)` is symmetric if and only if A is symmetric

```
inv( t(A) )
```

```
##        [,1]    [,2]  [,3]
## [1,] 0.0625  0.6875  0.25
## [2,] 0.0625 -0.3125  0.25
## [3,] 0.1250 -0.6250 -0.50
```

```
is_symmetric_matrix(A)
```

```
## [1] FALSE
```

```
is_symmetric_matrix( inv( t(A) ) )
```

```
## [1] FALSE
```

Here is a symmetric case:

```
B <- matrix( c(4, 2, 2,
               2, 3, 1,
               2, 1, 3), nrow=3, byrow=TRUE)
inv(B)
```

```
##       [,1]  [,2]  [,3]
## [1,]  0.50 -0.25 -0.25
## [2,] -0.25  0.50  0.00
## [3,] -0.25  0.00  0.50
```

```
inv( t(B) )
```

```
##       [,1]  [,2]  [,3]
## [1,]  0.50 -0.25 -0.25
## [2,] -0.25  0.50  0.00
## [3,] -0.25  0.00  0.50
```

```
is_symmetric_matrix(B)
```

```
## [1] TRUE
```

```
is_symmetric_matrix( inv( t(B) ) )
```

```
## [1] TRUE
```

```
all.equal( inv(B), inv( t(B) ) )
```

```
## [1] TRUE
```

More properties of matrix inverse

1. inverse of diagonal matrix = diag(1/ diagonal)

In these simple examples, it is often useful to show the results of matrix calculations as fractions, using `MASS::fractions()`.

```
D <- diag(c(1, 2, 4))
inv(D)
```

```
##      [,1] [,2] [,3]
## [1,]    1  0.0 0.00
## [2,]    0  0.5 0.00
## [3,]    0  0.0 0.25
```

```
MASS::fractions( diag(1 / c(1, 2, 4)) )
```

```
##      [,1] [,2] [,3]
## [1,]   1    0    0
## [2,]   0  1/2    0
## [3,]   0    0  1/4
```

2. Inverse of an inverse: `inv(inv(A)) = A`

```
A <- matrix(c(1, 2, 3,  2, 3, 0,  0, 1, 2), nrow=3, byrow=TRUE)
AI <- inv(A)
inv(AI)
```

```
##      [,1] [,2] [,3]
## [1,]    1    2    3
## [2,]    2    3    0
## [3,]    0    1    2
```

3. inverse of a transpose: `inv(t(A)) = t(inv(A))`

```
inv( t(A) )
```

```
##       [,1] [,2]  [,3]
## [1,]  1.50 -1.0  0.50
## [2,] -0.25  0.5 -0.25
## [3,] -2.25  1.5 -0.25
```

```
t( inv(A) )
```

```
##       [,1] [,2]  [,3]
## [1,]  1.50 -1.0  0.50
## [2,] -0.25  0.5 -0.25
## [3,] -2.25  1.5 -0.25
```

4. inverse of a scalar * matrix: `inv( k*A ) = (1/k) * inv(A)`

```
inv(5 * A)
```

```
##      [,1]  [,2]  [,3]
## [1,]  0.3 -0.05 -0.45
## [2,] -0.2  0.10  0.30
## [3,]  0.1 -0.05 -0.05
```

```
(1/5) * inv(A)
```

```
##      [,1]  [,2]  [,3]
## [1,]  0.3 -0.05 -0.45
## [2,] -0.2  0.10  0.30
## [3,]  0.1 -0.05 -0.05
```

5. inverse of a matrix product: `inv(A * B) = inv(B) %*% inv(A)`

```
B <- matrix(c(1, 2, 3, 1, 3, 2, 2, 4, 1), nrow=3, byrow=TRUE)
C <- B[, 3:1]
A %*%  B
```

```
##      [,1] [,2] [,3]
## [1,]    9   20   10
## [2,]    5   13   12
## [3,]    5   11    4
```

```
inv(A %*%  B)
```

```
##      [,1]  [,2]  [,3]
## [1,]  4.0 -1.50 -5.50
## [2,] -2.0  0.70  2.90
## [3,]  0.5 -0.05 -0.85
```

```
inv(B) %*%  inv(A)
```

```
##      [,1]  [,2]  [,3]
## [1,]  4.0 -1.50 -5.50
## [2,] -2.0  0.70  2.90
## [3,]  0.5 -0.05 -0.85
```

This extends to any number of terms: the inverse of a product is the product of the inverses in reverse order.

```
(ABC <- A %*% B %*% C)
```

```
##      [,1] [,2] [,3]
## [1,]   77  118   49
## [2,]   53   97   42
## [3,]   41   59   24
```

```
inv(A %*% B %*% C)
```

```
##      [,1]  [,2]   [,3]
## [1,]  1.5 -0.59  -2.03
## [2,] -4.5  1.61   6.37
## [3,]  8.5 -2.95 -12.15
```

```
inv(C) %*% inv(B) %*% inv(A)
```

```
##       [,1]  [,2]   [,3]
## [1,]  1.5 -0.59  -2.03
## [2,] -4.5  1.61   6.37
## [3,]  8.5 -2.95 -12.15
```

```
inv(ABC)
```

```
##       [,1]  [,2]   [,3]
## [1,]  1.5 -0.59  -2.03
## [2,] -4.5  1.61   6.37
## [3,]  8.5 -2.95 -12.15
```

6. $\det(A^{-1}) = 1/\det(A) = [\det(A)]^{-1}$

The determinant of an inverse is the inverse (reciprocal) of the determinant

```
det(AI)
```

```
## [1] 0.25
```

```
1 / det(A)
```

```
## [1] 0.25
```

Geometric interpretations

Some of these properties of the matrix inverse can be more easily understood from geometric diagrams. Here, we take a 2×2 non-singular matrix A,

```
A <- matrix(c(2, 1,
              1, 2), nrow=2, byrow=TRUE)
A
```

```
##      [,1] [,2]
## [1,]    2    1
## [2,]    1    2
```

```
det(A)
```

```
## [1] 3
```

The larger the determinant of A, the smaller is the determinant of A^{-1}.

```
AI <- inv(A)
MASS::fractions(AI)
```

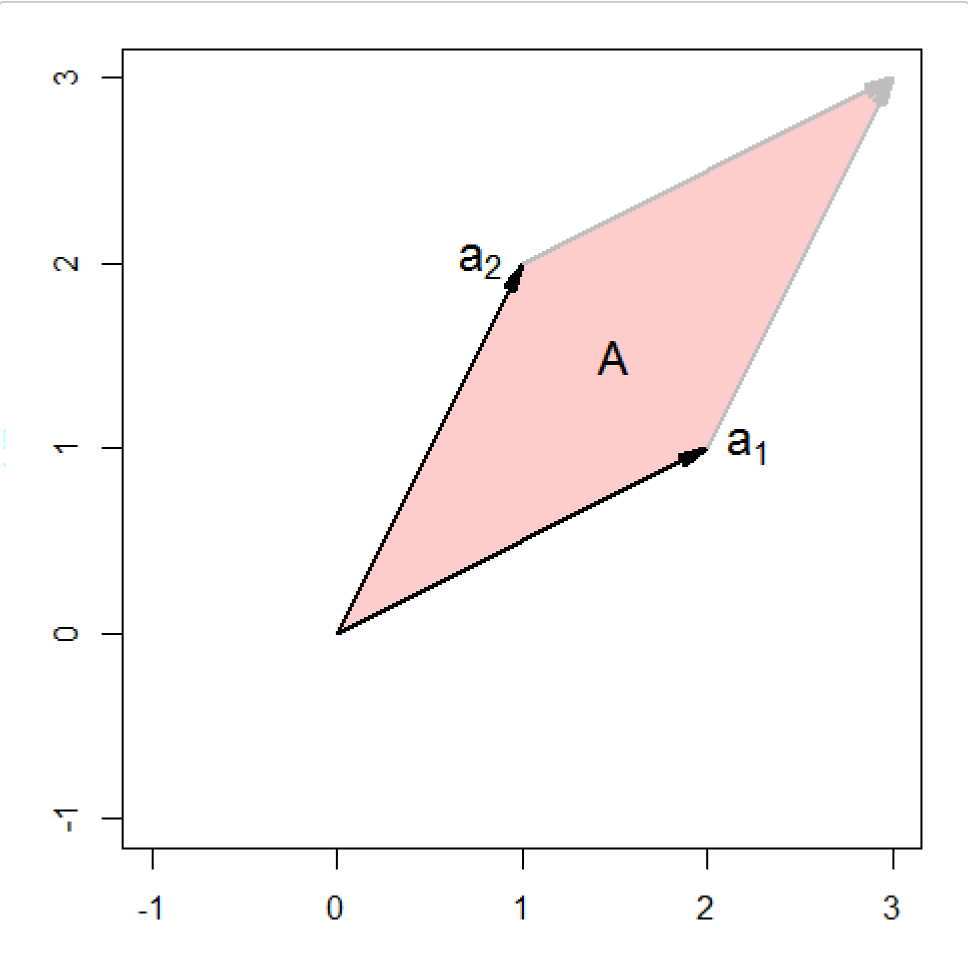
```
##      [,1] [,2]
## [1,]  2/3 -1/3
## [2,] -1/3  2/3
```

```
det(AI)
```

```
## [1] 0.3333
```

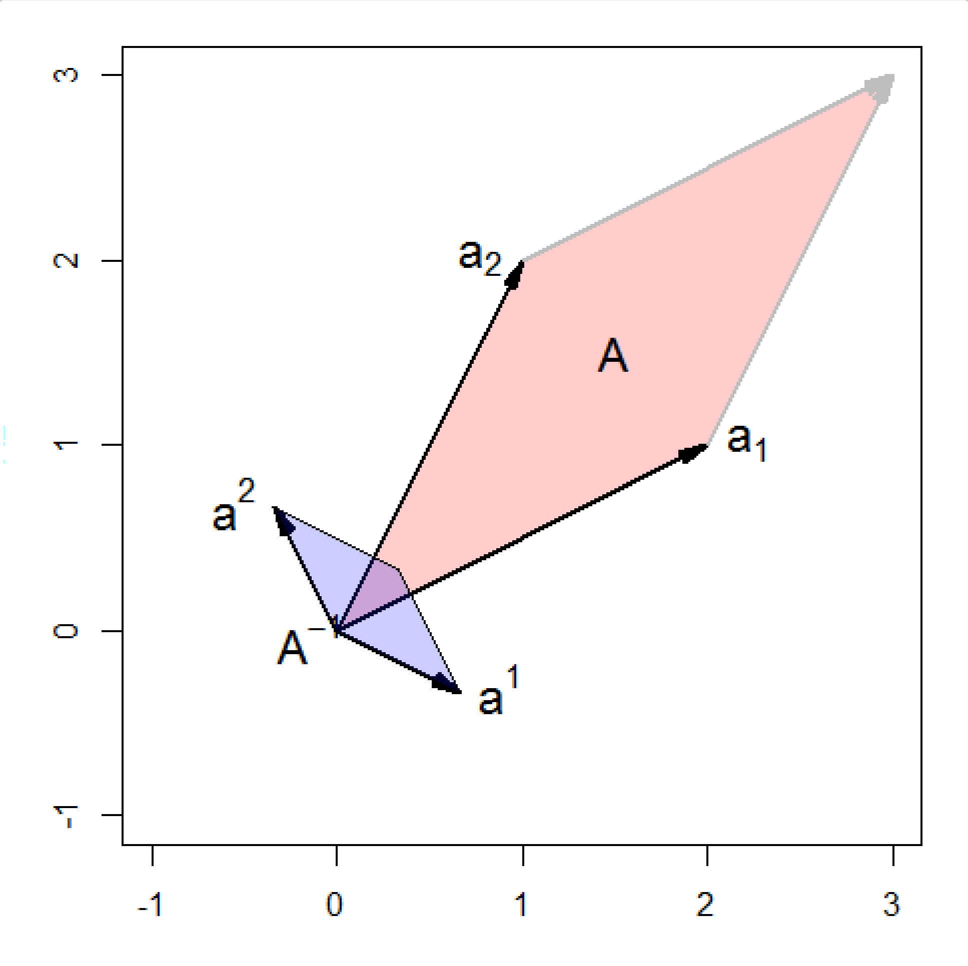
Now, plot the rows of A as vectors a_1, a_2 from the origin in a 2D space. As illustrated in `vignette("det-ex1")`, the area of the parallelogram defined by these vectors is the determinant.

```
par(mar=c(3,3,1,1)+.1)
xlim <- c(-1,3)
ylim <- c(-1,3)
plot(xlim, ylim, type="n", xlab="X1", ylab="X2", asp=1)
sum <- A[1,] + A[2,]
# draw the parallelogram determined by the rows of A
polygon( rbind(c(0,0), A[1,], sum, A[2,]), col=rgb(1,0,0,.2))
vectors(A, labels=c(expression(a[1]), expression(a[2])), pos.lab=c(4,2))
vectors(sum, origin=A[1,], col="gray")
vectors(sum, origin=A[2,], col="gray")
text(mean(A[,1]), mean(A[,2]), "A", cex=1.5)
```



The rows of the inverse A^{-1} can be shown as vectors a^1, a^2 from the origin in the same space.

```
vectors(AI, labels=c(expression(a^1), expression(a^2)), pos.lab=c(4,2))
sum <- AI[1,] + AI[2,]
polygon( rbind(c(0,0), AI[1,], sum, AI[2,]), col=rgb(0,0,1,.2))
text(mean(AI[,1])-.3, mean(AI[,2])-.2, expression(A^{-1}), cex=1.5)
```



Thus, we can see:

- The shape of A^{-1} is a 90^o rotation of the shape of A.
- A^{-1} is small in the directions where A is large.
- The vector a^2 is at right angles to a_1 and a^1 is at right angles to a_2
- If we multiplied A by a constant k to make its determinant larger (by a factor of k^2), the inverse would have to be divided by the same factor to preserve $AA^{-1} = I$.

One might wonder whether these properties depend on symmetry of A, so here is another example, for the matrix `A <- matrix(c(2, 1, 1, 1), nrow=2)`, where $\det(A) = 1$.

```
(A <- matrix(c(2, 1, 1, 1), nrow=2))
```

```
##      [,1] [,2]
## [1,]    2    1
## [2,]    1    1
```

```
(AI <- inv(A))
```

```
##      [,1] [,2]
## [1,]    1   -1
## [2,]   -1    2
```

The areas of the two parallelograms are the same because $\det(A) = \det(A^{-1}) = 1$.