Table of Contents

What is Probability Theory?

- Probability Theory is a **mathematical** framework for computing the probability of complex events.
- Under the assumption that **we know the probabilities of the basic events.**
- What is the precise meaning of "probability" and "event"?
- We will give precise definitions later in the class.
- For now, we'll rely on common sense.

A simple (?) question

We all know that if one flips a fair coin then the outcome is "heads" or "tails" with equal probabilities.

What does that mean?

It means that if we flip the coin k times, for some large value of k, say $k = 10,000$,

Then the number of "heads" is **about** $\frac{k}{2} = \frac{10{,}000}{2} = 5,000$

What do we mean by **about** ??

Simulating coin flips

We will use the pseudo random number generators in `numpy` to simulate the coin flips.

instead of "Heads" and "Tails" we will use $x_i = 1$ or $x_i = -1$ and consider the sum $S_{10000} = x_1 + x_2 + \cdots + x_{10000}$.

If the number of heads is about 5,000 then $S_{10000} \approx 0$

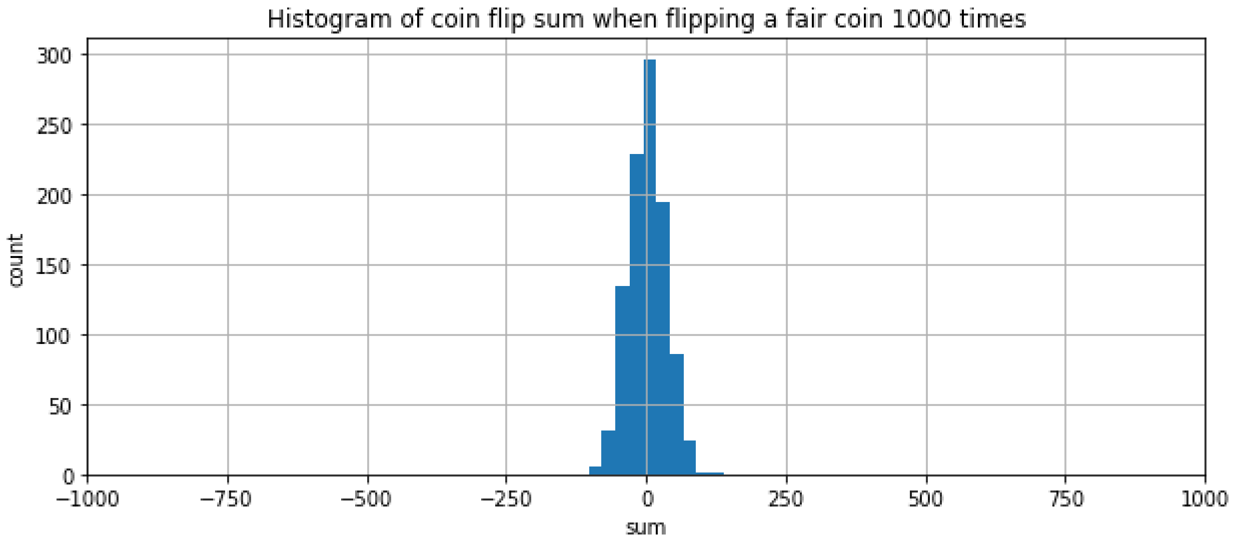
We will vary the number of coin flips, which we denote by k

In [2]:

```python
# Generate the sum of k coin flips, repeat that n times
def generate_counts(k=1000,n=100):
    X=2*(random.rand(k,n)>0.5)-1  # generate a kXn matrix of +-1 random numbers
    S=sum(X,axis=0)
    return S
```

In [3]:

```
k=1000
n=1000
counts=generate_counts(k=k,n=n)
figure(figsize=[10,4])
hist(counts);
xlim([-k,k])
xlabel("sum")
ylabel("count")
title("Histogram of coin flip sum when flipping a fair coin %d times"%k)
grid()
```



Note that the sum S_{1000} is not **exactly** 0, it is only **close to** 0.

Using **probability theory** we can calculate **how small** is $\left|S_k\right|$

In a later lesson we will show that the probability that

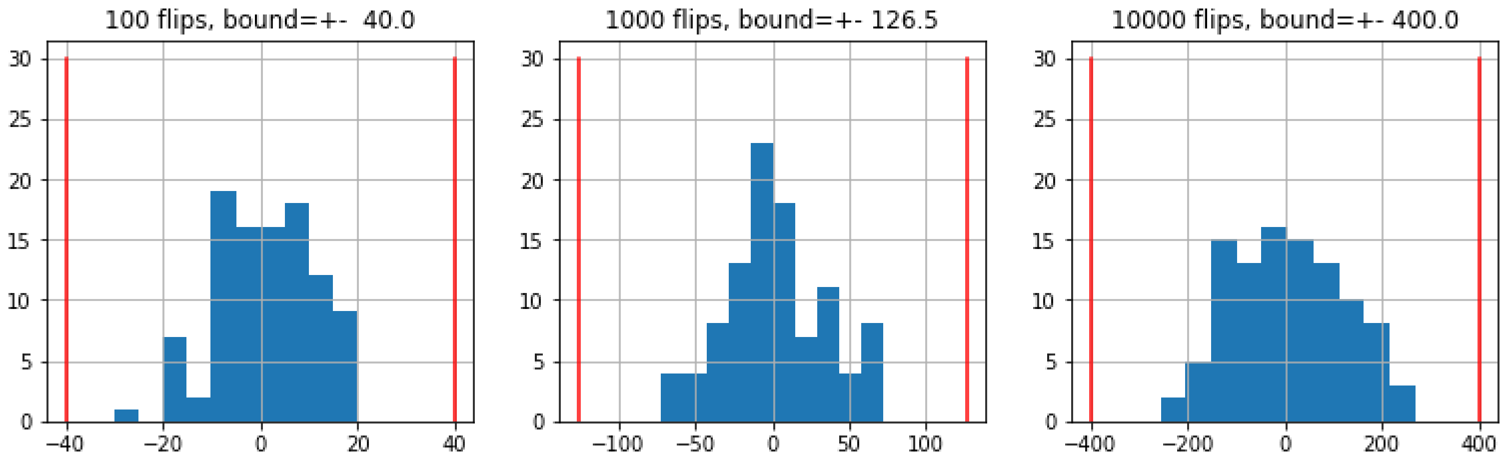
$$\left|S_k\right| \geq 4\sqrt{k}$$

is smaller than 2×10^{-8} which is 0.000002%

Let's use our simulation to demonstrate that this is the case:

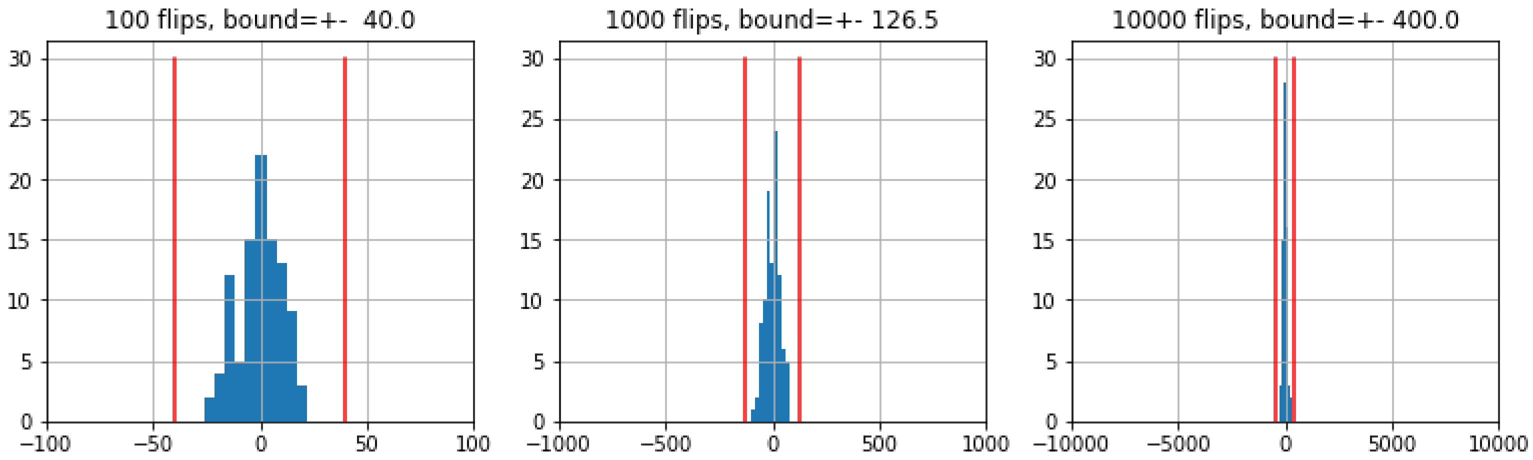
In [4]:

```
from math import sqrt
figure(figsize=[13,3.5])
for j in range(2,5):
    k=10**j
    counts=generate_counts(k=k,n=100)
    subplot(130+j-1)
    hist(counts,bins=10);
    d=4*sqrt(k)
    plot([-d,-d],[0,30],'r')
    plot([+d,+d],[0,30],'r')
    grid()
    title('%d flips, bound=+-%6.1f'%(k,d))
```



In [5]:

```
figure(figsize=[13,3.5])
for j in range(2,5):
    k=10**j
    counts=generate_counts(k=k,n=100)
    subplot(130+j-1)
    hist(counts,bins=10);
    xlim([-k,k])
    d=4*sqrt(k)
    plot([-d,-d],[0,30],'r')
    plot([+d,+d],[0,30],'r')
    grid()
    title('%d flips, bound=+-%6.1f'%(k,d))
```



Summary

We did some experiments summing k random numbers: $S_k = x_1 + x_2 + \cdots + x_k$

$x_i = -1$ with probability $1/2$, $x_i = +1$ with probability $1/2$

Our experiments show that the sum S_k is (almost) always in the range $\left[-4\sqrt{k}, +4\sqrt{k}\right]$

$$\text{If } k \to \infty, \quad \frac{4\sqrt{k}}{k} = \frac{4}{\sqrt{k}} \to 0$$

Therefor if $\quad k \rightarrow \infty, \dfrac{S_k}{k} \rightarrow 0$

What is probability theory?

It is the math involved in **proving** (a precise version of) the statements above.

In most cases, we can **approximate** probabilities using simulations (Monte-Carlo simulations)

Calculating the probabilities is better because:

- It provides a precise answer
- It is much faster than Monte Carlo simulations.

Up Next: What is Statistics ?