Bil 108

Introduction to the Scientific and Engineering Computing with MATLAB

Lecture 9

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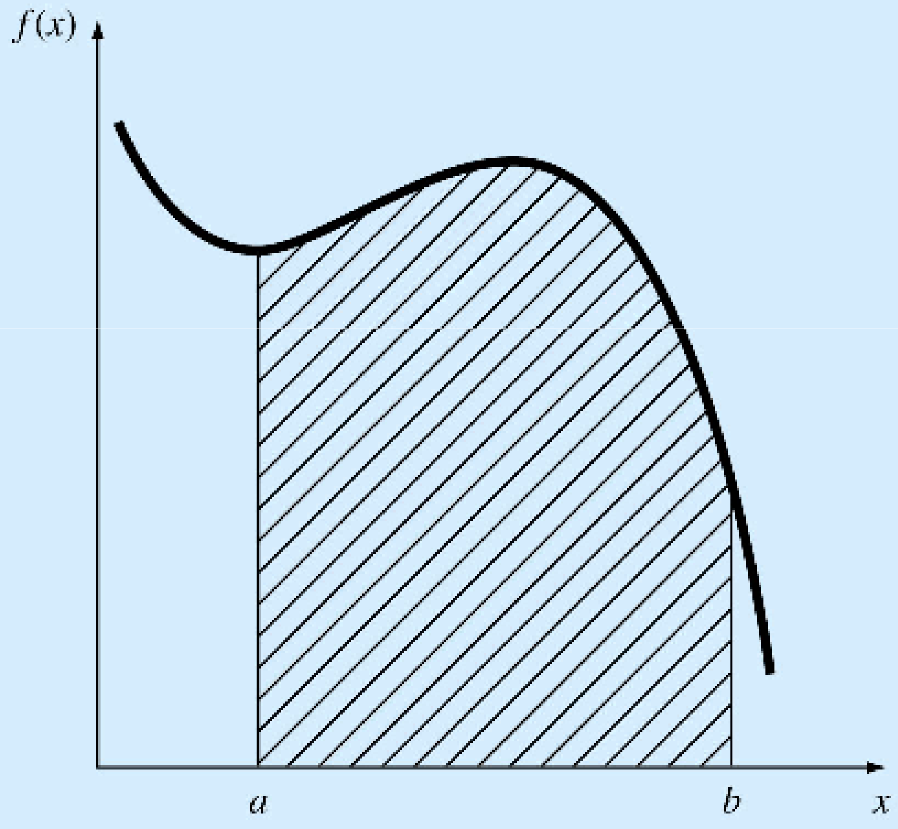
Numerical Integration

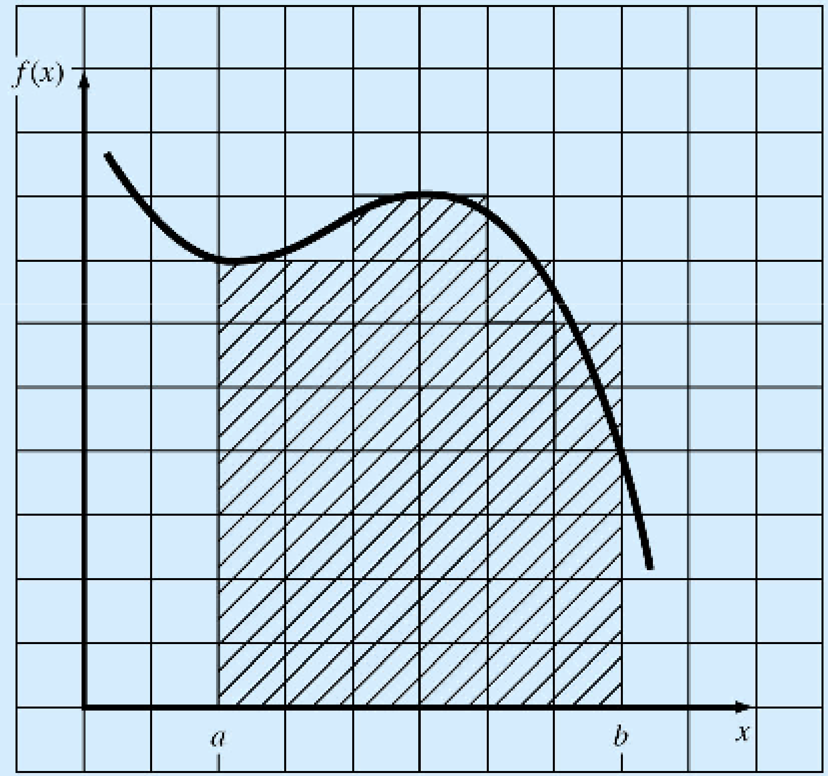
- Common mathematical operation in science and engineering

  - Calculating area, volume, velocity from accelaration, work from force and displacement are just few examples where integration is used.

  - Integration of simple functions can be done analytically

Graphical Representation of Integral





Integral = area under the curve

Use of a grid to approximate an integral

Basic Quadrature Rules

- The word “quadrature" reminds us of an elementary technique for finding the area

  - plot the function on graph paper and count the number of little squares that lie underneath the curve.

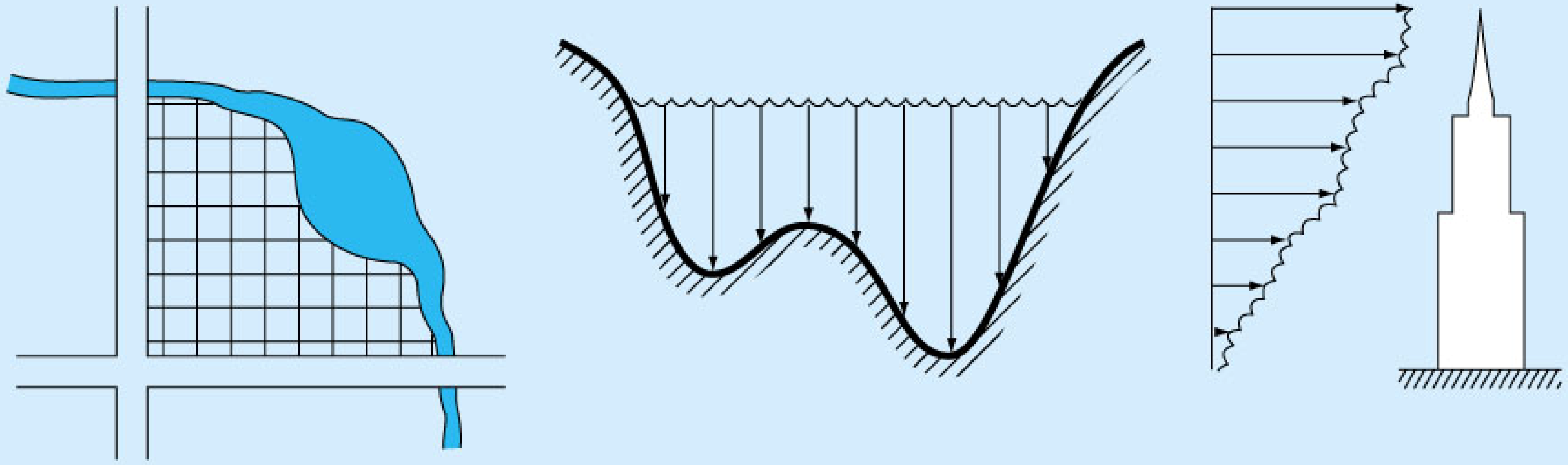
Piecewise integration

- Piecewise integration is superior to global integration.

- On each panel relatively low degree polynomial approx. to f(x) created. Integration the polynomial approx. on the panel gives **BASIC RULE**

- A basic rule involves just enough (x,f(x)) pairs to define one segment of piecewise polynomial.
- Applying the basic rule to each of the N panels and adding together the results gives what is called a **COMPOSITE RULE**

Numerical Integration



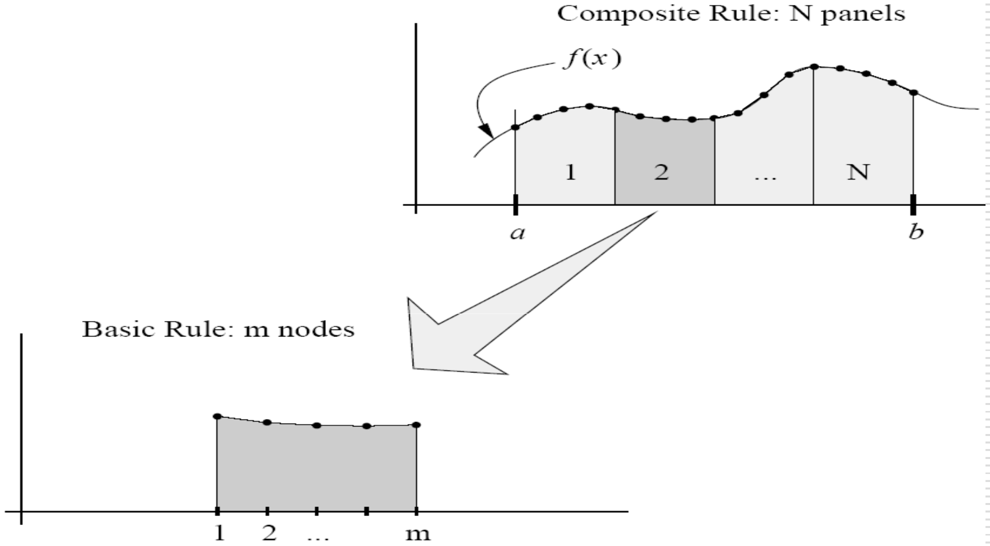
(a) (b) (c)

Survey of land area of an irregular lot

Cross-sectional area and volume flowrate in a river

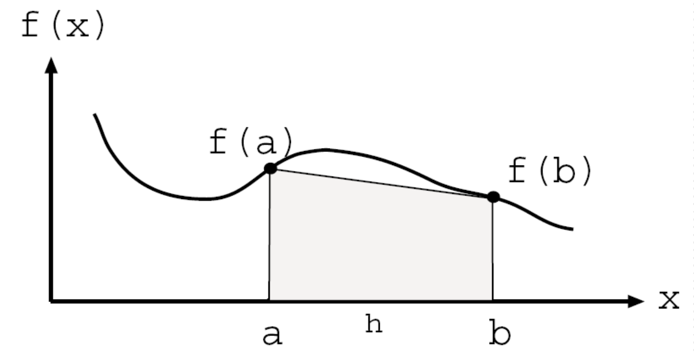
Net force against a skyscraper

Piecewise Integration



- If the location and number of nodes within a panel are equally spaced the resulting integration formulas are known as **NEWTON COTES RULES**

Trapezoid rule



The trapezoid rule, *T*, approximates the integral by the area of a trapezoid with base *h* and sides equal to the values of the integrand at the two end points.

Based on the strategy of replacing a complicted function or tabulated data with a polynomial that is easy to integrate

$$I = \int_a^b f(x)dx \cong \int_a^b f_n(x)dx$$

Where $f_n(x)$ is a polynomial of the form

$$f_n(x)=a_0+a_1X+.....+a_{n-1}X^{n-1}+ a_nX^n$$

Trapezoid rule

- The trapezoid rule is the first of the Newton-Cotes closed integration formulas. It corresponds to the case where the polynomial is the first order:

$$I = \int_a^b \left[f(a) + \frac{f(b) - f(a)}{b - a}(x - a) \right] dx$$

$$I = (b - a)\frac{f(a) + f(b)}{2}$$

Example:Trapezoidal Rule

- Evaluate the integral $\int_0^4 xe^{2x}dx$

- Exact solution

$$\int_0^4 xe^{2x}dx = \left[\frac{x}{2}e^{2x} - \frac{1}{4}e^{2x}\right]_0^4$$

$$= \frac{1}{4}e^{2x}(2x-1)\Big|_0^1 = 5216.926477$$

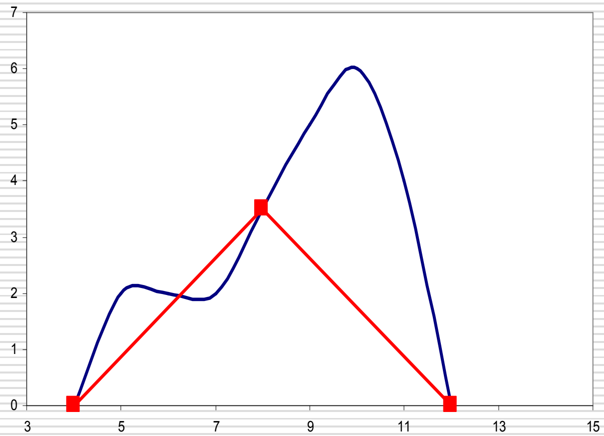
- Trapezoidal Rule

$$I = \int_0^4 xe^{2x}dx \approx \frac{4-0}{2}\left[f(0)+f(4)\right] = 2(0+4e^8) = 23847.66$$

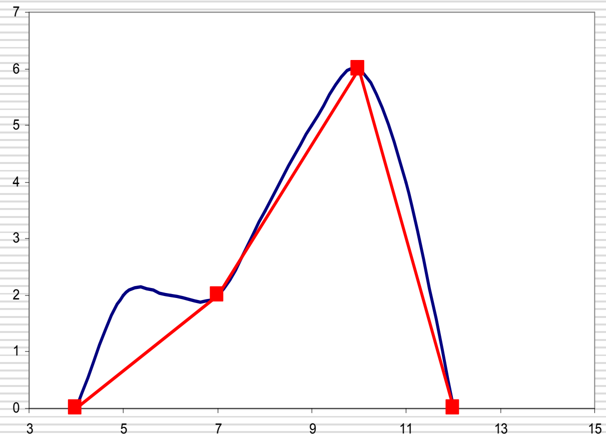
$$\varepsilon = \frac{5216.926 - 23847.66}{5216.926} = -357.12\%$$

Apply trapezoidal rule to multiple segments over integration limits

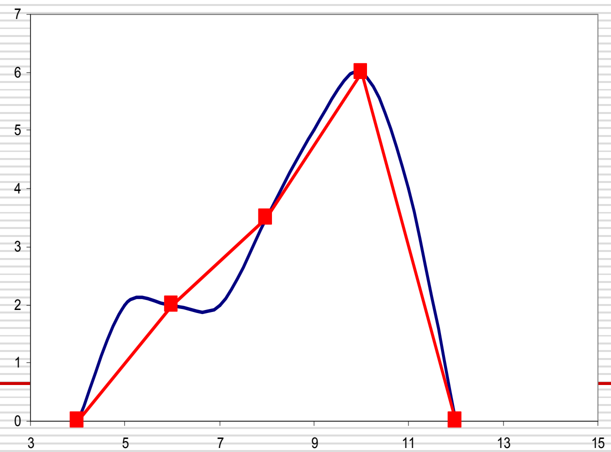
Two segments



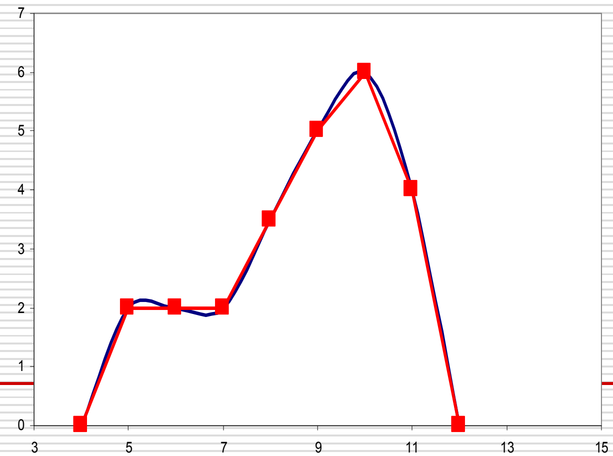
Three segments



Four segments



Many segments



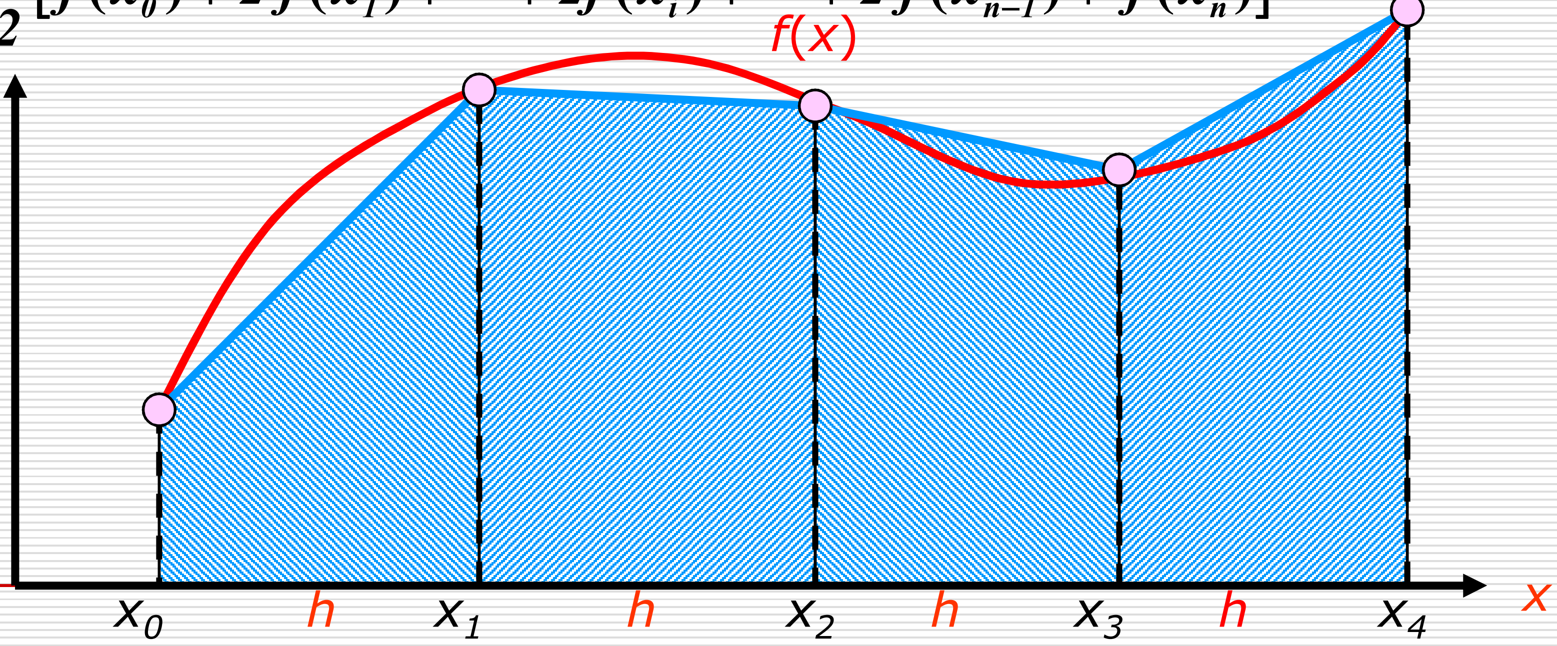
Composite Trapezoidal Rule

$$\int_a^b f(x)dx = \int_{x_0}^{x_1} f(x)dx + \int_{x_1}^{x_2} f(x)dx + \cdots\cdots + \int_{x_{n-1}}^{x_n} f(x)dx$$

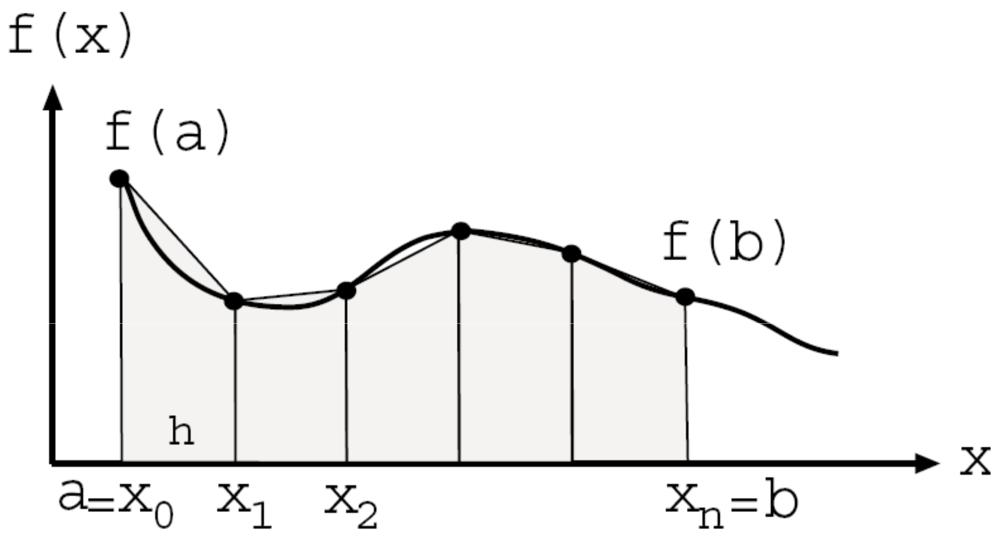
$$= \frac{h}{2}\left[f(x_0) + f(x_1)\right] + \frac{h}{2}\left[f(x_1) + f(x_2)\right] + \cdots + \frac{h}{2}\left[f(x_{n-1}) + f(x_n)\right]$$

$$= \frac{h}{2}\left[f(x_0) + 2f(x_1) + \cdots + 2f(x_i) + \cdots + 2f(x_{n-1}) + f(x_n)\right]$$

$$h = \frac{b-a}{n}$$



Composite Trapezoidal Rule



$$\int_a^b f(x)dx = \frac{h}{2}\left[f(x_0) + 2\sum_{i=1}^{n-1} f(x_i) + f(x_n) \right]$$

Composite Trapezoidal Rule

- Evaluate the integral $I = \int_0^4 xe^{2x}\,dx$

$$n = 1, h = 4 \Rightarrow I = \frac{h}{2}\left[f(0) + f(4)\right] = 23847.66 \qquad \varepsilon = -357.12\%$$

$$n = 2, h = 2 \Rightarrow I = \frac{h}{2}\left[f(0) + 2f(2) + f(4)\right] = 12142.23 \qquad \varepsilon = -132.75\%$$

$$n = 4, h = 1 \Rightarrow I = \frac{h}{2}\left[f(0) + 2f(1) + 2f(2) + 2f(3) + f(4)\right] = 7288.79 \qquad \varepsilon = -39.71\%$$

$$n = 8, h = 0.5 \Rightarrow I = \frac{h}{2}\left[f(0) + 2f(0.5) + 2f(1) + 2f(1.5) + 2f(2) + 2f(2.5) + 2f(3) + 2f(3.5) + f(4)\right] = 5764.76 \qquad \varepsilon = -10.50\%$$

$$n = 16, h = 0.25 \Rightarrow I = \frac{h}{2}\left[f(0) + 2f(0.25) + 2f(0.5) + \cdots + 2f(3.5) + 2f(3.75) + f(4)\right] = 5355.95 \qquad \varepsilon = -2.66\%$$

Simpson’s 1/3 rule

A more accurate estimate of an integral is to use higher order polynomials to connect the points. Simpson’s 1/3 rule corresponds to the case where the polynomial is second order:

$$I = \int_{x_0}^{x_2} \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} f(x_0) + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} f(x_1)$$

$$+ \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} f(x_2) d(x)$$

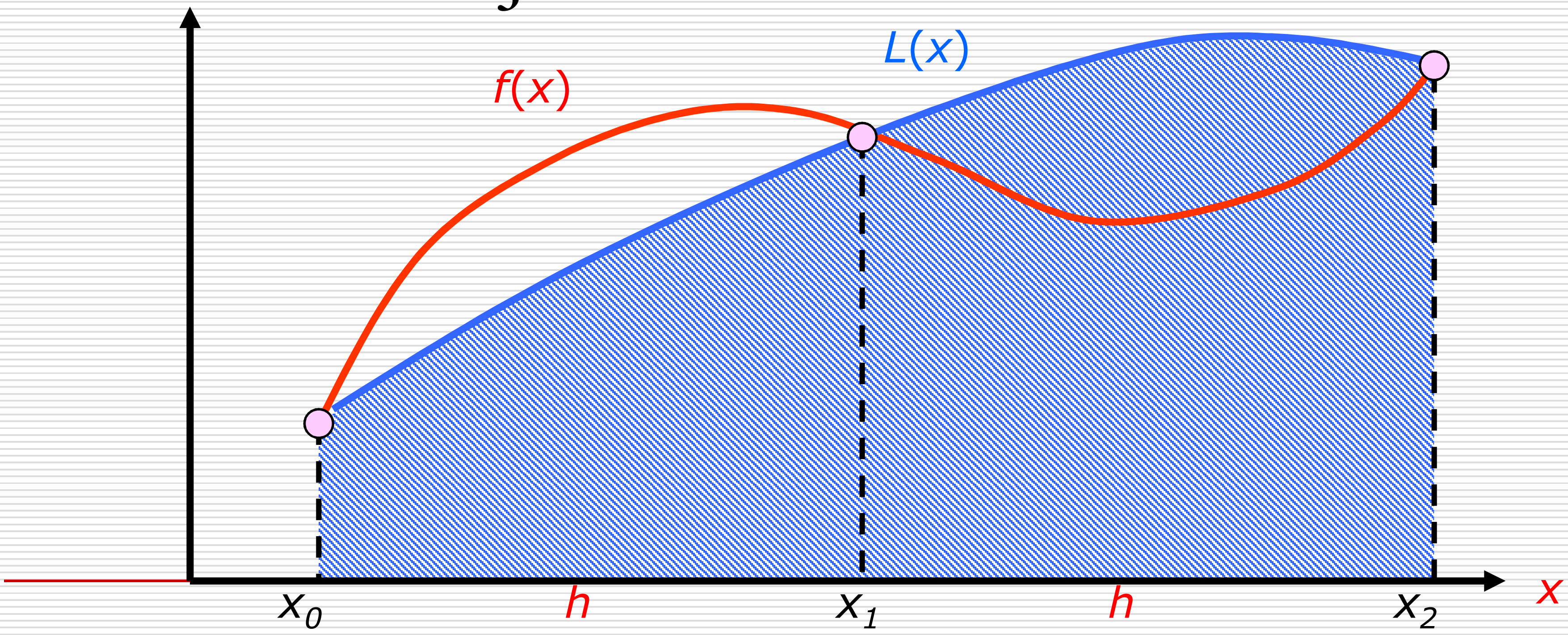
$$\int_{x_1}^{x_3} f(x)dx \approx h\left[\frac{1}{3} f_1 + \frac{4}{3} f_2 + \frac{1}{3} f_3\right]$$

Simpson's 1/3-Rule

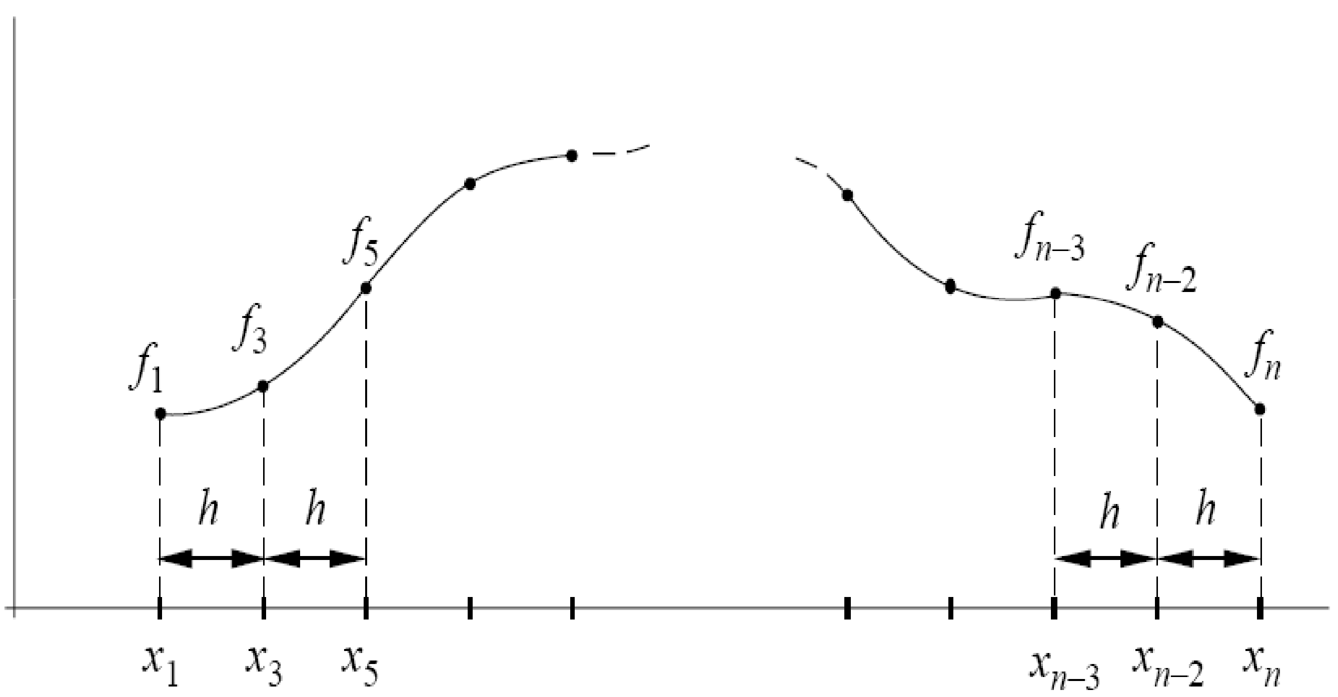
- Approximate the function by a parabola

$$\int_a^b f(x)dx \approx \sum_{i=0}^{2} c_i f(x_i) = c_0 f(x_0) + c_1 f(x_1) + c_2 f(x_2)$$

$$= \frac{h}{3}\left[f(x_0) + 4f(x_1) + f(x_2) \right]$$



Composite Simpson rule



$$\int_{x_1}^{x_n} f(x)dx = \frac{h}{3}\left(f_1 + 4\sum_{i=2,4,..}^{n-1} f_i + 2\sum_{i=3,5,..}^{n-2} f_i + f_n\right)$$

Example

$$f(x) = 0.2 + 25x - 200x^2 + 675x^3 - 900x^4 + 400x^5$$

From a=0 to b=0.8 Note that the exact value of the integral can be determined analytically to be 1.640533.

a) Simpson’s Rule
b) Composite Simpson’s Rule

IMPORTANT: This rule can be applied to the even number of segments and an odd number of points.

Evaluating definite

- _ MATLAB can evaluate definite
- _ This is provided that the integrand f(x) be available as a *function*, not an array of numbers

How does Matlab do it?

- _ The primary function for evaluating definite integrals is quad - The adaptative Simpson method

q=quad('function',a,b)

- The user has to make sure that the function does not have a vertical asymtote between a and b.
- It calculates the integral with an absolute error that is smaller than 1.0e-6. This number can be changed by adding optional tol argument to the command:

q=quad('function',a,b,tol)

The quadl command

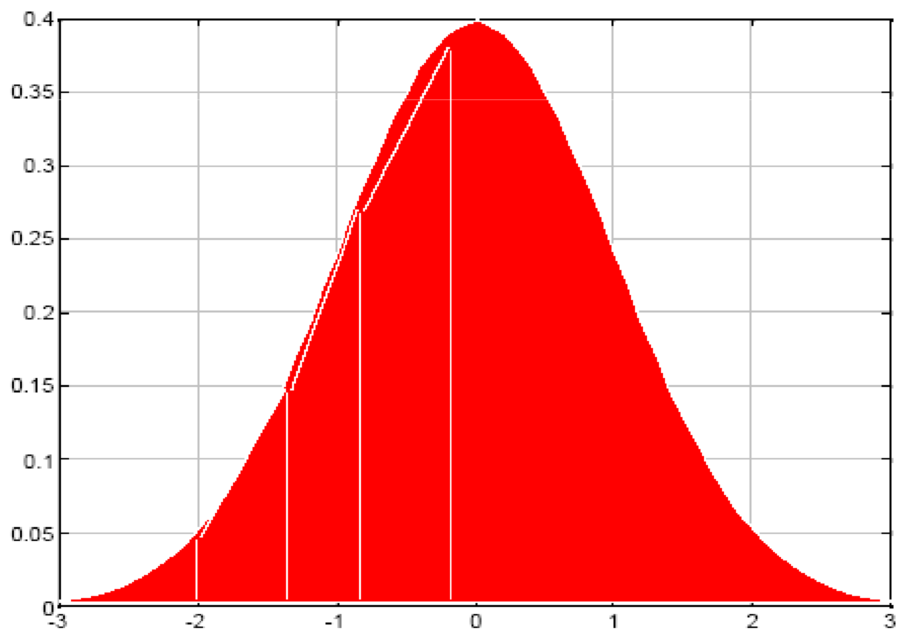
- Numerically evaluate integral, adaptive Lobatto quadrature
- quadl has the following syntax

  q=quadl(‘function’,a,b)

  q = quadl(fun,a,b,tol)
- tol is a number that defines the maximum error. With larger tol the integral is calculated less accurately but faster
- More efficient for high accuracies and smooth integrals

Approximation to integrals - *trapz*

- Function *trapz* uses areas of trapezoids to approximate the area under the curve



area=trapz(x,y)

Example

Evaluate the following $$\int_0^{3\pi/2} \cos(x)dx$$

Since cosine is a built-in MATLAB function;

```
y=quadl('cos',0,3*pi/2)
```

Example

- Use numerical integration to calculate the following integral:

$$\int_0^8 (xe^{-x^{0.8}} + 0.2)dx$$

```
>>quad(‘x.*exp(-x.^0.8)+0.2’,0,8)
>>quadl('exponent',0,8)
>>quad('exponent',0,8)
```

Trapezoid rule algorithm

```
function I=trapezoid(fun,a,b,npanel)
n=npanel+1;     %total number of nodes
h=(b-a)/(n-1);   %stepsize
x=a:h:b;          %divide the interval
f=feval(fun,x);    %evaluate the integrand
I=h*(0.5*f(1)+sum(f(2:n-1))+0.5*f(n));
```

Trapezoid rule sample

- The sample program below illustrates how the trapezoid rule is used to approximate the definite integral of the function *x^2* over the interval *x=0, 1*.

```
% Integration using the trapezoid rule.
a= 0;
b= 1;                % a,b limits of integration
N = 10;              % N is the number of sub-intervals
h=(b-a)/N;           % h width of each sub-interval
x=[a:h:b];           % x variable of integration
TL=a^2;              % TL, TR contribution from left and right endpoints
TR=b^2;
TI=0;                %TI contribution from the intermediate points
for n=2:N
f = x(n)^2;
TI=TI+f;             % f integrand
end
format long
I=(TL+2*TI+TR)*h*0.5
```

Simpson Algorithm

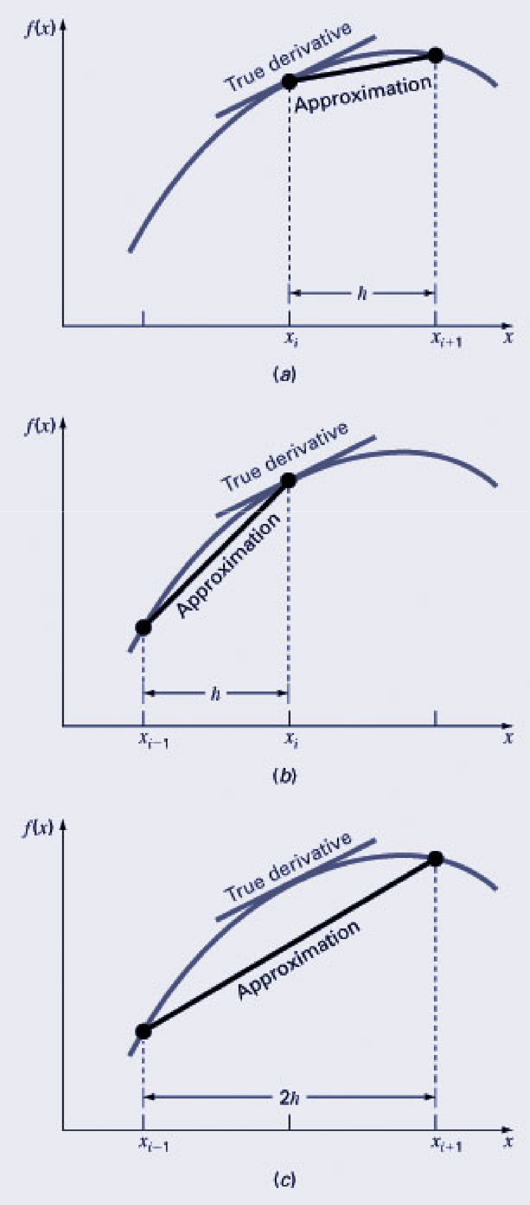
```
function I=simpson(fun,a,b,npanel)
n=2*npanel+1; %total number of nodes
h=(b-a)/(n-1); %stepsize
x=a:h:b; %divide the interval
f=feval(fun,x); %evaluate integrand
I=(h/3)*(f(1)+4*sum(f(2:2:n-1))+2*sum(f(3:2: n-2))+f(n));
```

Numerical Differentiation

- The derivative of a function f(x) is defined as a function f'(x), which is equal the rate of change of f(x) with respect to x.
- Can be expressed

$$f'(x) = \frac{df(x)}{dx}$$

Eg. Velocity is the rate of change of position and acceleration is the rate Of change of velocity etc.



Difference Expressions

- Numerical differentiation techniques estimate the derivative of a function at a point x_k by approximating the slope of the tangent line at x_k using values of the function at points near x_k. The approximation can be done in several ways

$$f'(x_k) = \frac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}}$$

BACKWARD DIFFERENCE

$$f'(x_k) = \frac{f(x_{k+1}) - f(x_k)}{x_{k+1} - x_k}$$

FORWARD DIFFERENCE

$$f'(x_k) = \frac{f(x_{k+1}) - f(x_{k-1})}{x_{k+1} - x_{k-1}}$$

CENTRAL DIFFERENCE

diff Function

- **diff(x)** Returns a new vector containing differences between adjacent values in the vector x. Returns a matrix containing differences between adjacent values in the columns if x is a matrix.
- The quantity diff(y)./diff(x) is an approximate derivative.

Example

$$f(x) = x^5 - 3x^4 - 11x^3 + 27x^2 + 10x - 24$$

```
%evaluate f(x) and f'(x) using backward differences
x=-4:0.1:5;
f=x.^5-3*x.^4-11*x.^3+27*x.^2+10*x-24;
df=diff (f) ./diff(x)
```

Plot both the polynomial and its derivative !!!!