EE 475 HW0: Random Variables and Processes

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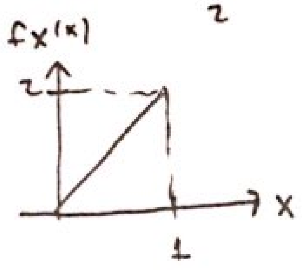
① X continuous RV

$$f_X(x) = \begin{cases} kx & 0<x<1 \\ 0 & \text{o.w.} \end{cases}$$

$$\int_{x=0}^{x=1} f_X(x)\,dx = 1$$

$$\Big|_{x=0}^{x=1} \frac{k.x^2}{2} = 1$$

$$\frac{k}{2} = 1 \quad k=2$$

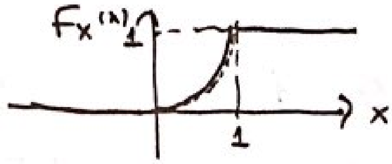


b) $F_X(x) = \int f_X(x).dx$

$F_X(x) = x^2 + K$

$F_X(1) = 1 = 1+K \quad k=0$

$F_X(x) = x^2$



c) $P(\frac{1}{4} < X \leq 2)$

$= F_X(2) - F_X(\frac{1}{4})$

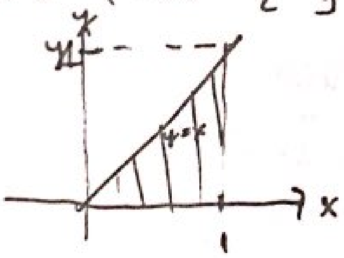
$= 1 - \frac{1}{16} = \frac{15}{16}$

② $E[X] = \int_{x=0}^{x=1} 2x.x\,dx = \frac{2x^3}{3}\Big|_{x=0}^{x=1} = 2/3$

$E[X^2] = \int_{x=0}^{x=1} 2x.x^2\,dx = \frac{2x^4}{4}\Big|_{x=0}^{x=1} = 1/2$

$V[X] = E[X^2] - (E[X])^2 = \frac{1}{2} - \frac{4}{9} = \frac{1}{18}$

③ $f_{XY}(x,y)$



$1 = \int_0^1 \int_0^x k\,dy\,dx = \int_0^1 kx\,dx$

$= k.\frac{x^2}{2}\Big|_0^1$

$= \frac{k}{2} = 1 \rightarrow k=2$

b) $f_X(x) = \int_{y=0}^{y=x} 2\,dy = 2x \quad \text{for} \quad y \leq x < 1$

$f_Y(y) = \int_{x=y}^{x=1} 2\,dx = 2.(1-y) \quad \text{for} \quad 0 \leq y \leq x$

c) $P(0<X<1/2, 0<Y<1/2) = \int_0^{1/2}\int_0^x 2\,dy\,dx = \int_0^{1/2} 2x = x^2\Big|_0^{1/2} = \frac{1}{4}$

d) $f_{X|Y}(x|y) = \frac{f_{XY}(x,y)}{f_Y(y)} = \frac{2}{2.(1-y)} = \frac{1}{1-y} \quad \text{for } 0 \leq y \leq x$

$$f_{Y|X}(y|x) = \frac{f_{XY}(x,y)}{f_X(x)} = \frac{2}{2x} = \frac{1}{x} \quad \text{for } y < x \le 1$$

c) $$E(Y/x) = \int_0^x y \cdot \frac{1}{x} \cdot dy = \frac{y^2}{2x} \Big|_{y=0}^{y=x} = \frac{x}{2} \quad \text{for } y \le x < 1$$

$$E(X|y) = \int_{x=y}^{x=1} \frac{x}{1-y} \cdot dx = \frac{x^2}{2 \cdot (1-y)} \Big|_{x=y}^{x=1} = \frac{1-y^2}{2 \cdot (1-y)} = \frac{1+y}{2} \quad \text{for } 0 \le y \le x$$

④ $X = \cos\Theta$ $Y = \sin\Theta$ Θ RV uniformly dist. on $(0, 2\pi)$

$$\mathrm{Cov}(X,Y) = E\left[(X - E[X]) \cdot (Y - E[Y])\right]$$

$$E[X] = E[\cos\theta] = \int_{\theta=0}^{\theta=2\pi} \frac{1}{2\pi} \cdot \cos\theta \cdot d\theta = 0$$

$$E[Y] = E[\sin\theta] = 0$$

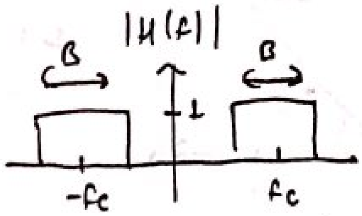
$$\mathrm{cov}(X,Y) = E[XY] = E[\sin\theta \cdot \cos\theta] = \frac{1}{2} E[\sin 2\theta]$$

$$E[\sin 2\theta] = \int_{\theta=0}^{\theta=2\pi} \frac{1}{2\pi} \sin 2\theta \cdot d\theta = -\frac{\cos 2\theta}{2 \cdot 2\pi} \Big|_{\theta=0}^{\theta=2\pi} = 0$$

$\mathrm{cov}(X,Y) = 0 \implies X, Y$ uncorrelated

b) $X^2 + Y^2 = 1$
$X^2 = 1 - Y^2$ it's clear that they are not independent

⑤ $R_X(\tau) = \frac{1}{2} N_0 \cdot \delta(\tau)$



F.T.

$$S_X(f) = \frac{N_0}{2}$$

$$S_Y(f) = S_X(f) \cdot |H(f)|^2 = \frac{N_0}{2} \cdot |H(f)|^2$$

$$P = \int_{-\infty}^{+\infty} S_Y(f)\, df = \frac{N_0}{2} \cdot \int_{-\infty}^{+\infty} |H(f)|^2 = \frac{N_0}{2} \cdot (2B) = N_0 \cdot B$$

(6) $S(f) = \begin{cases} N_0, & |f| < B \\ 0, & \text{o.w.} \end{cases}$

time domain → CTFT domain

$\dfrac{\sin wt}{\pi t} \rightarrow X(jw) = \begin{cases} 1, & |w| < \\ 0, & |w| > \end{cases}$

$w = 2\pi f$

$|f| < B$ means

$B = 2$

$\dfrac{w}{2\pi} = B$

$2\pi \cdot B = w$

PSD and auto cor. of $Y(t) = X^2(t)$

$$S(f) \xrightarrow[(FT)^{-1}]{} N_0 \cdot \frac{\sin(2\pi B t)}{\pi t}$$

$$R_{XX}(\tau) = E[X(t+\tau) \cdot X(t)]$$

$$R_{YY}(\tau) = E[Y(t+\tau) \cdot Y(t)] = E[X^2(t+\tau) \cdot X^2(t)]$$

Fact → $E(X_1 X_2 X_3 X_4) = E(X_1 X_2) \cdot E(X_3 X_4) + E(X_1 X_3) \cdot E(X_2 X_4) + E(X_1 X_4) \cdot E(X_2 X_3)$

$X_1 = X_2 = X(t+\tau)$

$X_3 = X_4 = X(t)$

then

$$R_{YY}(\tau) = E(X^2(t+\tau)) \cdot E(X^2(t)) + E[X(t+\tau) \cdot X(t)] \cdot E[X(t+\tau) \cdot X(t)] + E[X(t+\tau) \cdot X(t)] \cdot E[X(t+\tau) X$$

$$\underbrace{E(X^2(t+\tau)) = E(X^2(t))}_{\text{stationarity}}$$

$$E(X^2(t)) = R_{XX}(0)$$

$$R_{YY}(\tau) = (R_{XX}(0))^2 + 2 \cdot (R_{XX}(\tau))^2$$

$$R_{XX}(\tau) = N_0 \cdot \frac{\sin(2\pi B t)}{\pi t} \qquad (R_{XX}(\tau))^2 = N_0^2 \cdot \frac{\sin^2(2\pi B t)}{\pi^2 t^2}$$

$$R_{XX}(0) = N_0 \cdot 2B \left(\frac{\sin(2\pi B t)}{2\pi B t} \right)$$

$$R_{XX}(0) = N_0 \cdot 2B \cdot 1$$

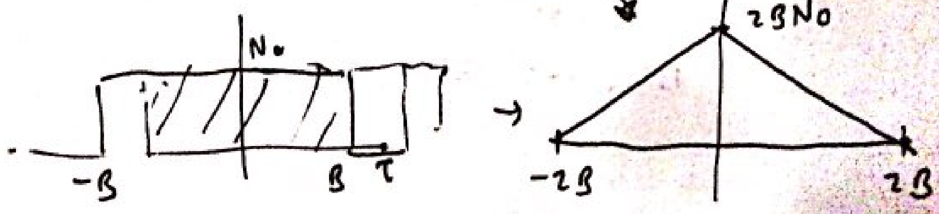
$$R_{YY}(\tau) = 4B^2 \cdot N_0^2 + N_0^2 \cdot \frac{\sin^2(2\pi B \tau)}{\pi^2 \cdot \tau^2}$$

$$R_{YY}(\tau) = (R_{XX}(0))^2 + 2 \cdot (R_{XX}(\tau))^2 \qquad R_{XX}(\tau) \xrightarrow[FT]{} S_{XX}(f)$$

$\downarrow FT \qquad \downarrow FT$

$$S_{YY}(f) = (R_{XX}(0))^2 \cdot \delta(f) + 2 \cdot S_{XX}(f) \circledast S_{XX}(f)$$

$$S_{YY}(f) = 4B^2 \cdot N_0^2 \cdot \delta(f) + 2 \cdot \underbrace{[S_{XX}(f) \circledast S_{XX}(f)]}$$



⑦ $S_x(f) = \begin{cases} 1, & |f| < W \\ 0 & \text{o.w.} \end{cases}$

$X(t)$ sampled at rate $f_s = 1/T$ to yield a disc.-time process $X(n) = X(nT)$

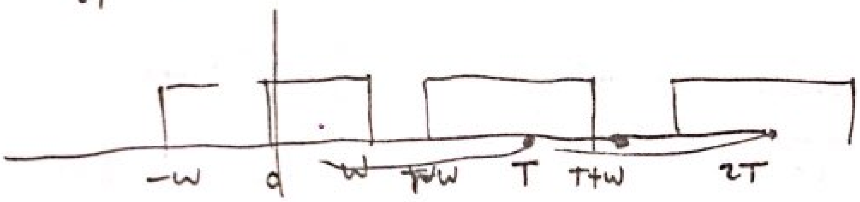
a) $S_x(f) \xrightarrow[(F.T)^{-1}]{} R_{XX}(\tau) = \dfrac{\sin 2\pi W\tau}{\pi\tau}$

$\tau = nT$

$R_{XX}(nT) = \dfrac{\sin 2\pi W.nT}{\pi.nT}$

$R_{XX}(n) = \dfrac{\sin(2\pi WnT)}{\pi nT}$

b) min T for spectrally flat sequence



no aliasing and for flat sequence $T = 2W$

⑧ $N(t)$ zero mean, stationary, narrow band process
equivalent lowpass process $Z(t) = X(t) + jY(t)$

$E[Z(t+\tau).Z(t)] = E[(X(t+\tau) + jY(t+\tau)).(X(t) + j.Y(t))]$

$= E[X(t+\tau).X(t)] - E[Y(t+\tau).Y(t)] + j.E[Y(t+\tau).X(t)] + j.E[X(t+\tau).Y(t)]$

$= R_{XX}(\tau) - R_{YY}(\tau) + j.[R_{YX}(\tau) + R_{XY}(\tau)]$

$= R_{XX}(\tau) - R_{YY}(\tau)$ | known: $R_{YX}(\tau) = -R_{XY}(\tau)$

Facts in a lowpass equivalent process

$X_i(t)$ and $X_q(t)$ have the same power spectral density,
$x(t)$ $y(t)$ which means they also have same autocorrelation function.

hence $R_{XX}(\tau) = R_{YY}(\tau)$

then

$E[Z(t+\tau).Z(t)] = 0$

b) $R_Z(\tau) = N_o\, \delta(\tau) = E[Z^*(t).Z(t+\tau)]$

$V = \int_0^T Z(t)dt$

$E[V^2] = E.\left[\left(\int_0^T .Z(t).dt\right).\left(\int_0^T Z(t).dt\right)\right] = E\left[\int_0^T\int_0^T Z(M).Z(n)\, dM dn\right]$

$E[V^2] = 0 = \int_0^T\int_0^T E[Z(M)Z(n)].dm\, dn$

from a) it's clear that

$E[Z(M).Z(n)] = 0$

for all M and n values.

$E[|V|^2] = E[V.V^*]$

$= E\left[\int_0^T\int_0^T Z(m).Z^*(n)\, dm dn\right]$

$\Rightarrow \quad E[Z(M).Z^*(n)] = \cancel{E[Z(n-M}$

$= R_Z(m-n) = N_o.\delta(M-n)$

$= \int_0^T\int_0^T N_o.\delta(M-n).dM\, dn$

$= \quad \delta(M-n) = 1 \quad$ when $M = n$

knowing that $m = n$ hence $\int_0^T N_o\, \delta(m-n).dM = N_o$

$E[|V|^2] = \int_0^T N_o.dn = N_o.T$