① $H(X) =$ $\quad x: 0 \to 1/3 \quad x: 1 \to 2/3$

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$$p(x=0) \cdot \log \frac{1}{p(x=0)} + p(x=1) \cdot \log \frac{1}{p(x=1)} =$$

$$= \frac{1}{3} \log 3 + \frac{2}{3} \cdot \log 3/2 = 0.9183$$

$$H(Y) = p(y=0) \cdot \log \frac{1}{p(y=0)} + p(y=1) \cdot \log \frac{1}{p(y=1)}$$

$y: 0 \to 1/6$
$\quad = 0.65$
$y: 1 \to 5/6$

$$H(X|Y) = \underbrace{H(X/Y=0)}_{0} \cdot \overset{1/6}{p(y=0)} + \underbrace{H(X/Y=1)}_{1} \cdot \overset{5/6}{p(y=1)}$$

$p(x=0/Y=1) = 1/5$

$p(x=1/Y=1) = 4/5$

$$\frac{5}{6} \cdot \left[\frac{1}{5} \cdot \log 5 + \frac{4}{5} \cdot \log 5/4 \right] = 0.6016$$

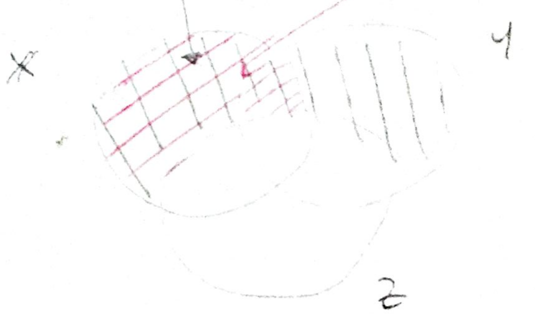
$$H(Y|X) = \underbrace{H(Y/X=0)}_{1} \cdot \underbrace{p(x=0)}_{1/3} + \underbrace{H(Y/X=1)}_{0} \cdot p(x=1)$$

$$H(Y|X) = 1/3$$

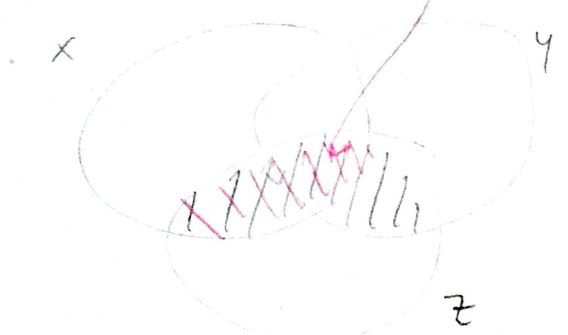
$$H(X,Y) = -\left[\frac{1}{6} \log \frac{1}{6} + \frac{1}{6} \log \frac{1}{6} + \frac{2}{3} \log \frac{2}{3} \right] = 1.2516$$

$$I(X;Y) = H(Y) - H(Y|X) \cong 0.32$$

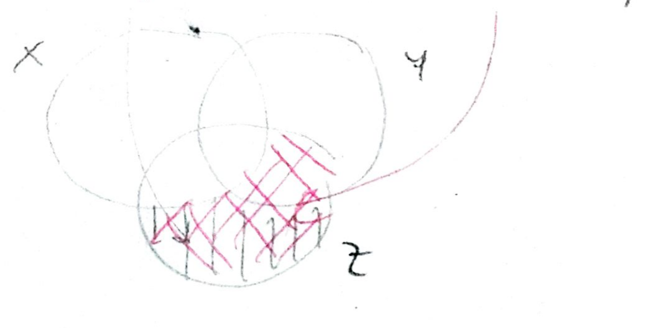
② a) $H(X,Y|Z) \geq H(X|Z)$



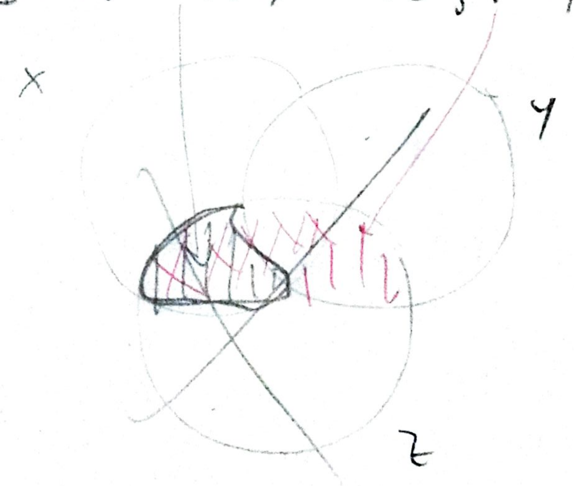
b) $I(X,Y;Z) \geq I(X;Z)$



c) $H(X,Y,Z) - H(X,Y) \leq H(X,Z) - H(X)$



d) $I(X;Z|Y) \geq I(Z;Y|X) - I(Z;Y) + I(X;Z)$



$I(X;Z|Y) \geq I(Z;Y|X) - I(Z;\cancel{X}) + I(X;Z)$

$H(Z|Y) - H(Z,Y|X) \geq H(Y|X) - H((Y|\cancel{X})|Z) - H(Y) + H(Y|Z) + H(Z) - H(Z|X)$

$\Rightarrow H(Y,Z) - H(Z|Y,X) \geq H(Y,X) - H(X) - H(Y|X,Z) + H(Y,Z) - H(Z) + H(Z) - H(Z|X) + H(X)$

$\Rightarrow H(Y|X,Z) - H(Z|Y,X) \geq H(Y,X) - H(Z,X)$

$H(Y|X,Z) + H(X,Z) \geq H(Y,X) + H(Z|Y,X)$

$H(X,Y,Z) \geq H(X,Y,Z)$

(3) a 10 — prefix property: no codeword is prefix of other

b 00 — $\{11, 110\}$ violates the prefix property.

c 11

d 110

b) uniquely decodable.

$\boxed{11}0 \to$ check codewords with prefix $\{0\} + \{00\}$

$\{00\}$ has the suffix $\{0\}$, which is not a codeword.

(4) $H(P) = \frac{1}{2}\log 2 + \frac{1}{4}\log 4 + \frac{1}{8}\log 8 + \frac{1}{16}\log 16 = 15/8$

$H(q) = \frac{1}{2}\log 2 + \frac{4}{8}\cdot\log 8 = \frac{1}{2} + \frac{3}{2} = 2$

$D(p||q) = H_p(q) - H(p)$

$= \sum_x p(x)\cdot\log\frac{1}{q(x)} = \frac{1}{2}\log 2 + \frac{1}{4}\log 8 + \frac{1}{8}\log 8 + \frac{2}{16}\cdot\log 8$

$\frac{4}{8} + \frac{6}{8} + \frac{3}{8} + \frac{3}{8} = 2$

$D(p||q) = 2 - \frac{15}{8} = \boxed{\frac{1}{8}}$

$D(q||p) = H_q(p) - H(q)$

$\sum_x q(x)\cdot\log\frac{1}{p(x)} = \frac{1}{2}\log 2 + \frac{1}{8}\log 4 + \frac{1}{8}\log 8 + \frac{1}{8}\log 16 + \frac{1}{8}\log 16$

$\frac{4}{8} + \frac{2}{8} + \frac{3}{8} + \frac{4}{8} + \frac{4}{8} = \frac{17}{8}$

$17/8 - 2 = \boxed{1/8}$

b) prove the optimality of C_1 and C_2 on X under p and q respectively.

$$\frac{1}{2}\cdot 1+\frac{1}{4}\cdot 2+\frac{1}{8}\cdot 3+\frac{1}{16}\cdot 4+\frac{1}{16}\cdot 4=1.875$$

expected codeword length of C_1 under p.

$$\frac{1}{2}+\frac{3}{8}+\frac{3}{8}+\frac{3}{8}+\frac{3}{8}=2$$ → expected codeword length of C_2 under q.

$1.875 = H(p)$
$2 = H(q)$

C_1 and C_2 are optimal for p and q respectively.

c) avg. codeword length of C_2 with prob. of p.

$$\frac{1}{2}\cdot 1+\left(\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{16}\right)\cdot 3=2$$

$H(p) = 15/8$

the difference is $\underline{1/8}$

d) avg. codeword length of C_1 with prob. of q.

$$\frac{1}{2}\cdot 1+\frac{1}{8}\cdot(2+3+4+4)=\frac{13}{8}+\frac{1}{2}=\frac{17}{8}$$

$H(q)=2$

the difference is $\underline{1/8}$

Q5

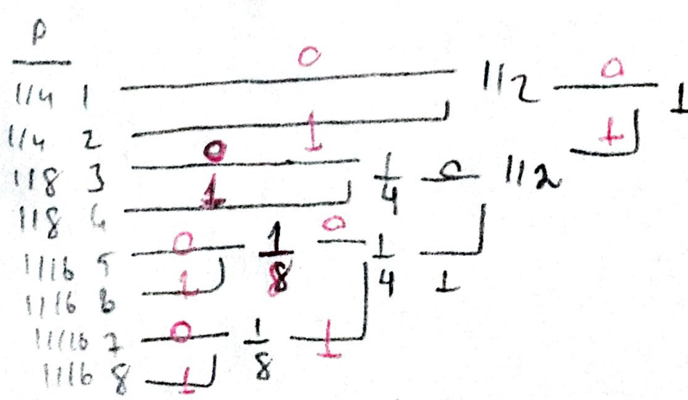
a) $p=\{1/2, 1/4, 1/4\}$

b) no. (110) can be replaced by (11) and gives shorter avg length for any p-distribution.

c) no. $C=\{0,1\}$ gives shorter avg codeword length

Q6

a)

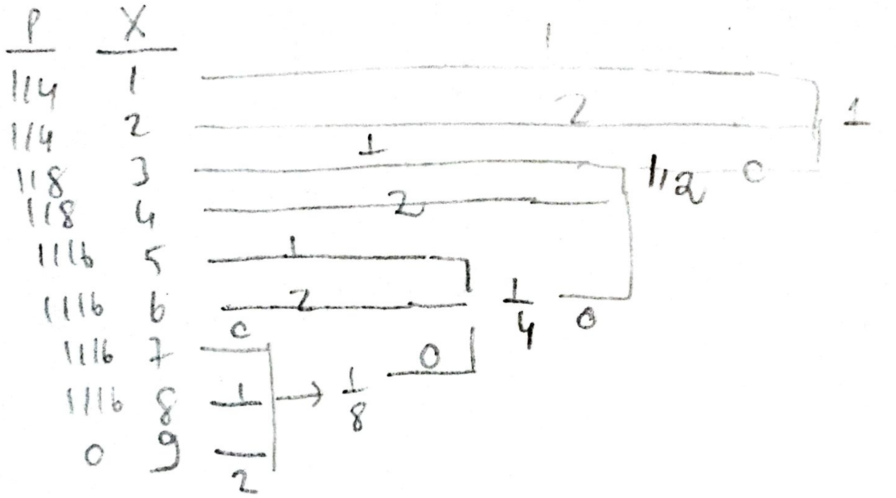


codeword	
00	1
01	2
100	3
101	4
1100	5
1101	6
1110	7
1111	8

b) avg length $2\times\left(\frac{1}{4}\cdot 2\right)+2\times\left(\frac{1}{8}\times 3\right)+4\times\left(\frac{1}{16}\cdot 4\right)=11/4$

$$H(X)=2\cdot\frac{1}{4}\log 4+2\cdot\frac{1}{8}\log 8+4\cdot\frac{1}{16}\log 16=11/4$$

c) Huffman 3ary code



X	codewords
1	1
2	2
3	01
4	02
5	001
6	002
7	0000
8	0001
9	0002 — p=0

Q7

$$h(X)=-\int_0^\infty (\lambda e^{-\lambda x})\cdot\log(\lambda e^{-\lambda x})\,dx \qquad \int_0^\infty \lambda e^{-\lambda x}dx=1$$

$$h(X)=-\int_0^\infty \log(\lambda)\,\lambda e^{-\lambda x}dx-\int_0^\infty(-\lambda x)\,\lambda e^{-\lambda x}dx \qquad E[X]=\int_0^\infty x\cdot\lambda e^{-\lambda x}dx=\frac{1}{\lambda}$$

$$=-\log\lambda-(-\lambda)E[X]=-\log\lambda-(-\lambda)\cdot\frac{1}{\lambda}=1-\log\lambda$$

b) $$h(X)=-\int_{-\infty}^{\infty} f(x)\cdot\log\left(\frac{\lambda}{2}e^{-\lambda|x|}\right)dx \qquad E[X]=\int_{-\infty}^{+\infty} x\cdot\frac{1}{2}\lambda e^{-\lambda|x|}dx$$

$$=-\int_{-\infty}^{\infty} f(x)\log\left(\frac{\lambda}{2}\right)dx-\int_{-\infty}^{\infty} f(x)\cdot(-\lambda|x|)\,dx \qquad 1=\int_{-\infty}^{+\infty}\frac{1}{2}\lambda e^{-\lambda|x|}dx$$

$$=-\log\left(\frac{\lambda}{2}\right)-\int_{-\infty}^{+\infty}\frac{\lambda}{2}e^{-\lambda|x|}\cdot(-\lambda)|x|\,dx \qquad \int_0^\infty x e^{-\lambda x}dx=\frac{1}{\lambda^2}$$

$$=-\log\left(\frac{\lambda}{2}\right)+2\cdot\int_0^\infty\frac{\lambda}{2}e^{-\lambda x}(\lambda x)\,dx$$

$$=-\log\left(\frac{\lambda}{2}\right)+2\cdot\frac{\lambda}{2}\cdot\lambda\cdot\frac{1}{\lambda^2}=1-\log\left(\frac{\lambda}{2}\right)$$

Gaussian entropy

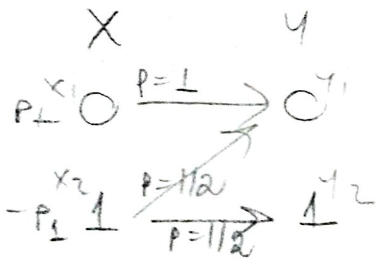
$$h(x) = -\int_{-\infty}^{+\infty} f(x)\cdot \log\left(\frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}\right) dx$$

$$h(x) = -\int_{-\infty}^{+\infty} f(x)\cdot \log\left(\frac{1}{\sqrt{2\pi\sigma^2}}\right) - \int_{-\infty}^{+\infty} f(x)\cdot\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) dx$$

$$= \frac{1}{2}\log(2\pi\sigma^2) + \frac{1}{2\sigma^2}\cdot\sigma^2 = \frac{1}{2}\cdot(\log(2\pi e\sigma^2))$$

therefore $\underbrace{h(x_1+x_2)}_{\text{Gaussian with } N(\mu_1+\mu_2,\ \sigma_1^2+\sigma_2^2)} = \frac{1}{2}\log(2\pi e(\sigma_1^2+\sigma_2^2))$

G8



$$I(X;Y) = H(Y) - H(Y|X)$$

$$H(Y) =$$

$H(Y|X=0) = 0$

$H(Y|X=1) = 1$

$H(Y|X) = p_2$

$p(y_1) = p_1 + \frac{p_2}{2}$ $\quad p(y_2) = \frac{p_2}{2}$

$= 1 - \frac{p_2}{2}$

$$\left(1-\frac{p_2}{2}\right)\log_2\left(1-\frac{p_2}{2}\right) + \frac{p_2}{2}\log\frac{p_2}{2} + p_2 = I$$ | take derivative w.r.t. p_2

$$-\frac{1}{2}\log_2\left(1-\frac{p_2}{2}\right) - \frac{1}{2}\left(1-\frac{p_2}{2}\right)\cdot\frac{1}{\ln 2}\cdot\frac{1}{1-\frac{p_2}{2}} + \frac{1}{2}\log\frac{p_2}{2} + \frac{1}{2}\frac{p_2}{2}\frac{1}{\ln 2}\frac{1}{p_2/2} + 1$$

$$-\frac{1}{2}\log_2\left(1-\frac{p_2}{2}\right)+\frac{1}{2}\log_2\left(\frac{p_2}{2}\right)+1=0$$

$$\frac{1}{2}\log \frac{p_2}{2}\cdot\frac{2}{2-p_2} = -1 \rightarrow \log_2 \frac{p_2}{2-p_2} = -2$$

$$\frac{1}{4} = \frac{p_2}{2-p_2}$$

$$2-p_2 = 4p_2$$

$$5p_2 = 2$$

$$p_2 = 2/5$$

$$p_1 = 3/5$$

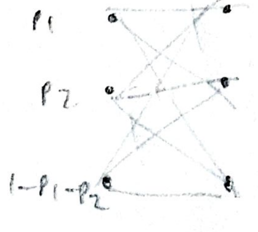
maximizes $I(X,Y)$

back to $I(X;Y) = H(Y) - H(Y|X)$

$$H(Y) = -\left[\frac{1}{5}\log\frac{1}{5}+\frac{4}{5}\log\frac{4}{5}\right] = 0.7219$$

$$H(Y) - H(Y|X) = 0.7219 - 0.4 = 0.3219$$

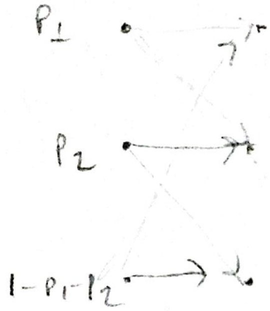
9 a) X Y



$$H(Y) = H\left(\left\{\frac{1}{3},\frac{1}{3},\frac{1}{3}\right\}\right) = \log 3$$

$$H(Y|X) = \log 3$$

$$I(Y;X) = \log 3 - \log 3 = 0$$



$I(X;Y) = H(Y) - H(Y|X)$

$H(Y) = H\left(\frac{1-p_2}{2}, \frac{p_1+p_2}{2}, \frac{1-p_1}{2}\right)$

$H(Y|X=c) = 1$

$H(Y|X) = p_1 + p_2 + 1 - p_1 - p_2 = 1$

$I(X;Y) = H(Y) - 1$

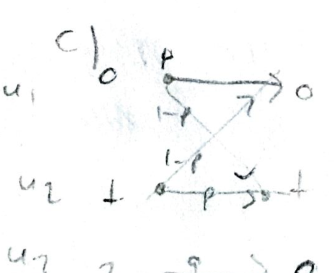
$H(Y)$ is maximized if $\underbrace{\frac{1-p_1}{2} = \frac{p_1+p_2}{2} = \frac{1-p_2}{2}}$

$\Downarrow$

$p_1 = p_2 = \frac{1}{3}$

hence $I(X;Y) = \log 3 - 1$

c)



effective alphabet size

$C_1 = 1 - H(p)$ $\quad 2^C = 2^{C_1} + 2^{C_2}$

$C = \log\left(2^{1-H(p)} + 2^{1-H(q)}\right)$

$C_2 = 1 - H(q)$ $\quad C = 1 + \log\left(2^{-H(p)} + 2^{-H(q)}\right)$

d)

$\underbrace{\longrightarrow}_{1}$

$C = H(Y) - H(Y|X)$
$= 0 - 0$
$= 0$

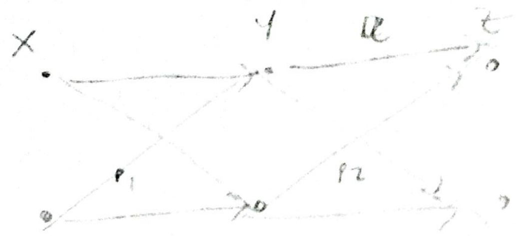
$2^C = 2^{C_1} + 2^{C_2}$

$C = \log(2^{C_1} + 2^{C_2})$

$C = \log(2^{C_1} + 1)$

$C = \log(2^{H(p)} + 1)$

Q10



$$P = P_1 \cdot P_2$$

$$P = \begin{bmatrix} 1-p & p \\ p & 1-p \end{bmatrix}^2$$

$$= \begin{bmatrix} 1-2p(1-p) & 2p(1-p) \\ 2p(1-p) & 1-2p(1-p) \end{bmatrix}$$

$$C(BSC) = 1 - H(p') = 1 - H(2p(1-p))$$