MATHS 120 – Assignment 2

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Contents

Question One

Let $S_1 \subseteq \mathbb{R}^3$ be the set of solutions to the following linear system:

$$3x + 2y + z = 10$$
$$x - y - z = -4$$

and $S_2 \subseteq \mathbb{R}^3$ be the set of solutions to this one:

$$x + y + z = 6$$
$$3x - 2y + z = 2$$

(a) What is S_1?

$$3x + 2y + z = 10$$
$$x - y - z = -4$$
$$\Downarrow$$
$$3x + 2y + z = 10$$
$$4x + y = 6$$
$$\Downarrow$$
$$3x + 2y + z = 10$$
$$5x - z = 2$$

Now we have an equation for both y and z in terms of x.
Let t be a parameter which can take any real value
Let $x = t$
$\Rightarrow 4t + y = 6$
$\Rightarrow y = 6 - 4t$
$\Rightarrow 5t - z = 2$
$\Rightarrow z = 5t - 2$
Since we have all 3 variables in terms of t we can find S_1 in terms of t.
$\therefore S_1 = \begin{bmatrix} t \\ 6-4t \\ 5t-2 \end{bmatrix}$

(b) What is S_2?

$$x + y + z = 6$$
$$3x - 2y + z = 2$$
$$\Downarrow$$
$$x + y + z = 6$$
$$2x - 3y = -4$$
$$\Downarrow$$
$$x + y + z = 6$$
$$5x + 3z = 14$$

Now we have an equation for both y and z in terms of x.
Let t be a parameter which can take any real value
Let $x = t$

$\Rightarrow 2t - 3y = -4$

$\Rightarrow y = \frac{2t+4}{3}$

$\Rightarrow 5t + 3z = 14$

$\Rightarrow z = \frac{14-5t}{3}$

Since we have all 3 variables in terms of t we can find S_2 in terms of t.

$\therefore S_2 = \begin{bmatrix} t \\ \frac{2t+4}{3} \\ \frac{14-5t}{3} \end{bmatrix}$

(c) What is $S_1 \cap S_2$?

The intersection between S_1 and S_2 will be the points that satisfy both S_1 and S_2.

That is, $S_1 = S_2 \Rightarrow \begin{bmatrix} t \\ 6-4t \\ 5t-2 \end{bmatrix} = \begin{bmatrix} t \\ \frac{2t+4}{3} \\ \frac{14-5t}{3} \end{bmatrix}$. This gives us 3 equations,

$$\begin{gathered} t = t \\ 6 - 4t = \frac{2t+4}{3} \\ 5t - 2 = \frac{14-5t}{3} \\ \Downarrow \\ t = t \\ 18 - 12t = 2t + 4 \\ 15t - 6 = 14 - 5t \\ \Downarrow \\ t = t \\ 14 = 14t \\ 20t = 20 \end{gathered}$$

The only values for t that satisfy this system is 1. This means that $S_1 = S_2$ when $t = 1$.

$$\begin{aligned} \therefore S_1 \cap S_2 &= \begin{bmatrix} 1 \\ 6-4(1) \\ 5(1)-2 \end{bmatrix} \\ &= \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \end{aligned}$$

□

Question Two

Let V be a vector space. Recall that a subset $S \subseteq V$ is said to be a **linear subspace** of V if it satisfies the following properties:

- $0 \in S$;
- For all $u, v \in S$, we have $u + v \in S$;
- For all $u \in S$ and all scalars a, we have $au \in S$.

Consider the linear system

$$a_{11}x_1 + ... + a_{1n}x_n = b_1$$
$$\vdots$$
$$a_{m1}x_1 + ... + a_{mn}x_n = b_m$$

where $a_{ij} \in \mathbb{C}, b_i \in \mathbb{C}$ for $1 \leq i \leq m$ and $i \leq j \leq n$. Let $S \subseteq \mathbb{C}^n$ denote the set of solutions to this linear system.

(a) If $\begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix} = 0$, prove that S is a linear subspace.

If the vector $\begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix} = 0$, that means the system of equations is homogenous, that is, all equations are equal to 0.

$$a_{11}x_1 + ... + a_{1n}x_n = 0$$
$$\vdots$$
$$a_{m1}x_1 + ... + a_{mn}x_n = 0$$

Now we need to show that the solution set, S, satisfies the properties of a linear subspace.

- $\mathbf{0} \in \boldsymbol{S}$: If we substitute in $x_1 = x_2 = ... = x_n = 0$ into any of the equations, we get $0 = 0$ as a result. Since this is true, we have 0 as a solution, thus $0 \in S$.
- **For all $\boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{S}$, we have $\boldsymbol{u} + \boldsymbol{v} \in \boldsymbol{S}$**: Suppose we take two vectors u, v from S. By definition, these two vectors satisfy all equations in the system. Thus we can write

$$a_{11}u + ... + a_{1n}u = 0$$
$$\vdots$$
$$a_{m1}u + ... + a_{mn}u = 0$$

and

$$a_{11}v + ... + a_{1n}v = 0$$
$$\vdots$$
$$a_{m1}v + ... + a_{mn}v = 0$$

If we add u and v and substitute it into the system, we get

$$a_{11}(u+v)+\ldots+a_{1n}(u+v)=0$$
$$\vdots$$
$$a_{m1}(u+v)+\ldots+a_{mn}(u+v)=0$$
$$\Downarrow$$
$$a_{11}u+a_{11}v+\ldots+a_{1n}u+a_{1n}v=0$$
$$\vdots$$
$$a_{m1}u+a_{m1}v+\ldots+a_{mn}u+a_{mn}v=0$$
$$\Downarrow$$
$$(a_{11}u+\ldots+a_{1n}u)+(a_{11}v+\ldots+a_{1n}v)=0$$
$$\vdots$$
$$(a_{m1}u+\ldots+a_{mn}u)+(a_{m1}v+\ldots+a_{mn}v)=0$$
$$\Downarrow$$
$$0+0=0$$
$$\vdots$$
$$0+0=0$$
$$\Downarrow$$
$$0=0$$
$$\vdots$$
$$0=0$$

Since we get 0 = 0 for every equation in the system, we know that $u+v$ must also be a solution, and thus $u+v \in S$.

- **For all $u \in S$ and all scalars a, we have $au \in S$**: Suppose we take a vector u from S, and we have a scalar a. By definition u satisfies every equation in the system. Thus, we can write

$$a_{11}u+\ldots+a_{1n}u=0$$
$$\vdots$$
$$a_{m1}u+\ldots+a_{mn}u=0$$

If we multiply our vector u by our scalar a and substitute it into the system we get

$$a_{11}au+\ldots+a_{1n}au=0$$
$$\vdots$$
$$a_{m1}au+\ldots+a_{mn}au=0$$
$$\Downarrow$$
$$a(a_{11}u+\ldots+a_{1n}u)=0$$
$$\vdots$$
$$a(a_{m1}u+\ldots+a_{mn}u)=0$$
$$\Downarrow$$
$$a(0)=0$$
$$\vdots$$
$$a(0)=0$$
$$\Downarrow$$
$$0=0$$
$$\vdots$$
$$0=0$$

Since we get $0 = 0$ for every equation in the system, we know that au must also be a solution to the system, thus $au \in S$.

Since every property for linear subspaces is satisfied, we can say that S is a linear subspace of V when $\begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix} = 0$

(b) If $\begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix} \neq 0$, prove that S is not a linear subspace.

If the vector $\begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix} \neq 0$, that means the system of equations is non-homogenous, that is, at least one of the equations is not equal to 0.

$$\begin{gathered} a_{11}x_1 + ... + a_{1n}x_n = b_1 \\ \vdots \\ a_{m1}x_1 + ... + a_{mn}x_n = b_m \end{gathered}$$

Now we need to show that the solution set, S, does not satisfy all the properties of a linear subspace.

- $\mathbf{0} \in \boldsymbol{S}$: If we substitute in $x_1 = x_2 = ... = x_n = 0$ into any of the equations, we get $0 = b_i$, for some i. This is not true, as $b_i \neq 0$. Therefore 0 is not a solution, thus $0 \notin S$.

$\therefore$ S is not a linear subspace of V as it does not satisfy the first property of a linear subspace.

Question Three

Let $X :\equiv \{v_1, v_2, v_3, v_4\}$ where

$$v_1 :\equiv \begin{bmatrix}1\\0\\2\end{bmatrix}, v_2 :\equiv \begin{bmatrix}0\\i\\-1\end{bmatrix}, v_3 :\equiv \begin{bmatrix}0\\-3i\\1\end{bmatrix}, v_4 :\equiv \begin{bmatrix}1\\2-2i\\i\end{bmatrix}.$$

(a) Show that span(X) = $\mathbb{C}^3$.

$$\text{Let } z = \begin{bmatrix}z_1\\z_2\\z_3\end{bmatrix} \in \mathbb{C}^3$$

$$\text{Suppose } z = c_1v_1 + c_2v_2 + c_3v_3 + c_4v_4 \mid c_1, c_2, c_3, c_4 \in \mathbb{R}$$

$$\Rightarrow \begin{bmatrix}z_1\\z_2\\z_3\end{bmatrix} = c_1\begin{bmatrix}1\\0\\2\end{bmatrix} + c_2\begin{bmatrix}0\\i\\-1\end{bmatrix} + c_3\begin{bmatrix}0\\-3i\\1\end{bmatrix} + c_4\begin{bmatrix}1\\2-2i\\i\end{bmatrix}$$

$$\Rightarrow \left[\begin{array}{cccc|c}1 & 0 & 0 & 1 & z_1\\0 & i & -3i & 2-2i & z_2\\2 & -1 & 1 & i & z_3\end{array}\right]$$

$$\Rightarrow \left[\begin{array}{cccc|c}1 & 0 & 0 & 1 & z_1\\0 & i & -3i & 2-2i & z_2\\0 & -1 & 1 & -2+i & -2z_1+z_3\end{array}\right](R_3 = R_3 - 2R_1)$$

$$\Rightarrow \left[\begin{array}{cccc|c}1 & 0 & 0 & 1 & z_1\\0 & 1 & -3 & -2-2i & -iz_2\\0 & -1 & 1 & -2+i & -2z_1+z_3\end{array}\right](R_2 = -iR_2)$$

$$\Rightarrow \left[\begin{array}{cccc|c}1 & 0 & 0 & 1 & z_1\\0 & 1 & -3 & -2-2i & -iz_2\\0 & 0 & -2 & -4-i & -2z_1-iz_2+z_3\end{array}\right](R_3 = R_3 + R_2)$$

$$\Rightarrow \left[\begin{array}{cccc|c}1 & 0 & 0 & 1 & z_1\\0 & 1 & -3 & -2-2i & -iz_2\\0 & 0 & 1 & 2+\frac{i}{2} & z_1+\frac{i}{2}z_2-\frac{1}{2}z_3\end{array}\right]\left(R_3 = -\tfrac{1}{2}R_3\right)$$

$$\Rightarrow \left[\begin{array}{cccc|c}1 & 0 & 0 & 1 & z_1\\0 & 1 & 0 & 4-\frac{1}{2}i & 3z_1+\frac{i}{2}z_2-\frac{3}{2}z_3\\0 & 0 & 1 & 2+\frac{i}{2} & z_1+\frac{i}{2}z_2-\frac{1}{2}z_3\end{array}\right](R_2 = R_2 + 3R_3)$$

Hence we have a free variable, c_4, and leading variables c_1, c_2, c_3.

Let $t = c_4$

$c_3 + \left(2 + \frac{i}{2}\right)t = z_1 + \frac{i}{2}z_2 - \frac{1}{2}z_3$

$c_3 \qquad\qquad = z_1 + \frac{i}{2}z_2 - \frac{1}{2}z_3 - 2t - \frac{i}{2}t$

$c_2 + \left(4 - \frac{1}{2}i\right)t = 3z_1 + \frac{i}{2}z_2 - \frac{3}{2}z_3$

$c_2 \qquad\qquad = 3z_1 + \frac{i}{2}z_2 - \frac{3}{2}z_3 - 4t + \frac{i}{2}t$

$c_1 + t = z_1$

$c_1 \quad = z_1 - t$

Hence

$$\begin{bmatrix}z_1\\z_2\\z_3\end{bmatrix} = (z_1 - t)\begin{bmatrix}1\\0\\2\end{bmatrix} + \left(3z_1 + \tfrac{i}{2}z_2 - \tfrac{3}{2}z_3 - 4t + \tfrac{i}{2}t\right)\begin{bmatrix}0\\i\\-1\end{bmatrix} + \left(z_1 + \tfrac{i}{2}z_2 - \tfrac{1}{2}z_3 - 2t - \tfrac{i}{2}t\right)\begin{bmatrix}0\\-3i\\1\end{bmatrix} + t\begin{bmatrix}1\\2-2i\\i\end{bmatrix}$$

for any $t \in \mathbb{C}$, thus X spans $\mathbb{C}^3$.

(b) Is X linearly independent? If not, pick one of the vectors in X and write it as a linear combination of the other vectors in X.

X is not linearly independent because there is a free variable, c_4, from the solution in a. This means we are able to obtain the zero vector in a non-unique way.

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & i & -3i & 2-2i \\ 2 & -1 & 1 & i \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & i & -3i & 2-2i \\ 0 & -1 & 1 & -2+i \end{array}\right] (R_3 = R_3 - 2R_1)$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & -3 & -2-2i \\ 0 & -1 & 1 & -2+i \end{array}\right] (R_2 = -iR_2)$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & -3 & -2-2i \\ 0 & 0 & -2 & -4-i \end{array}\right] (R_3 = R_3 + R_2)$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & -3 & -2-2i \\ 0 & 0 & 1 & 2+\frac{i}{2} \end{array}\right] \left(R_3 = -\frac{1}{2}R_3\right)$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 4-\frac{i}{2} \\ 0 & 0 & 1 & 2+\frac{i}{2} \end{array}\right] (R_2 = R_2 + 3R_3)$$

Therefore, we have that $c_1 = 1, c_2 = 4 - \frac{i}{2}, c_3 = 2 + \frac{i}{2}$.
This means that
$v_4 = c_1v_1 + c_2v_2 + c_3v_3$

$v_4 = 1v_1 + \left(4 - \frac{i}{2}\right)v_2 + \left(2 + \frac{i}{2}\right)v_3$

Thus v_4 is a linear combination of the other vectors, v_1, v_2, v_3

(c) Find a subset $B \subseteq X$ which is a basis for $\mathbb{C}^3$.
To find the basis, we only have to look at v_1, v_2, v_3, as v_4 is a free variable. That is,
$B :\equiv \{v_1, v_2, v_3\} \subseteq X$.
To prove B is a basis for X, we need to show that v_1, v_2, v_3 spans X or is linearly independent as B contains 3 vectors and is in $\mathbb{C}^3$ (By Corollary 2.2.24). However, we know v_1, v_2, v_3 is linearly independent as the only linear combination which yields the zero vector is the trivial one (Definition 2.2.15). This is because each equations pivot is exactly 1, the pivots are the only non-zero entry in the column (reduced echelon form), and there is no free variables. This means there is no way to construct the zero vector besides the trivial method and thus B is a basis for $\mathbb{C}^3$.

(d) Express an arbitrary vector $\begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{C}^3$ as a linear combination of the basis B that you found above.

From the coefficents from part (a), we have

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = x \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + \left(3x + \frac{i}{2}y - \frac{3}{2}z\right) \begin{bmatrix} 0 \\ i \\ -1 \end{bmatrix} + \left(x + \frac{i}{2}y - \frac{1}{2}z\right) \begin{bmatrix} 0 \\ -3i \\ 1 \end{bmatrix}$$

as an expression for an arbitrary vector in $\mathbb{C}^3$. $(x = z_1, y = z_2, z = z_3)$

Question Four

Consider the following points in $\mathbb{R}^3$

$$\mathrm{r}_0 :\equiv \begin{bmatrix}-1\\1\\2\end{bmatrix}, \mathrm{r}_1 :\equiv \begin{bmatrix}0\\1\\2\end{bmatrix}, \mathrm{R}_0 :\equiv \begin{bmatrix}1\\-1\\3\end{bmatrix}, \mathrm{R}_1 :\equiv \begin{bmatrix}-1\\0\\3\end{bmatrix}, \mathrm{R}_2 :\equiv \begin{bmatrix}1\\-1\\-2\end{bmatrix}.$$

(a) Find a parametric equation for the line L containing the points r_0 and r_1.
By Example 2.2.30, we know that if we have two points $u, w \in \mathbb{R}^n$, then $\{\lambda(u-w)+w : \lambda \in \mathbb{R}\}$ represents a unique line through u and w.
Therefore, the line L containing points r_0 and r_1 can be expressed as

$$\begin{aligned}\{\lambda(\mathrm{r}_0-\mathrm{r}_1)+\mathrm{r}_1 : \lambda \in \mathbb{R}\} &= \left\{\lambda\left(\begin{bmatrix}-1\\1\\2\end{bmatrix}-\begin{bmatrix}0\\1\\2\end{bmatrix}\right)+\begin{bmatrix}0\\1\\2\end{bmatrix} : \lambda \in \mathbb{R}\right\}\\ &= \left\{\lambda\begin{bmatrix}-1\\0\\0\end{bmatrix}+\begin{bmatrix}0\\1\\2\end{bmatrix} : \lambda \in \mathbb{R}\right\}\\ &= \left\{\begin{bmatrix}-\lambda\\1\\2\end{bmatrix} : \lambda \in \mathbb{R}\right\}\end{aligned}$$

(b) Find a parametric equation for the plane P containing the points $\mathrm{R}_0, \mathrm{R}_1, \mathrm{R}_2$.
By Example 2.2.32, we know that if we have three points $u, v, w \in \mathbb{R}^n$, then
$\{\lambda(u-w)+\mu(v-w)+w : \lambda, \mu \in \mathbb{R}\}$ represents a unique plane through all three points (provdided they are not colinear).
Therefore, the plane P can be represented as

$$\{\lambda(\mathrm{R}_0-\mathrm{R}_2)+\mu(\mathrm{R}_1-\mathrm{R}_2)+\mathrm{R}_2 : \lambda, \mu \in \mathbb{R}\}$$

$$= \left\{\lambda\left(\begin{bmatrix}1\\-1\\3\end{bmatrix}-\begin{bmatrix}1\\-1\\-2\end{bmatrix}\right)+\mu\left(\begin{bmatrix}-1\\0\\3\end{bmatrix}-\begin{bmatrix}1\\-1\\-2\end{bmatrix}\right)+\begin{bmatrix}1\\-1\\-2\end{bmatrix} : \lambda, \mu \in \mathbb{R}\right\}$$

$$= \left\{\lambda\begin{bmatrix}0\\0\\5\end{bmatrix}+\mu\begin{bmatrix}-2\\1\\5\end{bmatrix}+\begin{bmatrix}1\\-1\\-2\end{bmatrix} : \lambda, \mu \in \mathbb{R}\right\}$$

$$= \left\{\begin{bmatrix}-2\mu+1\\\mu-1\\5\lambda+5\mu-2\end{bmatrix} : \lambda, \mu \in \mathbb{R}\right\}$$

(c) Find the set of points where L and P intersect.
L and P will intersect when they are equal.

$$L = P$$

$$\left\{\begin{bmatrix}-\lambda_1\\1\\2\end{bmatrix} : \lambda_1 \in \mathbb{R}\right\} = \left\{\begin{bmatrix}-2\mu+1\\\mu-1\\5\lambda_2+5\mu-2\end{bmatrix} : \lambda_2, \mu \in \mathbb{R}\right\}$$

This gives us 3 equations:

$$-\lambda_1 = -2\mu + 1 \Rightarrow \lambda_1 = 2\mu - 1$$
$$1 = \mu - 1 \Rightarrow \mu = 2$$
$$2 = 5\lambda_2 + 5\mu - 2$$
$$\Downarrow$$
$$\lambda_1 = 2(2) - 1$$
$$2 = 5\lambda_2 + 5(2) - 2$$
$$\Downarrow$$
$$\lambda_1 = 3$$
$$\lambda_2 = -\tfrac{6}{5}$$

Therefore the plane P and line L will intercept at $\begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix}$.