COMPSCI 120 – Assignment 4

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Contents

Question 2

(a) Prove that $|x + 3| + |x - 7| \geq 10$ for every real number x.

For this enequality, we will have 3 cases. One when $x \geq 7$, another when $x \leq -3$, and another when $-3 < x < 7$.

For $x \geq 7$, we know that both $x + 3$ and $x - 7$ will be positive. This means that:

$$
\begin{aligned}
x + 3 + x - 7 &\geq 10 \\
2x - 4 &\geq 10 \\
2x &\geq 14 \\
x &\geq 7
\end{aligned}
$$

This is consistent with $x \geq 7$, so it is true for this case.

For $x \leq -3$, we know that both $x + 3$ and $x - 7$ will be negative. This means that:

$$
\begin{aligned}
-(x + 3) - (x - 7) &\geq 10 \\
-x - 3 - x + 7 &\geq 10 \\
-2x + 4 &\geq 10 \\
-2x &\geq 6 \\
x &\leq -3
\end{aligned}
$$

This is consistent with $x \leq -3$, so it is true for this case.

For $-3 < x < 7$, we know that $x + 3$ will be positive and $x - 7$ will be negative. This means that:

$$
\begin{aligned}
x + 3 - (x - 7) &\geq 10 \\
x + 3 - x + 7 &\geq 10 \\
10 &\geq 10
\end{aligned}
$$

This is always true, so we know that it is true for this case.

Since we have proved the inequality holds for all 3 cases, we know that the inequality will hold for all real numbers x.

(b) Prove that if all the vertices in a graph G have odd degree x, then the number of edges in G is a multiple of x.

Suppose our graph G has n vertices.
If all of the vertices are of the same (odd) degree, x, then the sum of the degrees of all the vertices is $n \cdot x$.

Claim 5.1 states that "Take any graph G. Then, the sum of the degrees of all of the vertices in G is always two times the number of edges in G.".

By Claim 5.1, the number of edges in G is given by $\frac{n \cdot x}{2}$.

Claim 7.3 states that "If G is a graph, then G must have an even number of vertices with odd degrees; that is, it is impossible to have a graph G with an odd number of vertices with odd degrees.".

Since every vertex of G has an odd degree, by Claim 7.3 we then know that there are an even amount of vertices, that is, n is even. This means that $n \cdot x$ will also be even, thus the number of edges, given by $\frac{n \cdot x}{2}$ is an integer. Therefore, the number of edges will be a multiple of x.

(c) Let G be a connected graph with n vertices, where $n > 1$. Prove that if no vertex in G has degree 1, then G has at least n edges.

We know G is a connected graph with n vertices, where $n > 1$ and every vertex has degree 1.
If G is connected and no vertex has degree 1, every vertex must be of at least degree 2.
Since every vertex is of at least degree 2, we can conclude that the sum of the degrees of all the vertices is at least $2 \cdot n$.
Claim 5.1 states that "Take any graph G. Then, the sum of the degrees of all of the vertices in G is always two times the number of edges in G.".
By Claim 5.1, we can then conclude that the number of edges is half of the sum of the degrees of all the vertices in G. Since we know the sum of the degrees is at least $2 \cdot n$, the number of edges is at least $\frac{2 \cdot n}{2} = n$.
Therefore, if G is a connected graph with n vertices and no vertex in G has degree 1, then G has at least n edges.

Question 3

(a) Prove that there does not exist a smallest positive rational number.

Assume there is a smallest positive rational number, n.
Since n is rational, $n = \frac{p}{q}$ where $p, q \in \mathbb{N} \mid q \neq 0$
Let $m = \frac{p}{q+1}$
Since we are assuming n to be the smallest rational number, we assume $n < m$.

$$\Rightarrow \frac{p}{q} < \frac{p}{q+1}$$
$$\Rightarrow p(q+1) < pq$$
$$\Rightarrow pq + p < pq$$
$$\Rightarrow p < 0$$

However we stated before that $p \in \mathbb{N}$, thus $p \nless 0$, meaning we have a contradiction and our assumption that $n < m$ is wrong. That means that m is smaller than n (as they obviously cannot be equal), and thus there is no smallest positive rational number.

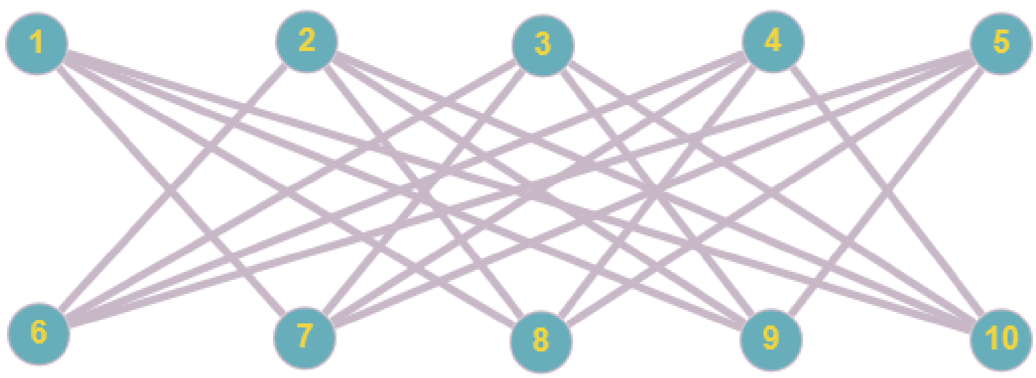
(b) Consider a tree, T . Prove that any edge added to T must produce a cycle in T.

Suppose you have a tree T, and a graph H, which is T with any edge added.
By theorem 6.3, we have that a tree T is connected and has $n - 1$ edges.
Since our graph H is our tree T with any edge added, it means that H has n edges and is connected, and by theorem 6.3 this means that H cannot be a tree.
By theorem 6.2, this them means that there is not exactly one path between any two vertices in H.
Since we know H is connected, there must be at least more than one way to reach at 1 vertex in H, thus meaning H contains a cycle.

(c) Prove or disprove the following statement:

There is a graph G with 10 vertices, in which every vertex has degree 4.

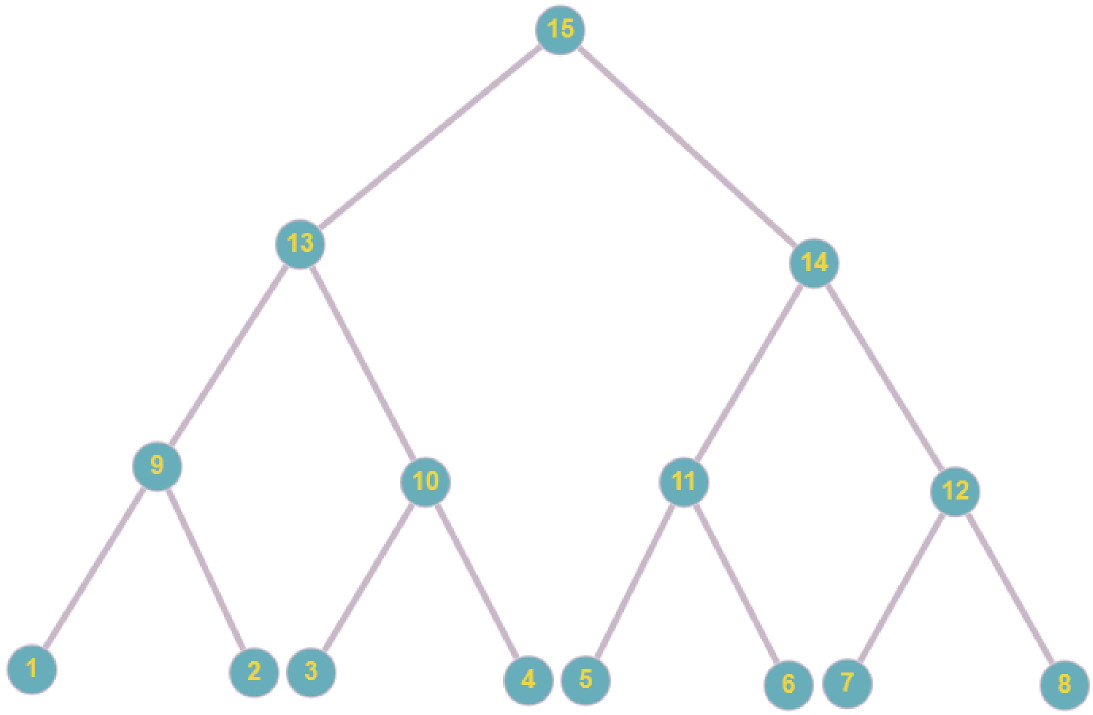
Create 2 rows of 5 nodes and connect every node to all 5 opposing nodes except the one directly opposite. This gives a graph with 10 vertices, in which every vertex is connected to 4 other vertices, and thus is of degree 4.



(d) Prove or disprove the following statement:

There is a binary tree T of height 3 with 9 leaves.

To find the maximum number of leaves in a binary tree of height 3, we can add the maximum number of children (2) until we reach height 3. This will produce a (full) binary tree of height 3 containing $2^3 = 8$ leaves. Thus it is impossible to have a binary tree of height 3 with 9 leaves.



Question 4

(a) Prove that $9^{2n} - 1$ is divisible by 80 for any positive integer n.

Base case: $n = 1$
$9^{2(1)} - 1 = 9^2 - 1$
$= 81 - 1 = 80$. Thus the equation is true for the base case.

Assume $9^{2n} - 1$ is divisible by 80 for some $n = k$, where k is a positive integer.
$\Rightarrow 9^{2k} - 1$ is divisible by 80
$\Rightarrow 9^{2k} - 1 = 80a$, where $a \in \mathbb{N}$.

We know that $9^{2n} - 1$ is divisible by 80 for any positive integer n if we can prove that this equation holds for some $n = k + 1$ where k is also a positive integer.

$n = k + 1$
$\Rightarrow 9^{2(k+1)} - 1 = 9^{2k+2} - 1$
$= 81 \cdot 9^{2k} - 1$
From our inductive hypothesis,
$81 \cdot 9^{2k} - 1 = 81 \cdot (80a + 1) - 1$
$= 81 \cdot 80a + 81 - 1$
$= 81 \cdot 80a + 80$
$= 80(81a + 1)$
Therefore, we have $9^{2(k+1)} - 1$ is a multiple of 80 for $n = k + 1$. This means that it is divisible by 80, thus we can conclude that $9^{2n} - 1$ is divisible by 80 for any positive integer n.

(b) Let $a_1 = 1$ and, for every integer $n > 1$, define an by the recurrence relation:

$$a_{n+1} = (n + 1)^2 - a_n$$

Prove that $a_n = \frac{n(n+1)}{2}$ for every positive integer n.

Base case: $n = 1$
$a_1 = \frac{1(1+1)}{2}$
$= \frac{1(2)}{2}$
$= 1$. Thus it holds for the base case.

Assume $a_n = \frac{n(n+1)}{2}$ for some $n = k$, where $k \geq 1$.
$\Rightarrow a_k = \frac{k(k+1)}{2}$
We also know that $a_{k+1} = (k + 1)^2 - a_k$.
$\Rightarrow a_{k+1} = (k + 1)^2 - \frac{k(k+1)}{2}$

We know that $a_n = \frac{n(n+1)}{2}$ for positive integer n if we can prove that this equation holds for some $n = k + 1$ where k is also a positive integer.

$n = k + 1$
$\Rightarrow a_{(k+1)+1} = ((k + 1) + 1)^2 - a_{k+1}$
$\Rightarrow a_{k+2} = (k + 2)^2 - a_{k+1}$
From our inductive hypothesis,

$$\begin{aligned}
a_{k+2} &= (k+2)^2 - \left((k+1)^2 - \tfrac{k(k+1)}{2}\right) \\
&= (k+2)^2 - (k+1)^2 + \tfrac{k(k+1)}{2} \\
&= k^2 + 4k + 4 - k^2 - 2k - 1 + \tfrac{k^2}{2} + \tfrac{k}{2} \\
&= \tfrac{k^2}{2} + \tfrac{5k}{2} + 3 \\
&= \tfrac{k^2+5k+6}{2} \\
&= \tfrac{(k+3)(k+2)}{2} \\
&= \tfrac{(k+2)((k+2)+1)}{2}
\end{aligned}$$

Therefore, the equation holds for $n = k + 1$, meaning that $a_n = \frac{n(n+1)}{2}$ for every positive integer n.

(c) Prove that $n! > n^2$ for $n \geq 4$.

Base case: $n = 4$
$4! > 4^2$

$24 > 16$. Thus it holds for the base case

Assume $n! > n^2$ for some $n = k$, where $k \geq 4$
$\Rightarrow k! > k^2$

We know that $n! > n^2$ for $n \geq 4$ if we can prove that this inequality holds for some $n = k + 1$ where $k \geq 4$.

$n = k + 1$

$\Rightarrow (k+1)! > (k+1)^2$

$\Rightarrow (k+1)k! > (k+1)^2$

From our inductive hypothesis, we know that $(k+1)k! > (k+1)k^2$.

Assume $k^2 > k + 1$
$\Rightarrow k > 1 + \frac{1}{k}$
Since $k \geq 4$, $\Rightarrow \frac{1}{k} \leq \frac{1}{4}$
$\therefore 1 + \frac{1}{k} \leq 1 + \frac{1}{4} < 2$
$\Rightarrow k > 2$
As $k \geq 4$, we know this is true, meaning $k^2 > k + 1$.

$\Rightarrow (k+1)k! > (k+1)k^2 > (k+1)^2$

$\Rightarrow (k+1)k^2 > (k+1)^2$

$\Rightarrow k^2 > k + 1$
We know this is true from the above proof, thus it holds that $(k+1)! > (k+1)^2$, and therefore $n! > n^2$ for all $n \geq 4$.