MATHS 120 – Assignment 3

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Question One

Consider the subset $B := \{\boldsymbol{b}_1, \boldsymbol{b}_2, \boldsymbol{b}_3\} \subset \mathbb{C}^3$

$$\boldsymbol{b}_1 := \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \quad \boldsymbol{b}_2 := \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}, \quad \boldsymbol{b}_3 := \begin{bmatrix} 2i \\ 0 \\ 1 \end{bmatrix}.$$

(a) *Prove that* B *is a basis for* $\mathbb{C}^3$.

We know by Corollary 2.2.24, 3 vectors will be a basis for $\mathbb{C}^3$ if they are either linearly independent or span $\mathbb{C}^3$.

By definition 2.5.2, we also know that if the determinant of a matrix $= 0$, then at least two of the columns are equal.

Let $Z := \begin{bmatrix} -1 & 0 & 2i \\ 0 & -1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$

Now, if the determinant of Z is non-zero, then we can be sure that the b_1, b_2, b_3 will be linearly independent, and thus B is a basis for $\mathbb{C}^3$.

By Lemma 2.5.7:

$$\begin{aligned} \det(Z) &= \det\left(\begin{bmatrix} -1 & 0 & 2i \\ 0 & -1 & 0 \\ 1 & 1 & 1 \end{bmatrix}\right) \\ &= \det\left(\begin{bmatrix} -1 & 0 & 2i \\ 0 & -1 & 0 \\ 0 & 1 & 1+2i \end{bmatrix}\right) \\ &= \det\left(\begin{bmatrix} -1 & 0 & 2i \\ 0 & -1 & 0 \\ 0 & 0 & 1+2i \end{bmatrix}\right) \end{aligned}$$

By Corollary 2.5.17, we can see the determinant is $1 + 2i$, thus the vectors are linearly independent, and B is a basis for $\mathbb{C}^3$.

(b) *Let* $l : B \to \mathbb{C}^2$ *be the function defined by*

$$l(\boldsymbol{b}_1) := \begin{bmatrix} i \\ -1 \end{bmatrix}, \quad l(\boldsymbol{b}_2) := \begin{bmatrix} 0 \\ 1-2i \end{bmatrix}, \quad l(\boldsymbol{b}_3) := \begin{bmatrix} 2 \\ -i \end{bmatrix},$$

and let $L : \mathbb{C}^3 \to \mathbb{C}^2$ *be its linear extension. Find an explicit formula for* $L(\boldsymbol{z})$*, where* $\boldsymbol{z} := \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} \in \mathbb{C}^3$ *is arbitrary.*

We can define $L(z)$ as:

$$L\left(z = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}\right) := \frac{ix + 2y + 2z}{3}\begin{bmatrix} i \\ -1 \end{bmatrix} - y\begin{bmatrix} 0 \\ 1-2i \end{bmatrix} + \frac{-ix + y + z}{3}\begin{bmatrix} 2 \\ -i \end{bmatrix}$$

(c) *Find the standard matrix of* L.

The standard matrix for L one where the columns are the images of the basis vectors under L.

$$\therefore [L] = \begin{bmatrix} \frac{-1-2i}{3} & \frac{2i+2}{3} & \frac{2i+2}{3} \\ \frac{-1-i}{3} & \frac{-5-5i}{3} & \frac{-2-i}{3} \end{bmatrix}$$

(d) *Let* $T : \mathbb{C}^2 \to \mathbb{C}^2$ *be the linear map whose standard matrix is*

$$[T] = \begin{bmatrix} -i & 1 \\ 0 & 2-2i \end{bmatrix}$$

Find the standard matrix of $T \circ L$.

The standard matrix of the composition of T and L is the product of the standard matrices of T and L.

Question Two

Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear map map. For each of the follow, either prove or disprove:

(a) *If for all $\boldsymbol{u} \in \mathbb{R}^n$ we have $\|T(\boldsymbol{u})\| = \|\boldsymbol{u}\|$, then T is injective.*

For T to be injective, $Ker(T) = \{0\}$

$\|T(u)\| = \|u\|$

$\Rightarrow \|0\| \quad = \|u\|$

$\Rightarrow u \quad = 0$

Hence, $\{u \in \mathbb{R}^n \mid T(u) = 0\}$, thus the statement that T is injective is true

(b) *The converse of part (a).*

The converse of part (a) is, If T is injective, for every $u \in \mathbb{R}^n$ we have $\|T(u)\| = \|u\|$
Take $T : \mathbb{R}^2 \to \mathbb{R}^2$, where T is defined as $T(x, y) = (2x, 2y)$.

$$
\begin{aligned}
\Rightarrow \|T(x, y)\| &= \|(2x, 2y)\| \\
&= \|2(x, y)\| \\
&= 2\|(x, y)\| \text{ by Proposition 2.2.41}
\end{aligned}
$$

$\therefore \|T(x, y)\| = 2\|u\| \quad \forall u \in \mathbb{R}$

Thus the converse is false.

Question Three

We define two new operations on complex matrices. Let $A \in Mat_{m\times n}(\mathbb{C})$.

- *The complex conjugate matrix $\overline{A}$ is defined entrywise, i.e.*

$$[\overline{A}]_{ij} := \overline{A_{ij}}.$$

- $A^\dagger := \overline{A}^\top$

Consider the complex matrix

$$U := \lambda \begin{bmatrix} 1 & 1 & 0 \\ -i & i & 0 \\ 0 & 0 & \sqrt{2}i \end{bmatrix},$$

where $\lambda \in \mathbb{C}$.

(a) *Find $U^\dagger$.*

First, transpose U,
$U^\top = \lambda \begin{bmatrix} 1 & -i & 0 \\ 1 & i & 0 \\ 0 & 0 & \sqrt{2}i \end{bmatrix}$
Now we can find the complex conjugate of $U^\top$,
$\overline{U}^\top = \overline{\lambda} \begin{bmatrix} 1 & i & 0 \\ 1 & -i & 0 \\ 0 & 0 & -\sqrt{2}i \end{bmatrix}$

(b) *Find all complex scalars $\lambda \in \mathbb{C}$ such that $U^\dagger = U^{-1}$.*

$$U^\dagger = U^{-1}$$
$$\Rightarrow U^\dagger U = I \text{ ,where } I \text{ is the identity matrix}$$
$$\Rightarrow \left(\overline{\lambda}\begin{bmatrix} 1 & i & 0 \\ 1 & -i & 0 \\ 0 & 0 & -\sqrt{2}i \end{bmatrix}\right)\left(\lambda \begin{bmatrix} 1 & 1 & 0 \\ -i & i & 0 \\ 0 & 0 & \sqrt{2}i \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
$$\Rightarrow \overline{\lambda}\lambda \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
$$\Rightarrow 2|\lambda|^2 = 1$$
$$\Rightarrow |\lambda|^2 = \frac{1}{2}$$
$$\Rightarrow |\lambda| = \frac{1}{\sqrt{2}}$$

Since λ is a complex scalar, λ is every complex number $\frac{1}{\sqrt{2}}$ away from the origin.
Suppose $\lambda = a + bi$, where $a, b \in \mathbb{R}$
$\therefore |\lambda| = |a + bi| = \frac{1}{\sqrt{2}}$

$$\begin{aligned}
\Rightarrow a^2 + b^2 &= \lambda^2 \\
\Rightarrow a^2 + b^2 &= \tfrac{1}{2} \\
\Rightarrow b^2 &= \tfrac{1}{2} - a^2
\end{aligned}$$

$$\Rightarrow b = \pm\sqrt{\tfrac{1}{2} - a^2}$$

$$\therefore \lambda = a \pm \sqrt{\tfrac{1}{2} - a^2}$$

Question Four

Let $u, v \in \mathbb{R}^n$ be arbitrary vectors, and let $A :\equiv [u, v]$ be the matrix with columns u and v. Show that

$$\sqrt{\det(A^\top A)} = \|u\|\|v\|\sin(\angle_{u,v}),$$

where $\angle_{u,v}$ is the angle between the vectors u and v.

$$\begin{aligned}
A^\top A &= [u, v]^\top [u, v] \\
&= \begin{bmatrix} u \cdot u & u \cdot v \\ u \cdot v & v \cdot v \end{bmatrix} \\
\therefore \det(A^\top A) &= \det\left(\begin{bmatrix} u \cdot u & u \cdot v \\ u \cdot v & v \cdot v \end{bmatrix}\right) \\
&= (u \cdot u)(v \cdot v) - (u \cdot v)^2 \\
&= \|u\|^2 \|v\|^2 - \left(\|u\|\|v\|\cos(\angle_{u,v})\right)^2 \\
&= \|u\|^2 \|v\|^2 - (\|u\|\|v\|)^2 \left(\cos(\angle_{u,v})\right)^2 \\
&= \|u\|^2 \|v\|^2 \left(1 - \cos(\angle_{u,v})\right)^2 \\
&= \|u\|^2 \|v\|^2 \sin^2(\angle_{u,v}) \\
\therefore \sqrt{\det(A^\top A)} &= \sqrt{\|u\|^2 \|v\|^2 \sin^2(\angle_{u,v})} \\
&= \|u\|\|v\|\sin(\angle_{u,v})
\end{aligned}$$