COMPSCI 120, S1, 2023

Assignment 4 **Due: 31 May 2023, 5.00 pm**

There are **four** problems listed below, covering material from chapters 6 and 7 from the coursebook. To get full credit on this assignment, pick any **three** of them to complete. If you submit more than three problems, markers will grade the first three problems that you submit and ignore the others. Each problem is worth 10 marks; complete and accurate answers to any three problems below will grant you full credit on this assignment.

Show all working!

Problems that do not show their work will receive reduced marks. Once you're done with it, submit a single PDF via Canvas before the due date. Hand-written submissions are fine as long as they are neat and legible. Please note that late assignments cannot be marked under any circumstances. With that said, if you find yourself unable to complete this assignment due to illness, injury, or personal/familial misfortune, please email Simone Linz at s.linz@auckland.ac.nz. We can come up with solutions to help you succeed in COMPSCI 120.

All of the questions in this assignment require a proof. Please note that you do **not** have to follow the suggested proof technique. Any correct proof is worth the maximum number of points.

1. **Trees**
   (a) (3 marks) A 5-ary tree is a rooted tree in which every vertex has at most 5 children. Suppose that T is a full 5-ary tree with ℓ leaves and p non-root parents. Prove that the number of leaves in T is $\ell = 4p + 5$.
   (b) (3 marks) An m-ary tree of height h is called *balanced* if all leaves are at level h or $h - 1$. Prove that there is a balanced rooted 5-ary tree with 3 non-root parents.
   (c) (4 marks) Prove that the maximum number of leaves in an m-ary tree of height h is m^h.
   Hint: Use induction on the height of the tree.

2. **Direct Proofs and Proof by Cases**
   (a) (4 marks) Prove that $|x + 3| + |x - 7| \geq 10$ for every real number x.
   (b) (3 marks) Prove that if all the vertices in a graph G have odd degree x, then the number of edges in G is a multiple of x
   (c) (3 marks) Let G be a connected graph with n vertices, where $n > 1$. Prove that if no vertex in G has degree 1, then G has at least n edges.

3. **Contradiction, construction, and disproof**
   (a) (3 marks) Prove that there does not exist a smallest positive rational number.
   (b) (3 marks) Consider a tree, T. Prove that any edge added to T must produce a cycle in T.
   (c) (2 marks) Prove or disprove the following statement:

   There is a graph G with 10 vertices, in which every vertex has degree 4.

(d) (2 marks) Prove or disprove the following statement:

There is a binary tree T of height 3 with 9 leaves.

4. **Induction**

(a) (3 marks) Prove that $9^{2n} - 1$ is divisible by 80 for any positive integer n

(b) (3 marks) Let $a_1 = 1$ and, for every integer $n > 1$, define a_n by the recurrence relation:

$$a_{n+1} = (n+1)^2 - a_n.$$

Prove that $a_n = \dfrac{n(n+1)}{2}$ for every positive integer n.

(c) (4 marks) Prove that $n! > n^2$ for $n \geq 4$.