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CIS 510: Introduction to Scientific Visualization

This document tries to evaluate the trade-offs of using Euler and RK4 (Runge-Kutta 4) ordinary differential equation solvers. The two main arguments of comparison for these two widely used methods of solving differential equations are performance and accuracy (rate of convergence in our case).

Both the systems (Euler and RK4) are convergent, i.e. they both tend to approach the exact solution as the step size approaches closer to 0. So, one thing that needs to be understood is that we may choose RK4 over Euler not because RK4 has no error, but the rate of convergence for RK4 is much higher than the rate of convergence for Euler method.

Before we argue which method has better accuracy, we must understand what an error is in case of approximation system. An approximation of a function leaves a possibility of deviation from the actual solution for the function. In numerical analysis, error is a characteristic of an approximation correctly performed[1]. The local truncation error of Euler's method has two factors of h and is given as $O(h^2)$. The local truncation error in case of RK4 method is $O(h^5)$. The error terms in the solution are obtained by the Tayler's series for both the systems. To get the essence of what these error terms actually represent, we can do a trivial comparison. Suppose to increase the accuracy we decrease the step size from h to h/2. In case of Euler method, the error term will be $O((h/2)^2)$, which compared to the original error term reduces the total error by 1/4. In case of RK4 the new error term will be $O((h/2)^5)$ which when compared to the original error term, reduces the total error by 1/32. This is what it means to converge fast as h gets closer to 0.

However, mathematically even if we show that RK4 has faster error convergence and for the same level of accuracy we can allow for a higher step size compared to Euler, it depends on the nature of the spatial discretization that we are dealing with. In case where the spatial discretization and characteristics of the space (in our case the vector field) are uniform and do not tend to change, the error in RK4 method can be as large as Euler term. However, were the characteristic of the space is highly variable, RK4 is much better in dealing with such fields.

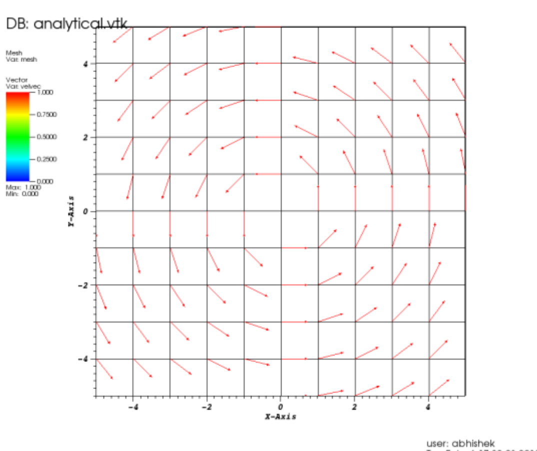

Figure 1. The circular vector field used to generate the streamlines.

Here, I have tried to device a small test to compare the results of particle advection under the same parameters. (step size and maximum number of steps to take). On the left is the dataset that I used to perform these tests. If you notice the nature of the vector field provided using the hedgehog plot, the actual solution ideally yields a trajectory that resembles a circle. Eventually, after a series of some finite steps, the particle should return to the same point where we start the advection.

[1] http://www.math.unl.edu/~gledder1/Math447/EulerError

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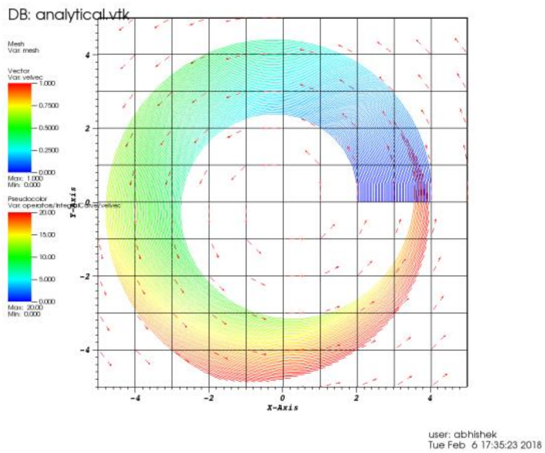

Figure 2. Streamlines generated using Euler solver

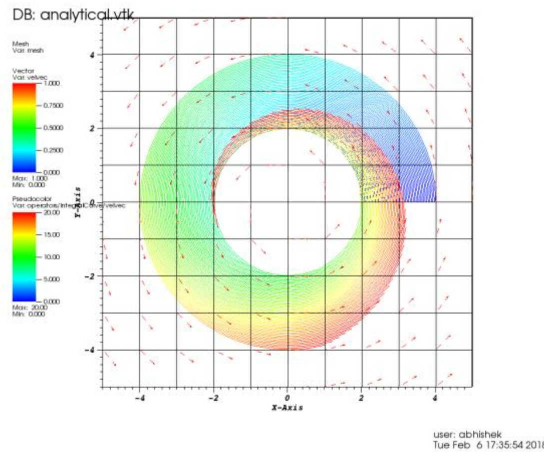

Figure 3. Streamlines generated using RK4 solver

The two figures above plot streamlines for the dataset that I introduced earlier. The blue extremities represent the starting positions of the particles, and the red extremities represent the ending positions. For the figure on the left which uses the Eulerian method for advection, we can see that the particles do not remotely end up close to where they started. For the figure on the right which used RK4 solver, we can see that the streamlines almost follow the perfect solution and the error is not noticeable. [VisIt was used to plot the streamlines, although I could have used my solution].

Although there are advantages of using RK methods in terms of accuracy of the advection, they require many more evaluation of velocities per step compared to Euler methods. 4 vs 1 in case of RK4. Decisions should be made to actually choose one over the other in terms of where more accuracy is required, and fast evaluation can be put on the back burner (priority wise). These decisions would change according to the resolution of the grid and the nature of the vector field.