THE LIMITING DISTRIBUTION OF CHARACTER SUMS

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Abstract. In this paper, we consider the distribution of the continuous paths of Dirichlet character sums modulo prime q on the complex plane. We also find a limiting distribution as $q \to \infty$ using Steinhaus random multiplicative functions, stating properties of this random process. This is motivated by Kowalski and Sawin's work on Kloosterman paths.

1. Introduction

Given a primitive Dirichlet character modulo q, we define the normalised partial character sum

$$S_\chi(t) := \frac{1}{\sqrt{q}} \sum_{n \leq qt} \chi(n),$$

for $t \in [0,1]$. Such character sums play a fundamental role in analytic number theory. Bober, Goldmakher, Granville and Koukoulopoulos [2] investigated the maximum of these sums, defining the distribution function

$$\Phi_q(\tau) := \frac{1}{\phi(q)} \# \left\{ \chi \mod q : \max_t |S_\chi(t)| > \frac{e^\gamma}{\pi}\tau \right\}. \tag{1}$$

In this paper we consider the paths traced out by the partial sums for a fixed character and study their distribution. We find that the limiting distribution of character sums is a random Fourier series which occurs in the description of $\Phi(\tau)$.

The character sum $S_\chi(t)$ is a step function, with jump discontinuities at every $qt \in \mathbb{Z}$. Adjusting the definition to form a continuous function, we define *Character Paths* as paths in the complex plane formed by drawing a straight line between the successive partial sums

$$S_\chi(x) = \tfrac{1}{\sqrt{q}} \textstyle\sum_{n \leq qx} \chi(n), \quad S_\chi(x + 1/q) = \tfrac{1}{\sqrt{q}} \textstyle\sum_{n \leq qx+1} \chi(n),$$

for $x \in [0,1)$ and $qx \in \mathbb{Z}$. We parameterise character paths by the function

$$f_\chi(t) := S_\chi(t) + \frac{\{qt\}}{\sqrt{q}} \chi\left(\lceil qt \rceil\right),$$

where $\{x\}$ is the fractional part of the number x. Character paths are polygonal, continuous and closed. See Figure 1 for examples of such paths.

We define the distribution of character paths by taking $\chi \bmod q \mapsto f_\chi(t)$ as a random process, choosing χ uniformly at random. We define the set of random processes

$$\mathcal{F}_q(t) := \{f_\chi(t) : \chi \bmod q\}.$$

We will find, in terms of Fourier series with multiplicative coefficients, the limiting distribution of the sequence $(\mathcal{F}_q)_{q \text{ prime}}$ as $q \to \infty$.

Definition 1.1. *Steinhaus random multiplicative functions X_n, $n \in \mathbb{N}$, are defined as*

$$X_n = \prod_{p^a \| n} X_p^a,$$

where $(X_p)_{p\ prime}$ are Steinhaus random variables, independent random variables uniformly distributed on the unit circle $\mathbb{U} = \{z : |z| = 1\}$. (Here $p^a \| n$ means p^a strictly divides n, so $p^a | n$ but $p^{a+1} \nmid n$).

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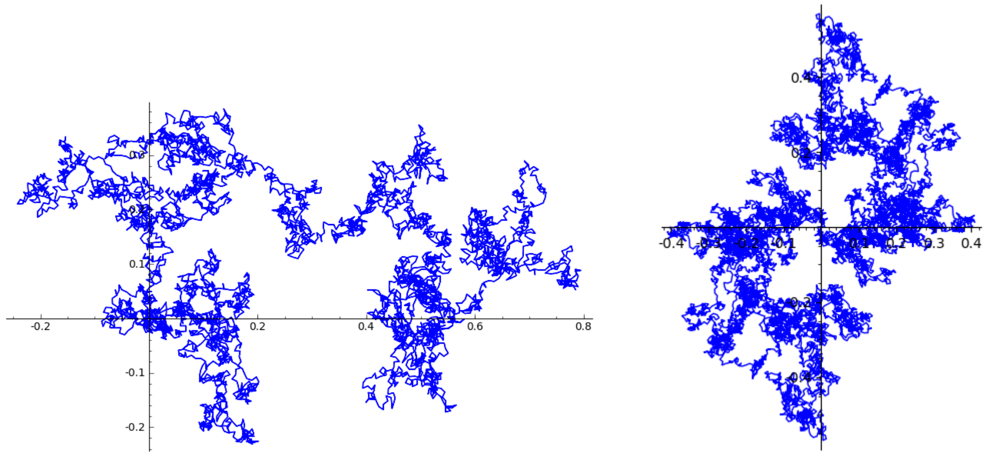


Odd path defined by $\chi(5) = e\left(\frac{1}{10007}\right)$. Even path defined by $\chi(5) = e\left(\frac{2}{10007}\right)$.

FIGURE 1. Character paths of an odd and even character modulo 10007.

The truncated Fourier series of a character sum is

$$\frac{\tau(\chi)}{2\pi i\sqrt{q}} \sum_{0<|k|<q} \frac{\overline{\chi}(k)}{k}(1 - e(-kt)) + O\left(\frac{\log q}{\sqrt{q}}\right). \tag{2}$$

This approximation is valid for character paths as

$$\max_t |S_\chi(t) - f_\chi(t)| \leq \frac{1}{\sqrt{q}}.$$

Splitting the distribution into odd and even characters, we use the method of moments to show the sequence of processes $(\mathcal{F}_{q,\pm})$ converges in distribution as $q \to \infty$ through the primes. We define $F(t)$ as the random Fourier series

$$F(t) := \frac{\eta}{\pi} \sum_{|k|>0} \frac{X_k}{k}(1 - e(kt)), \tag{3}$$

where X_k are Steinhaus random multiplicative functions for $k \geq 1$, $X_{-1} = \{\pm 1\}$ with equal probability, and η is a random variable uniformly distributed on $\mathbb{U}$. We further define $F_\pm(t)$ where we fix X_{-1} as $+1$ or -1.

Theorem 1.1. *Let q be an odd prime and $F_\pm$ be defined as above for $t \in [0,1]$. The sequence of the distributions of character paths $(\mathcal{F}_{q,\pm}(t))_q$ weakly converges to the process $F_\pm(t)$ as $q \to \infty$ through the primes. In other words, for any continuous map*

$$\psi : C([0,1]) \to \mathbb{C},$$

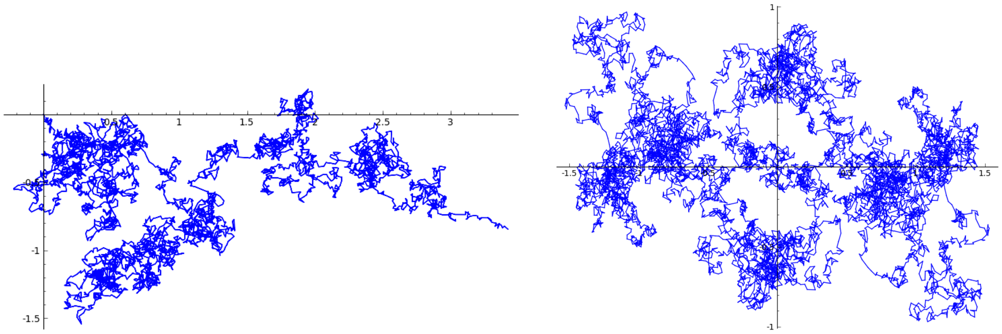
we have

$$\lim_{\substack{q\to\infty \\ q \text{ prime}}} \mathbb{E}\left(\psi\left(\mathcal{F}_{q,\pm}\right)\right) = \mathbb{E}\left(\psi(F_\pm)\right).$$

As referred to earlier, Bober, Goldmakher, Granville and Koukoulopoulos [2] investigated the distribution function $\Phi_q(\tau)$, defined in Equation (1). The limiting distribution of $(\Phi_{q,\pm}(\tau))_{q \text{ prime}}$ also used the process $F_\pm(t)$ from Theorem 1.1, showing $(\Phi_{q,\pm}(\tau))$ converges weakly to

$$\textbf{Prob}\left(\max_t |F_\pm(t)| > 2e^\gamma \tau\right),$$

as $q \to \infty$ through the primes. Theorem 1.1 can be used to recover the same result. They also showed that the infinite series defining the random process $F(t)$ converges almost surely. Examples of the paths formed by $F_\pm$ are shown in Figure 2.



A sample of $F_{-}(t)$. A sample of $F_{+}(t)$.

FIGURE 2. Samples of $F_{\pm}$ with 10007 terms.

The proof of Theorem 1.1 can be split into two parts: proving that the sequence $(\mathcal{F}_{q,\pm}(t))_q$ converges in finite distributions to the random process $F_{\pm}(t)$ and that the sequence of distributions is relatively compact. Convergence in finite distributions is proved in Section 2, using the method of moments. To prove relative compactness, it is sufficient to use Prohorov's Theorem [1, Theorem 5.1], which states that if a family of probability measures is tight, then it must be relatively compact. Section 3 proves the sequence of distributions $(\mathcal{F}_{q,\pm}(t))_q$ satisfies the tightness property, therefore proving Theorem 1.1.

We are interested in properties of the random process F. We rewrite the random process from Equation (3) as

$$F(t) = \frac{\eta}{\pi} \sum_{k \neq 0} \frac{X_k}{k} - \frac{\eta}{\pi} \sum_{k \neq 0} \frac{e(kt)}{k} X_k.$$

A priori, F_+ makes sense in $L^2([0,1])$, as the constant term disappears form $X_{-1} = 1$. In Section 4 we prove the constant term converges with probability 1 for $X_{-1} = -1$, therefore showing all samples of the random process F are almost surely in $L^2([0,1])$.

Theorem 1.2. *A sample of the random process $F(t)$ is almost surely a continuous function.*

This is significantly harder to prove than the $L^2([0,1])$ case. The idea behind the proof, shown in Section 4, is to consider the continuous y-smooth function

$$S_y(t) := \frac{c}{\pi} \sum_{\substack{k \neq 0 \\ P^+(|k|) \leq y}} \frac{1 - e(kt)}{k} \alpha_k,$$

where c, α_k are on the unit circle, α_k is completely multiplicative and $P^+(n)$ is the largest prime factor of y. The function S_y converges uniformly as $y \to \infty$ to a sample of the process F, proving all samples of F are in $C([0,1])$ with probability 1.

Remark 1.1. The definition of character paths is motivated by similar research by Kowalski and Sawin [11] [12]. In their papers they define *Kloosterman paths*, $K_p(t)$, view the paths as random variables, and find their limiting distribution as $p \to \infty$. Our work will continue in this vein to investigate the analogous limiting distribution of character paths. Due to the multiplicativity of Dirichlet characters, our random functions aren't independent. This leads to interesting properties shown in Section 4.

Remark 1.2. Steinhaus random multiplicative functions are completely multiplicative, with all values distributed on the unit circle $\mathbb{U}$. It is therefore reasonable that we compare partial sums of characters with partial sums of Steinhaus random multiplicative functions, assuming similar behaviour. However the periodicity of Dirichlet characters means this model is not ideal, so we instead consider X_n as Fourier coefficients. The sums of Steinhaus random multiplicative functions have a long history. See Harper [10] for an example of recent work on this.

Remark 1.3. Theorem 1.1 is restricted to q being prime, so a natural question is to consider composite q as well. Steinhaus random multiplicative functions are non zero so we need a high percentage of primitive characters modulo q. If we take q not being too smooth we might be able to relax this condition, as the contribution from imprimitive characters could potentially be included in the error terms already produced from the method. Future work could explore the generalised case when modulus of the characters is not prime.

Acknowledgements. I would like to thank my Ph.D. supervisor Jonathan Bober for his guidance.

Notation. We follow the usual convention of $e(x) = e^{2\pi i x}$ and $p^k \| n$ to be when $p^j | n$ for all $1 \leq j \leq k$ and $p^{k+1} \nmid n$. We take $d_N(x)$ as the Nth divisor function,

$$d_N(x) = \sum_{x_1 \cdots x_N = x} 1.$$

Given a positive integer n, we define $P^+(n)$ and $P^-(n)$ as the largest and smallest prime divisors of n. We take $P^+(1) = 1$ and $P^-(1) = \infty$ as in conventional notation.

2. Convergence of Finite-Dimensional Distributions of $\mathcal{F}_q$

In order to prove Theorem 1.1, we will first show convergence of finite-dimensional distributions. We take $(\mathcal{F}_{q,\pm}(t))_{q \text{ prime}}$ as the sequence of distributions of character paths modulo q dependent on the parity of the characters. Let $F_\pm(t)$ be the random processes defined by

$$F_+(t) := \frac{\eta}{\pi} \sum_{k \geq 1} \frac{X_k}{k} \sin(2\pi k t) \qquad \text{and} \qquad F_-(t) := \frac{\eta}{\pi} \sum_{k \geq 1} \frac{X_k}{k} (1 - \cos(2\pi k t)),$$

where X_n are Steinhaus random multiplicative functions and η is uniformly distributed on the unit circle.

Theorem 2.1. *Let q be an odd prime. The sequence of the distributions of character paths $(\mathcal{F}_{q,\pm}(t))_q$ converges to the process $F_\pm(t)$ in the sense of convergence of finite distributions. In other words, for every $n \geq 1$ and for every n-tuple $0 \leq t_1 < \cdots < t_n \leq 1$, the vectors*

$$(\mathcal{F}_{q,\pm}(t_1), \ldots, \mathcal{F}_{q,\pm}(t_n))$$

converge in law as $q \to \infty$ through the primes to

$$(F_\pm(t_1), \ldots, F_\pm(t_n)).$$

We prove this by the method of moments. We will define a moment M_q, of our distribution $\mathcal{F}_q$ and a moment M for the random process F. In Section 2.3, we prove M is determinate. Subsequently, in Section 2.4, we prove this sequence of moments M_q converges to M, the moment of F. This is sufficient to prove Theorem 2.1.

We are considering odd and even character paths separately. In this proof we will focus on results for odd character paths as the proof is analogous for the even character case. Where this is not the case, any changes will be addressed throughout the section.

2.1. Definitions of the Moments.

The Fourier series of the character path is

$$f_\chi(t) = \frac{\tau(\chi)}{2\pi i \sqrt{q}} \sum_{0 < |k| < q} \frac{\overline{\chi}(k)}{k} (1 - e(-kt)) + O\left(\frac{\log q}{\sqrt{q}}\right).$$

This results from truncating the Fourier series of the character sum $S_\chi(t)$ and the trivial inequality $|f_\chi(t) - S_\chi(t)| \leq \frac{1}{\sqrt{q}}$. The paths of odd and even characters are shown to differ greatly, exemplified in Figure 1, due the L-function $L(1,\chi) = 0$ when χ is even. As such, this paper will assess distributions of these character paths modulo odd prime q separately, dependent on parity. As a Fourier series we split this into

$$f_\chi(t) = \begin{cases} \frac{-\tau(\chi)}{\pi\sqrt{q}} \sum_{k=1}^{q} \frac{\overline{\chi}(k)}{k} \sin(2\pi k t) + O\left(\frac{\log q}{\sqrt{q}}\right), & \chi \text{ even}, \\ \frac{\tau(\chi)}{\pi i \sqrt{q}} \sum_{k=1}^{q} \frac{\overline{\chi}(k)}{k} (1 - \cos(2\pi k t)) + O\left(\frac{\log q}{\sqrt{q}}\right), & \chi \text{ odd}. \end{cases}$$

We define our moments M_q and M. In this section we will assume χ is odd as the proof is analogous to the even case. Therefore, taking a function from the odd distribution $\mathcal{F}_{q,-}$, we will take the character path modulo q as

$$f_\chi(t) = \frac{\tau(\chi)}{\pi i\sqrt{q}} \sum_{1\leq n\leq q} \frac{\overline{\chi}(n)}{n}(1-\cos(2\pi nt)) + O\left(\frac{\log q}{\sqrt{q}}\right).$$

We will also be considering the odd random series

$$F_-(t) = \frac{\eta}{\pi}\sum_{n\geq 1}\frac{X_k}{k}(1-\cos(2\pi nt)),$$

which for ease of notation will be referred to as $F(t)$ for the rest of this section.

Definition 2.1. *Let $k \geq 1$ be given and $\underline{t} = (t_1, \dots, t_k)$, where $0 \leq t_1 < \cdots < t_k \leq 1$, be fixed. Additionally fix $\underline{n} = (n_1, \dots, n_k)$ and $\underline{m} = (m_1, \dots, m_k)$, where $n_i, m_i \in \mathbb{Z}_{\geq 0}$. We define the moment sequence $M_q(\underline{n}, \underline{m})$ as*

$$M_q(\underline{n},\underline{m}) = \frac{2}{\phi(q)} \sum_{\chi\ odd} \prod_{i=1}^{k} f_\chi(t_i)^{n_i}\overline{f_\chi(t_i)}^{m_i},$$

and the moment $M(\underline{n},\underline{m})$ as

$$M(\underline{n},\underline{m}) = \mathbb{E}\left(\prod_{i=1}^{k} F(t_i)^{n_i}\overline{F(t_i)}^{m_i}\right).$$

$M(\underline{n},\underline{m})$ can be rewritten using methods from Bober and Goldmakher [3]. First, we use the Fourier expansion of $f_\chi(t)$, so

$$f_\chi(t_i)^{n_i}\overline{f_\chi(t_i)}^{m_i} = \frac{\tau(\chi)^{n_i}\overline{\tau(\chi)}^{m_i}}{(\pi\sqrt{q})^{n_i+m_i} i^{n_i-m_i}} \sum_{\substack{1\leq a\leq q^{n_i}\\ 1\leq b\leq q^{m_i}}} \overline{\chi}(a)\chi(b)\beta_{n_i,q,t_i}(a)\beta_{m_i,q,t_i}(b) + O\left(\frac{(\log q+1)^{n_i+m_i}}{\sqrt{q}}\right),$$

where $\beta_{N,q,t}$ is defined as

$$\beta_{N,q,t}(x) := \sum_{\substack{x_1\cdots x_N = x\\ x_i\leq q}} \prod_{k=1}^{N} \frac{(1-\cos(2\pi x_k t))}{x_k}. \tag{4}$$

Continuing to expand $M_q(\underline{n},\underline{m})$, we take a product of all $f_\chi(t_i)^{n_i}\overline{f_\chi(t_i)}^{m_i}$ for $i \in [1,k]$, showing

$$\prod_{i=1}^{k} f_\chi(t_i)^{n_i}\overline{f_\chi(t_i)}^{m_i} = \frac{\tau(\chi)^{n}\overline{\tau(\chi)}^{m}}{(\pi\sqrt{q})^{n+m} i^{n-m}} \sum_{\substack{1\leq a\leq q^{n}\\ 1\leq b\leq q^{m}\\ \text{for } i\in[1,k]}} \overline{\chi}(a)\chi(b)\mathcal{B}_{\underline{n},q,\underline{t}}(a)\mathcal{B}_{\underline{m},q,\underline{t}}(b) + O\left(\frac{(\log q)^{n+m}}{\sqrt{q}}\right),$$

where

$$n := n_1 + \cdots + n_k \text{ and } m := m_1 + \cdots + m_k$$

and

$$\mathcal{B}_{\underline{N},q,\underline{t}}(x) := \sum_{\substack{x_1\cdots x_k = x\\ x_i\leq q^{N_i}}} \prod_{i=1}^{k} \beta_{N_i,q,t_i}(x_i). \tag{5}$$

Furthermore, we take the average of this product over all odd Dirichlet characters χ to find

$$\begin{aligned} M_q(\underline{n},\underline{m}) =& \frac{1}{(\pi\sqrt{q})^{n+m} i^{n-m}} \sum_{\substack{1\leq a\leq q^{n}\\ 1\leq b\leq q^{m}}} \left(\mathcal{B}_{\underline{n},q,\underline{t}}(a)\mathcal{B}_{\underline{m},q,\underline{t}}(b)\right) \frac{2}{\phi(q)} \sum_{\substack{\chi \mod q\\ \chi\text{ odd}}} \overline{\chi}(a)\chi(b)\tau(\chi)^n\overline{\tau(\chi)}^m \\ &+ O\left(\frac{(\log q)^{n+m}}{\sqrt{q}}\right). \end{aligned} \tag{6}$$

This form is more useful for future calculations and will used to prove M_q tends to M as $q \to \infty$ through the primes.

We can additionally write $M(\underline{n}, \underline{m})$ in a similar form. $M(\underline{n}, \underline{m})$ is the expectation

$$\mathbb{E}\left(\prod_{i=1}^{k} F(t_i)^{n_i} \overline{F(t_i)}^{m_i}\right).$$

$F(t)$ is a random process, defined by the almost surely converging sum

$$F(t) = \frac{\eta}{\pi} \sum_{a \geq 1} \frac{X_a}{a}(1 - \cos(2\pi a t)).$$

The sum is not absolutely convergent, so justification is needed to manipulate the product $\prod_{i=1}^{k} F(t_i)^{n_i} \overline{F(t_i)}^{m_i}$. Firstly, the product can be expanded without changing the order of summation. We have

(7)

$$M(\underline{n}, \underline{m}) = \mathbb{E}\left(\frac{\eta^n \overline{\eta}^m}{\pi^{n+m}} \sum \cdots \sum \prod_{i=1}^{k} \prod_{j=1}^{n_i} \prod_{j'=1}^{m_i} \frac{X_{a_j} \overline{X_{b_{j'}}}}{a_j b_{j'}} (1 - \cos(2\pi a_j t_i))(1 - \cos(2\pi b_{j'} t_i))\right),$$

where $n = \sum n_i$ and $m = \sum m_i$ as above.

We want to swap the order of expectation with the order of summation. We can justify this with a corollary of Lebesgue's Dominated Convergence Theorem, which we quote from Gut.

Proposition 2.2. [9, Chapter 2, Corollary 5.3] *Suppose $\{Y_n : n \geq 1\}$ are random variables such that*

$$\left|\sum_{n=1}^{\infty} Y_n\right| \leq Y,$$

where Y is integrable. If $\sum_{n=1}^{\infty} Y_n$ converges almost surely, then $\sum Y_n$ and every Y_n are integrable and

$$\mathbb{E}\left(\sum_{n=1}^{\infty} Y_n\right) = \sum_{n=1}^{\infty} \mathbb{E}(Y_n).$$

We take $Y_n = \frac{X_n}{n}(1 - \cos(2\pi n t))$ and use results from Bober, Goldmakher, Granville and Koukoulopoulos. Their work [2, Theorem 1.4] states for all $\alpha > 0$

$$\frac{1}{\phi(q)} \#\left\{\chi \mod q : \max_{0 \leq t \leq 1} |f_\chi(t)| > \frac{e^\gamma}{\pi}\alpha\right\} \longrightarrow \mathbf{Prob}\left(\max_{0 \leq t \leq 1}\left|\sum_{n=1}^{\infty} \frac{X_n}{n}(1 - \cos(2\pi n t))\right| > 2e^\gamma \alpha\right),$$

as $q \to \infty$ through the primes. We combine this with their main theorem,

Proposition 2.3. [2, Theorem 1.1] *Let $\eta = e^{-\gamma} \log 2$ and $1 \leq \tau \leq \log\log q - M$ for some $M \geq 4$. Then*

$$\frac{1}{\phi(q)} \#\left\{\chi \mod q : \max_{0 \leq t \leq 1} |f_\chi(t)| > \tau\right\} \leq \exp\left\{-\frac{e^{\tau - 2 - \eta}}{\tau}\left(1 + O\left(\frac{\log \tau}{\tau}\right)\right)\right\}.$$

Taking $q \to \infty$, we see the probability of $\left|\sum_{n=1}^{\infty} \frac{X_n}{n}(1 - \cos(2\pi n t))\right|$ being large is negligible. Therefore the multivariate sum from Equation (7) converges almost surely and the absolute value of the sum is bounded. As a result, we can use Proposition 2.2 and swap the order of expectation and summation. The moment M therefore equals

$$\mathbb{E}\left(\frac{\eta^n \overline{\eta}^m}{\pi^{n+m}}\right) \sum \cdots \sum \mathbb{E}\left(X_a \overline{X_b}\right) \prod_{i=1}^{k} \prod_{j=1}^{n_i} \prod_{j'=1}^{m_i} \frac{(1 - \cos(2\pi a_j t_i))(1 - \cos(2\pi b_{j'} t_i))}{a_j b_{j'}},$$

where

$$a := \prod_{i=1}^{k} \prod_{j=1}^{n_i} a_j, \quad b := \prod_{i=1}^{k} \prod_{j'=1}^{m_i} b_{j'}.$$

Steinhaus random multiplicative functions X_n are orthogonal as n can always be written as a unique prime factorisation and $\mathbb{E}(X_p) = 0$ for all primes p. In other words,

$$\mathbb{E}\left(X_a\overline{X_b}\right) = \mathbb{1}_{a=b} := \begin{cases} 1, & a = b \\ 0, & \text{otherwise} \end{cases}.$$

Therefore we can rewrite the moment as follows,

$$M = \mathbb{E}\left(\frac{\eta^n\overline{\eta}^m}{\pi^{n+m}}\right)\sum\cdots\sum \mathbb{1}_{a=b}\prod_{i=1}^{k}\prod_{j=1}^{n_i}\prod_{j'=1}^{m_i}\frac{(1-\cos(2\pi a_j t_i))(1-\cos(2\pi b_{j'} t_i))}{a_j b_{j'}}.$$

The sums are now absolutely convergent, so we can swap the order of summation. As a result,

$$M(\underline{n},\underline{m}) = \mathbb{E}\left(\frac{\eta^n\overline{\eta}^m}{\pi^{n+m}}\right)\sum_{a\geq 1}\mathcal{B}_{\underline{n},\underline{t}}(a)\mathcal{B}_{\underline{m},\underline{t}}(a), \tag{8}$$

where

$$\mathcal{B}_{\underline{N},\underline{t}}(x) := \sum_{x_1\cdots x_k = x}\prod_{i=1}^{k}\beta_{N_i,t_i}(x_i)$$

and

$$\beta_{N_i,t_i}(x_i) = \sum_{y_1\cdots y_{N_i}=x_i}\prod_{j=1}^{N_i}\frac{(1-\cos(2\pi y_j t_i))}{y_j}.$$

Note that $\mathcal{B}_{\underline{N},\underline{t}}$ and β_{N_i,t_i} are the limits as $q\to\infty$ of $\mathcal{B}_{\underline{N},q,\underline{t}}$ from Equation (5) and β_{N_i,q,t_i} from Equation (4) respectively.

2.2. **Bounding the Moments.** Later in the paper we will be interested in bounding $\mathcal{B}_{\underline{N},q,\underline{t}}$ and $\mathcal{B}_{\underline{N},\underline{t}}$. The inequality we find is independent of q, so we can consider bounds at the same time. Therefore for this subsection we will work with $\mathcal{B}_{\underline{N},q,\underline{t}}$.

Recall,

$$\mathcal{B}_{\underline{N},q,\underline{t}}(x) = \sum_{\substack{x_1\cdots x_k = x\\ x_i\leq q^{N_i}}}\prod_{i=1}^{k}\beta_{N_i,q,t_i}(x_i),$$

where

$$\beta_{N,q,t}(x) = \sum_{\substack{x_1\cdots x_N = x\\ x_j\leq q}}\prod_{j=1}^{N}\frac{(1-\cos(2\pi x_j t))}{x_j}.$$

Since $|1-\cos(2\pi x_j t)|\leq 2$, we always have the bound

$$|\beta_{N,q,t}(x)|\leq\frac{2^N d_N(x)}{x},$$

where $d_N(x)$ is the Nth divisor function $\sum_{x_1\cdots x_N=x}1$. As a result,

$$\mathcal{B}_{\underline{N},q,\underline{t}}(x)\leq\frac{2^N}{x}\sum_{\substack{x_1\cdots x_k = x\\ x_i\leq q^{N_i}}}\prod_{i=1}^{k}d_{N_i}(x_i),$$

where $N = \sum N_i = |\underline{N}|$. To further bound $\mathcal{B}$ we next use the following lemma.

Lemma 2.4. *Let $d_{N_1}(x_1), d_{N_2}(x_2)$ be the N_1th and N_2th divisor function of $x_1, x_2\in\mathbb{N}$ respectively. We have the relation*

$$d_{N_1}(x_1)d_{N_2}(x_2)\leq d_{N_1+N_2}(x_1\cdot x_2).$$

Proof. We apply a combinatorial argument, where we view $d_N(x)$ as the number of ways of choosing N positive integers that multiply to x. Therefore $d_{N_1+N_2}(x_1 \cdot x_2)$ is at least the number of ways of choosing N_1 integers multiplying to x_1 times the number of ways of choosing N_2 integers multiplying to x_2. □

Using Lemma 2.4, we bound $\mathcal{B}_{\underline{N},q,\underline{t}}(x)$ by

$$\mathcal{B}_{\underline{N},q,\underline{t}}(x) \leq \frac{2^N d_N(x)}{x} \sum_{\substack{x_1\cdots x_k = x \\ x_i \leq q^{N_i}}} 1 \leq \frac{2^N d_N(x) d_k(x)}{x}.$$

In some places of the proof, it is sufficient to further bound $d_N(x)d_k(x)$ by $d^{N+k}(x)$ or $d_N(x)x^{\epsilon_k}$. However the main bound we will use is

$$\mathcal{B}_{\underline{N},q,\underline{t}}(x) \leq \frac{2^N d_N(x) d_k(x)}{x}. \tag{9}$$

This will be useful in future equations. Note that this is independent of q and $\underline{t}$, so the bounds hold for $\mathcal{B}_{\underline{N},\underline{t}} = \lim_{q\to\infty} \mathcal{B}_{\underline{N},q,\underline{t}}$.

2.3. **Proving Determinancy.** Our aim is to use method of moments to prove the distribution of character paths modulo q converges to $F(t)$ in the sense of finite distributions. For this we need to show the moment $M(\underline{n},\underline{m})$ has only one representing measure, otherwise known as determinate. It is sufficient to show M is a determinate complex moment sequence if it satisfies

$$\sum_{n=1}^{\infty} M(\underline{n},\underline{n})^{-1/2n} = \infty, \tag{10}$$

where $\underline{n} = (n_1, \ldots, n_k)$ and $n = \sum_i n_i$. This is also known as the Carleman condition [16, Theorem 15.11].

Lemma 2.5. *The moment $M(\underline{n},\underline{m})$ satisfies Equation* (10)*.*

Proof. This is shown using Equation (8) and taking $\underline{n} = \underline{m}$. Setting $n = \sum n_i = \sum m_i$, we have

$$M(\underline{n},\underline{m}) = \frac{1}{\pi^{2n}} \sum_{a\geq 1} \mathcal{B}_{\underline{n},\underline{t}}(a)^2.$$

We use the bound of $\mathcal{B}$ from Equation (9), taking $d_k(a) = a^{\epsilon_k}$ for small $\epsilon_k > 0$, so

$$M(\underline{n},\underline{n}) \leq \frac{2^{2n}}{\pi^{2n}} \sum_{a\geq 1} \frac{d_n(a)^2}{a^{2-2\epsilon_k}} =: \frac{2^{2n}}{\pi^{2n}} \sum_{a\geq 1} \frac{d_n(a)^2}{a^{2\sigma}},$$

taking $\sigma := 1 - \epsilon_k$. We can use Proposition 3.2 from Bober and Goldmakher [3], which states for $1/2 < \sigma \leq 1$,

$$\sum_{a=1}^{\infty} \frac{d_n(a)^2}{a^{2\sigma}} \leq \exp\left(2n\sigma \log\log(2n)^{1/\sigma} + \frac{(2n)^{1/\sigma}}{2\sigma - 1} + O\left(\frac{n}{2\sigma-1} + \frac{(2n)^{1/\sigma}}{\log(3(2n)^{1/\sigma - 1})}\right)\right). \tag{11}$$

We are interested in proving the Carleman condition, ie

$$\sum_{n=1}^{\infty} M(\underline{n},\underline{n})^{-1/2n} = \infty.$$

Here we have shown the sum has the lower bound

$$\sum_{n=1}^{\infty} M(\underline{n},\underline{n})^{-1/2n} \geq \frac{\pi}{2} \sum_{n=1}^{\infty} \exp\left(-\sigma \log\log\left((2n)^{1/\sigma}\right) - \frac{(2n)^{1/\sigma-1}}{2\sigma-1} + O\left(\frac{1}{2\sigma-1} + \frac{(2n)^{1/\sigma-1}}{\log(3(2n)^{1/\sigma-1})}\right)\right).$$

The lower bound can be rewritten as

$$\frac{\pi}{2}\sum_{n=1}^{\infty}\frac{\sigma^\sigma}{(\log 2n)^\sigma}\exp\left(-\frac{(2n)^{\frac{1-\sigma}{\sigma}}}{2\sigma-1}\right)\exp\left(O\left(\frac{1}{2\sigma-1}+\frac{(2n)^{1/\sigma-1}}{\log(3(2n)^{1/\sigma-1})}\right)\right).$$

Tending $\sigma = 1-\epsilon_k$ to 1, this sum diverges. Therefore

$$\sum_{n=1}^{\infty} M(\underline{n},\underline{n})^{-1/2n} > \infty,$$

and the Carleman condition holds. Therefore the claim is proved. □

2.4. **Convergence of Moments.** In this section we show the moment sequence M_q converges to the multivariate moment of F, therefore proving Theorem 2.1.

Lemma 2.6. *Let $k \geq 1$ be given and $0 \leq t_1 < \cdots < t_k \leq 1$ be fixed. Fix $\underline{n} = (n_1, \dots, n_k)$ and $\underline{m} = (m_1, \dots, m_k)$, where $n_i, m_i \in \mathbb{Z}_{\geq 0}$. Let*

$$M_q(\underline{n},\underline{m}) = \frac{2}{\phi(q)}\sum_{\chi\ odd}\prod_{i=1}^{k} f_\chi(t_i)^{n_i}\overline{f_\chi(t_i)}^{m_i}.$$

Then

$$M_q(\underline{n},\underline{m}) = M(\underline{n},\underline{m}) + O_{\underline{n},\underline{m},k}\left(\frac{(\log q)^{n+m}}{\sqrt{q}}\right),$$

where

$$M(\underline{n},\underline{m}) = \mathbb{E}\left(\prod_{i=1}^{k} F(t_i)^{n_i}\overline{F(t_i)}^{m_i}\right).$$

Importantly, $M_q(\underline{n},\underline{m}) \to M(\underline{n},\underline{m})$ as $q \to \infty$ through the primes.

This lemma is sufficient to prove Theorem 2.1, showing $(\mathcal{F}_q(t))_{q \text{ prime}}$ converges in finite distributions to $F(t)$. We prove Lemma 2.6 as a combination of the following 2 propositions.

Proposition 2.7. *Let $k \geq 1$ be given and $\underline{t} = (t_1, \dots, t_k)$, where $0 \leq t_1 < \cdots < t_k \leq 1$, be fixed. Fix $\underline{n} = (n_1, \dots, n_k)$ and $\underline{m} = (m_1, \dots, m_k)$, where $n_i, m_i \in \mathbb{Z}_{\geq 0}$ and*

$$n := n_1 + n_2 + \cdots + n_k = m_1 + \cdots + m_k.$$

The moment sequence defined in Lemma 2.6 can be expressed as

$$M_q(\underline{n},\underline{m}) = \frac{1}{\pi^{2n}}\sum_{a\geq 1}\mathcal{B}_{\underline{n},\underline{t}}(a)\mathcal{B}_{\underline{m},\underline{t}}(a) + O_{\underline{n},\underline{m},k}\left(\frac{(\log q)^{2n}}{\sqrt{q}}\right),$$

where $\mathcal{B}_{\underline{N},\underline{t}}$ is defined as

$$\mathcal{B}_{\underline{N},\underline{t}}(a) = \sum_{X_1\cdots X_k=a}\prod_{i=1}^{k}\left(\sum_{x_1\cdots x_N=X_i}\prod_{j=1}^{N_i}\frac{(1-\cos(2\pi x_j t))}{x_j}\right).$$

Proposition 2.8. *Let $k \geq 1$ be given and $\underline{t} = (t_1, \dots, t_k)$, where $0 \leq t_1 < \cdots < t_k \leq 1$, be fixed. Fix $\underline{n} = (n_1, \dots, n_k)$ and $\underline{m} = (m_1, \dots, m_k)$, where $n_i, m_i \in \mathbb{Z}_{\geq 0}$ and*

$$n := n_1 + n_2 + \cdots + n_k = m_1 + \cdots + m_k.$$

Then

$$M_q(\underline{n},\underline{m}) = \frac{1}{\pi^{2n}}\sum_{a\geq 1}\mathcal{B}_{\underline{n},\underline{t}}(a)\mathcal{B}_{\underline{m},\underline{t}}(a),$$

where $\mathcal{B}_{\underline{N},\underline{t}}$ is defined as in Proposition 2.7.

Before the proof of the propositions, we will use them to prove Lemma 2.6.

Proof of Lemma 2.6. Take $n = n_1 + n_2 + \cdots + n_k$ and $m = m_1 + \cdots m_k$. We split the proof into 2 cases: $n = m$ and $n \neq m$. The first case has already been shown by Propositions 2.7 and 2.8:

$$M_q(\underline{n}, \underline{m}) = \frac{1}{\pi^{2n}} \sum_{a \geq 1} \mathcal{B}_{\underline{n},\underline{t}}(a)\mathcal{B}_{\underline{m},\underline{t}}(a) + O_{\underline{n},\underline{m},k}\left(\frac{(\log q)^{2n}}{\sqrt{q}}\right)$$
$$= \mathbb{E}\left(\prod_{i=1}^{k} F(t_i)^{n_i} \overline{F(t_i)}^{m_i}\right) + O_{\underline{n},\underline{m},k}\left(\frac{(\log q)^{2n}}{\sqrt{q}}\right).$$

Therefore, the only case left to show is when $n \neq m$. As shown in Equation (8), the moment M is equivalent to

$$M(\underline{n}, \underline{m}) = \mathbb{E}\left(\frac{\eta^n \overline{\eta}^m}{\pi^{n+m}}\right) \sum_{a \geq 1} \mathcal{B}_{\underline{n},\underline{t}}(a)\mathcal{B}_{\underline{m},\underline{t}}(a).$$

Since η is uniformly distributed on the unit circle, $\mathbb{E}(\eta^n \overline{\eta}^m) = 0$ and the moment M vanishes. Therefore, to conclude the proof, we need to show the moment $M_q \to 0$ as $q \to \infty$. As shown in Equation (6), we can write $M_q(\underline{n}, \underline{m})$ as

$$M_q(\underline{n}, \underline{m}) = \frac{1}{(\pi\sqrt{q})^{n+m}} \sum_{\substack{1 \leq a \leq q^n \\ 1 \leq b \leq q^m}} \left(\mathcal{B}_{\underline{n},q,\underline{t}}(a)\mathcal{B}_{\underline{m},q,\underline{t}}(b)\right) \frac{2}{\phi(q)} \sum_{\chi} \overline{\chi}(a)\chi(b)\tau(\chi)^n \overline{\tau(\chi)}^m$$
$$+ O\left(\frac{(\log q)^{n+m}}{\sqrt{q}}\right).$$

Assuming $n > m$, we rewrite $\tau(\chi)^n \overline{\tau(\chi)}^m$ as $\sqrt{q}^m \tau(\chi)^{n-m}$. Therefore, taking $\chi(\overline{a}) := \overline{\chi}(a)$,

$$\frac{2}{\phi(q)} \sum_{\chi} \overline{\chi}(a)\chi(b)\tau(\chi)^n \overline{\tau(\chi)}^m = \frac{2\sqrt{q}^m}{\phi(q)} \sum_{\chi} \chi(\overline{a} \cdot b)\tau(\chi)^{n-m}.$$

Lemma 2.9. *For $N \in \mathbb{N}$,*

$$\frac{2}{\phi(q)} \left| \sum_{\substack{\chi \mod q \\ \chi(-1)=\sigma}} \chi(a)\tau(\chi)^N \right| \leq 2Nq^{(N-1)/2},$$

where $\sigma = \{1, -1\}$.

This lemma is a slight generalisation of a result by Granville and Soundararajan [8, Lemma 8.3]. Below follows Granville and Soundararajan's proof, with a modification to include when χ is even.

Proof. Firstly, we rewrite the sum as exponential sums, using orthogonality of characters and the definition of the Gauss sum $\tau(\chi)$:

$$\frac{2}{\phi(q)} \sum_{\substack{\chi \mod q \\ \chi(-1)=\sigma}} \chi(a)\tau(\chi)^N = \sum_{\substack{x_1,\ldots,x_N \mod q \\ x_1 \cdots x_N \equiv \overline{a} \mod q}} e\left(\frac{x_1 + \cdots + x_N}{q}\right) + \operatorname{sgn}(\sigma) \sum_{\substack{x_1,\ldots,x_N \mod q \\ x_1 \cdots x_N \equiv -\overline{a} \mod q}} e\left(\frac{x_1 + \cdots + x_N}{q}\right).$$

Then, using Deligne's bound [5]

$$\left| \sum_{\substack{x_1,\ldots,x_N \mod q \\ x_1 \cdots x_N \equiv b \mod q}} e\left(\frac{x_1 + \cdots + x_N}{q}\right) \right| \leq Nq^{(N-1)/2},$$

we have proved the lemma. □

As a result, we have the inequality

$$|M_q(\underline{n}, \underline{m})| \leq \frac{2(n-m)}{\pi^{(n+m)}\sqrt{q}^{m+1}} \sum_{\substack{1 \leq a \leq q^n \\ 1 \leq b \leq q^m}} |\mathcal{B}_{\underline{n},q,\underline{t}}(a)||\mathcal{B}_{\underline{m},q,\underline{t}}(b)| + O\left(\frac{(\log q)^{n+m}}{\sqrt{q}}\right).$$

We also have the bound on $\mathcal{B}$, as shown in Equation (9),

$$\mathcal{B}_{\underline{N},q,\underline{t}}(x) \leq \frac{2^N d_N(x) d_k(x)}{x}.$$

Therefore, trivially bounding both divisor functions by q^ϵ for $\epsilon > 0$,

$$\sum_{1\leq a\leq q^n} |\mathcal{B}_{\underline{n},q,\underline{t}}(a)| \ll 2^n q^\epsilon \sum_{1\leq a\leq q^n} \frac{1}{a} \leq 2^n q^\epsilon \log(q^n).$$

We get an analogous result for $\sum_{1\leq b\leq q^m} |\mathcal{B}_{\underline{m},q,\underline{t}}(b)|$. As a result,

$$M_q(\underline{n},\underline{m}) \ll \frac{2^{1+n+m}(n-m)q^\epsilon \log(q^n)\log(q^m)}{\pi^{n+m}\sqrt{q}^{m+1}} + O\left(\frac{(\log q)^{n+m}}{\sqrt{q}}\right),$$

which tends to zero as $q\to\infty$. By a similar method we can show this is also the case when $n<m$. Therefore Lemma 2.6 holds. $\square$

Having proven that Lemma 2.6, we will now prove Propositions 2.7 and 2.8, showing when $\sum n_i = \sum m_i$ both $\lim_{q\to\infty} M_q$ and M equal

$$\frac{1}{\pi^{2n}} \sum_{a\geq 1} \mathcal{B}_{\underline{n},\underline{t}}(a)\mathcal{B}_{\underline{m},\underline{t}}(a).$$

Note $\beta_{N,q,t}(x)$ is defined in Equation (5) as

$$\sum_{\substack{x_1\cdots x_k = x\\ x_i\leq q^{N_i}}} \prod_{i=1}^{k} \beta_{N_i,q,t_i}(x_i) = \sum_{\substack{x_1\cdots x_k = x\\ x_i\leq q^{N_i}}} \prod_{i=1}^{k} \sum_{\substack{x_1\cdots x_N = x\\ x_i\leq q}} \prod_{k=1}^{N} \frac{(1-\cos(2\pi x_k t))}{x_k}.$$

Proof of Proposition 2.7. Taking $n=m$, where

$$n := n_1+\cdots+n_k, \qquad m := m_1\cdots m_k,$$

we rewrite Equation (6) to

$$M_q(\underline{n},\underline{m}) = \frac{1}{\pi^{2n}} \sum_{1\leq a,b\leq q^n} \left(\mathcal{B}_{\underline{n},q,\underline{t}}(a)\mathcal{B}_{\underline{m},q,\underline{t}}(b)\right) \frac{2}{\phi(q)} \sum_{\chi} \overline{\chi}(a)\chi(b) + O\left(\frac{(\log q)^{2n}}{\sqrt{q}}\right),$$

where $\mathcal{B}_{\underline{n},q,\underline{t}}$ is defined as in Equation (5).

Using the orthogonality of χ, and noting we are only summing over odd characters χ modulo q, the moment sequence becomes

$$M_q(\underline{n},\underline{m}) = \frac{1}{\pi^{2n}}\Sigma_+(\underline{n},\underline{m}) - \frac{1}{\pi^{2n}}\Sigma_-(\underline{n},\underline{m}) + O\left(\frac{(\log q)^{2n}}{\sqrt{q}}\right), \tag{12}$$

where

$$\Sigma_+(\underline{n},\underline{m}) := \sum_{\substack{1\leq a,b\leq q^n\\ a\equiv b \mod q}} \mathcal{B}_{\underline{n},q,\underline{t}}(a)\mathcal{B}_{\underline{m},q,\underline{t}}(b), \quad \Sigma_-(\underline{n},\underline{m}) := \sum_{\substack{1\leq a,b\leq q^n\\ a\equiv -b \mod q}} \mathcal{B}_{\underline{n},q,\underline{t}}(a)\mathcal{B}_{\underline{m},q,\underline{t}}(b).$$

The method for the even character case would differ here, meaning the moment sequence would be

$$M_q(\underline{n},\underline{m}) = \frac{2}{\pi^{2n}} \sum_{\substack{1\leq a,b\leq q^n\\ a\equiv \pm b \mod q}} \mathcal{B}'_{\underline{n},q,\underline{t}}(a)\mathcal{B}'_{\underline{m},q,\underline{t}}(b) + O\left(\frac{(\log q)^{2n}}{\sqrt{q}}\right),$$

where

$$\mathcal{B}'_{\underline{N},q,\underline{t}}(a) = \sum_{X_1\cdots X_k = a} \prod_{i=1}^{k} \sum_{\substack{x_1\cdots x_N = X_i\\ x_j\leq q}} \prod_{j=1}^{N_i} \frac{\sin(2\pi x_j t_i)}{x_j}.$$

Staying with the odd character case, the aim is to get the main sum independent of q. Using ideas from Bober and Goldmakher [3, Proof of Lemma 4.1], we consider Σ_+ and Σ_- simultaneously. First, we split the sums into arithmetic progressions mod q,

$$\Sigma_\pm(\underline{n},\underline{m}) = \sum_{\substack{1\le a,b\le q\\ a\equiv\pm b \mod q}} \sum_{0\le\gamma_1,\gamma_2<q^{n-1}} \mathcal{B}_{\underline{n},q,\underline{t}}(a+\gamma_1 q)\mathcal{B}_{\underline{m},q,\underline{t}}(b+\gamma_2 q).$$

We simplify $\Sigma_\pm$ by splitting the inner sum into $\gamma_1=\gamma_2=0$, $\gamma_1\neq 0$, and $\gamma_2\neq 0$:

$$\sum_{0\le\gamma_1,\gamma_2<q^{n-1}} \mathcal{B}_{\underline{n},q,\underline{t}}(a+\gamma_1 q)\mathcal{B}_{\underline{m},q,\underline{t}}(b+\gamma_2 q)$$
$$= \big(\mathcal{B}_{\underline{n},q,\underline{t}}(a)\mathcal{B}_{\underline{m},q,\underline{t}}(b)\big) + \sum_{j=1}^{2}\sum_{\substack{0\le\gamma_1,\gamma_2<q^{n-1}\\ \gamma_j\neq 0}} \mathcal{B}_{\underline{n},q,\underline{t}}(a+\gamma_1 q)\mathcal{B}_{\underline{m},q,\underline{t}}(b+\gamma_2 q).$$

For ease of notation, we define the above latter sum as

$$\Omega = \sum_{j=1}^{2}\sum_{\substack{0\le\gamma_1,\gamma_2<q^{n-1}\\ \gamma_j\neq 0}} \mathcal{B}_{\underline{n},q,\underline{t}}(a+\gamma_1 q)\mathcal{B}_{\underline{m},q,\underline{t}}(b+\gamma_2 q).$$

We can bound Ω by using the bound of $\mathcal{B}$ shown in Section 2.2, so

$$\Omega \le 2^{2n}\sum_{j=1}^{2}\sum_{\substack{0\le\gamma_1,\gamma_2<q^{n-1}\\ \gamma_j\neq 0}} \frac{d_n(a+\gamma_1 q)d_k(a+\gamma_1 q)}{a+\gamma_1 q}\frac{d_n(b+\gamma_2 q)d_k(b+\gamma_2 q)}{b+\gamma_2 q}.$$

By bounding the divisor functions by $O(q^\epsilon)$, we can further bound the sum to

$$\Omega \ll_{n,k} 2^{2n}q^\epsilon\sum_{j=1}^{2}\sum_{\substack{0\le\gamma_1,\gamma_2<q^{n-1}\\ \gamma_j\neq 0}} \frac{1}{a+\gamma_1 q}\frac{1}{b+\gamma_2 q}.$$

We can use the bound on partial harmonic series,

$$\omega_x := \sum_{\gamma=1}^{q^{n-1}} \frac{1}{x+\gamma q} \le \frac{\log(q^{n-1})}{q},$$

to further bound Ω. As a result,

$$\sum_{j=1}^{2}\sum_{\substack{0\le\gamma_1,\gamma_2<q^{n-1}\\ \gamma_j\neq 0}} \frac{1}{a+\gamma_1 q}\frac{1}{b+\gamma_2 q} = \left(\frac{1}{a}+\omega_a\right)\omega_b + \omega_a\left(\frac{1}{b}+\omega_b\right) \le \frac{\log(q^{n-1})}{q}\left(\frac{1}{a}+\frac{1}{b}+\frac{2\log(q^{n-1})}{q}\right).$$

Therefore $\Sigma_\pm$ can be written as,

$$\Sigma_\pm = \sum_{\substack{1\le a,b\le q\\ a\equiv\pm b \mod q}} \big(\mathcal{B}_{\underline{n},q,\underline{t}}(a)\mathcal{B}_{\underline{m},q,\underline{t}}(b)\big) + O_{n,k}\left(\frac{2^{2n}q^\epsilon\log(q^{n-1})}{q}\sum_{\substack{1\le a,b\le q\\ a\equiv\pm b \mod q}}\left(\frac{1}{a}+\frac{1}{b}+\frac{2\log(q^{n-1})}{q}\right)\right).$$

For Σ_+ we have $a\equiv +b \mod q$ and $a,b\le q$. Therefore $a=b$ and we have

$$\Sigma_+ = \sum_{1\le a\le q}\big(\mathcal{B}_{\underline{n},q,\underline{t}}(a)\mathcal{B}_{\underline{m},q,\underline{t}}(a)\big) + O_{n,k}\left(\frac{2^{2n}q^\epsilon\log(q^{n-1})}{q}\sum_{1\le a\le q}\left(\frac{2}{a}+\frac{2\log(q^{n-1})}{q}\right)\right).$$

For Σ_- we have $a,b\le q$ and $a\equiv -b \mod q$. Therefore,

$$\Sigma_- = \sum_{1\le a\le q}\big(\mathcal{B}_{\underline{n},q,\underline{t}}(a)\mathcal{B}_{\underline{m},q,\underline{t}}(q-a)\big) + O_{n,k}\left(\frac{2^{2n}q^\epsilon\log(q^{n-1})}{q}\sum_{1\le a\le q}\left(\frac{1}{a}+\frac{1}{q-a}+\frac{2\log(q^{n-1})}{q}\right)\right).$$

We bound the partial harmonic series again by $\log q$ to simplify both errors for Σ_+ and Σ_-. Consequently both error terms above can be bounded by

$$O_{n,k}\left(\frac{2^{2n}\log(q^{n-1})}{q^{1-\epsilon}}\left(2\log q+2\log(q^{n-1})\right)\right).$$

By combining the error terms, the moment sequence from Equation (12) is

$$\begin{aligned}M_q(\underline{n},\underline{m}) =&\frac{1}{\pi^{2n}}\Sigma_+(\underline{n},\underline{m})-\frac{1}{\pi^{2n}}\Sigma_-(\underline{n},\underline{m})+O\left(\frac{(\log q)^{2n}}{\sqrt{q}}\right)\\ =&\frac{1}{\pi^{2n}}\sum_{1\le a\le q}\left(\mathcal{B}_{\underline{n},q,\underline{t}}(a)\mathcal{B}_{\underline{m},q,\underline{t}}(a)\right)+\frac{1}{\pi^{2n}}\sum_{1\le a\le q}\left(\mathcal{B}_{\underline{n},q,\underline{t}}(a)\mathcal{B}_{\underline{m},q,\underline{t}}(q-a)\right)\\ &+O_{n,k}\left(\frac{2^{2n+2}(\log q^{n-1})^2}{q^{1-\epsilon}}\right)+O\left(\frac{(\log q)^{2n}}{\sqrt{q}}\right).\end{aligned}$$

Our aim is to only have one main term,

$$\frac{1}{\pi^{2n}}\sum_{a\ge 1}\left(\mathcal{B}_{\underline{n},\underline{t}}(a)\mathcal{B}_{\underline{m},\underline{t}}(a)\right),$$

so we want to first bound the term

$$\frac{1}{\pi^{2n}}\sum_{1\le a\le q}\left(\mathcal{B}_{\underline{n},q,\underline{t}}(a)\mathcal{B}_{\underline{m},q,\underline{t}}(q-a)\right), \tag{13}$$

and then extend the sum

$$\sum_{1\le a\le q}\left(\mathcal{B}_{\underline{n},q,\underline{t}}(a)\mathcal{B}_{\underline{m},q,\underline{t}}(a)\right)$$

over all positive integers.

To bound Equation (13) we again use the bound of $\mathcal{B}$ from Section 2.2 to show

$$\begin{aligned}\sum_{1\le a\le q}\left(\mathcal{B}_{\underline{n},q,\underline{t}}(a)\mathcal{B}_{\underline{m},q,\underline{t}}(q-a)\right)&\le 2^{2n}\sum_{1\le a\le q}\frac{d_n(a)d_k(a)}{a}\frac{d_n(q-a)d_k(q-a)}{q-a}\\ &\le 2^{2n}q^{\epsilon}\sum_{1\le a\le q}\frac{1}{a(q-a)}\le 2^{2n}q^{\epsilon}\frac{2\log q}{q}.\end{aligned}$$

As a result,

$$M_q(\underline{n},\underline{n})=\frac{1}{\pi^{2n}}\sum_{1\le a\le q}\left(\mathcal{B}_{\underline{n},q,\underline{t}}(a)\mathcal{B}_{\underline{m},q,\underline{t}}(a)\right)+O_{n,k}\left(\frac{\log q}{q^{1-\epsilon}}\right)+O_{n,k}\left(\frac{(\log q^{n-1})^2}{q^{1-\epsilon}}\right)+O\left(\frac{(\log q)^{2n}}{\sqrt{q}}\right).$$

This can be simplified, as $\mathcal{B}_{\underline{n},q,\underline{t}}$ is equivalent to $\mathcal{B}_{\underline{n},\underline{t}}$ in this case, and we can combine the errors. Therefore

$$M_q(\underline{n},\underline{n})=\frac{1}{\pi^{2n}}\sum_{1\le a\le q}\left(\mathcal{B}_{\underline{n},\underline{t}}(a)\mathcal{B}_{\underline{m},\underline{t}}(a)\right)+O\left(\frac{(\log q)^{2n}}{\sqrt{q}}\right). \tag{14}$$

The final step is to extend the main sum to infinity. We rewrite the sum

$$\sum_{1\le a\le q}\left(\mathcal{B}_{\underline{n},\underline{t}}(a)\mathcal{B}_{\underline{m},\underline{t}}(a)\right)=\sum_{a\ge 1}\left(\mathcal{B}_{\underline{n},\underline{t}}(a)\mathcal{B}_{\underline{m},\underline{t}}(a)\right)-\sum_{a>q}\left(\mathcal{B}_{\underline{n},\underline{t}}(a)\mathcal{B}_{\underline{m},\underline{t}}(a)\right).$$

By bounding $\mathcal{B}$ as before, the second sum has the upper bound

$$\sum_{a>q}\left(\mathcal{B}_{\underline{n},\underline{t}}(a)\mathcal{B}_{\underline{m},\underline{t}}(a)\right)\le 2^{2n}\sum_{a>q}\left(\frac{d_n^2(a)d_k^2(a)}{a^2}\right).$$

We take $d_k(a)^2=O(a^{2\epsilon_k})=:O_k(a^{\epsilon})$, so

$$\sum_{a>q}\left(\mathcal{B}_{\underline{n},\underline{t}}(a)\mathcal{B}_{\underline{m},\underline{t}}(a)\right)\le 2^{2n}\sum_{a>q}\left(\frac{d_n^2(a)a^{\epsilon}}{a^2}\right). \tag{15}$$

This can be further bounded by using Rankin's trick, which is the inequality

$$\sum_{n>q} a_n \leq \sum_{n=1}^{\infty} a_n \left(\frac{n}{q}\right)^{\psi}$$

for some $\psi > 0$. Here it is sufficient to take $\psi = 2/3 - \epsilon$, the same $\epsilon = \epsilon_k$ as in Equation (15), so we have

$$\sum_{a>q} \left(\mathcal{B}_{\underline{n},\underline{t}}(a)\mathcal{B}_{\underline{m},\underline{t}}(a)\right) \leq 2^{2n} \sum_{a>q} \left(\frac{d_n^2(a)}{a^{2-\epsilon}}\right) \leq \frac{2^{2n}}{q^{2/3-\epsilon}} \sum_{a=1}^{\infty} \frac{d_n^2(a)}{a^{4/3}}.$$

By a simple comparison test we can see that the infinite sum converges. More explicitly we can use results from Bober and Goldmakher [3, Proposition 3.2], stated in Equation (11), where they bound $\sum d_n(a)a^{-2\sigma}$ by

$$\sum_{a=1}^{\infty} \frac{d_n(a)^2}{a^{2\sigma}} \leq \exp\left(2n\sigma \log\log(2n)^{1/\sigma} + \frac{(2n)^{1/\sigma}}{2\sigma - 1} + O\left(\frac{n}{2\sigma-1} + \frac{(2n)^{1/\sigma}}{\log(3(2n)^{1/\sigma-1})}\right)\right).$$

In our case, $\sigma := 2/3$, so our moment $M_q(\underline{n},\underline{n})$, shown last in Equation (14), is

$$\begin{aligned} M_q(\underline{n},\underline{n}) =& \frac{1}{\pi^{2n}} \sum_{a\geq 1} \left(\mathcal{B}_{\underline{n},\underline{t}}(a)\mathcal{B}_{\underline{m},\underline{t}}(a)\right) + O\left(\frac{(\log q)^{2n}}{\sqrt{q}}\right) \\ &+ O\left(\frac{2^{2n}}{\pi^{2n} q^{2/3-\epsilon}} \exp\left(\frac{4n}{3}\log\log(2n)^{3/2} + 3(2n)^{3/2} + O\left(3n + \frac{(2n)^{3/2}}{\log(3(2n)^{1/2})}\right)\right)\right). \end{aligned}$$

By combining errors, we have proved Proposition 2.7. □

To finish proving Lemma 2.6 we prove Proposition 2.8, showing how the expectation also equals the sum

$$\frac{1}{\pi^{2n}} \sum_{a\geq 1} \left(\mathcal{B}_{\underline{n},\underline{t}}(a)\mathcal{B}_{\underline{m},\underline{t}}(a)\right).$$

Proof of Proposition 2.8. We are interested in the expectation

$$M(\underline{n},\underline{m}) = \mathbb{E}\left(\prod_{i=1}^{k} F(t_i)^{n_i}\overline{F(t_i)}^{m_i}\right).$$

Using Equation (8) from Section 2.1, and $n := n_1 + \cdots n_k = m_1 + \cdots + m_k$, the moment is equivalent to

$$M(\underline{n},\underline{m}) = \mathbb{E}\left(\frac{\eta^n\overline{\eta}^n}{\pi^{2n}}\right) \sum_{a\geq 1} \mathcal{B}_{\underline{n},\underline{t}}(a)\mathcal{B}_{\underline{m},\underline{t}}(a).$$

Therefore we have

$$\mathbb{E}\left(\prod_{i=1}^{k} F_-(t_i)^{n_i}\overline{F_-(t_i)}^{m_i}\right) = \frac{1}{\pi^{2n}} \sum_{a\geq 1} \mathcal{B}_{\underline{n},\underline{t}}(a)\mathcal{B}_{\underline{m},\underline{t}}(a),$$

proving Proposition 2.8. □

Therefore we have shown the multivariate moment sequence

$$M_q(\underline{n},\underline{m}) = \frac{2}{\phi(q)} \sum_{\chi \text{ odd}} \prod_{i=1}^{k} f_\chi(t_i)^{n_i}\overline{f_\chi(t_i)}^{m_i}$$

converges, as $q \to \infty$ through the primes, to

$$\mathbb{E}\left(\prod_{i=1}^{k} F(t_i)^{n_i}\overline{F(t_i)}^{m_i}\right),$$

for all k-tuples $\underline{n}, \underline{m}$ and $0 \leq t_1 < \cdots < t_k \leq 1$. Therefore $(\mathcal{F}_{q,\pm})_{q \text{ prime}}$, the distribution of odd/even character paths f_χ modulo q, converges to $F_\pm$ as $q \to \infty$ in the sense of convergence in finite distributions.

3. Relative Compactness of the Sequence of Distributions

In the previous section we showed $(\mathcal{F}_q)$ converges in finite distributions to the process F as $q \to \infty$ through the primes. If we can prove the sequence of distributions is relatively compact, then it follows that $(\mathcal{F}_q)$ converges in distribution to F [1, Example 5.1]. This is much stronger than convergence of finite-dimensional distributions and concludes the proof of Theorem 1.1.

Prohorov's Theorem [1, Theorem 5.1] states that if a sequence of probability measures is tight, then it must be relatively compact. For this we use Kolmorogorov's tightness criterion, quoted from Revuz and Yor:

Proposition 3.1. [15, Th. XIII.1.8] *Let $(L_p(t))_{t\in[0,1]}$ be a sequence of $C([0,1])-$valued processes such that $L_p(0) = 0$ for all p. If there exist constants $\alpha > 0$, $\delta > 0$ and $C \geq 0$ such that for any p and any $s < t$ in $[0,1]$ we have*

$$\mathbb{E}(|L_p(t) - L_p(s)|^\alpha) \leq C|t - s|^{1+\delta}$$

then the sequence $(L_p(t))$ is tight.

For our sequence of processes $(\mathcal{F}_q(t))_{t\in[0,1]}$ we have $f_\chi(0) = 0$ for all q. We also have the trivial bound

$$|f_\chi(t) - f_\chi(s)| \leq \sqrt{q}|t - s|,$$

leading to

$$\mathbb{E}\left|f_\chi(t) - f_\chi(s)\right|^{2k} \leq q^k|t - s|^{2k}.$$

Therefore for $k \geq 1$ if we take $|t - s| < \frac{1}{q^{1-\epsilon}}$ for $\epsilon \in (0, \frac{k-1}{2k-1})$, the tightness criterion holds as

$$\mathbb{E}\left|f_\chi(t) - f_\chi(s)\right|^{2k} \leq |t - s|^{2k-\frac{k}{1-\epsilon}} =: |t - s|^{1+\delta_1}, \tag{16}$$

where $\delta_1 := \frac{k-1+\epsilon(1-2k)}{1-\epsilon}$.

Therefore if we show the tightness condition also holds for $|t - s| > \frac{1}{q^{1-\epsilon}}$ then $\mathcal{F}_q$ is everywhere relatively compact. Multiple mathematicians have found results bounding the average of the difference of character sums. For example, Cochrane and Zheng [4] prove for positive integers k and Dirichlet characters modulo prime q,

$$\frac{1}{q-1}\sum_{\chi\neq\chi_0}\left|\sum_{n=s+1}^{s+t}\chi(n)\right|^{2k} \ll_{\epsilon,k} q^{k-1+\epsilon} + |t - s|^k q^\epsilon.$$

To prove tightness however we need the $|t - s|$ term independent of q.

Lemma 3.2. *Let q be an odd prime. For all $\epsilon \in (0,1)$, there exists absolute constants $C_1(\epsilon), C_2$ independent of q such that for all $0 \leq s < t \leq 1$,*

$$\mathbb{E}\left|f_\chi(t) - f_\chi(s)\right|^4 \leq C_1(\epsilon)|t - s|^{1+\delta_2} + C_2\frac{(\log q)^4}{q},$$

where $\delta_2 := 1 - \epsilon$.

This lemma can be applied to characters of all moduli, not just primes, but for our work it is sufficient to look only at primitive characters. Clearly if $|t - s| \geq \frac{(\log q)^4}{q}$ then the equation becomes

$$\mathbb{E}\left|f_\chi(t) - f_\chi(s)\right|^4 \leq C|t - s|^{1+\delta},$$

which, combined with Equation (16) above, proves the sequence $(\mathcal{F}_q)$ is tight for all $s, t \in [0,1]$.

Lemma 3.2 is similar to a result of Bober and Goldmakher [3, Lemma 4.1] and we use parts of their work in the proof. Unlike Section 2.3, we will consider the odd and even case at the same time.

Proof of Lemma 2.3. Using the Fourier expansion of f_χ, the difference $(f_\chi(t) - f_\chi(s))$ can be written as

$$\frac{\tau(\chi)}{2\pi i\sqrt{q}} \sum_{1\le |n|\le q} \frac{\overline{\chi}(n)}{n} e(-sn)\left(1 - e(-(t-s)n)\right) + O\left(\frac{\log q}{\sqrt{q}}\right).$$

Consequently,

$$\left|f_\chi(t) - f_\chi(s)\right|^4 \le \frac{2^4}{\pi^4}\left| \sum_{1\le n\le q} \frac{\overline{\chi}(n)}{n} e(-sn)\left(1 - e(-(t-s)n)\right)\right|^4 + O\left(\frac{(3+\log q)^4}{q^2}\right).$$

Similar to Section 2.1 and [3, Lemma 4.1], we define

$$b(n) = \begin{cases} \sum_{\substack{n_1 n_2 = n \\ n_i \le q}} \prod_{j=1}^2 \left(\frac{e(-sn_j)}{n_j}\left(1 - e\left(-(t-s)n_j\right)\right)\right) & (n,q) = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore,

$$\left| \sum_{1\le n\le q} \frac{\overline{\chi}(n)}{n} e(-sn)\left(1 - e(-(t-s)n)\right)\right|^4 = \left|\sum_{1\le n\le q^2} \overline{\chi}(n) b(n)\right|^2.$$

The sum $b(n)$ can be bounded using $(1 - e(x)) \le \min\{2, x\}$, so

$$|b(n)| \le d(n) \min\left\{\frac{2^2}{n}, (2\pi(t-s))^2\right\}. \tag{17}$$

As a result, taking $n = a + mq$,

$$\mathbb{E}\left|\sum_{1\le n\le q^2} \chi(n) b(n)\right|^2 = \sum_{a=1}^{q} \left|\sum_{m=0}^{q} b(a+mq)\right|^2 \le 4\sum_{a=1}^{q} |b(a)|^2 + 4\sum_{a=1}^{q}\left|2^2 \sum_{m=1}^{q} \frac{d(a+mq)}{a+mq}\right|^2. \tag{18}$$

We are interested in bounding the latter inner sum,

$$\mathcal{D}_a := \sum_{m=1}^{q} \frac{d(a+mq)}{a+mq} = \sum_{\substack{m\le (a+q^2) \\ m\equiv a(q)}} \frac{d(m)}{m}.$$

By Abel summation this is

$$\frac{1}{a+q^2} \sum_{\substack{m\le (a+q^2) \\ m\equiv a(q)}} d(m) + \int_1^{(a+q^2)} \frac{1}{t^2} \sum_{\substack{m\le t \\ m\equiv a(q)}} d(a)dt.$$

We approximate the divisor sum by [7][14]

$$\sum_{\substack{n\le (a+q^2) \\ n\equiv a(q)}} d(n) = \frac{1}{\phi(q)} \sum_{\substack{n\le (a+q^2) \\ (n,q)=1}} d(n) + O((q^{1/2} + q^{2/3})q^\epsilon).$$

and can further bound the sum by [6, Equation 27.11.2]

$$\sum_{n\le (a+q^2)} d(n) = (a+q^2)\log(a+q^2) + (2\gamma - 1)(a+q^2) + O(q).$$

Combining these two equations,

$$\sum_{\substack{m\le x \\ m\equiv a(q)}} d(m) = O\left(\frac{x\log x}{\phi(q)}\right).$$

We take $x = (a+q^2)$ and $x = t$ to show

$$\mathcal{D}_a = \frac{1}{a+q^2} \sum_{\substack{m \le (a+q^2) \\ m \equiv a(q)}} d(m) + \int_1^{(a+q^2)} \frac{1}{t^2} \sum_{\substack{m \le t \\ m \equiv a(q)}} d(a) dt = O\left(\frac{\log q}{\phi(q)}\right) + O\left(\frac{(\log q)^2}{\phi(q)}\right).$$

As a result, Equation (18) becomes

$$\sum_{a=1}^{q} \left| \sum_{m=0}^{q} b(a+mq) \right|^2 \le 4 \sum_{a=1}^{q} |b(a)|^2 + O\left(\sum_{a=1}^{q} \left| \frac{(\log q)^2}{q} \right|^2 \right) = 4 \sum_{a=1}^{q} |b(a)|^2 + O\left(\frac{(\log q)^4}{q}\right).$$

Therefore the only sum left to evaluate is $\sum_{a \le q} |b(a)|^2$. Using the bound from Equation (17) and splitting the cases $\frac{1}{a} > \pi^2(t-s)^2$ and $\frac{1}{a} < \pi^2(t-s)^2$, we have

$$\sum_{a=1}^{q} |b(a)|^2 \le 2^4 \left(\pi^4 (t-s)^4 \sum_{a \le \pi^{-2}(t-s)^{-2}} d(a)^2 + \sum_{\pi^{-2}(t-s)^{-2} < a \le q} \frac{d(a)^2}{a^2} \right).$$

We combine the two sums by Rankin's trick. Taking $x = \pi^{-2}(t-s)^{-2}$,

$$\frac{1}{x^2} \sum_{a \le x} d(a)^2 \le \frac{x^{\sigma_1}}{x^2} \sum_{a=1}^{\infty} \frac{d(a)^2}{a^{\sigma_1}}, \qquad 1 < \sigma_1 < 2,$$

$$\sum_{a \ge x} \frac{d(a)^2}{a^2} \le \frac{1}{x^{\sigma_2}} \sum_{a=1}^{\infty} \frac{d(a)^2}{a^{2-\sigma_2}}, \qquad 0 < \sigma_2 < 1.$$

Note σ_1, σ_2 are bounded so that the sums converge and tend to zero as $x \to \infty$. These sums are one of Ramanujan's identities [13, Section 1.3.1, Question 5]. For $Re(s) > 1$,

$$\sum_{n=1}^{\infty} \frac{d(n)^2}{n^s} = \frac{\zeta(s)^4}{\zeta(2s)}.$$

Therefore

$$\sum_{a=1}^{q} |b(a)|^2 \le 2^4 \left(\frac{1}{x^{2-\sigma_1}} \frac{\zeta(\sigma_1)^4}{\zeta(2\sigma_1)} + \frac{1}{x^{\sigma_2}} \frac{\zeta(2-\sigma_2)^4}{\zeta(2(2-\sigma_2))} \right).$$

Taking $\sigma := \min(2-\sigma_1, \sigma_2) \in (0,1)$ and substituting back $\pi^2(t-s)^{-2} = x$,

$$\frac{1}{x^{2-\sigma_1}} \frac{\zeta(\sigma_1)^4}{\zeta(2\sigma_1)} + \frac{1}{x^{\sigma_2}} \frac{\zeta(2-\sigma_2)^4}{\zeta(2(2-\sigma_2))} \le \frac{C}{x^{\sigma}} = C\pi^{2\sigma}(t-s)^{2\sigma},$$

for some $C = C(\sigma) > 0$. As a result,

$$\mathbb{E}\left| f_\chi(t) - f_\chi(s) \right|^4 \le C(t-s)^{2\sigma} + O\left(\frac{(\log q)^4}{q}\right).$$

Taking $\sigma = 1 - \epsilon$ and therefore $2\sigma = 2 - \epsilon$, we have completed the proof. □

Lemma 3.2 shows that Kolmorogorov's tightness criterion holds for

$$|t-s|^{1+\delta_2} \gg \frac{(\log q)^4}{q},$$

where we take $\alpha = 4$ from Proposition 3.1. Therefore, combining with Equation (16), we have shown for constant K

$$\mathbb{E}\left| f_\chi(t) - f_\chi(s) \right|^4 \le \begin{cases} |t-s|^{1+\delta_1}, & |t-s| \le q^{-(1-\epsilon_1)} \\ C|t-s|^{1+\delta_2}, & |t-s| \ge (\log q)^4 q^{-1} \end{cases},$$

for $\delta_1 = \frac{1-3\epsilon_1}{1-\epsilon_1}$ and $\delta_2 = 1 - \epsilon_2$, where $\epsilon_1 \in (0, \frac{1}{3})$ and $\epsilon_2 \in (0,1)$.

For large enough q, we have

$$\frac{(\log q)^4}{q} < \frac{1}{q^{1-\epsilon_1}}.$$

Therefore taking $\delta := \min(\delta_1, \delta_2)$, Kolmorogorov's tightness criterion holds for all t, s and $(\mathcal{F}_q)$ is tight. As a result, $(\mathcal{F}_q)_{q \text{ prime}}$ converges in distribution to the random process F as $q \to \infty$, proving Theorem 1.1. This concurs with the result from Bober, Goldmakher, Granville and Koukoulopoulos for their distribution function

$$\Phi_q(\tau) := \frac{1}{\phi(q)} \#\left\{ \chi \mod q : \max_t |S_\chi(t)| > \frac{e^\gamma}{\pi}\tau \right\}$$

weakly converging to their limiting function [2, Theorem 1.4]

$$\Phi(\tau) := \mathbb{P}\left(\max_t |F(t)| > 2e^\gamma \tau \right).$$

4. Properties of $F(t)$

Recall the random process F, defined by the infinite sum

$$F(t) = \frac{\eta}{\pi} \sum_{n \neq 0} \frac{1 - e(nt)}{n} X_n,$$

where X_k are Steinhaus random multiplicative functions and η is a random variable uniformly distributed on the unit circle. In this section we deal with samples of the random process and prove some of their properties.

We define an arbitrary sample function as

$$G(t) := \frac{c}{\pi} \sum_{n \neq 0} \frac{1 - e(nt)}{n} \alpha_n,$$

for c, α_n on the unit circle and α_n completely multiplicative.

Lemma 4.1. *All samples of the random process $F(t)$ are in $L^2([0,1])$ with probability 1.*

Proof. Firstly, we split the sample function as

$$G(t) = \frac{c}{\pi} \sum_{n \neq 0} \frac{\alpha_n}{n} + \frac{c}{\pi} \sum_{n \neq 0} \frac{e(nt)}{n} \alpha_n =: \frac{c}{\pi} G_0 + \frac{c}{\pi} G_1(t).$$

The function $G_1(t)$ is in $L^2([0,1])$, as

$$\int_0^1 |G_1(t)|^2 dt = \sum_{n,m \neq 0} \frac{\alpha_n \overline{\alpha_m}}{mn} \int_0^1 e((n-m)t) dt = \sum_{n \neq 0} \frac{1}{n^2} < \infty.$$

Therefore we only need to prove constant term G_0 converges with probability 1. This can be shown by calculating the expectation,

$$\mathbb{E}(G_0) = \prod_p \mathbb{E}\left(1 - \frac{X_p}{p}\right)^{-1} = \prod_p \int_0^1 \left(1 - \frac{e(t_p)}{p}\right)^{-1} dt_p = \prod_p 1 = 1.$$

Then for all $x > 0$,

$$\mathbb{P}(G_0 > x) \leq \frac{\mathbb{E}(G_0)}{x} = \frac{1}{x}.$$

By letting $x \to \infty$, we see $\mathbb{P}(G_0 > x) \to 0$, so G_0 converges with probability 1.

Therefore since G is an arbitrary sample function, all samples of F are almost surely in $L^2([0,1])$. $\square$

Showing sample functions of F are almost surely continuous is non trivial. To show this we let $y > 1$ and consider the y-smooth and 'y-rough' parts of the infinite sum $G(t)$. Let

$$S_y := \sum_{\substack{n \neq 0 \\ P^+(|n|) \leq y}} \frac{1 - e(nt)}{n} \alpha_n \qquad \text{and } R_y := \sum_{\substack{n \neq 0 \\ P^+(|n|) > y}} \frac{1 - e(nt)}{n} \alpha_n,$$

where $P^+(n)$ is the largest prime factor of n. Note that $S_y + R_y = \frac{\pi}{c} G(t)$ and the two functions are not independent. These sums can also be seen as smooth and rough samples of the random process F.

Lemma 4.2. *For all $\epsilon > 0$ and sufficiently large $y > 1$,*

$$\mathbb{P}\left(\|R_y\|_\infty > \epsilon\right) \ll \exp\left\{-\frac{2\epsilon^2 y^{1/3}}{\log y}\left(\log \epsilon + 19 \log\log y\right)\right\}$$

independently of S_y, where $\|\cdot\|_\infty := \max_{t\in[0,1]} |\cdot|$. Notably for all $\epsilon > 0$, we have $\mathbb{P}\left(\|R_y\|_\infty > \epsilon\right) \to 0$ as $y \to \infty$.

Proof. For all $y \geq 1$,

$$\sum_{\substack{n\geq 1 \\ P^+(n)>y}} \frac{1-e(nt)}{n}\alpha_n = \sum_{\substack{n\geq 1 \\ P^+(n)\leq y}} \frac{\alpha_n}{n} \sum_{\substack{m>y \\ P^-(m)>y}} \frac{1-e(mnt)}{m}\alpha_m,$$

where $P^-(m)$ is the smallest prime factor of m. By setting

$$T(\alpha) := \max_{t\in[0,1]}\left|\sum_{\substack{m>y \\ P^-(m)>y}} \frac{1-e(mt)}{m}\alpha_m\right|,$$

we have

$$\|R_y\|_\infty := \max_{t\in[0,1]}\left|\sum_{\substack{n\neq 0 \\ P^+(n)>y}} \frac{1-e(nt)}{n}\alpha_n\right| \leq 2 \sum_{\substack{n\geq 1 \\ P^+(n)\leq y}} \frac{T(\alpha)}{n}.$$

By Merten's estimate (see for example [13, Lemma 7.5]),

$$\sum_{P+(n)\leq y} \frac{1}{n} \asymp \log y,$$

$\|R_y\|_\infty$ is bounded above by

$$\|R_y\|_\infty \ll T(\alpha)\log y.$$

Using results of Bober, Goldmakher, Granville and Koukoulopoulos for $q \to \infty$ [2, Proposition 5.2], we know for $k \geq 3$ and $y \geq k^3$,

$$\mathbb{E}\left[\left(\sum_{\substack{n>y \\ P^-(n)>y}} \frac{1-e(mt)}{m}\alpha_m\right)^{2k}\right] \ll \frac{1}{(\log y)^{40k}}.$$

We set ρ_y as the probability $T(\alpha)\log y > \epsilon$ and

$$k = \left\lfloor \frac{\epsilon^2 y^{1/3}}{\log y}\right\rfloor.$$

Therefore,

$$\rho_y \leq \left(\frac{\log y}{\epsilon}\right)^{2k} \mathbb{E}(T(\alpha)^{2k}) \leq \frac{\epsilon^{-2k}}{(\log y)^{38k}} \leq \left(\frac{\epsilon^{-1}}{(\log y)^{19}}\right)^{\frac{2\epsilon^2 y^{1/3}}{\log y}},$$

and consequently

$$0 \leq \lim_{y\to\infty} \rho_y \leq \lim_{y\to\infty}\left(\frac{\epsilon^{-1}}{(\log y)^{19}}\right)^{\frac{2\epsilon^2 y^{1/3}}{\log y}} = 0.$$

□

Subsequently we get the following corollary,

Corollary 4.3. *All samples of the random process F are almost surely continuous.*

Proof. Consider the sequence of functions $(S_y)_y$, defined by

$$S_y(t) := \frac{c}{\pi} \sum_{\substack{n\neq 0 \\ P^+(|n|)\leq y}} \frac{1-e(nt)}{n}\alpha_n,$$

where c, α_n are on the unit circle and $\{\alpha_n\}$ are completely multiplicative.

The function S_y is the $y-$smooth part of a sample of the random process F, which we call $G(t)$. We know that as $y \to \infty$, S_y converges to

$$G(t) = \frac{c}{\pi} \sum_{n\neq 0} \frac{1-e(nt)}{n}\alpha_n,$$

and the infinite sum defining G converges with probability 1 [2]. From Lemma 4.2 we have that

$$\mathbb{P}\left(\|R_y\|_\infty > \epsilon\right) \to 0$$

as $y \to \infty$. As a result, the sequence (S_y) uniformly converges to its limit, which by the Uniform Limit Theorem must be continuous. Therefore all samples of F are almost surely continuous. □

This proves Theorem 1.2, that every sample of $F(t) \in C([0,1])$ with probability 1.

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