Homework 2
Sky Liu

1. Suppose that one letter is to be selected at random from the 42 letters in the sentence, "The shortest distance between two points is a taxi." If Y denotes the number of letters in the word in which the selected letter appears, what is the value of E(Y)?

	The	shortest	distance	between	two	points	is	a	taxi.
value of Y	3	8	8	7	3	6	2	1	4
$P(Y=k)$	$\frac{3}{42}$	$\frac{8}{42}$	$\frac{8}{42}$	$\frac{7}{42}$	$\frac{3}{42}$	$\frac{6}{42}$	$\frac{2}{42}$	$\frac{1}{42}$	$\frac{4}{42}$

The distribution of Y:

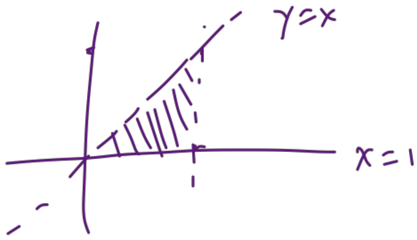
$P(Y=1)=\frac{1}{42}$, $P(Y=2)=\frac{2}{42}$, $P(Y=3)=\frac{3+3}{42}=\frac{6}{42}$

$P(Y=4)=\frac{4}{42}$, $P(Y=6)=\frac{6}{42}$, $P(Y=7)=\frac{7}{42}$, $P(Y=8)=\frac{8+8}{42}=\frac{16}{42}$

$$E(Y)=1\times\frac{1}{42}+2\cdot\frac{2}{42}+3\cdot\frac{6}{42}+4\cdot\frac{4}{42}+6\cdot\frac{6}{42}+7\cdot\frac{7}{42}+8\cdot\frac{16}{42}$$

$$=\frac{1+4+18+16+36+49+128}{42}=\frac{252}{42}=6$$

2. Suppose that $X$and Y have a continuous joint distribution for which the joint ppf is:
$f(x,y)=12y^2$ for $0\le y\le x\le 1$
Find the value of $E(XY)$.



$0\le X\le 1$, $0\le y\le x$

$$E(xy)=\int_0^1\int_0^x 12y^2\cdot xy\,dy\,dx$$

$$=\int_0^1\int_0^x 12xy^3\,dy\,dx$$

$$=\int_0^1 3xy^4\Big|_0^x\,dx$$

$$=\int_0^1 3x^5\,dx$$

$$=\frac{1}{2}x^6\Big|_0^1=\frac{1}{2}$$

3. Suppose that three random variables X_1, X_2, X_3 form a random sample from the uniform distribution on the interval $[0, 1]$. Find $E[(X_1 - 2X_2 + X_3)^2]$.

$X_1, X_2, X_3 \sim \text{Uniform}(0,1)$

$E(X_1) = E(X_2) = E(X_3) = \frac{1}{2}$

$E(X_1^2) = E(X_2^2) = E(X_3^2) = \int_0^1 \frac{1}{1-0} x^2 dx = \frac{1}{3} x^3 \Big|_0^1 = \frac{1}{3}$

$E(X_1 X_2) = E(X_2 X_3) = E(X_1 X_3) = \int_0^1 \int_0^1 x_1 x_3 \, dx_1 dx_3 = \frac{1}{4}$

$E[(X_1 - 2X_2 + X_3)^2] = E(X_1^2 - 4X_1X_2 + 2X_1X_3 + 4X_2^2 - 4X_2X_3 + X_3^2)$

$= \frac{1}{3} - 1 + \frac{1}{2} + \frac{4}{3} - 1 + \frac{1}{3} = \frac{1}{2}$

4. X has pdf

$f(x) = e^{-x}, \quad x > 0$

$Y = e^{\frac{3X}{4}}$

Find $E(Y)$

$f(x) = e^{-x}, \; x > 0. \quad Y = e^{\frac{3}{4}x}$

$E(Y) = \int_0^\infty e^{\frac{3}{4}x} f(x) \, dx = \int_0^\infty e^{\frac{3}{4}x} e^{-x} dx$

$= \int_0^\infty e^{-\frac{1}{4}x} dx = -4 e^{-\frac{1}{4}x} \Big|_0^\infty = 4$

5. X is the outcome of rolling a fair die.

$Y = g(X) = 2X^2 + 1$

Find $E(Y)$

X	Y	P
1	3	1/6
2	9	1/6
3	19	1/6
4	33	1/6
5	51	1/6
6	73	1/6

$E(Y) = \frac{1}{6} \times (3 + 9 + 19 + 33 + 51 + 73)$

$= \frac{1}{6} \times 188 = 31.3$

6. X has pdf

   $f(x) = 2(1-x), \quad 0 < x < 1$

   $Y = (2X+1)$

   Find $E(Y^2)$.

$$E(X) = \int_0^1 x \cdot 2(1-x)\,dx = \int_0^1 2x - 2x^2\,dx = x^2 - \tfrac{2}{3}x^3\Big|_0^1 = \tfrac{1}{3}$$

$$E(X^2) = \int_0^1 x^2 \cdot 2(1-x)\,dx = \int_0^1 2x^2 - 2x^3\,dx = \tfrac{2}{3}x^3 - \tfrac{1}{2}x^4\Big|_0^1 = \tfrac{1}{6}$$

$$E(Y^2) = E[(2X+1)^2] = E(4X^2 + 4X + 1)$$

$$= E(4X^2) + E(4X) + E(1)$$

$$= 4 \cdot \tfrac{1}{6} + 4 \cdot \tfrac{1}{3} + 1 = 3$$

7. Remember the binomial theorem: $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k \quad for n \in \mathbb{Z}^+$

   Show that $E[(ax+b)^n] = \sum_{i=0}^{n} \binom{n}{i} a^{n-i} b^i E(X^{n-i})$

$$E[(aX+b)^n] = E\left(\sum_{i=0}^{n} C_n^i\, aX^{n-i}\, b^i\right)$$

$$= \sum_{i=0}^{n} C_n^i\, a^{n-i} \cdot b^i\, E(X^{n-i})$$

8. The proportion of defective parts in a large shipment is p. A random sample of n parts is selected from the shipment. Let X denote the number of defective parts in the sample, and Y denote the number of good parts in the sample. Find $E(X-Y)$.

   If the sample size is 20 and p is 5%, what is $E(X-Y)$? Write out your answer as a complete sentence that expresses the meaning of your result.

defect rate: p.

good rate: $1-p$. $\Rightarrow$ $X \sim b(n, p)$ $\Rightarrow$ $E(X) = np$

sample size: n. $\quad Y \sim b(n, 1-p)$ $\quad E(Y) = n(1-p)$

$$E(X-Y) = E(X) - E(Y) = np - n(1-p) = 2np - n$$

let $n = 20$, $p = 0.05$.

$$E(X-Y) = 2 \cdot 20 \cdot 0.05 - 20 = -18$$

The expected value of defected parts out of good parts is -18.