Estimation Homework

1. Find the MLE of the proportion of Crunchy Oats cereal purchased by men. based on a rancom sample of 70 purchases, in which 58 were made by women and 12 were made by men. Find the M.L.E. of p.

$$\hat{p} = \frac{58}{70} = 82.86\%$$

2. Suppose that $X_1, \cdots, X_n$ form a random sample from the Bernoulli distribution with parameter θ, which is unknown, except that $0 < \theta < 1$. Show that the MLE of *theta* does not exist if every observed value is 0 or if every observed value is 1.

$X_1 \cdots X_n \sim \text{Bernoulli}(\theta)$

$f(x) = \theta^x (1-\theta)^{1-x}$

$$\begin{aligned} \mathcal{L}(\theta, X_1 \cdots X_n) &= \theta^{X_1}(1-\theta)^{1-X_1} \cdots \theta^{X_n}(1-\theta)^{1-X_n} \\ &= \theta^{\Sigma x} (1-\theta)^{n-\Sigma x} \\ &= \delta x \log(\theta) + (n - \Sigma x) \log(1-\theta) \end{aligned}$$

$$\frac{d\mathcal{L}(\theta, X_1 \cdots X_n)}{d\theta} = \frac{\Sigma x}{\theta} + \frac{n-\Sigma x}{1-\theta} = 0$$

$$\hat{\theta} = \frac{\Sigma x}{n}$$

when every observed value is 0. $\hat{\theta} = \frac{0}{n} = 0$

when every observed value is 1. $\hat{\theta} = \frac{n}{n} = 1$.

which is contradicted with $0 < \theta < 1$.

3. Suppose that $X_1, \cdots, X_n$ form a random sample from a Poisson distribution for which the mean λ is unknown, $(\lambda > 0)$.

- Determine the MLE of λ, assuming that at least one of the observed values is different from λ.
- Show that the MLE of λ does not exist if every observed value is 0.

$X_1 \dots X_n \sim \text{Poisson}(\lambda)$, λ is unknown. $\lambda > 0$

$$f(x) = \frac{\lambda^x e^{-\lambda}}{x!}$$

$$L(\lambda, x_1 \dots x_n) = \frac{\lambda^{x_1} e^{-\lambda}}{x_1!} \cdots \frac{\lambda^{x_n} e^{-\lambda}}{x_n!}$$

$$= \prod_{i=1}^{n} \frac{1}{x_i!} \lambda^{x_i} e^{-\lambda}$$

$$= \ell \; \log\lambda \, \Sigma x - n\lambda - \Sigma \log x!$$

$$\frac{d\,\mathcal{L}(\lambda, X_1 \dots X_n)}{d\lambda} = \frac{1}{\lambda} \Sigma X_i - n = 0$$

$$\hat{\lambda} = \frac{\Sigma X}{n} = \bar{X}$$

When every observed value is 0.

$\hat{\lambda} = 0$, which is sontradicted with $\lambda > 0$

4. Suppose that $X_1, \cdots, X_n$ form a random sample from a normal distribution for which the mean μ is known, but the variance σ^2 is unknown. Find the MLE of σ^2.

$X_1, \ldots X_n \sim N(\mu, \sigma^2)$. μ is known. σ^2 is unknown

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

$$L(\mu, \sigma; X_1 \ldots X_n) = \prod_{i=1}^{n} \frac{1}{\sigma\sqrt{2\pi}} e^{\left(-\frac{1}{2}\left(\frac{X_i-\mu}{\sigma}\right)^2\right)}$$

$$= \ell - n\log(\sigma) - \frac{n}{2}\log(2\pi) - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(X_i-\mu)^2$$

$$\frac{d\ell(\mu, \sigma; X_1 \ldots X_n)}{d\sigma} = -\frac{n}{\sigma} + \sigma^{-3}\sum_{i=1}^{n}(X_i-\mu)^2 = 0$$

$$\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i-\mu)^2$$