CSE 211: Discrete Mathematics	**(Due: 17/01/21)**

Homework #4

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Course Policy: Read all the instructions below carefully before you start working on the assignment, and before you make a submission.

- It is not a group homework. Do not share your answers to anyone in any circumstance. Any cheating means at least -100 for both sides.
- Do not take any information from Internet.
- No late homework will be accepted.
- For any questions about the homework, send an email to gizemsungu@gtu.edu.tr
- The homeworks (both latex and pdf files in a zip file) will be submitted into the course page of Moodle.
- The latex, pdf and zip files of the homeworks should be saved as "Name_Surname_StudentId".{tex, pdf, zip}.
- If the answers of the homeworks have only calculations without any formula or any explanation -when needed- will get zero.
- Writing the homeworks on Latex is strongly suggested. However, hand-written paper is still accepted **IFF** hand writing of the student is clear and understandable to read, and the paper is well-organized. Otherwise, the assistant cannot grade the student's homework.

Problem 1 (15+15=30 points)

Consider the non-homogeneous linear recurrence relation $a_n = 3a_{n-1} + 2^n$.

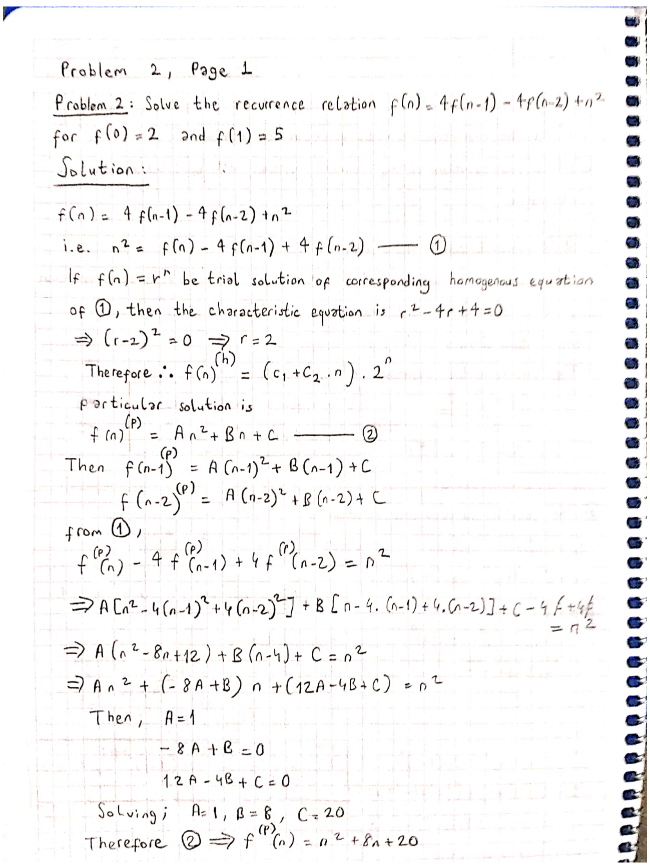
(a) Show that whether $a_n = -2^{n+1}$ is a solution of the given recurrence relation or not. Show your work step by step.
(Solution)
Given $a_n = -2^{n+1}$
$\therefore a_{n-1} = -2^{(n-1)+1} = -2^n$
Now $3a_{n-1} + 2^n = 3(-2^n) + 2^n = 2^n(-3+1)$
i.e. $3a_{n-1} + 2^n = 2^n(-2) = -2^{n+1}$
$\Rightarrow 3a_{n-1} + 2^n = a_n$
Thus, $a_n = -2^{n+1}$ satisfies the relation $a_n = 3a_{n-1} + 2^n$
$\therefore a_n = -2^{n+1}$ is a solution of given recurrence relation.

(b) Find the solution with $a_0 = 1$.
(Solution)
Now the equation can be written as
(i) $a_n - 3a_{n-1} = 2^n$
$\therefore$ The characteristic equation of associated homogenous equation $a_n - 3a_{n-1} = 0$ is $r - 3 = 0 \Rightarrow r = 3$
Hence, non-homogenous part is 2^n, so let the general solution be
$a_n = Ar^n + B2^n$
i.e. (ii) $a_n = A3^n + B2^n$ where A, B are constants.
Then $a_{n-1} = A3^{n-1} + B2^{n-1}$
Now, from (i) we have
$(A3^n + B2^n) - 3(A3^{n-1} + B2^{n-1}) = 2^n$
$\Rightarrow A3^n + B2^n - A3^n - \frac{3B}{2}2^n = 2^n$

$\Rightarrow (B - \frac{3B}{2})2^n = 2^n$
comparing both sides we get
$\frac{-B}{2} = 1 \Rightarrow$ B = -2
Now, using $a_0 = 1$ in (ii) we get
A+B = 1 $\Rightarrow$ A = 1-B $\Rightarrow$ A = 1-(-2) $\Rightarrow$ A = 3
Thus, the required solution of given recurrence relation is
$a_n = 3 * 3^n - 2 * 2^n$ from (ii)
i.e. $a_n = 3^{n+1} - 2^{n+1}$

Problem 2 (35 points)

Solve the recurrence relation f(n) = 4f(n-1) - 4f(n-2) + n^2 for f(0) = 2 and f(1) = 5.
(Solution)



Problem 2, Page 1

Problem 2: Solve the recurrence relation $f(n) = 4f(n-1) - 4f(n-2) + n^2$ for $f(0) = 2$ and $f(1) = 5$

Solution:

$f(n) = 4f(n-1) - 4f(n-2) + n^2$

i.e. $n^2 = f(n) - 4f(n-1) + 4f(n-2)$ —— (1)

If $f(n) = r^n$ be trial solution of corresponding homogenous equation of (1), then the characteristic equation is $r^2 - 4r + 4 = 0$

$\Rightarrow (r-2)^2 = 0 \Rightarrow r = 2$

Therefore $\therefore f(n)^{(h)} = (c_1 + C_2 . n) . 2^n$

particular solution is

$f(n)^{(P)} = An^2 + Bn + C$ —— (2)

Then $f(n-1)^{(P)} = A(n-1)^2 + B(n-1) + C$

$f(n-2)^{(P)} = A(n-2)^2 + B(n-2) + C$

from (1),

$f^{(P)}(n) - 4f^{(P)}(n-1) + 4f^{(P)}(n-2) = n^2$

$\Rightarrow A[n^2 - 4(n-1)^2 + 4(n-2)^2] + B[n - 4.(n-1) + 4.(n-2)] + C - 4\cancel{f} + 4\cancel{f} = n^2$

$\Rightarrow A(n^2 - 8n + 12) + B(n-4) + C = n^2$

$\Rightarrow An^2 + (-8A + B)n + (12A - 4B + C) = n^2$

Then, $A = 1$

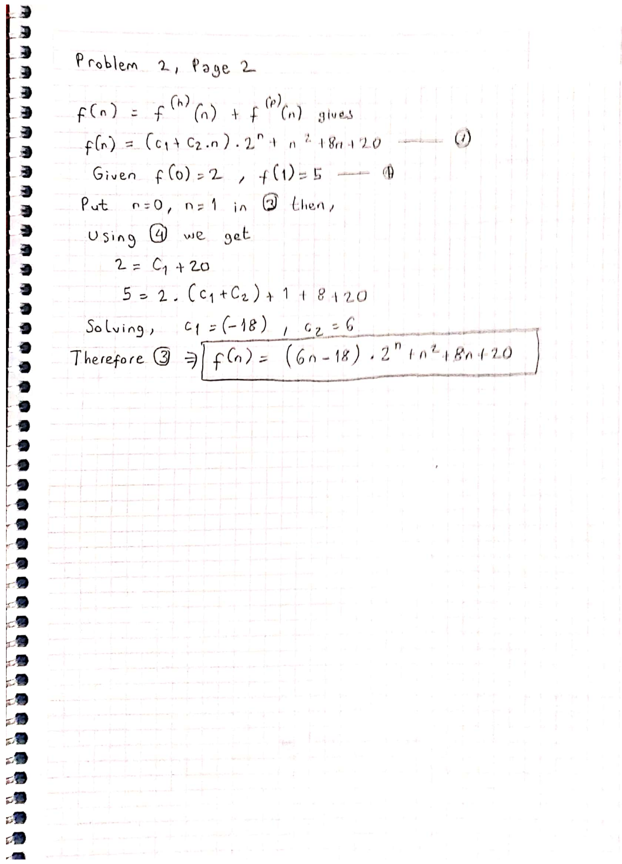
$-8A + B = 0$

$12A - 4B + C = 0$

Solving; $A = 1$, $B = 8$, $C = 20$

Therefore (2) $\Rightarrow f^{(P)}(n) = n^2 + 8n + 20$

Figure 1: Problem 2, Page 1



Problem 2, Page 2

$f(n) = f^{(h)}(n) + f^{(p)}(n)$ gives

$f(n) = (c_1 + c_2 . n) . 2^n + n^2 + 8n + 20$ — 3

Given $f(0) = 2$, $f(1) = 5$ — 4

Put $n = 0$, $n = 1$ in 3 then,

Using 4 we get

$2 = C_1 + 20$

$5 = 2 . (c_1 + C_2) + 1 + 8 + 20$

Solving, $c_1 = (-18)$, $c_2 = 6$

Therefore 3 $\Rightarrow$ $\boxed{f(n) = (6n - 18) . 2^n + n^2 + 8n + 20}$

Figure 2: Problem 2, Page 2

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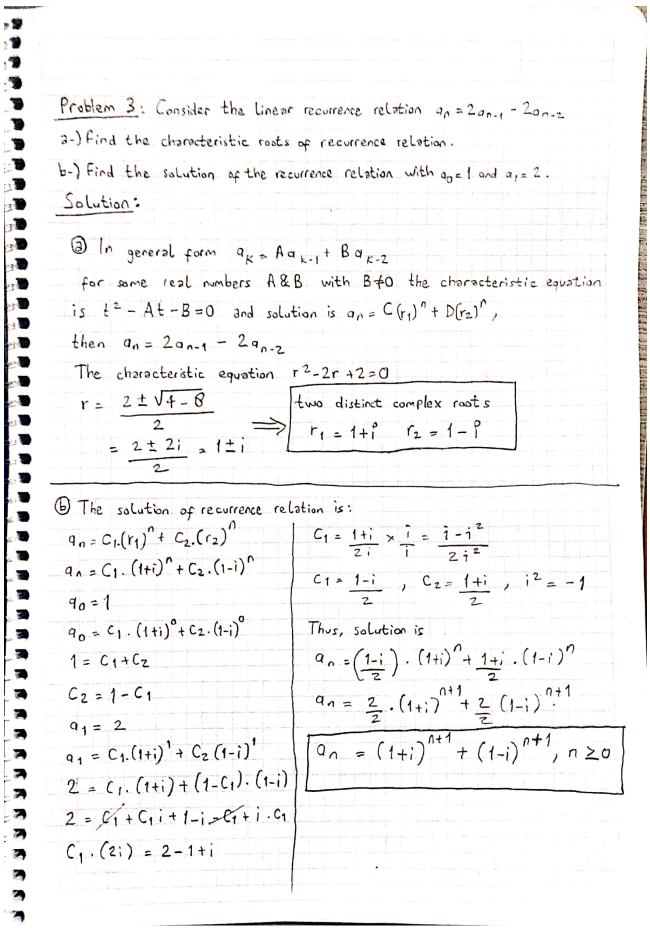
Problem 3 (20+15 = 35 points)

Consider the linear homogeneous recurrence relation $a_n = 2a_{n-1}$ - $2a_{n-2}$.
(a) Find the characteristic roots of the recurrence relation.
(Solution)

(b) Find the solution of the recurrence relation with $a_0 = 1$ and $a_1 = 2$.
(Solution)



Problem 3: Consider the linear recurrence relation $a_n = 2a_{n-1} - 2a_{n-2}$

a-) Find the characteristic roots of recurrence relation.

b-) Find the solution of the recurrence relation with $a_0 = 1$ and $a_1 = 2$.

Solution:

ⓐ In general form $a_K = Aa_{K-1} + Ba_{K-2}$

for some real numbers A & B with $B \neq 0$ the characteristic equation is $t^2 - At - B = 0$ and solution is $a_n = C(r_1)^n + D(r_2)^n$,

then $a_n = 2a_{n-1} - 2a_{n-2}$

The characteristic equation $r^2 - 2r + 2 = 0$

$r = \frac{2 \pm \sqrt{4-8}}{2} = \frac{2 \pm 2i}{2} = 1 \pm i \Rightarrow$ $\boxed{\text{two distinct complex roots } r_1 = 1+i \quad r_2 = 1-i}$

ⓑ The solution of recurrence relation is:

$a_n = C_1 \cdot (r_1)^n + C_2 \cdot (r_2)^n$

$a_n = C_1 \cdot (1+i)^n + C_2 \cdot (1-i)^n$

$a_0 = 1$

$a_0 = C_1 \cdot (1+i)^0 + C_2 \cdot (1-i)^0$

$1 = C_1 + C_2$

$C_2 = 1 - C_1$

$a_1 = 2$

$a_1 = C_1 \cdot (1+i)^1 + C_2(1-i)^1$

$2 = C_1 \cdot (1+i) + (1 - C_1) \cdot (1-i)$

$2 = \cancel{C_1} + C_1 i + 1 - i - \cancel{C_1} + i \cdot C_1$

$C_1 \cdot (2i) = 2 - 1 + i$

$C_1 = \frac{1+i}{2i} \times \frac{i}{i} = \frac{i - i^2}{2i^2}$

$C_1 = \frac{1-i}{2}$, $C_2 = \frac{1+i}{2}$, $i^2 = -1$

Thus, solution is

$a_n = \left(\frac{1-i}{2}\right) \cdot (1+i)^n + \frac{1+i}{2} \cdot (1-i)^n$

$a_n = \frac{2}{2} \cdot (1+i)^{n+1} + \frac{2}{2}(1-i)^{n+1}$

$\boxed{a_n = (1+i)^{n+1} + (1-i)^{n+1}, \; n \geq 0}$

Figure 3: Problem 3