Digital Speech Processing—Lecture 14

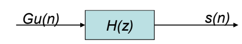
Linear Predictive Coding (LPC)-Lattice Methods, Applications

Prediction Error Signal

1. Speech Production Model

$$s(n) = \sum_{k=1}^{p} a_k s(n-k) + Gu(n)$$

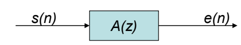
$$H(z) = \frac{S(z)}{U(z)} = \frac{G}{1 - \sum_{k=1}^{p} a_k z^{-k}}$$



2. LPC Model:

$$e(n) = s(n) - \tilde{s}(n) = s(n) - \sum_{k=1}^{p} \alpha_k s(n-k)$$

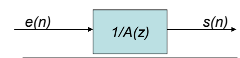
$$A(z) = \frac{E(z)}{S(z)} = 1 - \sum_{k=1}^{p} \alpha_k z^{-k}$$



3. LPC Error Model:

$$\frac{1}{A(z)} = \frac{S(z)}{E(z)} = \frac{1}{1 - \sum_{k=1}^{p} \alpha_k z^{-k}}$$

$$s(n) = e(n) + \sum_{k=1}^{p} \alpha_k s(n-k)$$



Perfect reconstruction even if a_k not equal to α_k

Lattice Formulations of LP

- both covariance and autocorrelation methods use two step solutions
  1. computation of a matrix of correlation values
  2. efficient solution of a set of linear equations
- another class of LP methods, called lattice methods, has evolved in which the two steps are combined into a recursive algorithm for determining LP parameters
- begin with Durbin algorithm--at the i^{th} stage the set of coefficients $\{\alpha_j^{(i)}, j = 1,2,\dots,i\}$ are coefficients of the i^{th} order optimum LP

Lattice Formulations of LP

- define the system function of the i^{th} order inverse filter (prediction error filter) as

$$A^{(i)}(z) = 1 - \sum_{k=1}^{i} \alpha_k^{(i)} z^{-k}$$

- if the input to this filter is the input segment $s_{\hat{n}}(m) = s(\hat{n}+m)w(m)$, with output $e_{\hat{n}}^{(i)}(m) = e^{(i)}(\hat{n}+m)$

$$e^{(i)}(m) = s(m) - \sum_{k=1}^{i} \alpha_k^{(i)} s(m-k)$$

- where we have dropped subscript $\hat{n}$ - the absolute location of the signal
- the z-transform gives

$$\boxed{E^{(i)}(z) = A^{(i)}(z)S(z) = \left(1 - \sum_{k=1}^{i} \alpha_k^{(i)} z^{-k}\right) S(z)}$$

Lattice Formulations of LP

- using the steps of the Durbin recursion

$$(\alpha_j^{(i)} = \alpha_j^{(i-1)} - k_i \alpha_{i-j}^{(i-1)}, \text{ and } \alpha_i^{(i)} = k_i)$$

 we can obtain a recurrence formula for $A^{(i)}(z)$ of the form

$$A^{(i)}(z) = A^{(i-1)}(z) - k_i z^{-i} A^{(i-1)}(z^{-1})$$

- giving for the error transform the expression

$$E^{(i)}(z) = A^{(i-1)}(z)S(z) - k_i z^{-i} A^{(i-1)}(z^{-1})S(z)$$

$$e^{(i)}(m) = e^{(i-1)}(m) - k_i b^{(i-1)}(m-1)$$

Lattice Formulations of LP

- where we can interpret the first term as the z-transform of the forward prediction error for an $(i-1)^{st}$ order predictor, and the second term can be similarly interpreted based on defining a backward prediction error

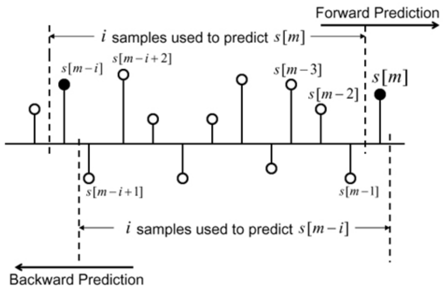
$$B^{(i)}(z) = z^{-i} A^{(i)}(z^{-1})S(z) = z^{-i} A^{(i-1)}(z^{-1})S(z) - k_i A^{(i-1)}(z)S(z)$$
$$= z^{-1} B^{(i-1)}(z) - k_i E^{(i-1)}(z)$$

- with inverse transform

$$b^{(i)}(m) = s(m-i) - \sum_{k=1}^{i} \alpha_k^{(i)} s(m+k-i) = b^{(i-1)}(m-1) - k_i e^{(i-1)}(m)$$

- with the interpretation that we are attempting to predict $s(m-i)$ from the i samples of the input that follow $s(m-i)$ => we are doing a ***backward*** prediction and $b^{(i)}(m)$ is called the ***backward prediction error sequence***

Lattice Formulations of LP



same set of samples is used to forward predict *s(m)* and backward predict *s(m-i)*

Lattice Formulations of LP

- the prediction error transform and sequence $E^{(i)}(z), e^{(i)}(m)$ can now be expressed in terms of forward and backward errors, namely

$$E^{(i)}(z) = E^{(i-1)}(z) - k_i z^{-1} B^{(i-1)}(z)$$

$$\boxed{e^{(i)}(m) = e^{(i-1)}(m) - k_i b^{(i-1)}(m-1)} \quad \boxed{*1}$$

- similarly we can derive an expression for the backward error transform and sequence at sample m of the form

$$B^{(i)}(z) = z^{-1} B^{(i-1)}(z) - k_i E^{(i-1)}(z)$$

$$\boxed{b^{(i)}(m) = b^{(i-1)}(m-1) - k_i e^{(i-1)}(m)} \quad \boxed{*2}$$

- these two equations define the forward and backward prediction error for an i^{th} order predictor in terms of the corresponding prediction errors of an $(i-1)^{th}$ order predictor, with the reminder that a zeroth order predictor does no prediction, so

$$\boxed{e^{(0)}(m) = b^{(0)}(m) = s(m) \quad 0 \le m \le L-1} \quad \boxed{*3}$$

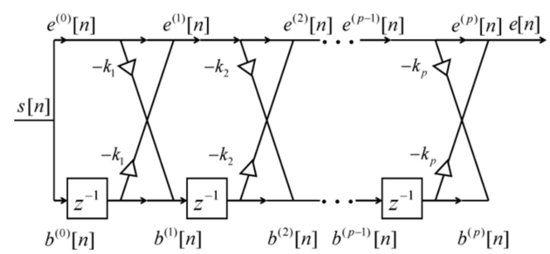
$$E(z) = E^{(p)}(z) = A(z) / S(z)$$

Lattice Formulations of LP

 Assume we know k_1 (from external computation):

1. we compute $e^{(1)}[m]$ and $b^{(1)}[m]$ from $s[m],\ 0 \le m \le L$ using Eqs. $*1$ and $*2$
2. we next compute $e^{(2)}[m]$ and $b^{(2)}[m]$ for $0 \le m \le L+1$ using Eqs. $*1$ and $*2$
3. extend solution (lattice) to p sections giving $e^{(p)}[m]$ and $b^{(p)}[m]$ for $0 \le m \le L+p-1$
4. solution is $e[n] = e^{(p)}[n]$ at the output of the p^{th} lattice section

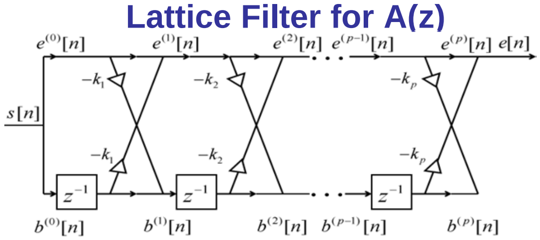
Lattice Formulations of LP



- lattice network with p sections—the output of which is the forward prediction error
- digital network implementation of the prediction error filter with transfer function *A(z)*

- no direct correlations
- no alphas
- k's computed from forward and backward error signals

Lattice Filter for A(z)



$$e^{(0)}[n] = b^{(0)}[n] = s[n] \quad 0 \le n \le L-1$$

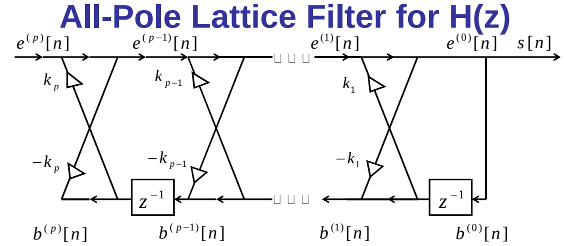
$$e^{(i)}[n] = e^{(i-1)}[n] - k_i b^{(i-1)}[n-1] \quad i = 1,2,\ldots,p,\ 0 \le n \le L-1+i$$

$$b^{(i)}[n] = -k_i e^{(i-1)}[n] + b^{(i-1)}[n-1] \quad i = 1,2,\ldots,p,\ 0 \le n \le L-1+i$$

$$e[n] = e^{(p)}[n], \quad 0 \le n \le L-1+p$$

$$\boxed{E(z) = A(z)S(z)}$$

All-Pole Lattice Filter for H(z)



$$e^{(p)}[n] = e[n], \quad 0 \le n \le L-1+p$$

$$e^{(i-1)}[n] = e^{(i)}[n] + k_i b^{(i-1)}[n-1]$$

$$i = p, p-1, \ldots, 1,\ 0 \le n \le L-1+i-1$$

$$b^{(i)}[n] = -k_i e^{(i-1)}[n] + b^{(i-1)}[n-1]$$

$$i = p, p-1, \ldots, 1, \quad 0 \le n \le L-1+i$$

$$s[n] = e^{(0)}[n] = b^{(0)}[n],\ 0 \le n \le L-1$$

$$\boxed{S(z) = \frac{1}{A(z)} E(z)}$$

All-Pole Lattice Filter for H(z)

1. since $b^{(i)}[-1]=0,\ \forall i,$ we can first solve for $e^{(i-1)}[0]$ for $i=p, p-1,\ldots,1,$ using the relationship: $e^{(i-1)}[0]=e^{(i)}[0]$
2. since $b^{(0)}[0]=e^{(0)}[0]$ we can then solve for $b^{(i)}[0]$ for $i=1,2,\ldots,p$ using the equation: $b^{(i)}[0]=-k_i\, e^{(i-1)}[0]$
3. we can now begin to solve for $e^{(i-1)}[1]$ as: $e^{(i-1)}[1]=e^{(i)}[1]+k_i\, b^{(i-1)}[0],\ i=p, p-1,\ldots,1$
4. we set $b^{(0)}[1]=e^{(0)}[1]$ and we can then solve for $b^{(i)}[1]$ for $i=1,2,\ldots,p$ using the equation: $b^{(i)}[1]=-k_i\, e^{(i-1)}[1]+b^{(i-1)}[0],\ i=1,2,\ldots,p$
5. we iterate for $n=2,3,\ldots,N-1$ and end up with $s[n]=e^{(0)}[n]=b^{(0)}[n]$

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Lattice Formulations of LP

- the lattice structure comes directly out of the Durbin algorithm
- the k_i parameters are obtained from the Durbin equations
- the predictor coefficients, α_k, do not appear explicitly in the lattice structure
- can relate the k_i parameters to the forward and backward errors via

$$\boxed{k_i=\frac{\sum_{m=0}^{L-1+i} e^{(i-1)}(m)b^{(i-1)}(m-1)}{\left\{\left[\sum_{m=0}^{L-1+i}[e^{(i-1)}(m)]^2\right]\left[\sum_{m=0}^{L-1+i}[b^{(i-1)}(m-1)]^2\right]\right\}^{1/2}}} \quad \boxed{*4}$$

- where k_i is a normalized cross correlation between the forward and backward prediction error, and is therefore called a partial correlation or PARCOR coefficient
- can compute predictor coefficients recursively from the PARCOR coefficients

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Direct Computation of *k* Parameters

- assume $s[n]$ non-zero for $0\le n\le L-1$
- assume k_i chosen to minimize total energy of the forward (or backward) prediction errors
- we can then minimize ***forward*** prediction error as:

$$E_{forward}^{(i)}=\sum_{m=0}^{L-1+i}\left[e^{(i)}(m)\right]^2=\sum_{m=0}^{L-1+i}\left[e^{(i-1)}(m)-k_i b^{(i-1)}(m-1)\right]^2$$

$$\frac{\partial E_{forward}^{(i)}}{\partial k_i}=0=-2\sum_{m=0}^{L-1+i}\left[e^{(i-1)}(m)-k_i b^{(i-1)}(m-1)\right]b^{(i-1)}(m-1)$$

$$k_i^{forward}=\frac{\sum_{m=0}^{L-1+i}\left[e^{(i-1)}(m)\cdot b^{(i-1)}(m-1)\right]}{\sum_{m=0}^{L-1+i}\left[b^{(i-1)}(m-1)\right]^2}$$

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Direct Computation of k Parameters

- we can also choose to minimize the ***backward*** prediction error

$$E_{backward}^{(i)}=\sum_{m=0}^{L-1+i}\left[b^{(i)}(m)\right]^2=\sum_{m=0}^{L-1+i}\left[-k_i e^{(i-1)}(m)+b^{(i-1)}(m-1)\right]^2$$

$$\frac{\partial E_{backward}^{(i)}}{\partial k_i}=0=-2\sum_{m=0}^{L-1+i}\left[-k_i e^{(i-1)}(m)+b^{(i-1)}(m-1)\right]e^{(i-1)}(m)$$

$$k_i^{backward}=\frac{\sum_{m=0}^{L-1+i}\left[e^{(i-1)}(m)b^{(i-1)}(m-1)\right]}{\sum_{m=0}^{L-1+i}\left[e^{(i-1)}(m)\right]^2}$$

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Direct Computation of k Parameters

- if we window and sum over all time, then

$$\sum_{m=0}^{L-1+i}\left[e^{(i-1)}(m)\right]^2=\sum_{m=0}^{L-1+i}\left[b^{(i-1)}(m-1)\right]^2$$

therefore

$$k_i^{PARCOR}=\sqrt{k_i^{forward}k_i^{backward}}=k_i^{forward}=k_i^{backward}$$

$$=\frac{\sum_{m=0}^{L-1+i} e^{(i-1)}(m)b^{(i-1)}(m-1)}{\left\{\sum_{m=0}^{L-1+i}\left[e^{(i-1)}(m)\right]^2\sum_{m=0}^{L-1+i}\left[b^{(i-1)}(m-1)\right]^2\right\}^{1/2}}$$

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Direct Computation of *k* Parameters

- minimize ***sum*** of forward and backward prediction errors over fixed interval (covariance method)

$$E_{Burg}^{(i)}=\sum_{m=0}^{L-1}\left\{\left[e^{(i)}(m)\right]^2+\left[b^{(i)}(m)\right]^2\right\}$$

$$=\sum_{m=0}^{L-1}\left[e^{(i-1)}(m)-k_i b^{(i-1)}(m-1)\right]^2+\sum_{m=-\infty}^{\infty}\left[-k_i e^{(i-1)}(m)+b^{(i-1)}(m-1)\right]^2$$

$$\frac{\partial E_{Burg}^{(i)}}{\partial k_i}=0=-2\sum_{m=0}^{L-1}\left[e^{(i-1)}(m)-k_i b^{(i-1)}(m-1)\right]b^{(i-1)}(m-1)$$

$$-2\sum_{m=0}^{L-1}\left[-k_i e^{(i-1)}(m)+b^{(i-1)}(m-1)\right]e^{(i-1)}(m)$$

$$k_i^{Burg}=\frac{2\sum_{m=0}^{L-1}\left[e^{(i-1)}(m)\cdot b^{(i-1)}(m-1)\right]}{\sum_{m=0}^{L-1}\left[e^{(i-1)}(m)\right]^2+\sum_{m=0}^{L-1}\left[b^{(i-1)}(m-1)\right]^2}$$

- $-1\le k_i^{Burg}\le 1$ ***always***

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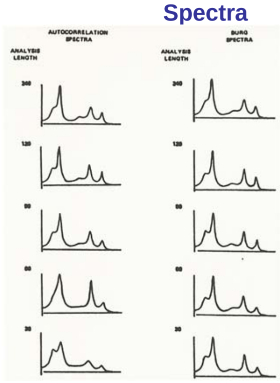
Comparison of Autocorrelation and Burg Spectra



• significantly less smearing of formant peaks using Burg method

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Summary of Lattice Procedure

- steps involved in determining the predictor coefficients and the k parameters for the lattice method are as follows
  1. initial condition, $e^{(0)}(m) = s(m) = b^{(0)}(m)$ from Eq. *3
  2. compute $k_1 = \alpha_1^{(1)}$ from Eq. *4
  3. determine forward and backward prediciton errors $e^{(1)}(m), b^{(1)}(m)$ from Eqs. *1 and *2
  4. set $i = 2$
  5. determine $k_i = \alpha_i^{(i)}$ from Eq. *4
  6. determine $\alpha_j^{(i)}$ for $j = 1, 2, \ldots, i-1$ from Durbin iteration
  7. determine $e^{(i)}(m)$ and $b^{(i)}(m)$ from Eqs. *1 and *2
  8. set $i = i+1$
  9. if i is less than or equal to p, go to step 5
  10. procedure is terminated
- predictor coefficients obtained directly from speech samples => without calculation of autocorrelation function
- method is guaranteed to yield stable filters ($|k_i|$<1) without using window

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Summary of Lattice Procedure

Lattice Algorithms

$\mathcal{E}^{(0)} = R[0]$ (1)
$e^{(0)}[n] = b^{(0)}[n] = s[n], \quad 0 \le n \le L-1$ (2)
for $i = 1, 2, \ldots, p$

compute k_i using either Eq. (9.125) or Eq. (9.128) (3)

compute $e^{(i)}[n], \quad 0 \le n \le L-1+i$ using Eq. (9.117b) (4a)
compute $b^{(i)}[n], \quad 0 \le n \le L-1+i$ using Eq. (9.117c) (4b)

$\alpha_i^{(i)} = k_i$ (5)
compute predictor coefficients
if $i > 1$ then for $j = 1, 2, \ldots, i-1$
$\alpha_j^{(i)} = \alpha_j^{(i-1)} - k_i \alpha_{i-j}^{(i-1)}$ (6)
end
compute mean-squared energy
$\mathcal{E}^{(i)} = (1 - k_i^2)\mathcal{E}^{(i-1)}$ (7)
end
$\alpha_j = \alpha_j^{(p)} \quad j = 1, 2, \ldots, p$ (8)
$e[n] = e^{(p)}[n], \quad 0 \le n \le L-1+p$ (9)

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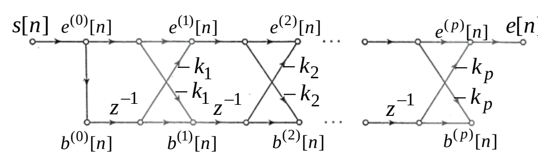
Summary of Lattice Procedure



$$\boxed{e^{(0)}(m) = b^{(0)}(m) = s(m)} \quad \boxed{*3}$$

$$\boxed{k_i = \frac{\sum_{m=0}^{L-1+i} e^{(i-1)}(m) b^{(i-1)}(m-1)}{\left\{ \left[\sum_{m=0}^{L-1+i} [e^{(i-1)}(m)]^2 \right] \left[\sum_{m=0}^{L-1+i} [b^{(i-1)}(m-1)]^2 \right] \right\}^{1/2}}} \quad \boxed{*4}$$

$$\boxed{e^{(i)}(m) = e^{(i-1)}(m) - k_i b^{(i-1)}(m-1)} \quad \boxed{*1}$$

$$\boxed{b^{(i)}(m) = b^{(i-1)}(m-1) - k_i e^{(i-1)}(m)} \quad \boxed{*2}$$

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Prediction Error Signal

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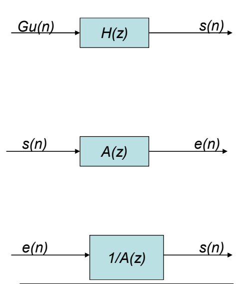
$$e(n) = s(n) - \tilde{s}(n) = s(n) - \sum_{k=1}^{p} \alpha_k s(n-k)$$

$$A(z) = \frac{E(z)}{S(z)} = 1 - \sum_{k=1}^{p} \alpha_k z^{-k}$$

3. LPC Error Model:

$$\frac{1}{A(z)} = \frac{S(z)}{E(z)} = \frac{1}{1 - \sum_{k=1}^{p} \alpha_k z^{-k}}$$

$$s(n) = e(n) + \sum_{k=1}^{p} \alpha_k s(n-k)$$



Perfect reconstruction even if a_k not equal to α_k

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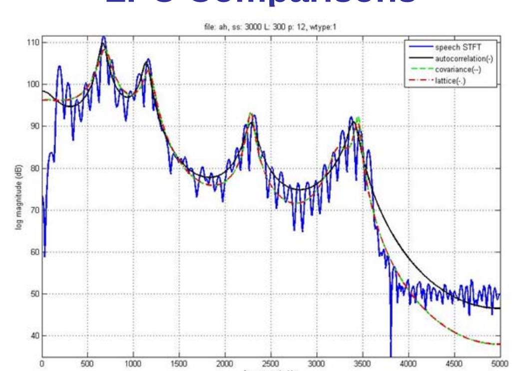
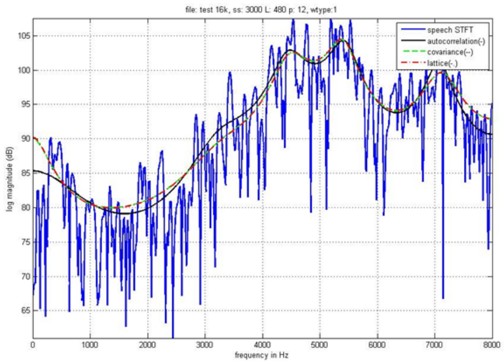
LPC Comparisons



LPC Comparisons



Comparisons Between LP Methods

- the various LP solution techniques can be compared in a number of ways, including the following:
  - computational issues
  - numerical issues
  - stability of solution
  - number of poles (order of predictor)
  - window/section length for analysis

LP Solution Computations

	Covariance Method (Cholesky Decomposition)	*Autocorrelation Method* (Durbin Method)	*Lattice Method* (Burg Method)
Storage			
Data	L_1	L_2	L_3
Matrix	$\sim p^2/2$	$\sim p$	—
Window	0	L_2	—
Computation (Multiplications)			
Windowing	0	L_2	—
Correlation	$\sim L_1 p$	$\sim L_2 p$	—
Matrix Solution	$\sim p^3$	$\sim p^2$	$5L_3 p$

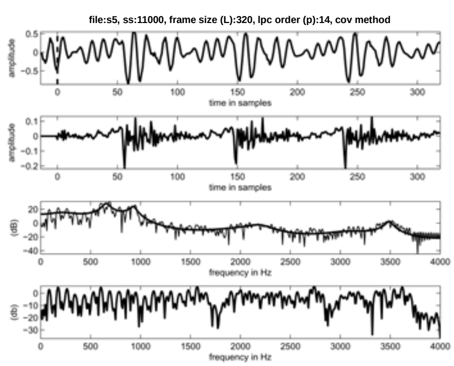
- assume $L_1 \approx L_2 >> p$; choose values of L_1=300, L_2=300, L_3=300, p=10
- computation for
  - covariance method ≈ $L_1 p + p^3$ ≈ 4000 *,+
  - autocorrelation method ≈ $L_2 p + p^2$ ≈ 3100 *,+
  - lattice method ≈ $5L_3 p$ ≈ 15000 *,+

LP Solution Comparisons

- stability
  - guaranteed for autocorrelation method
  - cannot be guaranteed for covariance method; as window size gets larger, this almost always makes the system stable
  - guaranteed for lattice method
- choice of LP analysis parameters
  - need 2 poles for each vocal tract resonance below $F_s/2$
  - need 3-4 poles to represent source shape and radiation load
  - use values of p ≈ 13-14

The Prediction Error Signal

LP Speech Analysis



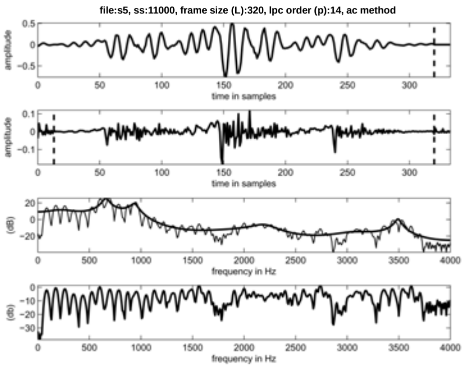
Top panel: speech signal

Second panel: error signal

Third panel: log magnitude spectra of signal and LP model

Fourth panel: log magnitude spectrum of error signal

LP Speech Analysis



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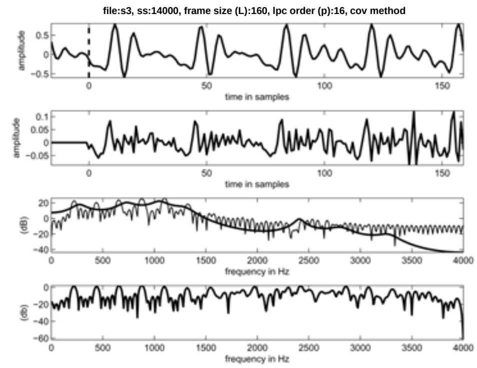
LP Speech Analysis



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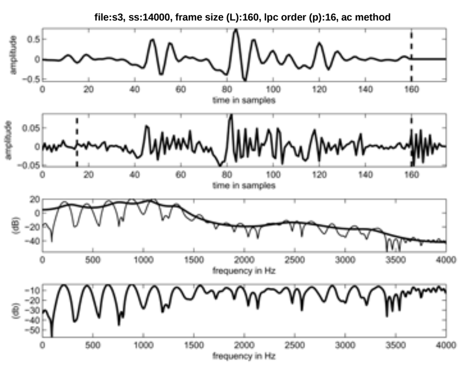
LP Speech Analysis



Top panel: speech signal

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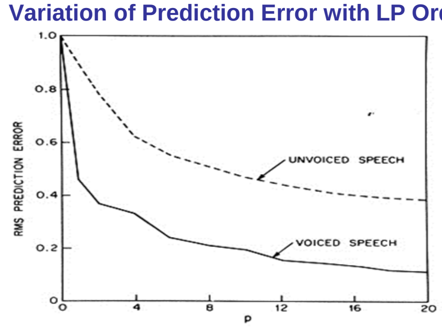
Variation of Prediction Error with LP Order



- prediction error flattens at around *p*=13-14
- normalized prediction error for unvoiced speech larger than for voiced speech => model isn't quite as accurate

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LP Solution Comparisons

- choice of LP analysis parameters
  - use small values of *L* for reduced computation
  - use *l* order of several pitch periods for reliable prediction-especially when using tapered window
  - use *l* from 100-400 samples at 10 kHz for autocorrelation
  - for lattice and covariance methods, *L* as small as 2 *p* has been used (within pitch periods); however if pitch pulse occurs within window, bad prediction results => use much larger values of *L*

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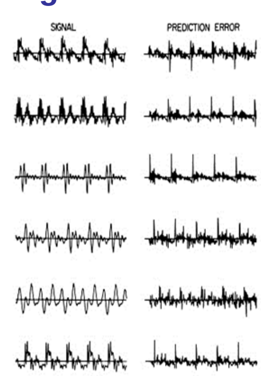
Prediction Error Signal Behavior

- the prediction error signal is computed as

$$e(n) = s(n) - \sum_{k=1}^{p} \alpha_k s(n-k) = Gu(n)$$

- $e(n)$ should be large at the beginning of each pitch period (voiced speech) => good signal for pitch detection
- can perform autocorrelation on $e(n)$ and detect largest peak
- error spectrum is approximately flat-so effects of formants on pitch detection are minimized



Normalized Mean-Squared Error

- for autocorrelation method

$$V_{\hat{n}} = \frac{\sum_{m=0}^{L+p-1} e_{\hat{n}}^2(m)}{\sum_{m=0}^{L-1} s_{\hat{n}}^2(m)}$$

- for covariance method

$$V_{\hat{n}} = \frac{\sum_{m=0}^{L-1} e_{\hat{n}}^2(m)}{\sum_{m=0}^{L-1} s_{\hat{n}}^2(m)}$$

- the prediction error sequence (defining $\alpha_0 = -1$) is

$$e_{\hat{n}}(m) = -\sum_{k=0}^{p} \alpha_k s_{\hat{n}}(m-k)$$

- giving many forms for the normalized error

$$V_{\hat{n}} = \sum_{i=0}^{p} \sum_{j=0}^{p} \alpha_i \frac{\phi_{\hat{n}}(i,j)}{\phi_{\hat{n}}(0,0)} \alpha_j$$

$$V_{\hat{n}} = -\sum_{i=0}^{p} \alpha_i \frac{\phi_{\hat{n}}(i,0)}{\phi_{\hat{n}}(0,0)}$$

$$V_{\hat{n}} = \prod_{i=1}^{p} (1-k_i^2)$$

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Experimental Evaluations of LPC Parameters

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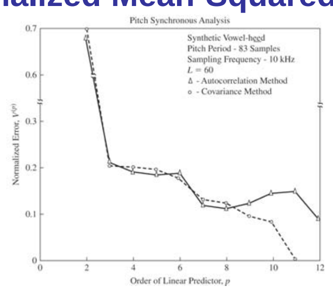
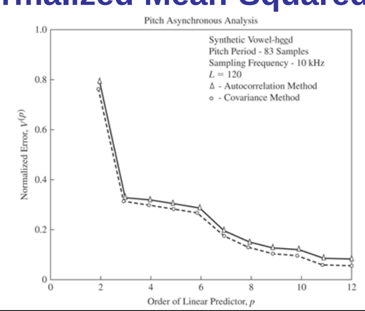
Normalized Mean-Squared Error



- V_n versus p for synthetic vowel, pitch period of 83 samples, L=60, pitch synchronous analysis
- covariance method-error goes to zero at p=11, the order of the synthesis filter
- autocorrelation method, V_n≈0.1 for p≥7 since error dominated by prediction at beginning of interval

Normalized Mean-Squared Error



- normalized error versus p for pitch asynchronous analysis, L=120, normalized error falls to 0.1 near p=11 for both covariance and autocorrelation methods

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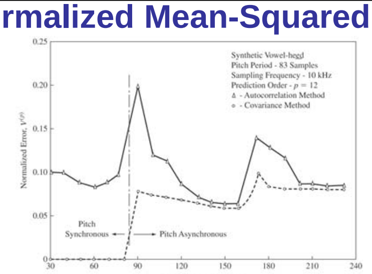
Normalized Mean-Squared Error



- normalized error versus L, p=12
- for L < pitch period (83 samples), covariance method gives smaller normalized error than autocorrelation method
- for values of L at or near multiples of pitch period, normalized error jumps due to large prediction error in vicinity of pitch pulse
- when L > 2 * pitch period, normalized error same for both autocorrelation and covariance methods

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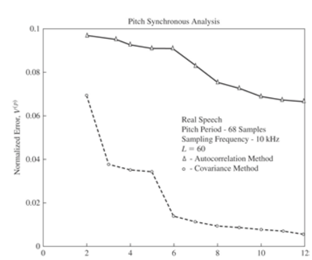
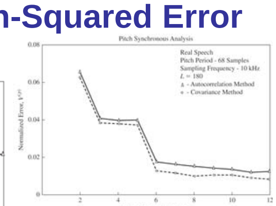
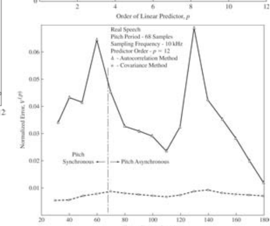
Normalized Mean-Squared Error

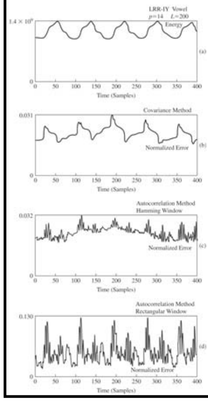






- results for natural speech

Normalized Mean-Squared Error



- variability of normalized error with positioning of frame
- sample-by-sample LP analysis of 40 msec of vowel /i/
- signal energy in part a; normalized error in part b for p=14 pole analysis using 20 msec frame size for covariance method; normalized error in part c for autocorrelation method using HW; normalized error in part d for autocorrelation method using RW
- average pitch period of 84 samples => 2.5 pitch periods in 20 msec window
- substantial variations in normalized error for covariance method-especially peaked when pitch pulses enter window => longer windows give larger normalized errors
- substantial high frequency variations in normalized error for autocorrelation method with some pitch modulations

Properties of the LPC Polynomial

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Minimum-Phase Property of $A(z)$

$A(z)$ has all its zeros inside the unit circle

Proof: Assume that z_o ($|z_o|^2 > 1$) is a zero (root) of $A(z)$

$$A(z) = (1 - z_o z^{-1})A'(z)$$

The minimum mean-squared error is

$$E_{\hat{n}} = \sum_{m=-\infty}^{\infty} e_{\hat{n}}[m]^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |A(e^{j\omega})|^2 |S_{\hat{n}}(e^{j\omega})|^2 \, d\omega$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|1 - z_o e^{-j\omega}\right|^2 \left|A'(e^{j\omega})\right|^2 \left|S_{\hat{n}}(e^{j\omega})\right|^2 d\omega > 0$$

$$\left|1 - z_o e^{-j\omega}\right|^2 = \left|z_o\right|^2 \left|1 - (1/z_o^*)e^{-j\omega}\right|^2$$

Thus, $A(z)$ could not be the optimum filter because we could replace z_o by $(1/z_o^*)$ and decrease the error.

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PARCORs and Stability

- prove that $|k_i| \geq 1 \Rightarrow |z_j^{(i)}| \geq 1$ for some j

$$A^{(i)}(z) = A^{(i-1)}(z) - k_i z^{-i} A^{(i-1)}(z^{-1}) = \prod_{j=1}^{i} (1 - z_j^{(i)} z^{-1})$$

It is easily shown that $-k_i$ is the coefficient of z^{-i} in $A^{(i)}(z)$, i.e., $\alpha_i^{(i)} = k_i$. Therefore,

$$|k_i| = \prod_{j=1}^{p} z_j^{(i)}$$

If $|k_i| \geq 1$, then either all the roots must be **on** the unit circle or at least one of them must be **outside** the unit circle.

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PARCORs and Stability

- if $|k_i| \geq 1$, then either all the roots must be **on** the unit circle or at least one of them must be **outside** the unit circle. Since this is true for all $A^{(i)}(z)$, $i = 1, 2, \ldots, p$, a necessary condition for the roots of $A^{(p)}(z)$ to be inside the unit circle is:

$$|k_i| < 1, \quad i = 1, 2, \ldots, p$$

- for the i^{th}-order optimum linear predictor,

$$E^{(i)} = (1 - k_i^2)E^{(i-1)} = \prod_{j=1}^{i} (1 - k_j^2)E^{(0)} > 0$$

so $|k_i| < 1$ and therefore $A^{(p)}(z)$ has all its roots inside the unit circle.

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Root Locations for Optimum LP Model

$$\tilde{H}(z) = \frac{G}{A(z)} = \frac{G}{1 - \sum_{i=1}^{p} \alpha_i z^{-i}}$$

$$= \frac{G}{\prod_{i=1}^{p} (1 - z_i z^{-1})} = \frac{G z^p}{\prod_{i=1}^{p} (z - z_i)}$$

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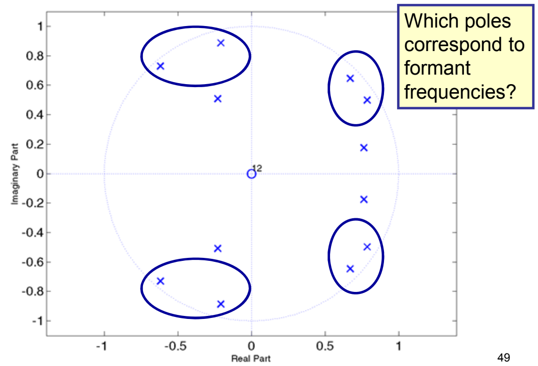
Pole-Zero Plot for Model



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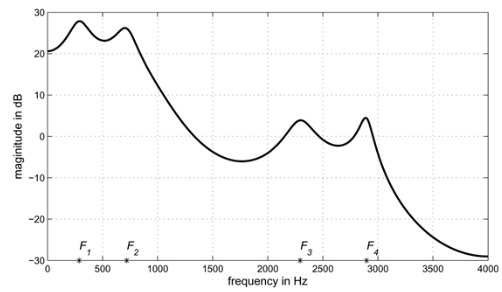
Pole Locations



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Pole Locations (F_S=10,000 Hz)

root magnitude	θ root angle(degrees)	F root angle (Hz)	formant
0.9308	10.36	288	F_1
0.9308	-10.36	-288	F_1
0.9317	25.88	719	F_2
0.9317	-25.88	-719	F_2
0.7837	35.13	976	
0.7837	-35.13	-976	
0.9109	82.58	2294	F_3
0.9109	-82.58	-2294	F_3
0.5579	91.44	2540	
0.5579	-91,44	-2540	
0.9571	104.29	2897	F_4
0.9571	-104.29	-2897	F_4

$$F = (\theta / 180) \cdot (F_S / 2)$$

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Estimating Formant Frequencies

- compute *A(z)* and factor it.
- find roots that are close to the unit circle.
- compute equivalent analog frequencies from the angles of the roots.
- plot formant frequencies as a function of time.

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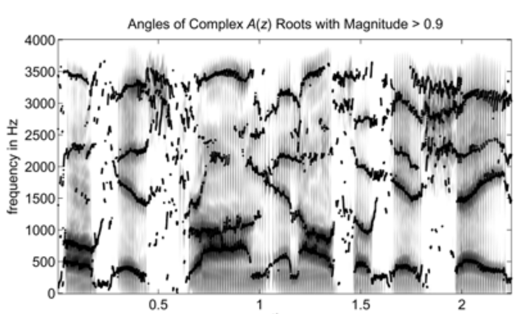
Spectrogram with LPC Roots



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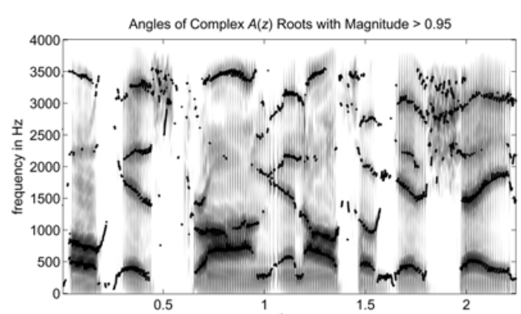
Spectrogram with LPC Roots



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Comparison to ABS Methods

- error measure for ABS methods is log ratio of power spectra, i.e.,

$$E' = \int_{-\pi}^{\pi} \left\{ \log \left[\frac{|S_n(e^{j\omega})|^2}{|H(e^{j\omega})|^2} \right] \right\}^2 d\omega$$

- thus for ABS minimization of E' is equivalent to minimizing mean squared error between two log spectra
- comparing E_{LPC} and E_{ABS} we see the following:
  - both error measures related to ratio of signal to model spectra
  - both tend to perform uniformly over whole frequency range
  - both are suitable to selective error minimization over specified frequency ranges
  - error criterion for LP modeling places higher weight on frequency regions

where $|S_n(e^{j\omega})|^2 < |H(e^{j\omega})|^2$, whereas the ABS error criterion places equal weight on both regions

$\Rightarrow$ for unsmoothed signal spectra (as obtained by STFT methods), LP error criterion yields better spectral matches than ABS error criterion

$\Rightarrow$ for smoothed signal spectra (as obtained at output of filter banks), both error criteria will perform well

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Relation of LP Analysis to Lossless Tube Models

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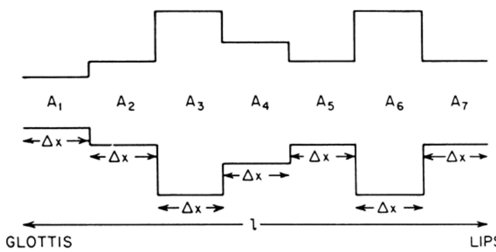
Discrete-Time Model - I



- Make all sections the same length with delay

$$\tau = \Delta x / c \quad \text{where} \quad \ell = N\Delta x$$

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Discrete-Time Model - II

- N-section lossless tube model corresponds to discrete-time system when:

$$F_S = \frac{cN}{2\ell}$$

where c is the velocity of sound, N is the number of tube sections, F_S is the sampling frequency, and ℓ is the total length of the vocal tract.

- The reflection coefficients $\{r_k,\ 1 \le k \le N-1\}$ are related to the areas of the lossless tubes by:

$$r_k = \frac{A_{k+1} - A_k}{A_{k+1} + A_k}$$

- Can find transfer function of digital system subject to constraints of the form $r_G = 1$,

$$r_N = r_L = \frac{\rho c / A_N - Z_L}{\rho c / A_N + Z_L}$$

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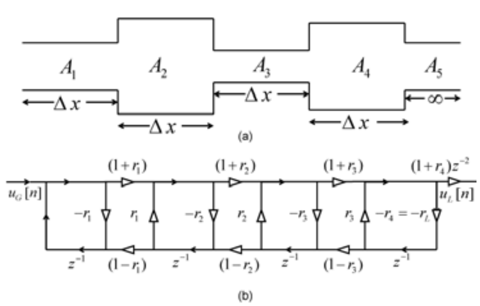
Discrete-Time Model - III



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Discrete-Time Model - IV

Given the system function:

$$V(z) = \frac{U_L(z)}{U_G(z)} = \frac{0.5(1+r_G)\prod_{k=1}^{N}(1+r_k) z^{-N/2}}{D(z)}$$

$$-1 \le r_k = \left(\frac{A_{k+1} - A_k}{A_{k+1} + A_k} \right) \le 1 \quad \text{reflection coefficient}$$

- if $r_G = 1$ (i.e., $R_G = \infty$) and $r_N = r_L = \dfrac{\rho c / A_N - Z_L}{\rho c / A_N + Z_L}$

then $D(z)$ satisties the recursion

$$\boxed{\begin{aligned} D_0(z) &= 1 \\ D_k(z) &= D_{k-1}(z) + r_k z^{-k} D_{k-1}(z^{-1}) \quad k = 1, 2, \ldots, N \\ D(z) &= D_N(z) \end{aligned}}$$

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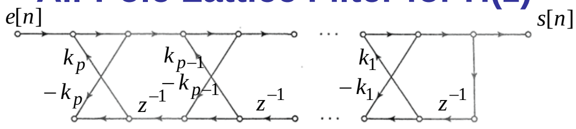
All-Pole Lattice Filter for H(z)



$$\begin{aligned} A^{(0)}(z) &= 1 \\ A^{(i)}(z) &= A^{(i-1)}(z) - k_i z^{-i} A^{(i-1)}(z^{-1}) \quad i = 1, 2, \ldots, p \\ A(z) &= A^{(p)}(z) \end{aligned}$$

$$S(z) = \frac{1}{A(z)} E(z)$$

If $r_i = -k_i$ and $N = p$, then $D(z) = A(z)$.

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Tube Areas from PARCORS

- Relation to areas:

$$-1 \le -k_i = r_i = \left(\frac{A_{i+1} - A_i}{A_{i+1} + A_i} \right) = \left(\frac{(A_{i+1}/A_i) - 1}{(A_{i+1}/A_i) + 1} \right) \le 1$$

- Solve for A_{i+1} in terms of A_i

$$A_{i+1} = \left(\frac{1 - k_i}{1 + k_i} \right) A_i > 0 \qquad \frac{A_{i+1}}{A_i} = \left(\frac{1 - k_i}{1 + k_i} \right) > 0$$

- Log area ratios (good for quantization)

$$g_i = \log \left(\frac{A_{i+1}}{A_i} \right) = \log \left(\frac{1 - k_i}{1 + k_i} \right)$$

Minimizes spectral sensitivity under uniform quantization

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Estimating Tube Areas from Speech (Wakita)

1. Sample speech $[s[n] = s_a(nT)]$ at sampling rate $F_s = 1/T = pc/(2\ell)$.
2. Remove effects of glottal source and radiation by pre-emphasis $x[n] = s[n] - s[n-1]$.
3. Compute the PARCOR coefficients on a short-time basis: k_i, $i = 1, 2, \ldots, p$.
4. Assuming $A_1 = 1$ (arbitrary), compute

$$A_{i+1} = \left(\frac{1 - k_i}{1 + k_i} \right) A_i, \quad i = 1, 2, \ldots, p - 1$$

H. Wakita, IEEE Trans. Audio and Electroacoustics, October, 1973.

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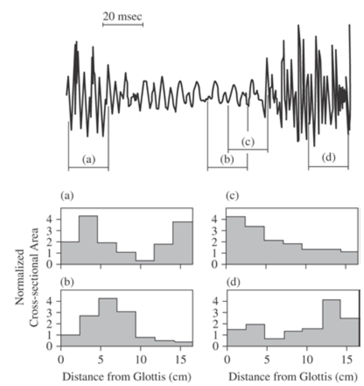
Estimation of Tube Areas



Analysis of syllable /IY/ /B/ /AA/
- /IY/ sound in part (a)
- /B/ sound in parts (b) and (c)
- /AA/ sound in part (d)

Using Wakita method to estimate tube areas from the speech waveform as shown in lower plot.

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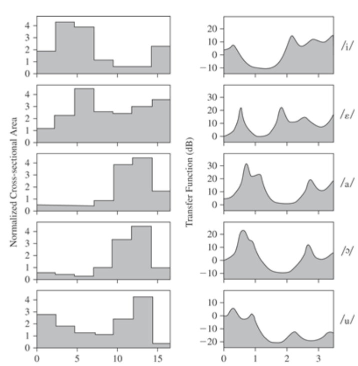
Estimations of Tube Areas



Use of Wakita method to estimate tube areas for the five vowels, /IY/, /EH/, /AA/, /AO/, /UW/

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Alternative Representations of the LP Parameters

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LP Parameter Sets

Parameter Set	*Representation*
LP Coefficients and Gain	$\{\alpha_k, 1 \le k \le p\}, G$
PARCOR Coefficients	$\{k_i, 1 \le i \le p\}$
Log Area Ratio Coefficients	$\{g_i, 1 \le i \le p\}$
Roots of Predictor Polynomial	$\{z_k, 1 \le k \le p\}$
Impulse Response of $H(z)$	$\{h[n], 0 \le n \le \infty\}$
LP Cepstrum	$\{\hat{h}[n], -\infty \le n \le \infty\}$
Autocorrelation of Impulse Response	$\{\tilde{R}(i), -\infty \le i \le \infty\}$
Autocorrelation of Predictor Polynomial	$\{R_a[i], -p \le i \le p\}$
Line Spectral Pair Parameters	$P(z), Q(z)$

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PARCORs to Prediction Coefficients

- assume that $k_i, i = 1, 2, \ldots, p$ are given. Then we can skip the computation of k_i in the Levinson recursion.

for $i = 1, 2, \ldots, p$
$\quad \alpha_i^{(i)} = k_i$
$\quad$ if $i > 1$, then for $j = 1, 2, \ldots, i-1$
$\quad\quad \alpha_j^{(i)} = \alpha_j^{(i-1)} - k_i \alpha_{i-j}^{(i-1)}$
$\quad$ end
end
$\alpha_j = \alpha_j^{(p)} \quad j = 1, 2, \ldots, p$

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Prediction Coefficients to PARCORs

- assume that $\alpha_j, j = 1, 2, \ldots, p$ are given. Then we can work backwards through the Levinson Recursion.

$\alpha_j^{(p)} = \alpha_j \quad$ for $j = 1, 2, \ldots, p$
$k_p = \alpha_p^{(p)}$
for $i = p, p-1, \ldots, 2$
$\quad$ for $j = 1, 2, \ldots, i-1$
$$\alpha_j^{(i-1)} = \frac{\alpha_j^{(i)} + k_i \alpha_{i-j}^{(i)}}{1 - k_i^2}$$
$\quad$ end
$\quad k_{i-1} = \alpha_{i-1}^{(i-1)}$
end

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LP Parameter Transformations

- roots of the predictor polynomial

$$A(z) = 1 - \sum_{k=1}^{p} \alpha_k z^{-k} = \prod_{k=1}^{p} \left(1 - z_k z^{-1}\right)$$

- where each root can be expressed as a z-plane or s-plane root, i.e.,

$$z_k = z_{kr} + j\, z_{ki}; \qquad s_k = \sigma_k + j\Omega_k$$
$$z_k = e^{s_k T}$$

giving

$$\Omega_k = \frac{1}{T} \tan^{-1}\left[\frac{z_{ki}}{z_{kr}}\right]; \qquad \sigma_k = \frac{1}{2T} \log\left(z_{kr}^2 + z_{ki}^2\right)$$

- important for formant estimation

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LP Parameter Transformations

- cepstrum of IR of overall LP system from predictor coefficients

$$\hat{h}(n) = \alpha_n + \sum_{k=1}^{n-1} \left(\frac{k}{n}\right) \hat{h}(k) \alpha_{n-k} \qquad 1 \le n$$

- predictor coefficients from cepstrum of IR

$$\alpha_n = \hat{h}(n) - \sum_{k=1}^{n-1} \left(\frac{k}{n}\right) \hat{h}(k) \alpha_{n-k} \qquad 1 \le n$$

where

$$H(z) = \sum_{n=0}^{\infty} h(n) z^{-n} = \frac{G}{1 - \sum_{k=1}^{p} \alpha_k z^{-k}}$$

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LP Parameter Transformations

- IR of all pole system

$$h(n) = \sum_{k=1}^{p} \alpha_k h(n-k) + G\delta(n) \qquad 0 \le n$$

- autocorrelation of IR

$$\tilde{R}(i) = \sum_{n=0}^{\infty} h(n) h(n-i) = \tilde{R}(-i)$$

$$\tilde{R}(i) = \sum_{k=1}^{p} \alpha_k \tilde{R}(|i-k|) \qquad 1 \le i$$

$$\tilde{R}(0) = \sum_{k=1}^{p} \alpha_k \tilde{R}(k) + G^2$$

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LP Parameter Transformations

- autocorrelation of the predictor polynomial

$$A(z) = 1 - \sum_{k=1}^{p} \alpha_k z^{-k}$$

with IR of the inverse filter

$$a(n) = \delta(n) - \sum_{k=1}^{p} \alpha_k \delta(n-k)$$

with autocorrelation

$$R_a(i) = \sum_{k=0}^{p-i} a(k)a(k+i) \qquad 0 \le i \le p$$

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LP Parameter Transformations

- log area ratio coefficients from PARCOR coefficients

$$g_i = \log\left[\frac{A_{i+1}}{A_i}\right] = \log\left[\frac{1-k_i}{1+k_i}\right] \qquad 1 \le i \le p$$

with inverse relation

$$k_i = \frac{1-e^{g_i}}{1+e^{g_i}} \qquad 1 \le i \le p$$

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Quantization of LP Parameters

- consider the magnitude-squared of the model frequency response

$$\left|H(e^{j\omega})\right|^2 = \frac{1}{|A(e^{j\omega})|^2} = P(\omega, g)$$

where g is a parameter that affects P.

- spectral sensitivity can be defined as

$$\frac{\partial S}{\partial g_i} = \lim_{\Delta g_i \to 0} \left| \frac{1}{\Delta g_i} \left[\frac{1}{2\pi} \int_{-\pi}^{\pi} \left| \log \frac{P(\omega, g_i)}{P(\omega, g_i + \Delta g_i)} \right| d\omega \right] \right|$$

which measures sensitivity to errors in the g_i parameters

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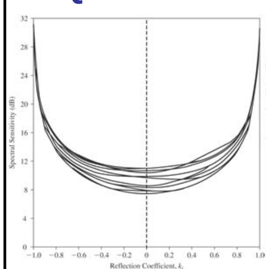
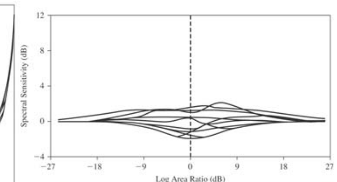
Quantization of LP Parameters





- spectral sensitivity for k_i parameters; low sensitivity around 0; high sensitivity around 1

- spectral sensitivity for log area ratio parameters, g_i – low sensitivity for virtually entire range is seen

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Line Spectral Pair Parameters

$$A(z) = 1 + \alpha_1 z^{-1} + \alpha_2 z^{-2} + \ldots + \alpha_p z^{-p}$$

$= $ all-zero prediction filter with all zeros, z_k, inside the unit circle

$$\tilde{A}(z) = z^{-(p+1)} A(z^{-1}) = \alpha_p z^{-1} + \ldots + \alpha_2 z^{-p+1} + \alpha_1 z^{-p} + z^{-(p+1)}$$

$= $ reciprocal polynomial with inverse zeros, $1/z_k$

- consider the following:

$$L(z) = \frac{\tilde{A}(z)}{A(z)} = \text{allpass system} \Rightarrow \left|L(e^{j\omega})\right| = 1, \text{ all } \omega$$

- form the symmetric polynomial $P(z)$ as:

$$P(z) = A(z) + \tilde{A}(z) = A(z) + z^{-(p+1)} A(z^{-1}) \Rightarrow P(z) \text{ has zeros for } L(z) = -1; (A(z) = -\tilde{A}(z))$$

$$\Rightarrow \arg\left\{L(e^{j\omega_k})\right\} = (k + 1/2) \cdot 2\pi, \; k = 0,1,\ldots,p-1$$

- form the anti-symmetric polynomial $Q(z)$ as:

$$Q(z) = A(z) - \tilde{A}(z) = A(z) - z^{-(p+1)} A(z^{-1}) \Rightarrow Q(z) \text{ has zeros for } L(z) = +1; (A(z) = \tilde{A}(z))$$

$$\Rightarrow \arg\left\{L(e^{j\omega_k})\right\} = k \cdot 2\pi, \; k = 0,1,\ldots,p-1$$

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Line Spectral Pair Parameters

- zeros of $P(z)$ and $Q(z)$ fall on unit circle and are interleaved with each other $\Rightarrow$ set of $\{\omega_k\}$ called Line Spectral Frequencies (LSF)
- LSFs are in ascending order
- stability of $H(z)$ guaranteed by quantizing LSF parameters

$$A(e^{j\omega}) = \frac{P(e^{j\omega}) + Q(e^{j\omega})}{2}$$

$$\left|A(e^{j\omega})\right|^2 = \frac{\left|P(e^{j\omega})\right|^2 + \left|Q(e^{j\omega})\right|^2}{4}$$

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Line Spectrum Pair (LSP) Parameters

- properties of LSP parameters
  1. $P(z)$ corresponds to a lossless tube, open at the lips and open ($k_{p+1}=1$) at the glottis
  2. $Q(z)$ corresponds to a lossless tube, open at the lips and closed ($k_{p+1}=-1$) at the glottis
  3. all the roots of $P(z)$ and $Q(z)$ are on the unit circle
  4. if p is an even integer, then $P(z)$ has a root at $z=+1$ and $Q(z)$ has a root at $z=-1$
  5. a necessary and sufficient condition for $|k_i|<1,\ i=1,2,...,p$ is that the roots of $P(z)$ and $Q(z)$alternate on the unit circle
  6. the LSP frequencies get close together when roots of $A(z)$ are close to the unit circle
  7. the roots of $P(z)$ are approximately equal to the formant frequencies

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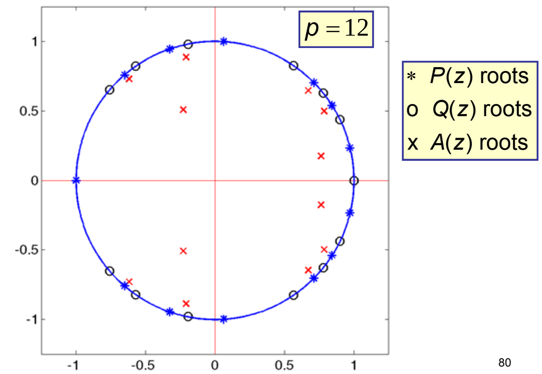
LSP Example



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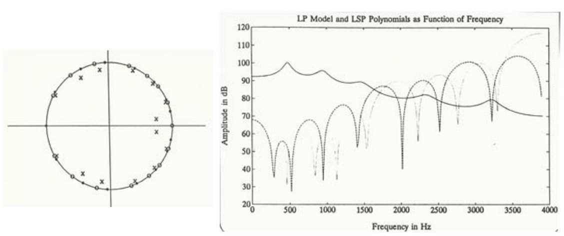
Line Spectrum Pair (LSP) Parameters



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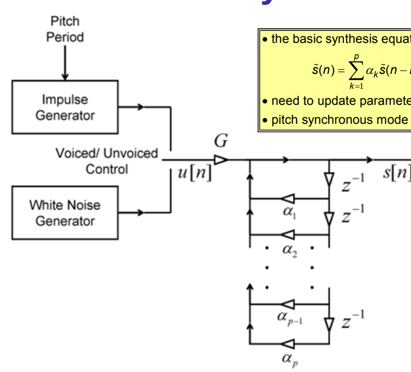
LPC Synthesis



- the basic synthesis equation is

$$\tilde{s}(n)=\sum_{k=1}^{p}\alpha_k\tilde{s}(n-k)+Gu(n)$$

- need to update parameters every 10 msec or so
- pitch synchronous mode works best

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LPC Analysis-Synthesis

1. Extract α_k parameters properly
2. Quantize α_k parameters properly so that there is little quantization error
   - Small number of bits go into coding the α_k coefficients
3. Represent *e(n)* via:
   - Pitch pulses and noise—LPC Coding
   - Multiple pulses per 10 msec interval—MPLPC Coding
   - Codebook vectors—CELP
     - Almost all of the coding bits go into coding of *e(n)*

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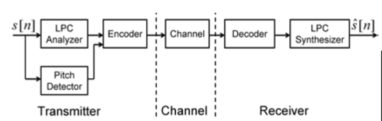
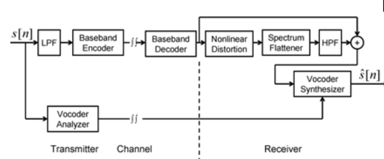
LPC Vocoder

header



- bit rates of 72 F_S where F_S=100, 67, and 33 for bit rates of 7200 bps, 4800 bps and 2400 bps



3600 bps

2400 bps

VEV LPC

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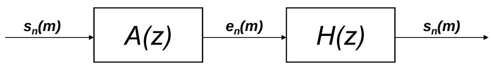
LPC Basics



$$A(z) = 1 - \sum_{k=1}^{p} \alpha_k z^{-k} = 1 - P(z) = \frac{E(z)}{S(z)};$$ prediction error filter

$$e_{\hat{n}}(m) = s_{\hat{n}}(m) - \sum_{k=1}^{p} \alpha_k s_{\hat{n}}(m-k);$$ prediction error

$$H(z) = \frac{1}{A(z)} = \frac{1}{1 - \sum_{k=1}^{p} \alpha_k z^{-k}} = \frac{1}{1 - P(z)};$$ all pole model

$$E_{\hat{n}} = \sum_{m=-\infty}^{\infty} \left[e_{\hat{n}}(m)\right]^2 = \sum_{m=-\infty}^{\infty} \left(s_{\hat{n}}(m) - \sum_{k=1}^{p} \alpha_k s_{\hat{n}}(m-k) \right)^2$$

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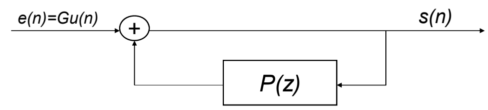
LPC Basics-Speech Model



$$H(z) = \frac{G}{1 - \sum_{k=1}^{p} \alpha_k z^{-k}} = \frac{G}{1 - P(z)} = \frac{S(z)}{U(z)}$$

$$H(e^{j\omega}) = \frac{G}{1 - \sum_{k=1}^{p} \alpha_k e^{-j\omega k}}$$

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Summary

- the LP model has many interesting and useful properties that follow from the structure of the Levinson-Durbin algorithms
- the different equivalent representations have different properties under quantization
  - polynomial coefficients (bad)
  - polynomial roots (okay)
  - PARCOR coefficients (okay)
  - lossless tube areas (good)
  - LSP root angles (good)
- almost all LPC representations can be used with a range of compression schemes and are all good candidates for the technique of Vector Quantization

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