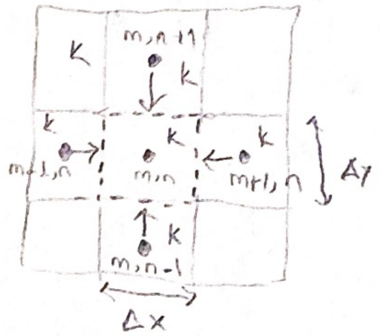
– Finite Difference Equations for all node types –

Assume: → no internal heat gen
↳ steady state
↳ $\Delta x = \Delta y$ (grid sizes)
↳ assume all q'' into center node

use energy balance in simplified: $(E_{in} = 0)$ form,

1) Interior Nodes

→ Inner Interior / Outer Interior
→ standard 5 point stencil



recall Fourier's Law of conduction $q_x = kA\frac{\Delta T}{L}$

$$q_{(m+1,n)\to(m,n)} = k(\Delta y\cdot 1)\frac{(T_{m+1,n} - T_{m,n})}{\Delta x}$$

$$q_{(m-1,n)\to(m,n)} = k(\Delta y\cdot 1)\frac{(T_{m-1,n} - T_{m,n})}{\Delta x}$$

$$q_{(m,n+1)\to(m,n)} = k(\Delta x\cdot 1)\frac{(T_{m,n+1} - T_{m,n})}{\Delta y}$$

$$q_{(m,n-1)\to(m,n)} = k(\Delta x\cdot 1)\cdot\frac{T_{(m,n-1)} - T_{m,n}}{\Delta y}$$

and from i) energy balance, ii) $\Delta x = \Delta y$

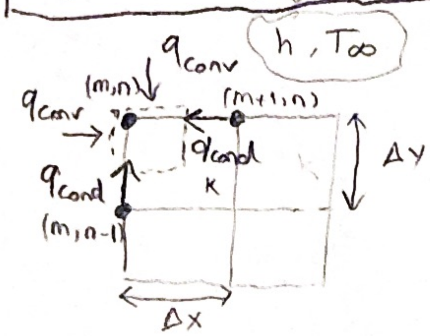
$$q_{in} = \sum_{i=1}^{4} q_{i\to m,n} = 0 =$$

$$\frac{k(\Delta y\cdot 1)(T_{m+1,n} - T_{m,n})}{\Delta x} + \text{-----} = 0 \quad \text{gives}$$

$$\boxed{T_{m,n+1} + T_{m,n-1} + T_{m+1,n} + T_{m-1,n} - 4T_{m,n} = 0}$$

Same for outer interior nodes ~~★★★~~

2) External Corner Nodes with Convection



$$\sum q_{in} = k\left(\frac{\Delta y}{2}\cdot 1\right)\cdot \frac{T_{m+1,n} - T_{m,n}}{\Delta x/2} + k\left(\frac{\Delta x \cdot 1}{2}\right)\frac{T_{m,n-1} - T_{m,n}}{\Delta y/2}$$

$$+ h(\Delta x \cdot 1)\cdot(T_\infty - T_{m,n}) + h(\Delta y \cdot 1)(T_\infty - T_{m,n}) = 0$$

in general: Conduction from two adjacent nodes + Convection from two exposed surfaces = 0

for $\Delta x = \Delta y$ gives

$$k(T_{m+1,n} - T_{m,n}) + k(T_{(m,n-1)} - T_{(m,n)})$$

$$+ 2(h\Delta x(T_\infty - T_{(m,n)})) = 0 \rightarrow$$

$$k(T_{m+1,n} + T_{m,n-1}) + 2h\Delta x T_\infty + (-2k + 2h\Delta x)T_{m,n} = 0$$

$$\underbrace{(T_{m+1,n} + T_{m,n-1})}_{\text{adjacent nodes}} + 2\frac{h\Delta x}{k}T_\infty - 2\left(\frac{h\Delta x}{k} + 1\right)T_{m,n} = 0$$

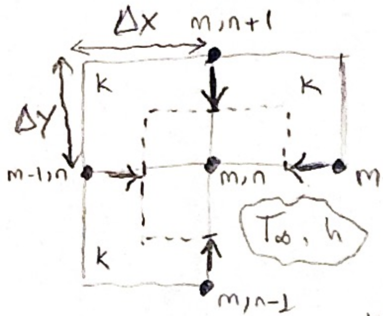
So in general form

$$\rightarrow (T_{adj-1} + T_{adj-2}) + \frac{2h\Delta x}{k}T_\infty - 2\left(\frac{h\Delta x}{k} + 1\right)T_{m,n} = 0$$

which is applicable for all
- top left
- top right
- bottom left
- bottom right

external corners

3) Inner Corner Nodes with convection



$\Sigma q_{in} =$ Conduction from 2 interior adjacent nodes + Conduction from 2 adjacent surface nodes + Convection from fluid from 2 exposed surfaces $= 0$

$$\frac{k(\Delta x \cdot 1)(T_{m,n+1} - T_{m,n})}{\Delta y} + \frac{k(\Delta y \cdot 1)(T_{m-1,n} - T_{m,n})}{\Delta x} +$$

$$\frac{k(\Delta y/2 \cdot 1)(T_{m+1,n} - T_{m,n})}{\Delta x} + \frac{k(\frac{\Delta x}{2} \cdot 1)(T_{m,n-1} - T_{m,n})}{\Delta y} +$$

$$h\left(\frac{\Delta x}{2} \cdot 1\right) \cdot (T_\infty - T_{m,n}) + h\left(\frac{\Delta y}{2} \cdot 1\right)(T_\infty - T_{m,n}) = 0$$

$$k(T_{m,n+1} + T_{m-1,n}) - 2k\,T_{m,n} + \frac{k}{2}(T_{m+1,n} + T_{m,n-1}) - k\,T_{m,n} +$$

$$h\,\Delta x\,T_\infty - h\,\Delta x\,T_{m,n} = 0 \rightarrow$$

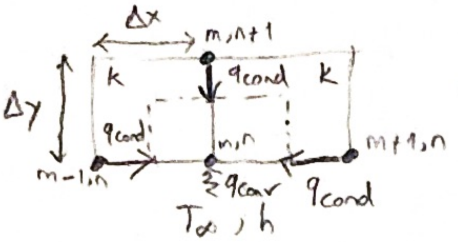
$$2(T_{m,n+1} + T_{m-1,n}) + (T_{m+1,n} + T_{m,n-1}) + \frac{h\,\Delta x\,T_\infty}{k} -$$

$$2\left(3 + \frac{h\,\Delta x}{k}\right)T_{m,n} = 0$$

in general form covering all possible 4 directions;

$$\boxed{2\left(T^1_{inner,adj} + T^2_{inner,adj}\right) + \left(T^1_{surf,adj} + T^2_{surf,adj}\right) + \frac{h\,\Delta x\,T_\infty}{k} - 2\left(3 + \frac{h\,\Delta x}{k}\right)T_{m,n} = 0}$$

4) Inner Surface Nodes with Convection



$$\Sigma q_{in} = \text{Conduction from an interior node} + \text{Conduction from 2 surface nodes} + \text{Convection from fluid from the exposed surface} = 0$$

$$\frac{k(\Delta x \cdot 1)(T_{m,n+1} - T_{m,n})}{\Delta y} + \frac{k\left(\frac{\Delta y}{2} \cdot 1\right)(T_{m-1,n} - T_{m,n})}{\Delta x} + \frac{k\left(\frac{\Delta y}{2} \cdot 1\right) \cdot (T_{m+1,n} - T_{m,n})}{\Delta x} +$$

$$h(\Delta x \cdot 1)(T_\infty - T_{m,n}) = 0$$

$$k(T_{m,n+1}) + \frac{k}{2}\left(T_{m-1,n} + T_{m+1,n}\right) + h\Delta x T_\infty - (2k + h\Delta x)T_{m,n} = 0$$

$$2(T_{m,n+1}) + (T_{m-1,n} + T_{m+1,n}) + \frac{2h\Delta x T_\infty}{k} - 2\left(2 + \frac{h\Delta x}{k}\right)T_{m,n} = 0$$

in general form covering all 4 directions;

$$2(T_{inner,adj}) + (T^1_{surf,adj} + T^2_{surf,adj}) + \frac{2h\Delta x T_\infty}{k} - 2\left(2 \frac{h\Delta x}{k}\right)T_{m,n} = 0$$

where $h = h_{hot,air} = 90\ W/m^2K$

$T_\infty = T_{\infty,\ hot\ air} = 300^\circ C$

$k = k_{inner\ material} = 45\ W/mK$

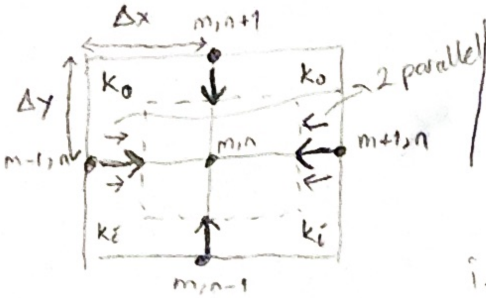
5) Outer Surface Nodes with convection;

Same general form eq'n as (4) but with data:

$$h = \begin{matrix} h_{cold\ air} = 25\ W/m^2K \\ h_{inside\ air} = 4\ W/m^2K \end{matrix}, \quad T_\infty = \begin{matrix} T_{\infty,cold\ air} = 4^\circ C \\ T_{\infty,\ inside\ air} = 24^\circ C \end{matrix}$$

$k = k_{outer\ material} = 15\ W/mK.$

6) Interface surface Nodes



if horizontal interface as depicted in the left

→ harmonic mean Conduction in Y-dir : $\frac{k_1 k_2}{k_1 + k_2}$

↳ average conduction in X-dir: $\frac{k_1 + k_2}{2}$

if vertical interface

→ harmonic mean Conduction in X-dir

↳ average Conduction in Y-dir.

$$\sum q_{in} = k_o \frac{(\Delta x \cdot 1)(T_{m,n+1} - T_{m,n})}{\Delta Y/2} + k_i \frac{(\Delta x \cdot 1)(T_{m,n-1} - T_{m,n})}{\Delta Y/2} +$$

$$\left(k_o(\Delta Y/2 \cdot 1) + k_i\left(\frac{\Delta Y}{2} \cdot 1\right)\right)\left(T_{m+1,n} - T_{m,n}\right) + \left(k_o\left(\frac{\Delta Y}{2} \cdot 1\right) + k_i\left(\frac{\Delta Y}{2} \cdot 1\right)\right)\left(T_{m-1,n} - T_{m,n}\right)$$

simplification yields;

for horizontal interface;

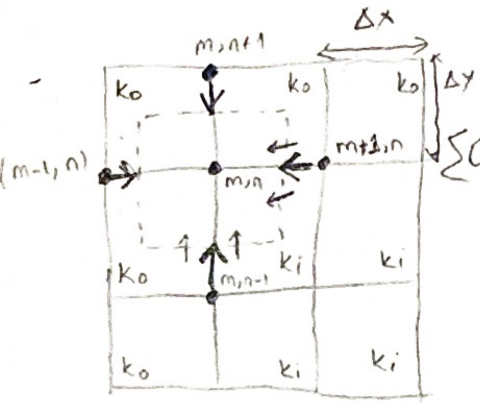
$$4k_o k_i \underbrace{\left[T_{m,n+1} + T_{m,n-1}\right]}_{\text{homogenous nodes}} + (k_o + k_i)^2 \underbrace{\left[T_{m-1,n} + T_{m+1,n}\right]}_{\text{interface nodes}} + \left[4k_i k_o + 2(k_i + k_o)^2\right] T_{m,n} = 0$$

similarly for vertical interface

wher $k_i = 45 \ W/mK$

$k_o = 15 \ W/mK$.

7) Interface Corner Nodes without convection



$\sum q_{in} =$ Conduction thru 2 homogenus material sections + Conduction thru 2 interface materials (parallel flow) $= 0$

$$\frac{k_o(\Delta x \cdot 1)(T_{m,n+1} - T_{m,n})}{\Delta y/2} + \frac{k_o(\Delta y \cdot 1)(T_{m-1,n} - T_{m,n})}{\Delta x/2} +$$

$$\frac{(k_o + k_i)\left(\frac{\Delta y}{2} \cdot 1\right)(T_{m+1,n} - T_{m,n})}{\Delta x/2} + \frac{(k_o + k_i)\left(\frac{\Delta x}{2} \cdot 1\right)(T_{m,n-1} - T_{m,n})}{(\Delta y/2)} = 0$$

where $k_i = 45\,W/mK$

$k_o = 15\,W/mK$