1 Orthogonality and vector spaces

1.1 Orthogonal vectors

Two vectors$x$, y are said to be orthogonal if they satisfy the equation:

$$|x|^2 + |y|^2 = |x + y|^2 \tag{1}$$

This criteria is not the most convenient criteria. hence, the need for a more practical one. Let's consider the following:

$$\begin{aligned}
|x|^2 &= x^T \cdot x && \text{another formula } |x|^2 \\
|x|^2 + |y|^2 &= |x + y|^2 && \text{condition for orthogonality} \\
\iff x^T \cdot x + y^T \cdot y &= (x + y)^T \cdot (x + y) \\
x^T \cdot y + y^T \cdot x &= 0 \\
(x^T \cdot y)^T &= (x^T \cdot y) && x^T \cdot y \text{ is a scalar} \\
\iff (x^T \cdot y)^T &= 0
\end{aligned}$$

Thus, we end up with the final criterion:

$$x \perp y \iff x^T \cdot y = y^T \cdot x = 0 \tag{2}$$

1.1.1 Remark

The only vector orthogonal to itself is the zero vector.

1.2 Orthogonal vector spaces

1.2.1 definition

Two vector spaces S and T are said to be orthogonal if $\forall x \in S$ and $\forall y \in T$ we have $x \perp y$
if we have $S \perp T$ then $S \cap T = Z$ as any vector$i$ in the intersection would have to be orthogonal to itself, thus it is the null vector.

1.2.2 Orthogonal complements

S is the orthogonal complement of T (and vice versa) if S satisfies the following condition:

$$x \in S, x \perp y, \implies y \in T, \forall x \in S \tag{3}$$

More informally,T contains all the vectors orthogonal to every vector in S and not only some of them.

1.2.3 The fundamental four spaces and orthogonality

Let's prove that $R(A)$ and $N(A)$ are indeed orthogonal complements in $\mathbb{R}^n$.

$$\begin{aligned} R(A) &= r_1, r_2, ..r_k && \text{one basis of } R(A) \\ x \in N(A) \iff A \cdot x = 0 &\implies r_i^T \cdot x = 0 && \text{for } i = 1, 2, ..k \\ y \in R(A) \iff y &= \sum_{i=1}^{k} a_i \cdot r_i \\ y^T \cdot x &= \sum_{i=1}^{k} a_i \cdot r_i \cdot x = \sum_{i=1}^{k} a_i \cdot r_i^T \cdot x = 0 \end{aligned}$$

Thus $R(A)$ and $C(A)$ are indeed orthogonal. The only intersection of these two spaces is the zero vector. A vector in $R(A)$ or $C(A)$ belongs to a larger vector space which is R^n. We recall the important result:

$$dim(C(A)) = rank(A) \quad dim(N(A)) = n - rank(A) \tag{4}$$

leading to:

$$dim(C(A)) + dim(N(A)) = n \tag{5}$$

Thus a basis formed out of these two subspaces span the entire R^n. More interestingly, if $x \neq 0 \in R^n$ belongs strictly to either $C(A)$ or $N(A)$ for any matrix A with n columns. Thus, if we have $x \in C(A)$ and $y, x^T \cdot y$ then $y \in N(A)$ as y cannot belong to $N(A)$ unless it is the zero vector which also belongs to $N(A)$. The same argument is applied to prove the other direction. As A can be any matrix then the same result applies for A^T. Hence:

$N(A)$ is the orthogonal complement of $C(A)$ in $\mathbb{R}^n$
$N(A^T)$ is the orthogonal complement of $C(A^T)$ in $\mathbb{R}^m$

1.2.4 Proving the two-directions of the orthogonality complement

if we have S is the orthogonal complement of T then the other way should also be correct. why ?. Let $v_1, v_2, ...v_k$ be a basis of S. Then let

$$V = \begin{bmatrix} v_1 & v_2 & .. & v_k \end{bmatrix} \tag{6}$$

Thus, $S = C(V)$. Then $T = N(V)$ which proves that T is the orthogonal complement of S

2 Projections

2.1 A and $A^T \cdot A$

There are two major results:

$$N(A) = N(A^T \cdot A)$$
$$rank(A) = rank(A^T \cdot A)$$

let's consider the equation:

$$\begin{aligned}
A^T \cdot A \cdot x &= 0 && A \cdot x \in N(A^T) \\
A \cdot x \in C(A) \quad \forall x &\in R^n && \text{property of matrix multiplication} \\
\implies A \cdot x &\in (C(A) \cap N(A^T) = Z) \\
\implies A \cdot x &= 0
\end{aligned}$$

Thus, we just proved the first direction. Additionally,

$$A \cdot x = 0 \implies A^T \cdot A \cdot x = 0 \tag{7}$$

which means that $x \in N(A) \iff x \in N(A^T)$ and consequently

$$N(A) = N(A^T) \tag{8}$$

As we have

$$n - rank(AA^T) = dim(N(AA^T)) = dim(N(A)) = n - rank(A)$$
$$rank(AA^T) = rank(A)$$

2.2 Projection

2.2.1 Projection in 2D

consider a vector b in the plane. and another vector a such that a and b are independent. From geometry the closest vector in the line directed by a to the vector b is the ***projection*** of b on a. let's denote such vector as p

$$\begin{aligned}
p &= a \cdot x && p \text{ parallel to } a \\
p^T \cdot (b - p) &= 0 && p \text{ is the projection of } b \text{ on } a \\
x \cdot a^T \cdot (b - a \cdot x) &= 0 && \text{substitute for } p \\
a^T \cdot b - a^T \cdot a \cdot x &= 0 && \text{expand} \\
x &= \frac{a \cdot a^T}{a^T \cdot a} \cdot b && a^T \cdot a \text{ is a scalar}
\end{aligned}$$

2.3 Projection in general

Let S a subspace represented as $C(A)$ for some matrix A. We have a projection vector p of vector b on subspace S. Thus, we have $a^T \cdot (b - p) = 0$ for all $a_i \in A$. Having A as $\begin{bmatrix} a_1 \\ a_2 \\ .. \\ .. \\ a_n \end{bmatrix}$, then we have $A^T \cdot (b - p) = 0$. Keeping in mind that $p \in C(A)$ then $\exists\ \alpha$ such that $p = A \cdot \alpha$.

$$\begin{aligned}
A^T(A \cdot \alpha - b) &= 0 && \text{orthogonality condition} \\
A^T \cdot A \cdot \alpha - A^T \cdot b &= 0 && \text{expand} \\
\alpha &= (A^T \cdot A)^{-1} \cdot A^T && \text{find } \alpha \\
P &= A \cdot (A^T \cdot A)^{-1} \cdot A^T && \text{the projection matrix } P
\end{aligned}$$

2.3.1 Properties of P

P is quite a powerful as it has a number of properties:

- $P^T = P$
- $P^2 = P$
- projects any vector onto the vector space.

2.3.2 Results

- if $b \in C(A)$ then $P \cdot b = b$
- if $b \in C(A) \implies b = A \cdot c$ for some vector c. Then $P \cdot b = A \cdot (A^T \cdot A)^{-1} \cdot A^T \cdot A \cdot c = A \cdot c = b$
- if $b \perp C(A)$ then $P \cdot b = 0$
- if $b \perp C(A)$ then $A^T \cdot b = 0$ then $P \cdot b = A \cdot (A^T \cdot A)^{-1} \cdot A^T \cdot b = 0$