1 Matrix Inverse

A matrix inverse (in general) is defined only for square matrices. The inverse of matrix A denoted as A^{-1} is the matrix satisfying the equation:

$$A^{-1} \cdot A = I \tag{1}$$

A matrix with an inverse is referred to as ***invertible, non-degenerate, non singular***.

2 Properties of Inverse

A matrix inverse satisfies the following properties:

- right inverse is the same as left inverse:
$$A^{-1} \cdot A = A \cdot A^{-1} = I \tag{2}$$
- uniqueness, if $A \cdot B = A \cdot C = I$ then $B = C$
- $$(A \cdot B)^{-1} = B^{-1} \cdot A^{-1} \tag{3}$$
- if a matrix is invertible then:
$$A \cdot x = 0 \iff x = 0 \tag{4}$$

The last property can be proved easily. Assuming we have $A^{-1} \cdot A = I$ and $A \cdot x = 0$ then $A^{-1} \cdot A \cdot x = 0 \implies (A^{-1} \cdot A) \cdot x = 0 \implies x = 0$. It directly follows that:

$$\text{if } A \cdot x = 0 \text{ for } x \neq 0, \text{ then } A \text{ is non-invertible}$$

3 calculating the inverse of a matrix

Let A be the matrix in question. We multiply all the elementary matrices used in the elimination algorithm. until we obtain the matrix I. Let's denote the result of all the elementary matrices by T. We have $T \cdot A = I$. then T is indeed the inverse of the matrix A.
It is crucial to point that the elimination will break permanently if and only if the matrix is ***non-invertible***

4 LU factorization

4.1 A = LU

an invertible matrix A admits another basic factorization referred to as ***LU*** where L is a lower triangular matrix and U is an upper one. This is done by applying elimination on A until converting it to a lower triangular matrix. There should be no rows exchanges.

4.2 PA = LU

There might be perfectly invertible matrices A for which the elimination breaks temporarily which requires row exchanges: applying th permutation matrices. Having P as the product of all the permutation matrices used in the factorization. The equation is slightly modified to $PA = LU$

5 Matrix Transpose

a matrix A with dimensions (a, b) has a transpose matrix with dimensions (b, a) denoted as A^T such that $A^T_{ij} = A^T_{ji}$

5.1 Properties

- $(A^T)^T = A$
- $(AB)^T = B^T \cdot A^T$
- $A^T = A \iff$ A symmetric
- $A^T \cdot A$ is always symmetric.