1 Eigenvectors and Eigenvalues

1.1 Definition

for a square matrix A, eigenvectors are special vectors that satisfy the condition:

$$A \cdot x = \lambda \cdot x \tag{1}$$

for some value λ. λ's are referred to as the ***eigenvalues***

1.2 calculations

we have

$$\begin{aligned} A \cdot x &= \lambda \cdot x \\ A \cdot x &= I \cdot \lambda \cdot x \\ (A - I \cdot \lambda) \cdot x &= 0 \qquad\qquad \text{with } x \neq 0 \\ \implies |(A - I \cdot \lambda)| &= 0 \end{aligned}$$

Eigenvalues values are the solutions of the last equation. Finding the corresponding eigenvectors can achieved by elimination.

1.3 trace and product

There are two important results:

- The sum of eigenvalues is proved to be the sum of values in the diagonal positions.
- The product of eigenvalues is equal to $|A|$

1.4 Additional Observations

1. if vector x is an Eigenvector then $c \cdot x$ is also an Eigenvector
2. if $x_1, x_2, ..x_n$ are eigenvalues of A then $x_1 + b, x_2 + b, ..x_n + b$ are eigenvalues of $A + I \cdot b$
3. $A^2 \cdot x = A \cdot Ax = \lambda A \cdot x = \lambda^2 \cdot x$, more generally, $\lambda(A^k) = \lambda(A)^k$ and $Eig(A^k) = Eig(A)$
4. $Ax = \lambda_1 x$ and $Bx = \lambda_2 x$ does not imply that $(A + B) \cdot (\lambda_1 + \lambda_2) = (x + y)$
5. a symmetric matrix has only purely real eigenvalues with orthogonal eigenvectors:
6. and anti-symmetric has only purely complex eigenvalues
7. each matrix has n Eigenvectors, generally they are independent. Yet, with repeated eigenvalues, some eigenvectors are repeated.

2 Diagonalization

Assuming that matrix A has n independent Eigenvectors denoted as $x_1, x_2, ..x_n$ respectively. Consider the following:

$$\begin{aligned}
S &= \begin{bmatrix} x_1 & x_2 & .. & x_n \end{bmatrix} && \text{define matrix } S \\
A \cdot S &= A \cdot \begin{bmatrix} x_1 & x_2 & .. & x_n \end{bmatrix} \\
&= \begin{bmatrix} \lambda_1 \cdot x_1 & \lambda_2 \cdot x_2 & .. & \lambda_n \cdot x_n \end{bmatrix} \\
&= \begin{bmatrix} \lambda_1 & 0 & 0 & .. & 0 \\ 0 & \lambda_2 & 0 & .. & 0 \\ ... \\ 0 & 0 & 0 & .. & \lambda_n \end{bmatrix} \cdot S && \text{diagonalize} \\
A \cdot S &= S \cdot \Lambda \\
A &= S \cdot \Lambda \cdot S^{-1} && \text{Assuming } n \text{ independent vectors}
\end{aligned}$$

3 Applications

3.1 Powers of a matrix

Assuming A has n independent eigenvectors, $A = S \cdot \Lambda \cdot S^{-1}$.

$$A^2 = A \cdot A = S \cdot \Lambda \cdot S^{-1} \cdot S \cdot \Lambda \cdot S^{-1} = S \cdot \Lambda^2 \cdot S^{-1}$$

With simple induction the general result can be proved:

$$A^k = S \cdot \Lambda^k \cdot S^{-1} \qquad \forall k \geq 0$$

3.1.1 Eigenvalues and diagonalization

$$n \text{ different } \lambda\text{'s} \implies n \text{ different eigenvectors}$$
$$\text{repeated } \lambda\text{'s might or might not lead to dependent eigenvectors}$$

One example is the identity matrix I, it has only one eigenvalue 1. Yet, we can find n independent eigenvectors.

3.2 $u_k = A \cdot u_{k-1}$

Consider the following sequence:

$$\begin{aligned} u_{k+1} &= A \cdot u_k \\ u_0 &= constant \end{aligned}$$

The general expression for u_k is:

$$u_k = A^k \cdot u_0$$

As A has n independent eigenvectors, then $S \cdot c = u_0$ admits a unique solution. Then:

$$u_k = \Lambda^k \cdot S \cdot c$$

3.3 Differential Equations

3.3.1 Manual process

The differential equation

$$\frac{du}{dt} = A$$

when solved manually follows a general procedure. The solution can be expressed as:

$$u = \sum_{i=1}^{n} c_i \cdot e^{\lambda_i \cdot t} \cdot x_i$$

So, the main procedure is to find $Eig(A)$. which directly gives us a solution to the differential equation, but not the Initial value problem. The coefficients c_i can be deducted from

$$u(0) = \sum_{i=1}^{n} c_i \cdot x_i$$

$$u(0) = S \cdot \begin{bmatrix} c_1 \\ c_2 \\ .. \\ c_n \end{bmatrix}$$

Assuming A has n independent eigenvectors, then the last equation admits a unique solution.

3.3.2 Mathematical process

Consider the differential equation:

$$\frac{du}{dt} = A \cdot u$$

Let's find the general solution for this equation. Assuming:

$$u(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \\ .. \\ u_n(t) \end{bmatrix}$$

we can express u as $u = S \cdot v$ where S is the Eigenvector matrix of A. Then

$$\begin{aligned}
\frac{du}{dt} &= A \cdot u \\
S\frac{dv}{dt} &= A \cdot S \cdot v \\
\frac{dv}{dt} &= S^{-1} A \cdot S \cdot v \\
\frac{dv}{dt} &= \Lambda \cdot v \\
\frac{dv}{dt} &= \begin{bmatrix} \lambda_1 & 0 & 0 & .. & 0 \\ 0 & \lambda_2 & 0 & .. & 0 \\ ... & & & & \\ 0 & 0 & 0 & .. & \lambda_n \end{bmatrix} \cdot v
\end{aligned}$$

Due to the diagonal nature of the matrix Λ, we are moving from one differential equation to n uncoupled differential equations:

$$\frac{dv_i}{dt} = \lambda_i \cdot v_i \qquad\qquad \forall\ i \in [1, n]$$

The solution for each of these equations is:

$$v_i = e^{\lambda_i \cdot t} \cdot v_{i0}$$

which leads to the solution of the initial problem:

$$\begin{aligned}
v &= e^{\Lambda \cdot t} \cdot v_0 && \text{solve for } v \\
S \cdot v &= S \cdot e^{\Lambda \cdot t} \cdot v_0 && \\
u &= S \cdot e^{\Lambda \cdot t} \cdot S^{-1} \cdot u_0 && \text{using } u_0 = S \cdot v_0
\end{aligned}$$

According to the values of λ, we can classify the solutions to this differential equation:

- stable:

$$\lim_{t \to \infty} u(t) = 0 \qquad\qquad Real(\lambda_i) < 0$$

- steady state:

$$\lim_{t \to \infty} u(t) = cte \quad Real(\lambda_i) <= 0, \text{ but at least } Real(\lambda_j) = 0 \text{ for some } j$$

- blow up: diverge:

$$\lim_{t \to \infty} u(t) = \infty \qquad\qquad Real(\lambda_j) > 0 \text{ for some } j$$

3.3.3 e^{At}

just like any function, the Taylor expansion can be applied to e^{At}. We have:

$$e^{At} = \sum_{i=0}^{\infty} \frac{(At)^i}{i!}$$

assuming that A can be diagonalized, $A = S \cdot \Lambda \cdot S^{-1}$, then:

$$\begin{aligned}
e^{At} &= \sum_{i=0}^{\infty} \frac{(S\Lambda S^{-1})^i \cdot t^i}{i!} && \text{use } A\text{'s diagonalization} \\
&= S \cdot \sum_{i=0}^{\infty} \frac{(\Lambda)^i \cdot t^i}{i!} \cdot S^{-1} && \text{factorized } S \text{ and } S^{-1} \\
e^{At} &= S \cdot e^{\Lambda t} \cdot S^{-1}
\end{aligned}$$

Let's understand $e^{\Lambda t}$ better

$$\begin{aligned}
e^{\Lambda t} &= \sum_{i=0}^{\infty} \frac{(\Lambda)^i \cdot t^i}{i!} \\
&= \sum_{i=0}^{\infty} \frac{(\Lambda)^i \cdot t^i}{i!} \implies \sum_{i=0}^{\infty} \frac{(\lambda_j)^i \cdot t^i}{i!} && \forall\ j \in [1, n] \\
\sum_{i=0}^{\infty} \frac{(\lambda_j)^i \cdot t^i}{i!} &= e^{\lambda_j t} && \text{Taylor expansion} \\
e^{\Lambda t} &= \begin{bmatrix} e^{\lambda_1 t} & 0 & 0 & .. & 0 \\ 0 & e^{\lambda_2 t} & 0 & .. & 0 \\ ... \\ 0 & 0 & 0 & .. & e^{\lambda_n t} \end{bmatrix} && \text{due to the diagonal nature of } \Lambda
\end{aligned}$$

4 Markov Matrices

a Markov matrix A is a square matrix satisfying two conditions:

- $a_{ij} > 0\ \forall\ (i, j)$
- $\sum_{i=1}^{n} a_{ik} = 0\ \forall k \in [1, n]$

More informally, a Markov matrix:

- has all elements larger than 0
- all columns add up to 1

Out of these two properties, other more significant results are satisfied:

1. A^k is a Markov Matrix $\forall\ k \in \mathbb{N}$

2. $\lambda = 1$ is an eigenvalue of A

3. $|\lambda_i| <= 1$

4. if $x_i \in Eig(A)$, then x_i have all elements larger than 0

4.1 Proofs

1.

$$A = \begin{bmatrix} a_{11} & a_{12} & .. & .. & a_{1n} \\ a_{21} & a_{22} & .. & .. & a_{2n} \\ .. & .. & .. & .. & .. \\ a_{n1} & a_{n2} & .. & .. & a_{nn} \end{bmatrix}$$

$$A^2 = \begin{bmatrix} a_{11} & a_{12} & .. & .. & a_{1n} \\ a_{21} & a_{22} & .. & .. & a_{2n} \\ .. & .. & .. & .. & .. \\ a_{n1} & a_{n2} & .. & .. & a_{nn} \end{bmatrix} \cdot \begin{bmatrix} a_{11} & a_{12} & .. & .. & a_{1n} \\ a_{21} & a_{22} & .. & .. & a_{2n} \\ .. & .. & .. & .. & .. \\ a_{n1} & a_{n2} & .. & .. & a_{nn} \end{bmatrix}$$

$$c_{A^2-1} = \sum_{i=1}^{n} c_{A-i} \cdot a_{1i} \qquad c_{M-i}\text{: } i\text{-th column of matrix } M$$

$$c_{A^2-1} = \begin{bmatrix} a_{11} \\ a_{21} \\ .. \\ a_{n1} \end{bmatrix} \cdot a_{11} + \begin{bmatrix} a_{12} \\ a_{22} \\ .. \\ a_{n2} \end{bmatrix} \cdot a_{12} + .. + \begin{bmatrix} a_{1n} \\ a_{2n} \\ .. \\ a_{nn} \end{bmatrix} \cdot a_{1n} \qquad \text{first column in } A^2 \text{ explicitly}$$

The sum of the first column is equal to $\sum_{i=1}^{n} a_{i1} = 1$, which can be generalized to any column k and since all elements in A are positive, multiplication and additions of positive elements produce positive elements Thus the claim is proved.

2. $\lambda = 1$ is an eigenvalue of A is equivalent to proving that $A - I$ is singular. the matrix A has columns adding up to 1. since we are subtracting 1 from each diagonal then $A - I$ has all columns adding up to 0. Therefore,

$$\begin{bmatrix} 1 & 1 & .. & 1 \end{bmatrix} \cdot \begin{bmatrix} a_{11}-1 & a_{12} & .. & .. & a_{1n} \\ a_{21} & a_{22}-1 & .. & .. & a_{2n} \\ .. & .. & .. & .. & .. \\ a_{n1} & a_{n2} & .. & .. & a_{nn}-1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ .. \\ 0 \end{bmatrix} \qquad \text{rows add up to 0}$$

Therefore, we have $dim(N(A^T)) \geq 1$ and since the matrix is square, $dim(N(A)) \geq 1$ and the matrix is singular.