# 1 Vector Space

a vector space S is set of vectors with the following constraints:

- for every vector $v \in S$, the vector $c * v \in S$
- for any two vectors $v, w \in S$, the vector $v + w \in S$

Therefore a Vector Space can be defined as a set where for any two vectors $v, w \in S$ the vector $c * v + d * w \in S$. A subspace of $S$, is vector space whose elements belong to $S$ such as:

- $\mathbf{R}^2$ and a line passing through the origin
- $\mathbf{R}^3$ and a plane / line passing through the origin.

One straight-forward conclusion can be drawn:

the zero vector is a subspace of any vector space.

Given vector spaces $S_1$ and $S_2$, the following:

- $S_1 \cup S_2$ is not (necessarily) a subspace
- $S_1 \cap S_2$ is a vector space.

## 1.1 Proof

- consider a line and a plane such that the line is not part of the latter. A sum of two vectors of different spaces would result in a vector that belongs to neither spaces, thus does not belong to their union.
- if $x_1 \in S_1 S_1 \cap S_2 \implies x_1 \in S_1$ and $x_1 \in S_2 \implies c_1 \cdot x_1 + c_2 \cdot x_2 \in S_1$ and $c_1 \cdot x_1 + c_2 \cdot x_2 \in S_2$ which implies $c_1 \cdot x_1 + c_2 \cdot x_2 \in S_1 \cap S_2$

# 2 Column and Null spaces

## 2.1 Definitions

**The Column space** of a matrix $A$ generally referred to as $C(A)$ is the set vector resulting of linear combinations of the column vectors of matrix $A$.

**Null Space** of a matrix $A$ are the vectors $x$ that satisfy the equation $A \cdot x = 0$.

## 2.2 $A \cdot x = 0$

such equation is of major importance as its solution provides a great deal of information about the matrix $A$. Let's consider the algorithm to solve it:

**Algorithm 1** Solve $A \cdot x = 0$

```
while i < rows and j < columns do
  if A[i][j] == 0 then
    check for row exchange
    if if a non-zero row exists then
      perform row exchange
    else
      add column to non-pivot columns
      j ← j + 1
      continue
    end if
  end if
  for k in [i + 1, row] do
    eliminate downwards
  end for
  add column to pivot columns
  j ← j + 1
  i ← i + 1
end while
{the column left out are necessarily non-pivot columns}
for i in [j + 1, col] do
  add col i to non-pivot columns
end for
{reduce the matrix to Reduced Row echelon form}
for i in pivot cols do
  perform elimination upwards
end for
{Now the matrix has zeros above and below the pivots}
diagonalize the matrix: set the pivots to 1
```

{The matrix now is of the from $\begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix}$}

**return** $\begin{bmatrix} -F \\ I \end{bmatrix}$

## 2.3 $A \cdot x = b$, $b \neq 0$

such an equation is quite related to $C(A)$. However, it heavily relies on the results from $A \cdot x = 0$. The ***RREF*** of a matrix can be obtained through a series of elementary matrices multiplication. Thus it is easier to deal with $E \cdot A \cdot x = E \cdot b$ than with the original equation.

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**Algorithm 2** Solve $A \cdot x = b$

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transforming the equation $A \cdot x = b$ to $R \cdot x = c$
find a particular solution $x_p$:

- check if the zero rows in $R$ correspond to zero values in $c$, if they do not, then the equation has no solution
- if they do, then set free variables: corresponding to free pivots to 0
- the solution $x_p$ can be found directly by broadcasting $c$ to match the dimensions $(rows, 1)$ with 0's

**return** $x_p + N(A)$

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## 2.4 solutions to equations

we define the ***rank*** of a matrix $A$ generally denoted as $r(A)$ as the number of pivot columns. Assuming a matrix $A$ is of dimensions $m, n$ then we consider the following cases:

- $r = m = n$ the matrix is invertible. The equation $A \cdot x = b$ has one and only solution for every vector in $\mathbb{R}^n$
- $r = m < n$ the equation has infinitely many solutions
- $r = n < m$ the equation has 0 or 1 solution.
- $r < n, r < m$ the equation has 0 or infinitely many.

## 2.5 Independence, Basis and dimension

for a matrix $A(m \cdot n)$ where $m < n$, the equation $A * x = 0$ has at least one non-zero solution since at least one variable will be free.

vectors $x_1, x_2, ...x_n$ are independent $\iff$ the only solution for $\sum_{i=1}^{n} x_i \cdot c_i = 0$ is $c_i = 0$ for $i = 1, 2..n$

Let's perceive the definition from a different perspective. if $v_1, v_2, v_3...vn$ are considered columns of a matrix, then they are independent $\iff$ $N(A)$ is the zero vector $\iff$ rank(A) == number of columns.

vectors $v_1, v_2, v_3 ... vn$ span a vector space when the contains all and only all the linear combinations of those vectors.

A basis of a vector space $S$ is a sequence of vectors $x_1, x_2, ... x_n$ with two properties

- they are independent
- they span the vector space

There are infinitely many basis for a single vector space. Yet, they all have one single property in common: the number of vectors known as the vector space's dimension.

$$dim(C(A)) = rank(A) \tag{1}$$

## 2.6 The 4 fundamental subspaces

### 2.6.1 dimensional properties

Each matrix $A(m \cdot n)$ has 4 remarkable vectors spaces. They are characterized by a number of properties listed below.

- column space:
  - C(A): set of vectors of linear combination of $A$'s column vectors.
  - it is a subspace of $R^m$
  - dim = rank(A)
- null space :
  - N(A): set of solutions of the equation $A \cdot x = 0$
  - it is a subspace of $R^n$
  - dim = n - rank(A)
- row space:
  - set of linear combinations of rows, also denoted as $C(A_T)$
  - it is a subspace of $R^n$
  - dim = rank(A)
- left null space:
  - $N(A^T)$
  - it is a subspace of $R^m$
  - dim = m - rank(A)

### 2.6.2 Basis

The subspaces mentioned above have predefined efficient algorithms (steps) to find (a) basis for each one of them. They are listed below:

- Column space $C(A)$:
  1. reduce the matrix $A$ to its ***RREF*** form denoted by $R$
  2. $R$ has pivot columns. These pivot columns are basis for $C(A)$
- Null Space $N(A)$: The procedure is explained in detail in the section $A \cdot x = 0$
- Row Space $C(A^T)$: as the definition indicate, it might be tempting to transpose the matrix $A$ and then look for the column space of the transpose. Yet, it is unnecessarily expensive.
  1. reduce the matrix $A$ to its ***RREF*** form denoted by $R$
  2. The form should have $r(A)$ first rows non-zero rows. These rows represent the basis.

  The elementary matrix responsible for the RREF form might include permutation matrices. Thus, it is not always true that the first $r(A)$ in the original matrix $A$ constitute a basis.
- left null space: $N(A^T)$
  1. reduce the matrix $A$ to its ***RREF*** form denoted by $R$ while tracking the process in matrix $E$.
  2. Matrix $E$ satisfies the equation $E \cdot A = R$. Considering the fact that $R$ has $m - r$ zero rows at the bottom of the matrix, then the last $m - r$ rows in $E$ represent the basis for as they are independent, (belong to an invertible matrix $E$) and by matrix multiplication (the row way) we have

     $$R_i = \sum_{k}^{n} E_{ik} \cdot A_k \tag{2}$$

     with $M_i$ represents the $i$-th row in matrix $M$

## 2.7 Rank 1 matrices

A matrix $A$ with $r(A) = 1$, can be written as $u \cdot v^T$ where $u$ and $v$ are vectors.

Since $r(A) = 1$, there is only one independent column vector and the rest are linear combination of that column vector. Assuming $A$ has $n$ columns, then we have $n - 1$ real values $c_1, c_2, ...c_{n-1}$ such that:

$$A = \begin{bmatrix} u & c1 \cdot u & c_2 \cdot u & .. & .. & c_{n-1} \cdot u \end{bmatrix} = u \cdot \begin{bmatrix} 1 & c_1 & c_2 & .. & .. & c_{n-1} \end{bmatrix} = u \cdot v^T \tag{3}$$