1 Matrix multiplication

1.1 vector and matrix multiplication

1.1.1 multiplication by row

We consider the multiplication of matrix A of shape (m, n) and a vector v of shape $(n, 1)$. The result is a $(m, 1)$ vector.

$$\begin{bmatrix} a_{11} & a_{12} & .. & a_{1n} \\ a_{21} & a_{22} & .. & a_{2n} \\ .. & .. & .. & .. \\ .. & .. & .. & .. \\ a_{m1} & a_{m2} & .. & a_{mn} \end{bmatrix} \cdot \begin{bmatrix} v_1 \\ v_2 \\ .. \\ .. \\ v_m \end{bmatrix} = v1 \cdot a_1 + v2 \cdot a2 + ... + v_m \cdot a_m \tag{1}$$

1.1.2 multiplication by column

We consider the multiplication of vector v_1 of shape $(1, m)$matrix A of shape (m, n). The result is a vector of shape $(1, n)$.

$$\begin{bmatrix} v_1 & v_2 & .. & .. & v_m \end{bmatrix} \cdot \begin{bmatrix} a_{11} & a_{12} & .. & a_{1n} \\ a_{21} & a_{22} & .. & a_{2n} \\ .. & .. & .. & .. \\ .. & .. & .. & .. \\ a_{m1} & a_{m2} & .. & a_{mn} \end{bmatrix} = v_1 \cdot r_1 + v_2 \cdot r_2 + .. + v_m \cdot r_m \tag{2}$$

where r_i is the ith row.

1.2 Matrix and Matrix multiplication

The matrix multiplication operation is only valid with matrices of the following form: $A(m, n)$ and $B(n, p)$. Thus, it is obvious that $A \cdot B \neq B \cdot A$

$$A \cdot B = \begin{bmatrix} a_{11} & a_{12} & .. & a_{1n} \\ a_{21} & a_{22} & .. & a_{2n} \\ .. & .. & .. & .. \\ .. & .. & .. & .. \\ a_{m1} & a_{m2} & .. & a_{mn} \end{bmatrix} \cdot \begin{bmatrix} b_{11} & b_{12} & .. & b_{1p} \\ b_{21} & b_{22} & .. & b_{2p} \\ .. & .. & .. & .. \\ .. & .. & .. & .. \\ b_{n1} & b_{m2} & .. & b_{np} \end{bmatrix} \tag{3}$$

let's refer to the ith column in matrix A as c_{ai} and ith row as r_{ai}. Similarly we have c_{bi} and r_{bi}. Then $A \cdot B$ can be seen as:

- $$\begin{bmatrix} A \cdot c_{b1} & A \cdot c_{b2} & .. & A \cdot c_{bp} \end{bmatrix}$$

- $$\begin{bmatrix} r_{a1} \cdot B \\ r_{a2} \cdot B \\ .. \\ .. \\ r_{am} \cdot B \end{bmatrix}$$

- C such that

$$C_{ij} = \sum_{k=1}^{n} a_{ik} \cdot b_{kj} \tag{4}$$

2 Elementary Matrices

Elementary matrices are simple square matrices that occur frequently from an algorithmic perspective in Linear Algebra.

2.1 Identity matrix

It is the multiplicative identity in the set of matrices. The n dimensional identity matrix can be expressed as follows:

$$I = \begin{bmatrix} 1 & 0 & 0 & .. & 0 \\ 0 & 1 & 0 & .. & 0 \\ .. & .. & .. & .. & 0 \\ .. & .. & .. & .. & 1 \end{bmatrix}$$

For any square $A \cdot I = I \cdot A = A$. The same conclusion can be made with non-square matrices with the right side.

2.1.1 Inverse of Identity

The inverse of identity matrix is the identity matrix itself. $I \cdot I = I$

2.2 Elimination matrices

These matrices express the subtraction/addition steps in the elimination algorithm.

$$E_{ij}(x) = \begin{bmatrix} 1 & 0 & 0 & .. & 0 \\ 0 & 1 & 0 & .. & 0 \\ .. & .. & x & .. & 0 \\ .. & .. & .. & .. & 1 \end{bmatrix}$$

where the value x is in the ij entry: ith row and jth column. More formally:

$$E_{ij}(x) \cdot A = B \tag{5}$$

where B is the same as A with the exception of $r_{bj} = r_{aj} + x \cdot r_{ai}$

2.2.1 inverse of elimination matrices

The inverse of $E_{ij}(x)$ is $E_{ij}(-x)$.

2.3 Permutation matrix

Such matrices are used to swap other matrices' rows. A permutation matrix is basically an identity matrix with a different order of rows. In other words, there is a single 1 in every row and column.

2.3.1 inverse of permutation matrix

Any permutation matrix satisfy the following property:

$$P^T \cdot P = I \tag{6}$$