1 Properties of determinant

A determinant is a mathematical function : $M- > R$ where M is the set of all matrices. determinant, also denoted as $DET(A)$ or $|A|$ is characterized with a number of properties

1. $DET(I) = 1$
2. $|DET(PA)| = |DET(A)|$ where P is a permutation matrix. if the number of rows exchanged is even then $|PA| = |A|$, otherwise, $|PA| = -|A|$. It is to be noted that a permutation is either odd or even
3. (a) multiplying a single row of A by $K \in \mathbb{R}$ produces a matrix B such that $|B| = K \cdot |A|$

   (b) let $b = \begin{bmatrix} b_1 & b_2 & .. & b_n \end{bmatrix}$ and $A = \begin{bmatrix} a_{11} & a_{12} & .. & a_{1n} \\ a_{21} & a_{22} & .. & a_{2n} \\ .. \\ a_{n1} & a_{n2} & .. & a_{nn} \end{bmatrix}$

   $A+b = \begin{bmatrix} a_{11} & a_{12} & .. & a_{1n} \\ a_{21}+b_1 & a_{22}+b_2 & .. & a_{2n}+b_n \\ .. \\ a_{n1} & a_{n2} & .. & a_{nn} \end{bmatrix}$ $A^{'} = \begin{bmatrix} a_{11} & a_{12} & .. & a_{1n} \\ b_1 & b_2 & .. & b_n \\ .. \\ a_{n1} & a_{n2} & .. & a_{nn} \end{bmatrix}$

   $|A + b| = |A| + |A^{'}|$
4. if matrix A has two equal rows then $|A| = 0$:
   $A = \begin{bmatrix} a_1 \\ .. \\ a_j \\ .. \\ a_k = a_j \\ .. \\ a_n \end{bmatrix}$ we have $P \cdot A = A \implies |A| = -|A|$ (as it is only one row exchange) $\implies |A| = 0$ Using property 3.a we can easily see that is applies also on proportional rows.
5. a matrix A with a row of zeros has $|A| = 0$: apply property 3.a with $k = 0$
6. subtracting any factor of one row from another row does not change the

determinant.

$$DET(\begin{bmatrix} .. \\ row_i \\ .. \\ row_j - l \cdot row_i \\ .. \end{bmatrix}) = DET(\begin{bmatrix} .. \\ row_i \\ .. \\ row_j \\ .. \end{bmatrix}) - l \cdot DET(\begin{bmatrix} .. \\ row_i \\ .. \\ row_i \\ .. \end{bmatrix}) \quad \text{apply property 3.b}$$
$$= DET(\begin{bmatrix} .. \\ row_i \\ .. \\ row_j \\ .. \end{bmatrix}) \quad \text{apply property 4}$$

7. if D is a diagonal matrix, then $|D| = \prod(pivots)$: using property 3.a we can see that $|D| = \prod(pivots) \cdot |I|$

8. if A is upper or lower triangular then $|A| == \prod(pivots)$

9. $|A \cdot B| = |A| \cdot |B|$

10. $|A^T| = |A|$

2 Determinant's formulas

2.1 Big Formula

2.1.1 (2, 2) matrix

Let's prove the formula for a $(2, 2)$ matrix.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & 0 \\ c & d \end{bmatrix} + \begin{bmatrix} 0 & b \\ c & d \end{bmatrix} \quad \text{apply 3.a}$$
$$= \begin{bmatrix} a & 0 \\ c & 0 \end{bmatrix} + \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} + \begin{bmatrix} 0 & b \\ 0 & d \end{bmatrix} + \begin{bmatrix} 0 & b \\ c & 0 \end{bmatrix} \quad \text{apply 3.a on each of the two matrices}$$
$$= a \cdot c - b \cdot d \quad \text{apply 2, 4, 6}$$

2.1.2 (3, 3) matrix

let's extend the idea to $(3, 3)$ before generalizing.

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} = \begin{bmatrix} a_1 & 0 & 0 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} + \begin{bmatrix} 0 & b_1 & 0 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} + \begin{bmatrix} 0 & 0 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$$

each of these 3 matrices will go through the same process applied to the 2-nd row resulting in a total of 9 matrices such each has only one non-zero value in the each of the first 2 rows. Each of these 9 matrix will produce other 3 matrix each having only 1 non-zero value in the last row. so the final number is $3^3 = 27$ matrices. The matrices with a non-zero determinant are those that have values in different columns and rows. for $n = 3$ they are the following:

$$\begin{aligned}
\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} &= \begin{bmatrix} a_1 & 0 & 0 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} + \begin{bmatrix} 0 & b_1 & 0 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} + \begin{bmatrix} 0 & 0 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \\
&= \begin{bmatrix} a_1 & 0 & 0 \\ 0 & b_2 & 0 \\ 0 & 0 & c_3 \end{bmatrix} + \begin{bmatrix} a_1 & 0 & 0 \\ 0 & 0 & b_3 \\ 0 & c_2 & 0 \end{bmatrix} + \begin{bmatrix} 0 & a_2 & 0 \\ b_2 & 0 & 0 \\ 0 & 0 & c_3 \end{bmatrix} \\
&+ \begin{bmatrix} 0 & a_2 & 0 \\ 0 & 0 & b_3 \\ c_1 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & a_1 \\ b_1 & 0 & 0 \\ 0 & c_2 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & a_2 \\ 0 & b_2 & 0 \\ c_1 & 0 & 0 \end{bmatrix}
\end{aligned}$$

2.1.3 (n, n) matrix

the Using the same idea, each matrix will produce n^n matrices, only $n!$ of them have non-zero determinant as they must have non-zero values in different rows and columns. The final formula can be expressed as:

$$|A| = \sum a_{1\alpha} \cdot a_{2\beta} \cdot a_{3\sigma} .. \cdot a_{n\omega}$$

where $\alpha, \beta, \sigma, ..\omega$ represent different permutations of $(1, 2, ..n)$

2.2 The co-factors formula

Given a matrix A we define M_{ij} as the matrix resulting from removing the i-th row and j-th column. and $C_{ij} = (-1)^{i+j} \cdot |M_{ij}|$ Another way to see $|A|$ is:

$$|A| = \sum_{i=1}^{n} a_{ki} \cdot C_{ki}$$

where k can be any value within the range $[1, n]$

3 Applications of determinants

3.1 Inverse of a matrix

Using cofactors, the inverse of a matrix A can be expressed as :

$$A^{-1} = \frac{C^T}{|A|}$$

Let's prove it:

$$A = \begin{bmatrix} a_{11} & a_{12} & .. & a_{1n} \\ a_{21} & a_{22} & .. & a_{2n} \\ ... \\ a_{n1} & a_{n2} & .. & a_{nn} \end{bmatrix} \qquad \text{define } A$$

$$C = \begin{bmatrix} C_{11} & C_{12} & .. & C_{1n} \\ C_{21} & C_{22} & .. & C_{2n} \\ ... \\ C_{n1} & C_{n2} & .. & C_{nn} \end{bmatrix} \qquad \text{define } C$$

$$|A| = \sum_{i=1}^{n} a_{ji} \cdot C_{ji} \qquad \text{for any } j \text{ in range } [1, n]$$

$$A \cdot C^T = \begin{bmatrix} a_{11} & a_{12} & .. & a_{1n} \\ a_{21} & a_{22} & .. & a_{2n} \\ ... \\ a_{n1} & a_{n2} & .. & a_{nn} \end{bmatrix} \cdot \begin{bmatrix} C_{11} & C_{21} & .. & C_{n1} \\ C_{12} & C_{22} & .. & C_{n2} \\ ... \\ C_{1n} & C_{2n} & .. & C_{nn} \end{bmatrix}$$

$$(A \cdot C^T)_{ii} = \sum_{i=1}^{n} a_{ji} \cdot C_{ji} = |A|$$

$$(A \cdot C^T)_{ij} = \sum_{k=1}^{n} a_{ik} \cdot C_{jk}$$

$$= DET(\begin{bmatrix} a_{11} & a_{12} & .. & a_{1n} \\ ... \\ a_{j1} & a_{j2} & .. & a_{jn} \\ ... \\ a_{j1} & a_{j2} & .. & a_{jn} \\ a_{n1} & a_{n2} & .. & a_{nn} \end{bmatrix}) = 0 \qquad \text{det of a matrix with 2 equal rows is 0}$$

$$(A \cdot C^T) = |A| \cdot I$$

$$A^{-1} = \frac{C^T}{|A|}$$