1 Symmetric Matrices

1.1 Properties

Symmetric matrices have quite powerful properties:

1. all its eigenvalues are real
2. its eigenvectors can be chosen orthonormal
3. the number of positive pivots is the same the number of positive Eigenvalues.

1.2 Proofs

This section includes certain proofs for a number of results stated in the previous section:

- Let$A$ be a symmetric matrix $\implies A^T = A$. let x_1 and x_2 be two independent eigenvectors.

  $$\begin{aligned} x_2^T \cdot A^T A \cdot x_1 &= (x_2 \cdot A)^T \cdot (A \cdot x_1) \\ x_2^T \cdot A^T A \cdot x_1 &= x_2^T x_1 \cdot \lambda_2 \cdot \lambda_1 \\ x_2^T \cdot A^2 \cdot x_1 &= x_2^T x_1 \cdot \lambda_2 \cdot \lambda_1 && \text{using symmetry} \\ x_2^T \cdot x_1 \cdot \lambda_2^2 &= x_2^T x_1 \cdot \lambda_2 \cdot \lambda_1 && \text{using result 3} \\ x_2^T \cdot x_1 &= 0 && x_1 \text{ and } x_2 \text{ independent} \implies \lambda_2 \neq \lambda_1 \end{aligned}$$

  Thus the claim is proved (the orthogonality part) for the case where the λ's are different.

- let's prove that λ's are real:

  $$\begin{aligned} A \cdot x &= \lambda \cdot x \\ A \cdot \bar{x} &= \bar{\lambda} \cdot \bar{x} && \text{conjugate both sides} \\ \bar{x}^T \cdot A^T &= \bar{x}^T \cdot \bar{\lambda} && \text{transpose} \\ \bar{x}^T \cdot A \cdot x &= \bar{x}^T \cdot x\bar{\lambda} && \text{multiply by } x \text{ both sides and use symmetry} \\ \bar{x}^T \cdot A \cdot x &= \bar{x}^T \lambda \cdot x && \text{multiply first equation by } \bar{x}^T \\ \bar{x}^T \lambda \cdot x &= \bar{x}^T \cdot x\bar{\lambda} && \text{use the last 2 equations} \\ \bar{x}^T \cdot x &\neq 0 && \text{complex vector length} \\ \bar{\lambda} &= \lambda \end{aligned}$$

  Thus the claim is proved.

The third statement is slightly more advanced theorem. Using the orthogonality property, A can be written as:

$$A = Q \cdot \Lambda \cdot Q^T$$

This equation can be expended to be written as follows:

$$A = \sum_{i=1}^{n} q_i \cdot q_i^T \cdot \lambda_i$$

2 Positive Definite Matrix

There is a special case of square matrices referred to as the ***Positive Definite*** matrix satisfying the following condition:

$$x^T \cdot A \cdot x > 0 \qquad\qquad \forall x \ \in \mathbb{R}^n$$

2.1 $x^T \cdot A \cdot x$

Assuming

$$A = \begin{bmatrix} a_{11} & a_{12} & .. & .. & a_{1n} \\ a_{21} & a_{22} & .. & .. & a_{2n} \\ .. & .. & .. & .. & .. \\ a_{n1} & a_{n2} & .. & .. & a_{nn} \end{bmatrix}$$

and

$$x = \begin{bmatrix} x_1 \\ x_2 \\ .. \\ x_n \end{bmatrix}$$

then

$$x^T \cdot A \cdot x = \sum_{i=1}^{n} a_{ii} \cdot x_i^2 + \sum_{(i,j), i \neq j} (a_{ij} + a_{ji}) \cdot x_i \cdot x_j$$

According to the expansion above, Any quadratic function can be expressed as $x^T \cdot A \cdot x$ where A is a symmetric matrix. Thus out focus is directed towards symmetric Positive definite.

2.2 Positive definite and symmetric

The positive definite condition does not impose symmetry as there is asymmetrical matrix satisfying the same condition. for example:

$$A = \begin{bmatrix} 2 & 0 \\ 2 & 2 \end{bmatrix}$$

$$x^T \cdot A \cdot x = 2(x_1^2 + x_2^2 + x_1 x_2) > 0$$

The positive definite matrices are supposed to be symmetric for the symmetry's desirable properties.

2.3 Equivalent conditions

The general condition provided in the definition has a number of equivalent mathematical conditions:

- all eigenvalues are strictly positive
- the inner determinants are positive

Let's prove the equivalence between the definition condition and the other two.

$$\begin{aligned}
A &= Q^T \cdot \Lambda \cdot Q && \text{A is symmetric} \\
x^T \cdot A \cdot A &= x^T \cdot Q^T \cdot \Lambda \cdot Q \cdot x && \\
x^T \cdot A \cdot x &= y^T \cdot \Lambda \cdot y && Q \text{ is invertible then } Q \cdot x \text{ can be any vector} \\
x^T \cdot A \cdot x &= \sum_{i=1}^{n} \lambda_i \cdot y_i^2 &&
\end{aligned}$$

Therefore

$$\begin{aligned}
x^T \cdot A \cdot x > 0 & \qquad \forall x \ \in \mathbb{R}^n \\
\iff \sum_{i=1}^{n} \lambda_i \cdot y_i^2 > 0 & \qquad \forall y \ \in \mathbb{R}^n \\
\iff \lambda_i > 0 & \qquad \text{for } i = 1, 2.., n
\end{aligned}$$

the last equivalence can be easily proved and thus the first equivalence is proved. As for the second equivalence, we need to recall that for any symmetric matrix the number of positive eigenvalues is the same as the number of positive pivots. Thus a matrix is positive definite if all its pivots are positive. This condition is equivalent to: all the inner upper left determinants are positive.

A number of observations can be made about positive definiteness:

- if A and B are positive definite then so is $A + B$
- if $A = R^T R$ is positive semi definite for all matrices R, positive definite for all R with independent columns.

2.4 Minima application

for a function f with n variables: $x_1, x_2, .., x_n$. then the criteria for a minima are as follows:

1. $\frac{\delta f}{\delta x_i} = 0 \ \forall x_i$
2. the matrix of 2nd derivatives is positive definite

where the matrix of 2nd derivatives can be expressed as:

$$\frac{\delta^2 f}{\delta^2 x} = \begin{bmatrix} f_{x_1 x_1} & f_{x_1 x_2} & .. & .. & a_{x_1 x_n} \\ f_{x_2 x_1} & f_{x_2 x_2} & .. & .. & f_{x_2 x_n} \\ .. & .. & .. & .. & .. \\ f_{x_n x_1} & f_{x_n x_2} & .. & .. & f_{x_n x_n} \end{bmatrix}$$