Discrete Mathematics, Tutorial VI

1. (a) Find a recurrence relation for the number of strictly increasing sequences of positive integers that have 1 as their first term and *n* as their last term, where *n* is a positive integer. That is, sequences $a_1, a_2, ..., a_k$, where $a_1 = 1, a_k = n$, and $a_j < a_{j+1}$ for $j = 1, 2, ..., k-1$.
   (b) What are the initial conditions?
   (c) How many sequences of the type described in (a) are there when n is an integer with $n \geq 2$?
2. (a) Find a recurrence relation for the number of bit strings of length *n* that contain three consecutive 0s.
   (b) What are the initial conditions?
   (c) How many bit strings of length seven contain three consecutive 0s?
3. (a) Find a recurrence relation for the number of bit strings of length n that contain the string 01.
   (b) What are the initial conditions?
   (c) How many bit strings of length seven contain the string 01?
4. A string that contains only 0s, 1s, and 2s is called a **ternary string**.
   (a) Find a recurrence relation for the number of ternary strings of length n that contain two consecutive symbols that are the same.
   (b) What are the initial conditions?
   (c) How many ternary strings of length six contain consecutive symbols that are the same?
5. Let *T(m, n)* denote the number of onto functions from a set with *m* elements to a set with *n* elements. Show that *T(m, n)* satisfies the recurrence relation

   $$T(m,n) = n^m - \sum_{k=1}^{n-1} C(n,k)T(m,k)$$

   whenever $m \geq n$ and $n > 1$, with the initial condition $T(m,1) = 1$.
6. (a) Let $S(n,k)$ denote the number of ways of partitioning n distinct elements into k disjoint non-empty subsets. Give a combinatorial proof that $S(n,k) = S(n-1,k-1) + kS(n-1,k)$.
   (b) What will be the value of $S(n+1,n)$?
7. Give a combinatorial proof that $\sum_{k=1}^{n} k\binom{n}{k}^2 = n\binom{2n-1}{n-1}$.
8. Let (x_i, y_i), $i = 1, 2, 3, 4, 5$, be a set of five distinct points with integer coordinates in the xy plane. Show that the midpoint of the line joining at least one pair of these points has integer coordinates.
9. (a) Show that if five integers are selected from the first eight positive integers, there must be a pair of these integers with a sum equal to 9.
   (b) Is the conclusion in part (a) true if four integers are selected rather than five?
10. Show that for every integer n there is a multiple of n that has only 0s and 1s in its decimal expansion.
11. Show that every sequence of $n^2 + 1$ distinct real numbers contains a subsequence of length $n + 1$ that is either strictly increasing or strictly decreasing.
12. There are 9 people, aged from 18 to 58, at a family reunion. Show by pigeonhole principle that it is possible to choose two disjoint groups of these people in such a way that the sums of the ages of the people in each group are equal.

13. Let $A = \{1, 2, \dots, 2n\}$ and let $B \subset A$ be any subset of A, such that $|B| = n + 1$. Using pigeon-hole principle, show that there exists two integers $a_i, a_j \in B$, such that either a_i divides a_j or a_j divides a_i.

14. How many solutions are there to the equation

    $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 29$,

    where $x_i, i = 1, 2, 3, 4, 5, 6$, is a non-negative integer such that

    (a) $x_i > 1$ for $i = 1, 2, 3, 4, 5, 6$?

    (b) $x_1 \geq 1, x_2 \geq 2, x_3 \geq 3, x_4 \geq 4, x_5 > 5$, and $x_6 \geq 6$?

    (c) $x_1 \leq 5$?

    (d) $x_1 < 8$ and $x_2 > 8$?