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Homework 5

EEE 573
Medical Imaging

Homework 5 Report

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Contents

Question 1

This question asks us to find the longitudinal and transverse magnetization components in an MR sequence when two RF pulses are applied with tip angle α. We assume $M_z(0^-) = M_0$ and $M_{xy}(0^-) = 0$, where M_0 is the equilibrium magnetization. We assume that $TR >> T_2$, meaning the T_2 relaxation will fully happen between two pulses, we do not approximate this with T_1.

a. $M_z(0^+)$ and $M_{xy}(0^+)$
The first pulse happens.

$$M_z(0^+) = M_0 cos(\alpha) \tag{1}$$
$$M_{xy}(0^+) = M_0 sin(\alpha)e^{j\phi} \tag{2}$$

b. $M_z(TR^-)$ and $M_{xy}(TR^-)$

$$M_z(t) = M_0(1 - e^{-t/T_1}) + M_0 cos(\alpha)e^{-t/T_1} \tag{3}$$
$$M_z(TR^-) = M_0(1 - e^{-TR/T_1}) + M_0 cos(\alpha)e^{-TR/T_1} \tag{4}$$
$$= M_0 - (M_0 - M_0 cos(\alpha))e^{-TR/T_1} \tag{5}$$
$$M_{xy}(t) = M_0 sin(\alpha)e^{-j(2\pi\nu_0 t - \phi)}e^{-t/T_2} \tag{6}$$
$$M_{xy}(TR^- >> T_2) = 0 \tag{7}$$

c. $M_z(TR^+)$ and $M_{xy}(TR^+)$

$$M_z(TR^+) = (M_0 - (M_0 - M_0 cos(\alpha))e^{-TR/T_1})cos(\alpha) \tag{8}$$
$$M_{xy}(TR^+) = (M_0 - (M_0 - M_0 cos(\alpha))e^{-TR/T_1})sin(\alpha)e^{j\phi} \tag{9}$$

d. $M_z(t)$ and $M_{xy}(t)$ for $t > TR$

$$M_z(t) = M_0(1 - e^{-t/T_1}) + M_z(TR^-)cos(\alpha)e^{-t/T_1} \tag{10}$$
$$= M_0(1 - e^{-t/T_1}) + (M_0 - M_0 cos(\alpha))e^{-TR/T_1})cos(\alpha)e^{-t/T_1} \tag{11}$$
$$M_{xy}(t) = (M_0 - (M_0 - M_0 cos(\alpha))e^{-TR/T_1})sin(\alpha)e^{-j(2\pi\nu_0 t - \phi)}e^{-t/T_2} \tag{12}$$

Question 2

In this question, we are asked to derive the steady state longitudinal magnetization for sequential pulses like presented in the previous question. Let us define, for any iteration n of pulses. M_0 is defined as in the previous question.

$$M_z(nTR^-) = M_z \tag{13}$$

Then we can say that magnetization after the RF pulse would be as follows.

$$M_z(nTR^+) = M_z cos(\alpha) \tag{14}$$

Then, we can find the magnetization before the next RF pulse.

$$M_z((n+1)TR^-) = M_0 - (M_0 - M_z(nTR^+))e^{-TR/T_1} \tag{15}$$

which can be rewritten as,

$$M_z((n+1)TR^-) = M_0 - (M_0 - M_z cos(\alpha))e^{-TR/T_1} \tag{16}$$

Then, for any steady-state, we can assume that $M_z(nTR^-) = M_z((n+1)TR^-) = {M_z}^{ss}$:

$$M_z = M_0 - (M_0 - M_z cos(\alpha))e^{-TR/T_1} \tag{17}$$

We can rearrange the terms as follows to obtain the ${M_z}^{ss}$ value.

$$M_z(1 - cos(\alpha)e^{-TR/T_1}) = M_0(1 - e^{-TR/T_1}) \tag{18}$$

$$M_z = {M_z}^{ss} = M_0 \frac{1 - e^{-TR/T_1}}{1 - cos(\alpha)e^{-TR/T_1}} \tag{19}$$

Question 3

In this question we are asked to draw a pulse diagram that covers a circular trajectory in k-space. The trajectory we are expected to cover is presented in Figure 1.

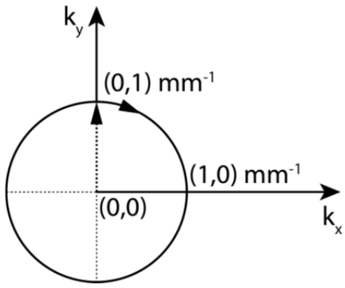


Figure 1: Trajectory in k-space

Here, we are told that the data acquisition occurs in the circular trajectory and the maximum G is $4G/cm$. We are expected to cover the trajectory in the smallest time possible. The proposed pulse sequence diagram is given in Figure 2

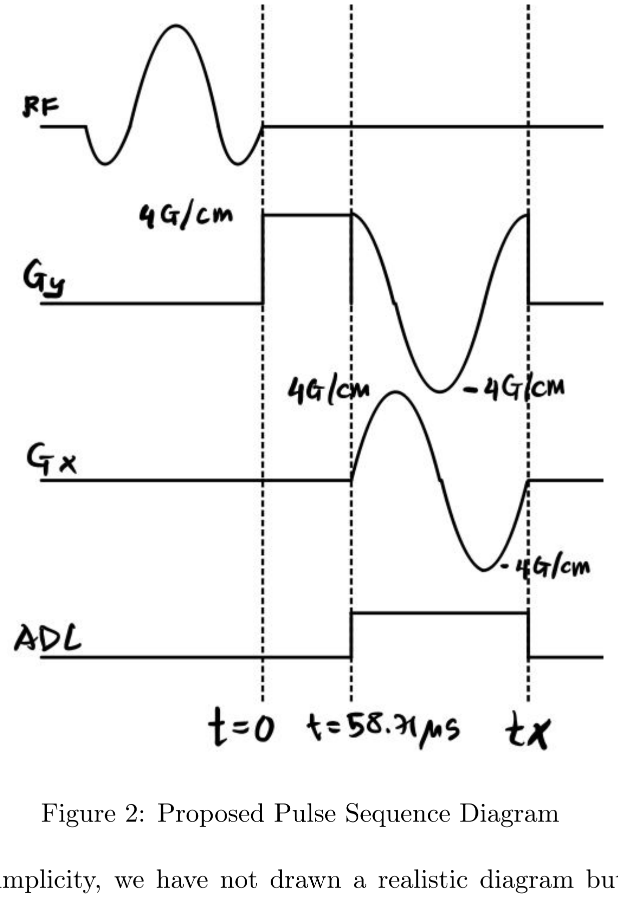


Figure 2: Proposed Pulse Sequence Diagram

We can see that for simplicity, we have not drawn a realistic diagram but an ideal one. We first applied the RF pulse, and no slice selection is applied. Hence, we adjust the y-gradient to move up in the k_y direction. After that, for the circular trajectory, we have applied sinusoidal gradients, one being sine and the other one being cosine such that the trajectory follows the same radius with varying r. This is the window that we take the data in as we see with the ADC. The time applied before acquisition can be found through $gyromagneticratio \Delta G_y \Delta t = \Delta k_y$. Also the given t_x can be given via integration of one of the gradients with respect to time.

Question 4

This question asks us about the estimation of the longitudinal relaxation (t_1) from two pairs of MRI images, one being a noise-free set while the other is not. The first two parts of the question asks about the underlying theory, while the other questions are about practice over MATLAB.

a. This part asks us to convert the signal level during echo time TE for as steady-state pulse sequence to the expression given below.

$$\frac{f(x,y)}{\sin\alpha} = e^{-TR/T_1}\frac{f(x,y)}{tan\alpha} + AM_0e^{-TE/T_2}(1-e^{-TR/T_1}) \tag{20}$$

We are given the equation,

$$f(x,y) = AM_0\sin\alpha e^{-TE/T_2}\frac{1-e^{-TR/T_1}}{1-\cos\alpha e^{-TR/T_1}} \tag{21}$$

For that, we first rewrite $f(x,y)/sin\alpha$ as follows.

$$\frac{f(x,y)}{\sin\alpha} = AM_0e^{-TE/T_2}\frac{1-e^{-TR/T_1}}{1-\cos\alpha e^{-TR/T_1}} \tag{22}$$

Then, we can also write $f(x,y)/tan\alpha$.

$$\frac{f(x,y)}{tan\alpha} = AM_0e^{-TE/T_2}\frac{1-e^{-TR/T_1}}{1-\cos\alpha e^{-TR/T_1}}\cos\alpha \tag{23}$$

which, can be rewritten as,

$$\frac{f(x,y)}{tan\alpha} = AM_0e^{-TE/T_2}(1-e^{-TR/T_1})\frac{\cos\alpha}{1-\cos\alpha e^{-TR/T_1}} \tag{24}$$

Then, we can say that,

$$\frac{f(x,y)}{tan\alpha}e^{-TR/T_1} = AM_0e^{-TE/T_2}(1-e^{-TR/T_1})\frac{\cos\alpha e^{-TR/T_1}}{1-\cos\alpha e^{-TR/T_1}} \tag{25}$$

Also, we can rearrange the equation as given below and see if the result is similar.

$$\frac{f(x,y)}{\sin\alpha} - AM_0e^{-TE/T_2}(1-e^{-TR/T_1}) \tag{26}$$

$$= AM_0e^{-TE/T_2}\frac{1-e^{-TR/T_1}}{1-\cos\alpha e^{-TR/T_1}} - AM_0e^{-TE/T_2}(1-e^{-TR/T_1}) \tag{27}$$

$$= AM_0e^{-TE/T_2}(1-e^{-TR/T_1})\left(\frac{1}{1-\cos\alpha e^{-TR/T_1}} - 1\right) \tag{28}$$

$$= AM_0e^{-TE/T_2}(1-e^{-TR/T_1})\frac{\cos\alpha e^{-TR/T_1}}{1-\cos\alpha e^{-TR/T_1}} \tag{29}$$

$$= \frac{f(x,y)}{tan\alpha}e^{-TR/T_1} \tag{30}$$

Hence, we can confirm the original equality can be written in terms of this one.

b. This part requires us to show that an estimate of T_1, denoted as $\tilde{T_1}$, can be obtained using the following relation,

$$\tilde{T_1} = \frac{TR}{ln\left(\frac{f_2}{tan\alpha_2} - \frac{f_1}{tan\alpha_1}\right) - ln\left(\frac{f_2}{\sin\alpha_2} - \frac{f_1}{\sin\alpha_1}\right)} \tag{31}$$

Here, we start from the equation given in the previous part and assume $TE >> T_2$, in order to remove the effect of T_2 decay and simplify the equation. Then, the equality becomes,

$$\frac{f(x,y)}{\sin\alpha} = e^{-TR/T_1}\frac{f(x,y)}{tan\alpha} \tag{32}$$

For two $f_i, i = 1, 2$, we can substitute one from the other as follows.

$$\left(\frac{f_2}{tan\alpha_2} - \frac{f_1}{tan\alpha_1}\right) = e^{TR/T_1}\left(\frac{f_2}{\sin\alpha_2} - \frac{f_1}{\sin\alpha_1}\right) \tag{33}$$

Then we take the natural logarithm of the two sides.

$$ln\left(\frac{f_2}{tan\alpha_2} - \frac{f_1}{tan\alpha_1}\right) = \frac{TR}{T_1} + ln\left(\frac{f_2}{\sin\alpha_2} - \frac{f_1}{\sin\alpha_1}\right) \tag{34}$$

Then, we can use algebra to find T_1:

$$\frac{TR}{T_1} = ln\left(\frac{f_2}{tan\alpha_2} - \frac{f_1}{tan\alpha_1}\right) - ln\left(\frac{f_2}{\sin\alpha_2} - \frac{f_1}{\sin\alpha_1}\right) \tag{35}$$

$$T_1 = \frac{TR}{ln\left(\frac{f_2}{tan\alpha_2} - \frac{f_1}{tan\alpha_1}\right) - ln\left(\frac{f_2}{\sin\alpha_2} - \frac{f_1}{\sin\alpha_1}\right)} \tag{36}$$

c. This part asks us to display the noise free images, namely *image1* and *image2*, and find the T_1 map using the approximation derived above. These are images of the same patient and derived from the same axial slice with the same T_1 weighted images using the same TE and TR, which are $15ms$ and $600ms$ respectively, the only difference being the tip angle. Tip angle (α) is 45° in image 1 while 90° in image 2. The images are given in Figure 3. Hence, using the equation, in the

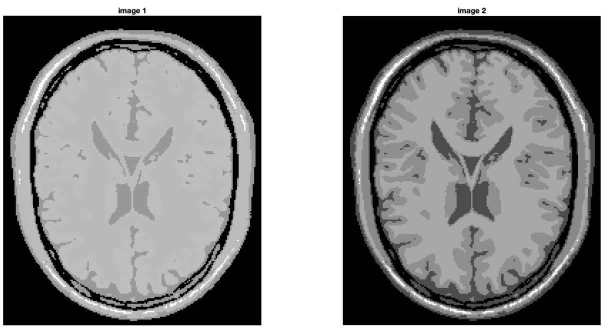


Figure 3: Noise-Free Images

previous part, the T_1 map is as given in Figure 4.

Figure 4: T_1 Map for the Noise-Free Images

d. After that, we were required to estimate the T_1 of the white matter in the image, using the **roiellipse.m**, which is an elliptic selection tool that is manually applied on the image that results in a binary mask of the selected region. For white matter, we have placed the ellipse as follows (see Fig. 5).

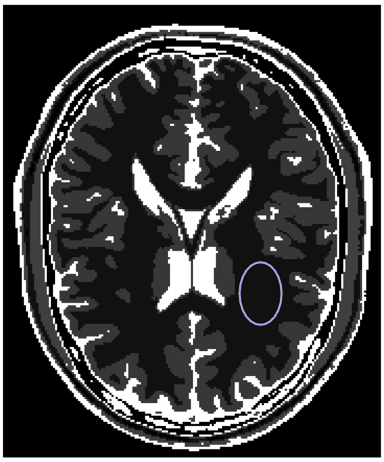
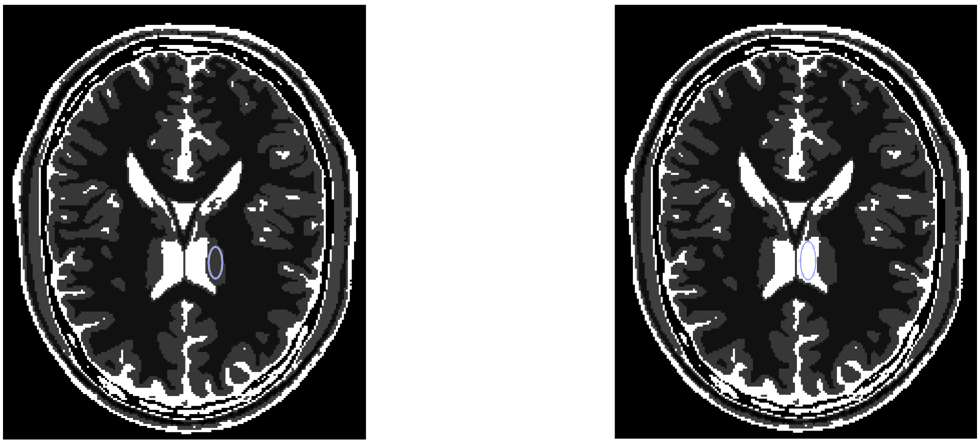


Figure 5: ROI Selection to Capture White Matter on T_1 Map

Then, we take the mean of the T_1 map by applying the mask on the map, and the result is given as $500.00ms$.

e. In this part, we are asked to repeat the procedure followed in the previous part for the gray matter and cerebrospinal fluid (CSF). The applied ROIs are given below.



(a) ROI Selection to Capture Gray Matter

(b) ROI Selection to Capture CSF

Figure 6: Selected Elliptic ROIs on T_1 Map

The T_1 estimate for the Gray Matter is found as $833.00ms$ while it is $2569.00ms$ for the CSF.

f. This part asks us to run the same calculations we have done in Part C with the noisy images. Since the images contain noise, we have displayed the absolute values of the T_1 map components and also taken the real part into account in case the data is complex. Also, we have displayed the image to be in the range $[0, 3000]ms$. The noisy images are presented in Figure 7.

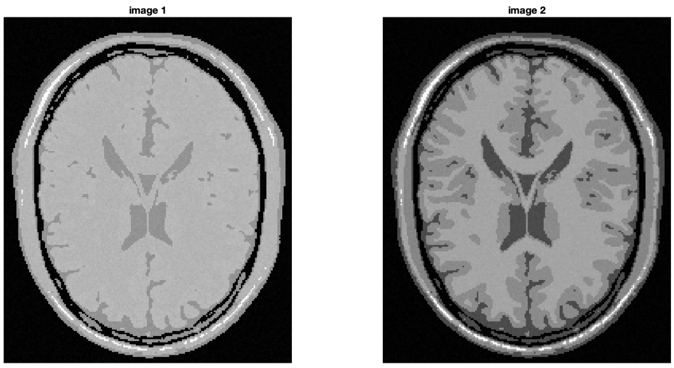


Figure 7: Noisy Images

Then, we can again show the map as given in Figure 8

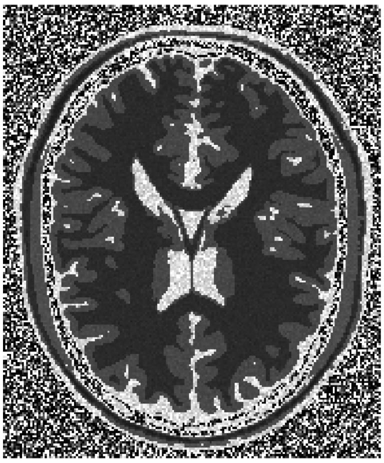
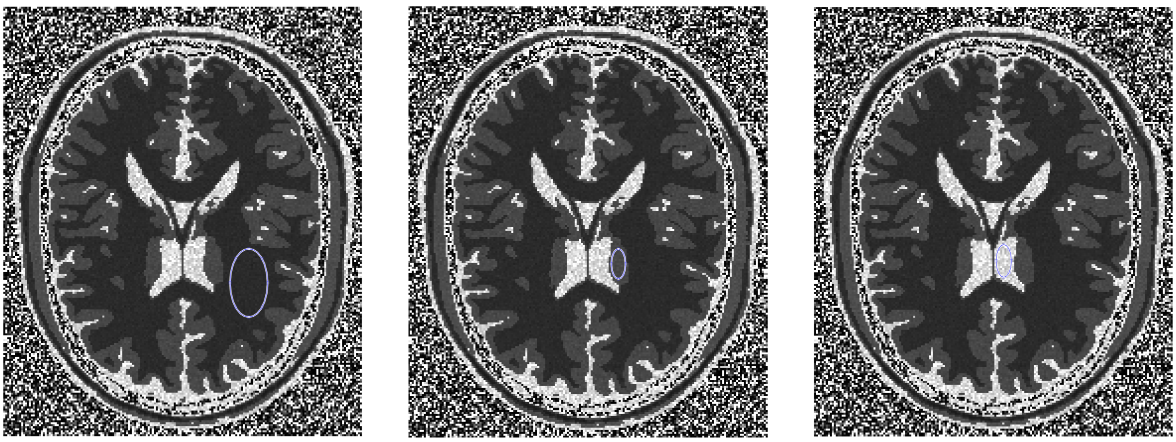


Figure 8: T_1 Map for the Noisy Images

g. Here, we are required to find the T_1 estimates as we have in the previous parts for all tissue types. The ROIs are as given in Figure 9.



(a) ROI Selection for Gray Matter (b) ROI Selection for White Matter (c) ROI Selection for CSF

Figure 9: Selected Elliptic ROIs on T_1 Map

The resulting means estimates gave $500.3234ms$, 837.6729 and 2537.8 for White Matter, Gray Matter and CSF respectively.

h. The biggest deviation of the T_1 estimates of the noisy images from their noise-free counterparts has come from the Cerebrospinal Fluid (CSF). We can prove why this happens mathematically. As we have a definition to approximate the noise-free condition, we can use a noisy definition.

$$\tilde{T_1}' = \frac{TR}{ln\left(\frac{f_2+\epsilon_2}{tan\alpha_2} - \frac{f_1+\epsilon_1}{tan\alpha_1}\right) - ln\left(\frac{f_2+\epsilon_2}{\sin\alpha_2} - \frac{f_1+\epsilon_1}{\sin\alpha_1}\right)} \tag{37}$$

where ϵ_i is a random matrix with the same shape as the image, drawn from an unknown distribution. This makes the the images themselves random matrices and T_1 a random variable. Then we can draw the original $\tilde{T_1}$ from this equation using the following logarithm feature.

$$ln(a+b) = ln(b) + ln(1+\frac{b}{a}) \tag{38}$$

We can use this as follows;

$$ln\left(\frac{f_2+\epsilon_2}{tan\alpha_2} - \frac{f_1+\epsilon_1}{tan\alpha_1}\right) = ln\left(\frac{f_2}{tan\alpha_2} - \frac{f_1}{tan\alpha_1}\right) + ln\left[1 + \frac{\left(\frac{\epsilon_2}{tan\alpha_2} - \frac{\epsilon_1}{tan\alpha_1}\right)}{\left(\frac{f_2}{tan\alpha_2} - \frac{f_1}{tan\alpha_1}\right)}\right] \tag{39}$$

We can simply define the noise related segment as E_1. Similarly,

$$ln\left(\frac{f_2+\epsilon_2}{sin\alpha_2} - \frac{f_1+\epsilon_1}{sin\alpha_1}\right) = ln\left(\frac{f_2}{sin\alpha_2} - \frac{f_1}{sin\alpha_1}\right) + ln\left[1 + \frac{\left(\frac{\epsilon_2}{sin\alpha_2} - \frac{\epsilon_1}{sin\alpha_1}\right)}{\left(\frac{f_2}{sin\alpha_2} - \frac{f_1}{sin\alpha_1}\right)}\right] \tag{40}$$

And we can define the noise section as E_2. If we define E as $E_1 + E_2$, we can reiterate the earlier definition as follows.

$$\tilde{T_1}' = \frac{TR}{E + ln\left(\frac{f_2}{tan\alpha_2} - \frac{f_1}{tan\alpha_1}\right) - ln\left(\frac{f_2}{\sin\alpha_2} - \frac{f_1}{\sin\alpha_1}\right)} \tag{41}$$

Hence, we can say that

$$\frac{1}{\tilde{T_1}'} = \frac{E}{TR} + \frac{1}{\tilde{T_1}} = \frac{E\tilde{T_1} + TR}{TR\tilde{T_1}} \tag{42}$$

We would like to measure variation, which can be defined as $\Delta\tilde{T_1} = |\tilde{T_1}' - \tilde{T_1}|$. We can express this as follows.

$$\Delta\tilde{T_1} = |\frac{TR\tilde{T_1}}{E\tilde{T_1} + TR} - \tilde{T_1}| \tag{43}$$

$$= |\frac{TR\tilde{T_1} - \tilde{T_1}^2 E - TR\tilde{T_1}}{E\tilde{T_1} + TR}| \tag{44}$$

$$= |\frac{-\tilde{T_1}^2 E}{E\tilde{T_1} + TR}| = \frac{\tilde{T_1}^2 E}{E\tilde{T_1} + TR} \tag{45}$$

$$= \frac{\tilde{T_1}^2}{\tilde{T_1} + TR/E} \tag{46}$$

We can say that E is in the order of 10^{-4} which makes TR/E a very large value mostly. This means that we can approximate it as follows.

$$\Delta\tilde{T_1} = \frac{\tilde{T_1}^2}{TR/E} \tag{47}$$

In the end we can say that,

$$\Delta\tilde{T_1} \propto \tilde{T_1}^2 \tag{48}$$

Here, we can nondeterministically see that the value of T_1 directly contributes to the change in its value in the presence of noise. Hence, we can say that with higher T_1 values, it is expected that the deviation to get even bigger from the noise-free values proportional to $\tilde{T_1}^2$. This outcome also coincides with the intuitive solution. Since, the higher values of T_1 show brighter in the T_1 weighted contrast MRI, they are more susceptible to noise when compared to the other tissue types. Visually, we can see that from all parts, CSF is the fluid that looks more disturbed when compared with the other tissues.

MATLAB Code

Code that is used in Question 4 are presented below.

hw5.m

```
close all; clear;
load('brainT1_mri.mat');
% convert angles to radians
flip_degree = flip_degree .* pi ./ 180;
%% c) Display noise-free images and find T1
figure()
sgtitle('Noise-Free Images')

subplot(1,2,1)
imshow(image1, [])
title('image 1')
subplot(1,2,2)
imshow(image2, [])
title('image 2')

div = (log(image2./tan(flip_degree(2))-image1./tan(flip_degree(1)))- ...
log(image2./sin(flip_degree(2))-image1./sin(flip_degree(1))));
T1map = real(TR./div);
figure()
imshow(T1map,[])
title('T1 Map')
%% d)
figure;
imshow(T1map,[]);
BW = roiellipse;
T1_est = mean(T1map(BW));
disp(T1_est)
% T1est for White Matter = 500.00
%% e)
figure;
imshow(T1map,[]);
BW = roiellipse;
T1_est = mean(T1map(BW));
disp(T1_est)
% T1est for Gray Matter = 833.00
figure;
imshow(T1map,[]);
BW = roiellipse;
T1_est = mean(T1map(BW));
disp(T1_est)
% T1est for CSF = 2569.00
%% f) Display noisy images and find T1
figure()
sgtitle('Noisy Images')

subplot(1,2,1)
imshow(image1_noisy, [])
title('image 1')
subplot(1,2,2)
imshow(image2_noisy, [])
title('image 2')

div = (log(image2_noisy./tan(flip_degree(2))-image1_noisy./tan(flip_degree(1))) ...
-log(image2_noisy./sin(flip_degree(2))-image1_noisy./sin(flip_degree(1))));
T1map = real(TR./div);
figure()
imshow(abs(T1map),[0 3000])
title('T1 Map')
%% g)
figure;
imshow(abs(T1map),[0 3000]);
BW = roiellipse;
```

```
T1_est = mean(T1map(BW));
disp(T1_est)
% T1est for White Matter = 500.3234
figure;
imshow(abs(T1map),[0 3000]);
BW = roiellipse;
T1_est = mean(T1map(BW));
disp(T1_est)
% T1est for Gray Matter = 837.6729
figure;
imshow(abs(T1map),[0 3000]);
BW = roiellipse;
T1_est = mean(T1map(BW));
disp(T1_est)
% T1est for White Matter = 2537.8
```