Homework 3 Report

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November 18, 2020

Contents

Question 1

This question is about imaging some tissue types with parallel x-ray imaging. Given the following scheme, we are first asked to mathematically formulate the target and background intensities for the diseased and its surrounding tissue. Then we compute the local contrast around the diseased tissue, which is given as,

$$localcontrast = \frac{f_t - f_b}{f_b} \tag{1}$$

where, f_t is the target intensity, and f_b is the background intensity. In the other part, we are given two models for the shield component of the system with their width and linear attenuation coefficients. We are asked if one of the two components provide at least 95% attenuation, or at most 5% transmission.

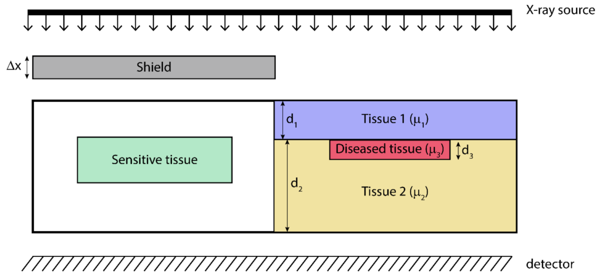


Figure 1: Provided System

a. We first compute the target and background intensities. Since this is a mono-energetic and narrow-beam system, we can assume all of the X-Ray photons are orthogonal to the source axis. Hence the photons only travel through the given material entering from 90 degrees and moving through others, which makes the subsequent layers of tissues behave like cascade systems. Hence we can define the intensities as follows. Also, we can ignore the geometric effects since there are none: As $\theta = \pi/2$, $cos(\theta) = 1$ for all photons.

$$\begin{aligned} I_t &= I_0 \exp\{-(\mu_1 d_1 + \mu_3 d_3 + \mu_2(d_2 - d_3))\} && (2)\\ &= I_0 \exp\{-((0.15)1.5 + (0.75)1 + (0.45)3)\} = I_0 0.09778 && (3)\\ I_b &= I_0 \exp\{-(\mu_1 d_1 + \mu_2 d_2)\} && (4)\\ &= I_0 \exp\{-((0.15)1.5 + (0.45)4)\} = I_0 0.13199 && (5)\\ & && (6) \end{aligned}$$

Hence, we can find the local contrast,

$$localcontrast = \frac{I_0(0.09778 - 0.13199)}{I_0 0.13199} = -0.25918 \tag{7}$$

b. Here, we calculate the attenuation caused by the two shields A and B using the imaging equation.

$$I_d = I_0 exp\{-\mu_a \Delta x_a\} = I_0 exp\{-6.5 * 0.5\} = I_0(0.03877) \tag{8}$$

$$\rightarrow \frac{I_d}{I_0} * 100 = 3.877\% \tag{9}$$

We see that this allows 3.877% transmission, that corresponds to 96.12% attenuation, which is sufficient for our criteria. Then we examine B.

$$I_d = I_0 exp\{-\mu_b \Delta x_b\} = I_0 exp\{-17 * 0.15\} = I_0(0.07808) \tag{10}$$

$$\rightarrow \frac{I_d}{I_0} * 100 = 7.808\% \tag{11}$$

This corresponds to a 92.12% attenuation, which does not fit our criteria even though the material has higher linear attenuation coefficient. It didn't produce enough protection due to its thickness. Hence, we can conclude that we can use **a** but **not b** to shield this type of tissue.

Question 2

This question asks us to find the imaging of a uniform hexagon sliced on the x axis. Side of the uniform hexagon is given as 40 cm and it is imaged $z_1 cm$ away from the point source with a total distance of $200cm$ between the source and the detector. The linear attenuation coefficient (μ_0) is given as $0.03cm^{-1}$. We are required to obtain the detected intensities for $z_1 = 50cm$ and $z_1 = 150cm$. The graphical definition is provided below.

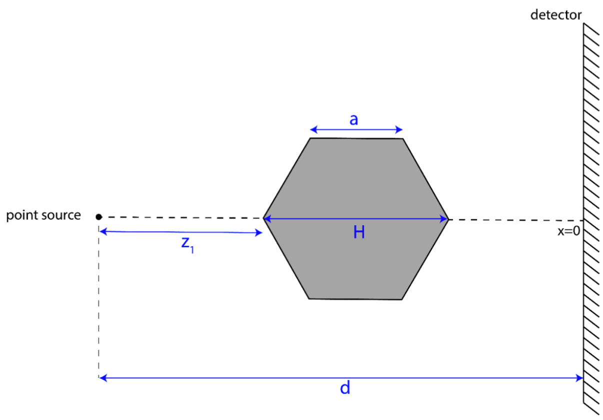


Figure 2: Problem Definition

Since this represents a symmetric image with respect to the origin. We will be using the upper part and then take the absolute value of x so that the imaging equations stay correct. We will solve this using the geometric approach and will solve for general distance of the object then we will place the values to determine the particular solutions. We will engage the question using the possible combinations of attenuation with respect to the geometry of the hexagon. Hence there are three possibilities:

1. No interaction with the object.

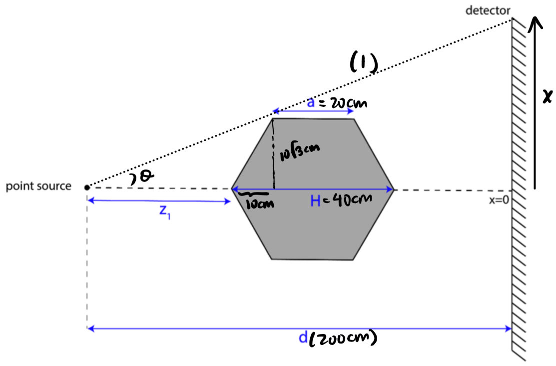


Figure 3: 1st Possibility

Then we can define the image intensity as,

$$I_d(x,0) = I_0 cos^3(\theta), \quad cos(\theta) = \frac{200}{\sqrt{200^2 + x^2}} \tag{12}$$

This is the case when,

$$\frac{10\sqrt{3}}{|x|} \leq \frac{z_1 + 10}{200} \rightarrow |x| \geq \frac{2000\sqrt{3}}{z_1 + 10} \tag{13}$$

2. Ray comes from the side and exits from through the top.

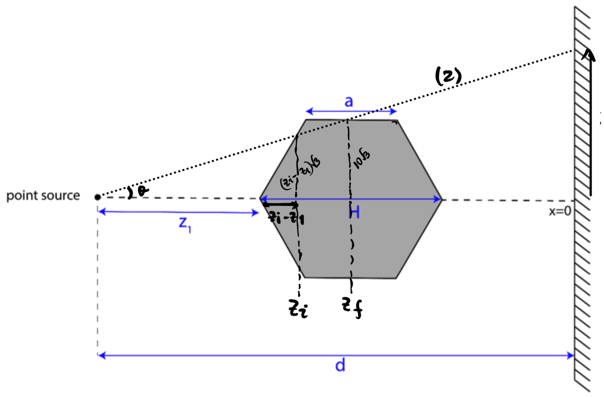


Figure 4: 2nd Possibility

Here, we should label the entrance and exist points, for that I have chosen to use the notation z_i for z initial and z_f as z final. Then the intensity on the detector becomes,

$$I_d(x,0) = I_0 cos^3(\theta) \exp\left\{-\frac{\mu_0}{cos(\theta)}[z_f - z_i]\right\} \tag{14}$$

Then we should find what z_i and z_f corresponds to when written in terms of x, which is why we use the triangle similarity equations.

$$\frac{(z_i - z_1)\sqrt{3}}{|x|} = \frac{z_i}{200} \tag{15}$$

$$\rightarrow z_i = \frac{z_1 200\sqrt{3}}{200\sqrt{3} - |x|} \tag{16}$$

$$\frac{z_f}{200} = \frac{10\sqrt{3}}{|x|} \tag{17}$$

$$\rightarrow z_f = \frac{2000\sqrt{3}}{|x|} \tag{18}$$

Hence, the intensity becomes,

$$I_d(x,0) = I_0 cos^3(\theta)\exp\left\{-\frac{\mu_0}{cos(\theta)}\left[\frac{2000\sqrt{3}}{|x|} - \frac{z_1 200\sqrt{3}}{200\sqrt{3} - |x|}\right]\right\} \tag{19}$$

This happens when,

$$\frac{2000\sqrt{3}}{z_1 + 30} \leq |x| < \frac{2000\sqrt{3}}{z_1 + 10} \tag{20}$$

3. Ray comes from one side and exits from the other.

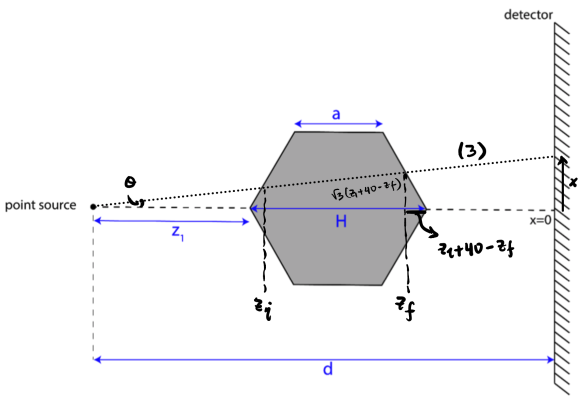


Figure 5: 3rd Possibility

Here, the same rule for z_i applies, however, we find z_f which is given as below.

$$\frac{z_f}{200} = \frac{(z_1 + 40 - z_f)\sqrt{3}}{|x|} \tag{21}$$

$$\rightarrow z_f = \frac{(z_1 + 40)200\sqrt{3}}{|x| + 200\sqrt{3}} \tag{22}$$

Hence, the intensity becomes,

$$I_d(x,0) = I_0 cos^3(\theta)\exp\left\{-\frac{\mu_0}{cos(\theta)}\left[\frac{(z_1 + 40)200\sqrt{3}}{|x| + 200\sqrt{3}} - \frac{z_1 200\sqrt{3}}{200\sqrt{3} - |x|}\right]\right\} \tag{23}$$

We can easily see that this happens when,

$$|x| \leq \frac{2000\sqrt{3}}{z_1 + 30} \tag{24}$$

Hence, we can derive the overall equation for general parameters (μ_0, z_0) as follows.

$$I_d(x,0) = \begin{cases} I_0 cos^3(\theta) & |x| \geq \frac{2000\sqrt{3}}{z_1+10} \\ I_0 cos^3(\theta) \exp\left\{-\frac{\mu_0}{cos(\theta)}\left[\frac{2000\sqrt{3}}{|x|} - \frac{z_1 200\sqrt{3}}{200\sqrt{3}-|x|}\right]\right\} & \frac{2000\sqrt{3}}{z_1+30} \leq |x| < \frac{2000\sqrt{3}}{z_1+10} \\ I_0 cos^3(\theta) \exp\left\{-\frac{\mu_0}{cos(\theta)}\left[\frac{(z_1+40)200\sqrt{3}}{|x|+200\sqrt{3}} - \frac{z_1 200\sqrt{3}}{200\sqrt{3}-|x|}\right]\right\} & |x| \leq \frac{2000\sqrt{3}}{z_1+30} \end{cases}$$

$$where \quad cos(\theta) = \frac{200}{\sqrt{200^2 + x^2}}$$

Hence, we can extract the individual solutions from this equation for $\mu_0 = 0.03$ and $z_1 = \{50, 150\}$.

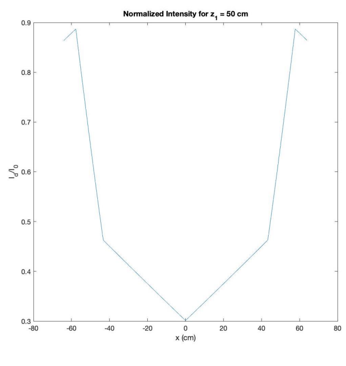
$$I_{d z_1=50, \mu_0=0.03}(x,0) = \begin{cases} I_0 cos^3(\theta) & |x| \geq \frac{100\sqrt{3}}{3} \\ I_0 cos^3(\theta) \exp\left\{-\frac{1}{cos(\theta)}\left[\frac{60\sqrt{3}}{|x|} - \frac{300\sqrt{3}}{200\sqrt{3}-|x|}\right]\right\} & 25\sqrt{3} \leq |x| < \frac{100\sqrt{3}}{3} \\ I_0 cos^3(\theta) \exp\left\{-\frac{1}{cos(\theta)}\left[\frac{540\sqrt{3}}{|x|+200\sqrt{3}} - \frac{300\sqrt{3}}{200\sqrt{3}-|x|}\right]\right\} & |x| \leq 25\sqrt{3} \end{cases}$$

$$I_{d z_1=150, \mu_0=0.03}(x,0) = \begin{cases} I_0 cos^3(\theta) & |x| \geq \frac{25\sqrt{3}}{2} \\ I_0 cos^3(\theta) \exp\left\{-\frac{1}{cos(\theta)}\left[\frac{60\sqrt{3}}{|x|} - \frac{900\sqrt{3}}{200\sqrt{3}-|x|}\right]\right\} & \frac{100\sqrt{3}}{9} \leq |x| < \frac{25\sqrt{3}}{2} \\ I_0 cos^3(\theta) \exp\left\{-\frac{1}{cos(\theta)}\left[\frac{1140\sqrt{3}}{|x|+200\sqrt{3}} - \frac{900\sqrt{3}}{200\sqrt{3}-|x|}\right]\right\} & |x| \leq \frac{100\sqrt{3}}{9} \end{cases}$$

Question 3

In this question, we are asked to plot the imaging equations we have found on MATLAB. Then we compared the plots and commented on their similarities/differences.

a. The plotted intensities are presented below. The code can be found in Appendix A.



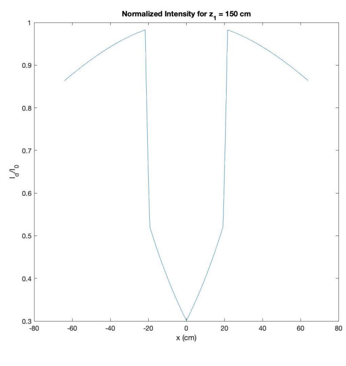


Figure 6: $I(x,0)$ for $z_1 = 50cm$ and $z_1 = 150cm$

b. When we look at these two plots, we see that they both resemble the attenuation that is caused by a hexagonal structure, so the exercise seems to have achieved its purpose. In both parts, we can clearly observe the effect of obliquity and inverse square law as the lines seems like they are multiplied with a cosine. As, we can predict, the intensities were lower in the inner region (region (3)) since it faced most attenuation in that region, and the attenuation decreased towards the outer areas as the distance travelled in the hexagon also decreased.

When we compare these plots, we can see that the $z_1 = 50$ plot has increased object magnification when compared with the $z = 150$, which is expected as object magnification is represented as d/z. Also, we can see that after theta is out of range for both objects, we can see they present the same line, which is also expected as after that point, there is no attenuation, only obliquity and inverse square law effects are observed. Also, we can see that the minimum and maximum values are the same for both of the plots. This is due to the fact that although the magnification changes, the object does not change shape, hence the attenuation, doesn't change, the plots seems like one is a more stretched out version of the other. ın order to observe the x-ray image more easily, we can take the inverse of this image with respect to the x-axis, which makes the more attenuated parts brighter and transmitted parts darker.

Since the objective is to image the diagonal, we can say that neither of the plots cover the objective perfectly, due to the geometric effects. However, we should say that $z_1 = 150$ is affected less when compared with $z_1 = 50$ since it is closer to the detector plane. Hence, we can say that the best image can be observed if $z_1 = 160$, which is essentially `hugging the detector`.

Question 4

this question asks us to find the 2D Radon transform of the 2D Gaussian function given as,

$$f(x,y) = e^{-\frac{x^2+y^2}{2}} \tag{25}$$

Since, this is a circularly symmetric function, we an define it on the polar coordinates as $f(r) = e^{-r^2/2}$ to see it more clearly. Hence, as this is not a θ dependent function, we can define its Radon transform as,

$$g(l,\theta) = g(l)_{\theta=0} \tag{26}$$

Hence, we can reiterate the radon transform as follows,

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\delta(x\cos\theta + y\sin\theta - l)dxdy \tag{27}$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\delta(x - l)dxdy \tag{28}$$

Here, we use the sifting property of the impulse function.

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(l,y)\delta(x-l)dxdy \tag{29}$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\frac{l^2+y^2}{2}}\delta(x-l)dxdy \tag{30}$$

$$= \int_{-\infty}^{\infty} e^{-\frac{l^2+y^2}{2}}dy \int_{-\infty}^{\infty}\delta(x-l)dx \tag{31}$$

$$= e^{-l^2/2}\int_{-\infty}^{\infty} e^{-y^2/2}dy \tag{32}$$

$$g(l,\theta) = \sqrt{\frac{\pi}{2}}e^{-l^2/2} \tag{33}$$

Question 5

In this question, we are asked to find and sketch $g(l,\theta)$ for $\theta = \{0, \pi/2\}$. The object is provided in graphical form shown below and the linear attenuation coefficients $\mu_1 = 0.25cm^{-1}$, $\mu_2 = 0.05cm^{-1}$, $\mu_3 = 0.35cm^{-1}$ and $r = 15cm$. The object is presented below.

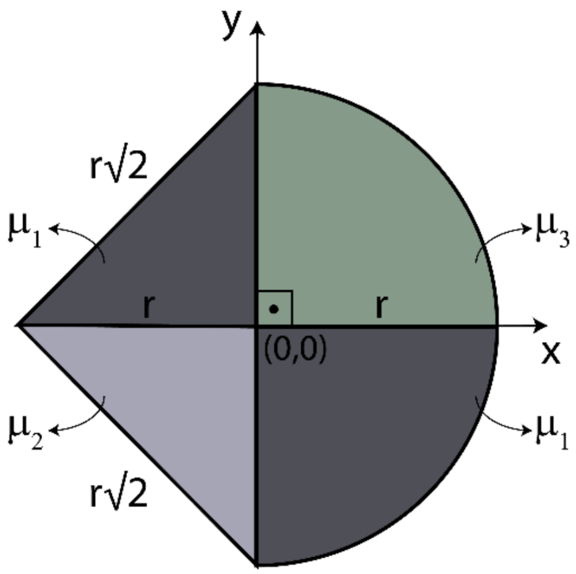


Figure 7: Object that will be CT Scanned

a. Here, we decompose the separate components separately for $\theta = 0$ and define the projection.

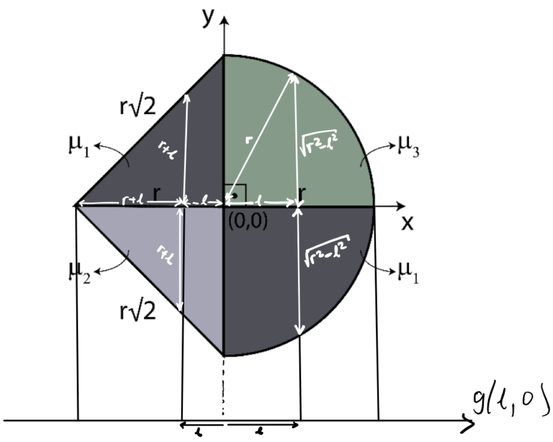


Figure 8: Projection for $\theta = 0$

The function becomes,

$$g(l,0) = \begin{cases} \mu_1(r+l) + \mu_2(r+l) & -r < l < 0 \\ \mu_3\sqrt{r^2-l^2} + \mu_1\sqrt{r^2-l^2} & 0 < l < r \\ 0 & else \end{cases}$$

which can be rewritten as,

$$g(l,0) = \begin{cases} (\mu_1 + \mu_2)(r+l) & -r < l < 0 \\ (\mu_3 + \mu_1)\sqrt{r^2-l^2} & 0 < l < r \\ 0 & else \end{cases}$$

Then, putting in the input values, projection and its sketch becomes as follows.

$$g(l,0) = \begin{cases} 0.3(15+l) & -15 < l < 0 \\ 0.6\sqrt{15^2-l^2} & 0 < l < 15 \\ 0 & else \end{cases}$$

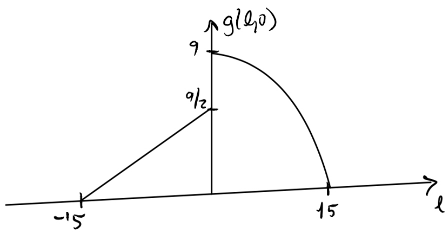


Figure 9: Sketch of $(l, 0)$

b. We repeat the same procedure for $\theta = \pi/2$.

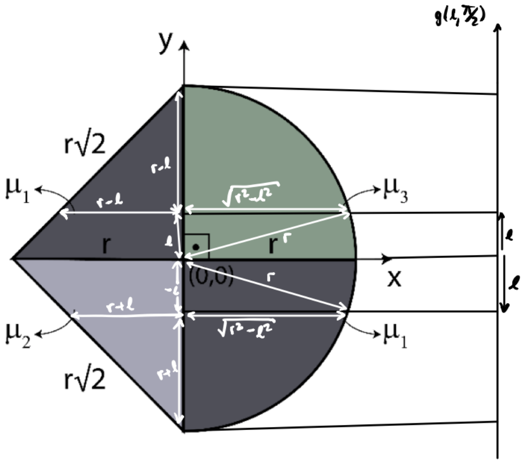


Figure 10: Projection for $\theta = \pi/2$

The function becomes,

$$g(l,0) = \begin{cases} \mu_2(r+l) + \mu_1\sqrt{r^2 - l^2} & -r < l < 0 \\ \mu_2(r-l) + \mu_3\sqrt{r^2 - l^2} + & 0 < l < r \\ 0 & else \end{cases}$$

Placing the coefficients and radius:

$$g(l,0) = \begin{cases} 0.05(15+l) + 0.25\sqrt{15^2 - l^2} & -15 < l < 0 \\ 0.25(15-l) + 0.35\sqrt{15^2 - l^2} & 0 < l < 15 \\ 0 & else \end{cases}$$

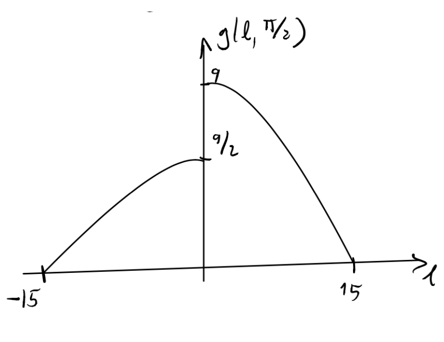


Figure 11: Sketch of $(l, \pi/2)$

MATLAB Code

Code that is used in Question 3 are presented below.

hw3.m

```
close all; clear all;

TOTAL_PIXEL = 512;
DETECTOR_LEN = 128;
MU_0 = 0.03;
Z_1 = [50, 150];

x_range = linspace(-DETECTOR_LEN/2, DETECTOR_LEN/2, TOTAL_PIXEL);

figure();
for z=[1:2]
    subplot(1,2,z)
    plot(x_range, calcInt(x_range, MU_0, Z_1(z)));
    xlabel('x (cm)')
    ylabel('I_d/I_0')
    title(sprintf('Normalized Intensity for z_1 = %d cm', Z_1(z)));
end

function cos = cos_theta(t)
    cos = 200./sqrt(200.^2 + t.^2);
end

function int = intensity1(x)
    int = cos_theta(x).^3;
end

function int = intensity2(x, mu, z)
    int = intensity1(x) .* exp((- mu ./ cos_theta(x)) .* ...
                        ((2000.*sqrt(3)./abs(x)) - ...
                        (z.*200.*sqrt(3)./(200.*sqrt(3)-abs(x)))) );
end

function int = intensity3(x, mu ,z)
    int = intensity1(x) .* exp((- mu ./ cos_theta(x)) .* ...
                     (((z+40).*200.*sqrt(3)./(abs(x) + 200.*sqrt(3))) - ...
                     (z.*200.*sqrt(3)./(200.*sqrt(3)-abs(x)))) );
end

function int = intensity(x,mu,z)
    if(abs(x) >= 2000.*sqrt(3)/(z+10))
        int = intensity1(x);
    elseif 2000.*sqrt(3)/(z+30) <= abs(x) && abs(x) < 2000.*sqrt(3)/(z+10)
        int = intensity2(x,mu,z);
    elseif abs(x) < 2000.*sqrt(3)/(z+30)
        int = intensity3(x,mu,z);
        disp(int)
    end
end

function int_vec = calcInt(x_vec, mu, z)
    int_vec = [];
    for x = x_vec
```

```
        int_vec = [int_vec intensity(x,mu,z)];
    end
end
```