Homework 2 Report

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Question 1

This question is about modulation and modulation transfer function denoted as MTF. The question first asks us to compute the MTF of a system whose point spread function (PSF) is provided. Then it gives three different input signals for which we find the input and output modulations. Inputs are all provided in the format as below and are periodic.

$$f(x,y) = A + B\sin(2\pi(u_0x + v_0y)) \tag{1}$$

Hence, we can specify the input modulation (m_f) as B/A and output modulation since,

$$m_f = \frac{f_{max} - f_{min}}{f_{max} - f_{min}} = \frac{B + A - (A - B)}{B + A + (A - B)} = \frac{B}{A} \tag{2}$$

Then, we can define the output modulation as

$$m_g = m_f MTF(u_0, v_0) \tag{3}$$

a. The given response is given below, where $\sigma = 4$, which we will use at the very end of each question.

$$h(x,y) = \frac{1}{2\pi\sigma^2}e^{-\frac{x^2+4y^2}{2\sigma^2}} \tag{4}$$

MTF, by default is defined as follows,

$$MTF(u,v) = \frac{|H(u,v)|}{H(0,0)} \tag{5}$$

Hence, we need to first find the Fourier Transform of the PSF.

$$H(u,v) = \mathcal{F}_{2D}[f(x,y)] = \frac{1}{2\pi\sigma^2}\sqrt{2\pi}\sigma\frac{\sqrt{2\pi}\sigma}{2}e^{-\pi^2\sigma^2(2u^2+v^2/2)} \tag{6}$$

$$= \frac{1}{2}e^{-\pi^2\sigma^2(2u^2+v^2/2)} \tag{7}$$

Since, this function is always positive, we can find MTF as below.

$$MTF(u,v) = \frac{H(u,v)}{H(0,0)} = \frac{\frac{1}{2}e^{-\pi^2\sigma^2(2u^2+v^2/2)}}{\frac{1}{2}} = e^{-\pi^2\sigma^2(2u^2+v^2/2)} \tag{8}$$

The plots for $MTF(u,0)$ and $MTF(0,v)$ are provided.

$$MTF(u,0) = e^{-2\pi^2\sigma^2u^2} \xRightarrow[\sigma=4]{insert} e^{-32\pi^2u^2} \tag{9}$$

$$MTF(0,v) = e^{-\pi^2\sigma^2(v^2/2)} \xRightarrow[\sigma=4]{insert} e^{-8\pi^2v^2} \tag{10}$$

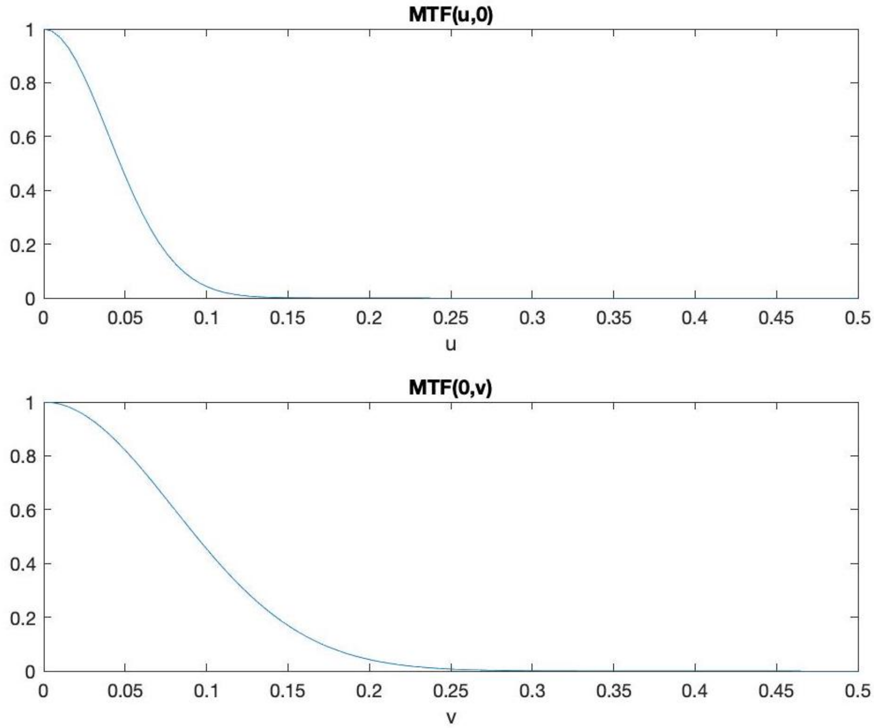

Figure 1: MTF(0,v) and MTF(u,0)

b. The input is given as follows for this part.

$$f(x,y) = 3 + 2\sin(\frac{2\pi x}{10}) = 3 + 2\sin(2\pi\frac{1}{10}x) \tag{11}$$

Here we can observe that spatial frequency is given as $u_0 = 1/10$. Hence, we can find the input and output modulations as given below.

$$m_f = B/A = 2/3 \tag{12}$$

$$m_g = m_f MTF(u_0, 0) = \frac{2}{3}e^{-\pi^2\sigma^2(2u_0^2)} = \frac{2}{3}e^{-\pi^2\sigma^2(2(1/10)^2)} \tag{13}$$

$$\xrightarrow[\sigma=4]{insert} \frac{2}{3}e^{-\pi^2 8/25} \tag{14}$$

c. Same technique is applied as the previous part.

$$f(x,y) = 3 + 2\sin(\frac{2\pi y}{10}) = 3 + 2\sin(2\pi\frac{1}{10}y) \tag{15}$$

Here we can observe that spatial frequency is given as $v_0 = 1/10$. Hence, we can find the input and output modulations as given below.

$$m_f = B/A = 2/3 \tag{16}$$

$$m_g = m_f MTF(0, v_0) = \frac{2}{3}e^{-\pi^2\sigma^2((1/2)v_0^2)} = \frac{2}{3}e^{-\pi^2\sigma^2(1/2(1/10)^2)} \tag{17}$$

$$\xrightarrow[\sigma=4]{insert} \frac{2}{3}e^{-\pi^2 2/25} \tag{18}$$

d. Same technique is applied as the previous parts using both spatial frequencies this time.

$$f(x,y) = 3 + 2\sin(\frac{2\pi(x+y)}{10}) = 3 + 2\sin(2\pi\frac{1}{10}x + 2\pi\frac{1}{10}y) \tag{19}$$

Here we can observe that spatial frequency is given as $u_0 = v_0 = 1/10$. Hence, we can find the input and output modulations as given below.

$$m_f = B/A = 2/3 \tag{20}$$

$$m_g = m_f MTF(u_0, v_0) = \frac{2}{3}e^{-\pi^2\sigma^2(2u_0^2+(1/2)v_0^2)} = \frac{2}{3}e^{-\pi^2\sigma^2(5/2(1/10)^2)} \tag{21}$$

$$\xrightarrow[\sigma=4]{insert} \frac{2}{3}e^{-\pi^2 2/5} \tag{22}$$

Question 2

This question provides us with two different PSFs for two different imaging systems and asks us to compute the FWHM (Full-Width of Half-Maximum) for those systems and asks us to determine the better system for specified input.

a. Before going through the calculations, we can observe the shapes of the functions. The triangle is fairly simple as it follows $y = 1-4x$ between 0 and 1/4 and $y = 4x+1$ between -1/4 and 0, zero else. Hence, we can inherently check the FWHM since we know that the maximum value is one, making the half-maximum 1/2. The corresponding x values then become $-1/8$ and $1/8$, making the FWHM 0.25.

However, we see that the shape of the second function is a two-sided exponential function, Hence we can calculate the max value, then find the half-maximum range. As the value at x=0 is 1, we can say that the maximum value is one and we can therefore declare half-max as 0.5.

$$0.5 = e^{-8\pi x^2} \rightarrow x = \pm\sqrt{\frac{ln(1/2)}{-8\pi}} \tag{23}$$

$$= \pm\sqrt{\frac{ln(2)}{8\pi}} \tag{24}$$

Then FWHM becomes,

$$FWHM_{h_2} = 2\sqrt{\frac{ln(2)}{8\pi}} = \sqrt{\frac{ln(2)}{2\pi}} \approx 0.33214 \tag{25}$$

Here, we can state that system to is better in terms of resolution since its FWHM is bigger.

b. Here, the question asks which of the modeled imaging techniques should be used if the given input is a sinusoidal wave given as $f(x) = \sin(8\pi x)$. In order to find that easily without consulting to convolution, we can work with the FWHMs of the systems. If we wold like to know if we can separate the responses, we need to first find the separation of the input signal. Since this is a sinusoidal wave, we can say that the separation is equal to the period.

$$f(x) = \sin(8\pi x) = \sin(2\pi 4x) \tag{26}$$

$$\rightarrow u_0 = 4 \tag{27}$$

$$\rightarrow T_0 = 1/4 = 0.25 \tag{28}$$

In order to be able to image, the FWHM of the system should be strictly greater than the separation of the input. Here, we cannot use h_1 as its FWHM is equal to the separation. Then, we should use h_2, as it holds the condition $FWHM_2 = 0.33214 > T_0 = 0.25$

Question 3

This question is about performance metrics, namely the prevalence, sensitivity, specificity, positive-predictive value (PPV) and negative-predictive value (NPV) of an experiment. After a classification task of any sort (machine-learning algorithm, biological test for a disease), a contingency table (confusion matrix) is constructed to show how successful the test is according to the ground truth. The binary contingency table we use here describes the number of cases that are correctly/ falsely classified with respect to their ground truths. Hence, we use statistical elements to quantify the success in different perspectives. We will use the following notation to refer to the table elements where the columns show the ground truth status (disease) and the rows show the test outcome.

$$\begin{array}{ccc} CT & + & - \\ + & a & b \\ - & c & d \end{array}$$

- **Prevalence** is the ratio of the number of subjects with disease to the number of the whole group, which gives information on the presence of the disease in the test group, and is calculated by,

$$prevalence = \frac{a+c}{a+b+c+d}$$

- **Sensitivity** (True Positive Rate - TPR) is the ratio of the subjects that are correctly identified to have the disease to all the subjects that have the disease. Gives relevant information on how accurate the positive classification is and is calculated by,

$$sensitivity = \frac{a}{a+c}$$

- **Specificity** (True Negative Rate - TNR) is the ratio of the subjects that are correctly identified to be healthy to all the subjects that are indeed healthy. Gives relevant information on how accurate the negative classification is and is calculated by,

$$specificity = \frac{d}{d+b}$$

- **PPV** is the ratio of the subjects that are correctly identified to have the disease to all subjects that are tested as positive. Shows the likelihood of having the disease if the test is positive.

$$PPV = \frac{a}{a+b}$$

- **NPV** is the ratio of the subjects that are correctly identified to be healthy to all subjects that are tested as negative. Shows the likelihood of being healthy if the test is negative.

$$NPV = \frac{d}{d+c}$$

		Disease	
		+	-
Test	+	183	72
	-	320	7425

Figure 2: Contingency Table for Threshold Level $145mmol/L$

a. Here, we are provided with the following contingency table and are asked to find the prevalence, sensitivity, specificity, PPV and NPV. With the information previously provided, we can use the following calculations to find the statistics. We have used the percentage results to make more sense out of the data.

$$prevalence = \frac{193+320}{193+320+7425+72} = 6.29\% \tag{29}$$
$$sensitivity = \frac{183}{183+320} = 36.38\% \tag{30}$$
$$specificity = \frac{7425}{7425+72} = 99.03\% \tag{31}$$
$$PPV = \frac{183}{183+72} = 71.76\% \tag{32}$$
$$NPV = \frac{7425}{7425+320} = 95.86\% \tag{33}$$

b. Here, we are provided with the following contingency table and are asked to find the prevalence, sensitivity, specificity, PPV and NPV, followed the same procedure with the previous part. With the information previously provided, we can use the following calculations to find the statistics. We have used the percentage results to make more sense out of the data.

		Disease	
		+	-
Test	+	143	39
	-	360	7458

Figure 3: Contingency Table for Threshold Level $185mmol/L$

$$prevalence = \frac{143+360}{143+360+7458+39} = 6.29\% \tag{34}$$
$$sensitivity = \frac{143}{143+360} = 28.43\% \tag{35}$$
$$specificity = \frac{7458}{7458+39} = 99.48\% \tag{36}$$
$$PPV = \frac{143}{143+39} = 78.57\% \tag{37}$$
$$NPV = \frac{7458}{7458+360} = 95.40\% \tag{38}$$

When we compare the results, we can say that prevalence stays constant, which makes sense since it is a metric based on the ground-truth only. The sensitivity decreases, meaning we can detect less true positives while already high sensitivity increases more. However, we see PPV increasing much more than NPV falling.

c. This part asks to determine the test that will be proceeded with if I were the patient taking the test. I would go for test (b) since, although the NPV stays nearly constant, PPV improves, meaning that if I test positive I would know $\sim 80\%$ that I am indeed positive, which is much more higher than the 1st test's 71%. If my results turn out to be negative, in both tests I would know that I am indeed $\sim 95\%$ healthy.

Question 4

This question is related to a hypothetical iron-oxide particle that is presented in an MPI setting. The question provides us with phantom tracers that are prepared with their point sources (two being industry standard Perimag and Resovist, one is the new one we have discovered) and gradually asks us to find the FWHM and SNR values of the tracers, then find an analogy between the imaged versions of a vessel with these tracers.

a. This part asks us to visualize the tracers and place their axes, where the x-axis changes in $-45mm \leq x \leq 45mm$ and y-axis changes through $-15mm \leq x \leq 15mm$. The image is 1500x500 pixels, and is presented below.

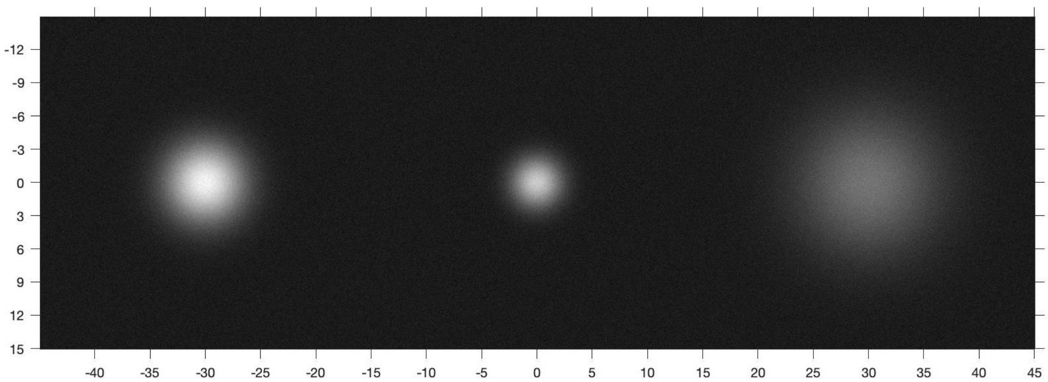

Figure 4: Phantom Tracers

b. In this part, we are asked to cover the FWHMs of these tracers. The method is simple as the tracers are isotropic. We can slice one axis (preferred the $v = 0$), find its maximum and half-maximum, and find the values that give the half-maximum with some tolerance. The tolerance we defined by trial and error equals 0.0068. The code is given below.

```
fwhm = [];
for i=0:2
    img = MPI_image(:,500*i+1:500*i+500);
    axis = img(:,250);
    [M,I] = max(axis);
    half_max = [];
```

```
        for i=1:size(axis)
            if(abs(axis(i) - M/2) < 0.0068 * M/2)
                half_max = [half_max i];
            end
        end
        add = (half_max(end) - half_max(1)).*3./50;
        fwhm = [fwhm add];
    end
```

Here, we find the result in terms of mm, that is why we multiply the values with (3/50), which is the value of one pixel in terms of distance in mm. Hence, we found that the Perimag has FWHM of 6.60mm, new tracer has FWHM of 4.1400mm, and Resovist has FWHM of 12.42mm.

c. This part wants us to find the Signal-to-Noise Ratio of the tracers. To do that, we have reserved a rectangular space in each tracer which contained no visible tracer effect and measured its standard deviation, which we referred to as noise. Then we divided the maximum value of the tracer to the noise component to measure SNR. The code is provided below.

```
    snrs = [];
    for i=0:2
        img = MPI_image(:,500*i+1:500*i+500);
        %selected upper left corner, a 50x50region
        noise_region = MPI_image(1:50,1:50);
        stdev = std(noise_region, 0, 'all');
        max_val = max(img, [], 'all');
        snr = max_val./stdev;
        snrs = [snrs snr];
    end
```

In the end, we have found that the SNR for Perimag, new tracer and Resovist are 44.7899, 37.4376 and 23.8727 respectively. meaning that the Perimag has the highest strength when compared with the other tracers.

d. In this part, we are given a vessel imaging data, we first show the image given below.

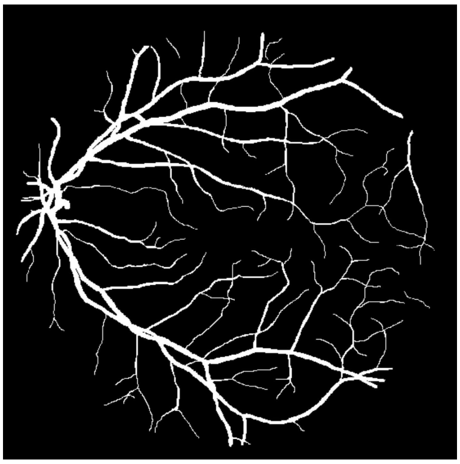

Figure 5: Vessel Imaging Data

Then we are given three different data, one of each are filled with one tracer from the previous parts. Our task is to find which of the images are filled with which of the tracers. To further understand, we first display the filled images.

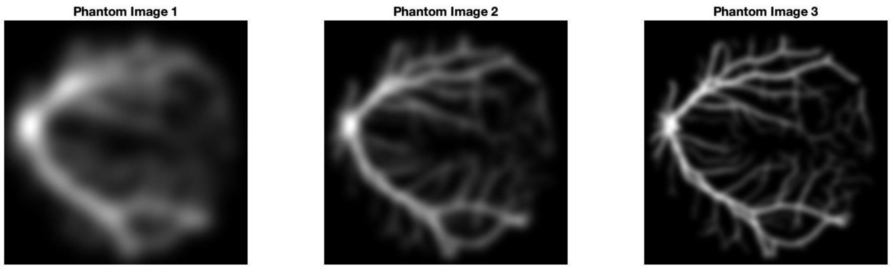

Figure 6: Vessel Imaging Data

In order to determine which tracer is used, I have developed a systematic approach although it is not perfect. To separate the tracer from the imaging data, we have taken their Fourier transforms and divided them with each other, which corresponds to a deconvolution in time domain. The mathematics behind is explained. We can decompose the image as follows. We will denote the phantom image with i_p and the initial vessel with i_v and the ideal tracer with i_t. First we assume that,

$$i_p = i_v * i_k \tag{39}$$

$$\xrightarrow{\mathcal{F}_{\in\mathcal{D}}} I_P = I_V I_K \tag{40}$$

$$\tag{41}$$

Then we can divide the transformed image to the initial image to ideally recover the kernel used in transformation.

$$I_K = I_P / I_V \tag{42}$$

$$i_k = \mathcal{F}_{2D}^{-1}(I_P / I_V) \tag{43}$$

The deconvolved images are given below.

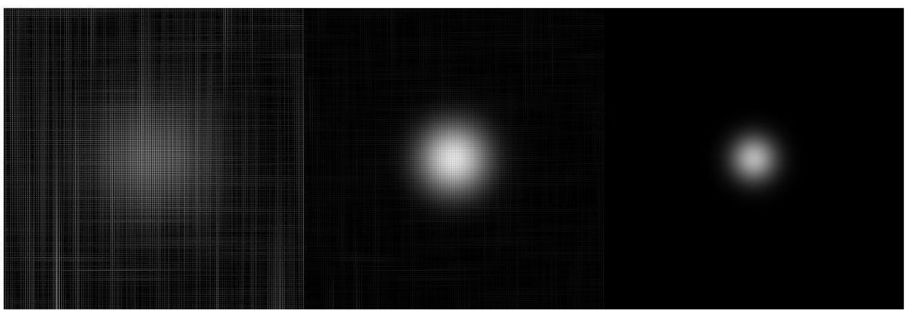

Figure 7: Phantom Images 1,2 and 3 respectively

Although there is noise present due to the aliasing, we can clearly see the relation between Figure 4. In the first phantom image, we can see that the Resovist tracer is used. The second image seems to be filled with the Perimag, and the third phantom image is filled with the new tracer.

MATLAB Code

Code that is used in Question 4 are presented below.

fft2c.m

```
function d = fft2c(im)
% d = fft2c(im)
% fft2c performs a centered fft2
d = fftshift(fft2(ifftshift(im)));
end
```

ifft2c.m

```
function im = ifft2c(d)
% im = ifft2c(d)
% ifft2c performs an centered ifft2
im = fftshift(ifft2(ifftshift(d)));
end
```

hw2.m

```
close all; clear all;

load('MPI_data.mat');
%% PART A
figure();
x = linspace(0,45,19);
y = linspace(0,15,11);
imshow(MPI_image, []);
axis('on');
xticks(x*(1500/45));
xticklabels(string(linspace(-45,45,19)));
yticks(y*(500/15));
yticklabels(string(linspace(-15,15,11)));

%% PART B
fwhm = [];
for i=0:2
    img = MPI_image(:,500*i+1:500*i+500);
    axis = img(:,250);

    [M,I] = max(axis);

    half_max = [];
    for i=1:size(axis)
        if(abs(axis(i) - M/2) < 0.0068 * M/2)
            half_max = [half_max i];
        end
    end
    add = (half_max(end) - half_max(1)).*3./50;
    fwhm = [fwhm add];
end

disp(fwhm)

%% PART C
snrs = [];
```

```
for i=0:2
    img = MPI_image(:,500*i+1:500*i+500);
    %selected upper left corner, a 50x50region
    noise_region = MPI_image(1:50,1:50);
    stdev = std(noise_region, 0, 'all');
    max_val = max(img, [], 'all');
    snr = max_val./stdev;
    snrs = [snrs snr];
end

disp(snrs)

%% PART D
load('vessel.mat');
figure()
imshow(vessel_phantom, []);
figure()
subplot(1,3,1)
imshow(vessel_image_1, []);
title('Phantom Image 1');
subplot(1,3,2)
imshow(vessel_image_2, []);
title('Phantom Image 2');
subplot(1,3,3)
imshow(vessel_image_3, []);
title('Phantom Image 3');

scale = [0,44.1412];
dec = ifft2c(fft2c(vessel_image_1)./fft2c(vessel_phantom));
dec1 = ifft2c(fft2c(vessel_image_2)./fft2c(vessel_phantom));
dec2 = ifft2c(fft2c(vessel_image_3)./fft2c(vessel_phantom));
merged = [dec(750:1250,750:1250) dec1(750:1250,750:1250) dec2(750:1250,750:1250)];
figure();
imshow(merged, [scale])
```