Homework 1 Report

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Contents

Question 1

This question requires us to find and label an example from six commonly used medical imaging techniques, being x-Ray, CT, PET, Ultrasound, MRI and MPI.

- x-Ray [1]

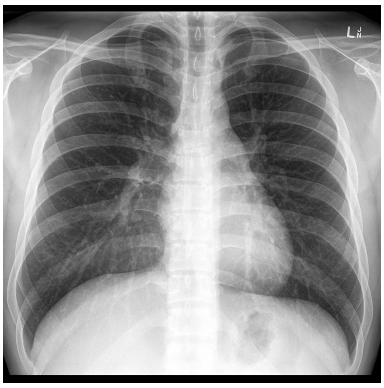

Figure 1: Reference Chest X-Ray Image

This is a frontal chest x-ray image of a healthy person. In X-Ray Imaging, we mostly observe the dense tissue structures as the x-ray rays refract while the soft tissues tend to pass the rays, the are observed as black. Here, we see the heart in the middle, also the vascular structure of the lungs with the bronchi together with the diaphragm. Also, we easily observe the bone structures: ribs, spine, trachea, and clavicle.

- Computerized Tomography (CT) [2]

Figure 2: Reference CT Image of the Abdomen

This is an axial CT image of abdomen of a healthy person. CT, is usually used to analyze the anatomical features of the organs, their size and/or if any pathological transformation exists. Here, from this cross-section, we are able to observe the liver (to the left), the

[1] https://radiopaedia.org/cases/normal-frontal-chest-x-ray
[2] https://radiopaedia.org/cases/normal-ct-abdomen

right kidney, the spleen, vertebrae and the spinal cord, and finally the upper section of the stomach with the aorta.

- Positron Emission Tomography (PET) [3]

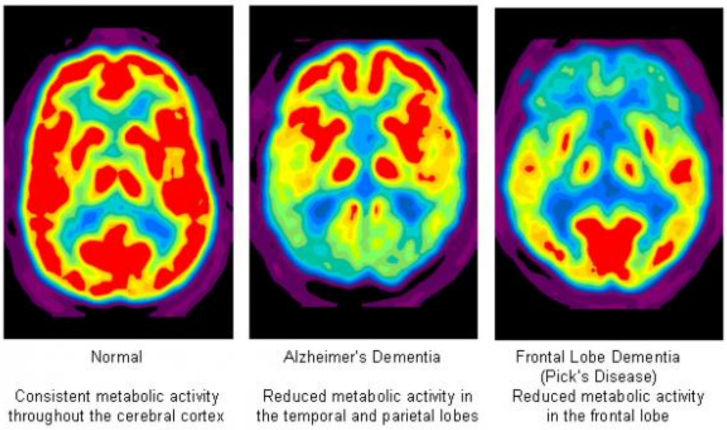

Figure 3: PET Brain Scans of Normal, Alzheimer's and Frontal Lobe Dementia

PET is an imaging system that is used to measure the functional activity of organs. Mainly, we usually scan the brain to determine diseases that change the way it functions, and to find metastatic activity. Here, we see the PET scans of a healthy individual, a subject with Alzheimer's, and Frontal Lobe Dementia. we can easily observe the functional loss in the temporal parts of the Alzheimer's, and frontal lobe in the Dementia case.

- Ultrasound Imaging [4]

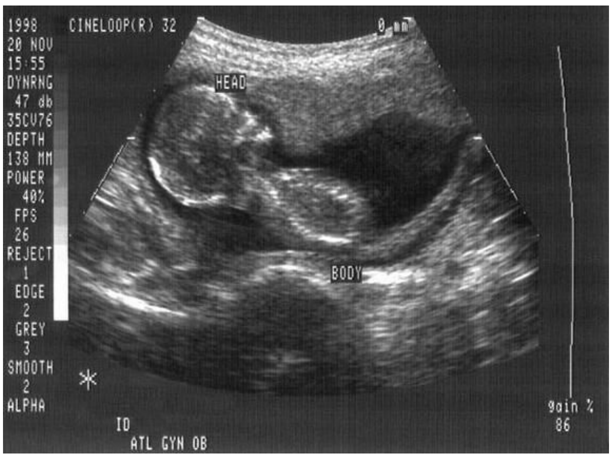

Figure 4: Ultrasound Image of a Pregnant Female's Uterus

This is the ultrasound image of a pregnant patient's uterus. We generally use ultrasound to observe the organs and tissues that are closer to the surface since ultrasound frequencies are a little higher than the audible spectrum. Here, we clearly see the baby in the middle.

[3] https://www.norcalscans.org/for-patients/clinical-info/brain-disorders/
[4] https://www.livescience.com/38426-ultrasound.html

- Magnetic Resonance Imaging (MRI) [5]

Figure 5: Sagittal MRI Image of Head

The MRI is usually used to image soft tissues like muscle and nerves. Here, we see the CNS elements clearly as we see the brain (cerebrum), the cerebellum, the spinal bulb (medulla oblangata). We see the mouth and nasal cavities and the esophagus.

- Magnetic Particle Imaging (MPI) [6]

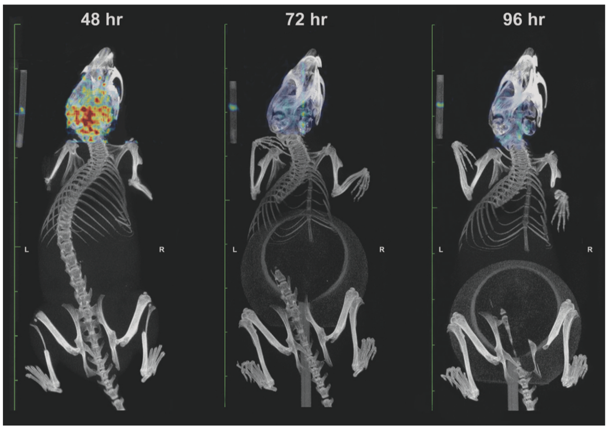

Figure 6: MPI Images of a Mouse which had Stroke, in 48,72,96 hours

MPI, being an emerging technique, focuses on using superparamagnetic iron oxide particles. As described by Wu et. al. , human scanners are being developed, which is the reason I was not able to recover a human image. However, I have found real-time imaging instances of a mouse who had a stroke. The figure shows the changes in brain activity in 48,72 and 96 hours. Here, we can see the high-resolution imaging of the bone structure of the mouse. Also, MPI is under development for vascular imaging, oncology, and cell tracking.

[5] https://www.aboutkidshealth.ca/article?contentid=1334&language=english
[6] Wu et. al, A Review of Magnetic Particle Imaging and Perspectives on Neuroimaging, DOI: 10.3174/ajnr.A5896

Question 2

This question requires us to investigate if the system is linear and shift-invariant given their input-output relationships. To determine linearity, we will construct a signal input that corresponds to the weighted sum of arbitrary signals and look if the system is linearizable, meaning we can rewrite it as the sum of individual outputs. This signal is defined as $f'(x,y) = \sum_t w_t f_t(x,y)$. Furthermore, to determine the shift-invariance, we will use two methods and see if the outputs are equal to each other. In one version, we will first shift the input signal and take its transform and in the other, we will do the reverse. The solutions are presented below for each question.

a. $g(x,y) = f(2x,1) - f(0,y/2)$

- Linearity

$$\mathcal{S}[f'(x,y)] = \mathcal{S}[\sum_t w_t f_t(x,y)] \tag{1}$$

$$= \sum_t \mathcal{S}[w_t f_t(x,y)] \tag{2}$$

$$= \sum_t w_t \mathcal{S}[f_t(x,y)] \tag{3}$$

$$= \sum_t w_t (f_t(2x,1) - f_t(0,y/2)) \tag{4}$$

Hence, we can see that the system is linear.

- Shift-Invariance

Transform first, shift second.

$$\mathcal{S}[f(x,y)] = f(2x,1) - f(0,y/2) \tag{5}$$

$$\xrightarrow[(x_0,y_0)]{Shift by} f(2x - x_0, 1 - y_0) - f(-x_0, y/2 - y_0) \tag{6}$$

Shift first, transform second.

$$\mathcal{S}[f(x - x_0, y - y_0)] = f(2x - 2x_0, 1) - f(0, (y - y_0)/2) \tag{7}$$

We can see that the expressions (6) and (7) do not match, meaning that the system is not shift-invariant.

b. $g(x,y) = \frac{\partial}{\partial x} f(x - x_0, y)$

- Linearity

$$\mathcal{S}[f'(x,y)] = \mathcal{S}[\sum_t w_t f_t(x,y)] \tag{8}$$

$$= \sum_t \mathcal{S}[w_t f_t(x,y)] \tag{9}$$

$$= \sum_t w_t \mathcal{S}[f_t(x,y)] \tag{10}$$

$$= \sum_t w_t \frac{\partial}{\partial x} f_t(x - x_0, y) \tag{11}$$

Hence, we can see that the system is linear, since derivative operation is a linear operator.

- Shift-Invariance
  Transform first, shift second.

$$\mathcal{S}[f(x,y)] = \frac{\partial}{\partial x} f(x - x_0, y) \tag{12}$$

$$\xrightarrow[(\alpha,\beta)]{Shiftby} \frac{\partial}{\partial x} f(x - x_0 - \alpha, y - \beta) \tag{13}$$

  Shift first, transform second.

$$\mathcal{S}[f(x-\alpha, y-\beta)] = \frac{\partial}{\partial x} f(x - x_0 - \alpha, y - \beta) \tag{14}$$

  We can see that the expressions (13) and (14) match, meaning that the system is shift-invariant. We can conclude that this system is an LSI system.

Question 3

This question requires us to derive the solutions to some signal operations given the signal $f(x,y) = x^2 + xy + y^3$.

a. $f(x,y)\delta(x-2, y+1)$
   Since this is a multiplication of the function with a shifted point impulse, we can say that the function takes its value at the location of the impulse. The solution then becomes,

$$f(x,y)\delta(x-2,y+1) = f(2,-1) = 2^2 + 2(-1) + (-1)^3 = 1 \tag{15}$$

b. $f(x,y) * \delta(x-3, y+1)$
   Here, we can define the convolution operation as follows for these two functions.

$$f(x,y) * \delta(x-2,y+1) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\xi,\eta)\delta(x-3-\xi, y+1-\eta)d\xi d\eta \tag{16}$$

   Hence, by using the sifting property we can say that the solution becomes,

$$f(x,y) * \delta(x-3,y+1) = f(x-3,y+1) = (x-3)^2 + (x-3)(y+1) + (y+1)^3 \tag{17}$$

c. $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(x-1, 2y) f(x,y) dxdy$
   We can rewrite this integral as,

$$\frac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(x-1,y)f(x,y)dxdy = \frac{1}{2}f(1,0)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(x,y)dxdy \tag{18}$$

   Then the solution becomes,

$$\frac{1}{2}f(1,0) = \frac{1}{2} \tag{19}$$

d. $f(x+2,-2) * \delta(x-2, y+1)$
Since the function's y value is constant, we can see this as a function that varies with x stretched onto the y axis. Then, we can reduce this function to a one-dimensional convolution operation as follows.

$$f_{1D}(x) = f(x,-2) \tag{20}$$
$$f_{1D}(x+2) * \delta(x-2) \tag{21}$$
$$= f_{1D}(x+2-2) = f_{1D}(x) \tag{22}$$

Then, we can re-transform into 2D space as in (20) and the solution becomes $f(x,-2) = x^2 - 2x - 8$.

Question 4

This question asks us to find the 2-Dimensional Fourier Transforms of the following signals. Also, we are given the ability to use any known formulation in the Fourier Transform table, hence some derivations will not be fully explained thoroughly. However for briefing, the 2-D Fourier Transform is defined as follows.

$$\mathcal{F}_{2D}\{f\}(u,v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)e^{-2\pi j(ux+vy)}dxdy \tag{23}$$

a. $g(x,y) = e^{2\pi j v_0 y}$
We can clearly say that multiplication with a complex exponential in the spatial domain results in a shift in the frequency domain. Hence the result becomes,

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} 1e^{-2\pi j(ux+(v-v_0)y)}dxdy \tag{24}$$
$$= \delta(u, v-v_0) \tag{25}$$

b. $g(x,y) = \delta(x-2, 3y)$
Here, we can define the transform integral as follows.

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(x-2,3y)e^{-2\pi j(ux+vy)}dxdy \tag{26}$$

Then, we can use the scaling and sifting properties of the impulse function and as the transform of impulse is well known as the constant function, we can reach the following result.

$$= \frac{1}{3}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(x-2,y)e^{-2\pi j(ux+vy)}dxdy \tag{27}$$
$$= \frac{1}{3}e^{-2\pi j2u} \tag{28}$$

c. $g(x, y) = sinc(2x, 4y - 1)$
Here, since sinc is a separable function, we can rewrite the integral in terms of two separate 1-D Fourier transforms as follows, and we can use the same ability since the Fourier Transform of sinc is the rectangle (rect) function, which is also a separable function.

$$= \int_{-\infty}^{\infty} sinc(2x)e^{-2\pi jux}dx \int_{-\infty}^{\infty} sinc(4(y - 1/4))e^{-2\pi jvy}dy \tag{29}$$

From this, we can separately find the results using the scaling property,

$$= \left(\frac{1}{2}rect(\frac{u}{2})\right)\left(\frac{1}{4}rect(\frac{v}{4})e^{-2\pi jv(1/16)}\right) \tag{30}$$

$$= \frac{1}{8}rect(\frac{u}{2}, \frac{v}{4})e^{-2\pi jv(1/16)} \tag{31}$$

d. $g(x, y) = rect(x/2, 8y)e^{2\pi j(u_0x+4v_0y)}$
Here, as in c., we can rearrange the transform into two as rect is a separable function.

$$= \int_{-\infty}^{\infty} rect(x/2)e^{-2\pi jx(u-u_0)}dx \int_{-\infty}^{\infty} rect(8y)e^{-2\pi jy(v-4v_0)}dy \tag{32}$$

$$= (2sinc(2(u - u_0)))\left(\frac{1}{8}sinc(\frac{v - 4v_0}{8})\right) \tag{33}$$

$$= \frac{1}{4}sinc(2(u - u_0)), \frac{v - 4v_0}{8}) \tag{34}$$

e. $g(x, y) = rect(2x - 3, 3y) * sin(\pi x - 6\pi y)$
Here, we will use the convolution property of the Fourier transform as convolution in spatial domain is equivalent to multiplication in Fourier domain. Hence, we will take the transforms separately and multiply them to achieve the result. we have applied the rect transform in the previous parts, and transform of sine is also given below.

$$\mathcal{F}_{2D}[sin(2\pi(u_0x + v_0y))] = \frac{1}{2j}(\delta(u - u_0, v - v_0) - \delta(u + u_0, v + v_0)) \tag{35}$$

$$G(x, y) = \mathcal{F}_{2D}[rect(2x - 3, 3y)]\mathcal{F}_{2D}[sin(\pi x - 6\pi y)] \tag{36}$$

$$= \left(\frac{1}{6}sinc(\frac{u}{2}, \frac{v}{3})e^{-2\pi ju(3/2)}\right)\left(\frac{1}{2j}(\delta(u - 1/2, v + 3) - \delta(u + 1/2, v - 3))\right) \tag{37}$$

$$= \frac{1}{12j}sinc(\frac{u}{2}, \frac{v}{3})(\delta(u - 1/2, v + 3) - \delta(u + 1/2, v - 3))e^{-2\pi ju(3/2)} \tag{38}$$

f. $g(x, y) = sinc(x - 1, 4y)cos(4\pi x - 2\pi y)$
Here, we split the function into two as follows.

$$g(x, y) = g_1(x, y)g_2(x, y) \tag{39}$$

$$g_1(x, y) = sinc(x - 1, 4y) \tag{40}$$

$$g_2(x, y) = cos(4\pi x - 2\pi y) \tag{41}$$

Then,

$$G(u, v) = G_1(u, v) * G_2(u, v) \tag{42}$$

Hence, if we find G_1 and G_2, we can find the solution.

$$G_1(u,v) = \frac{1}{4}rect(u, v/4)e^{-2\pi ju} \tag{43}$$
$$G_2(u,v) = \mathcal{F}_{2D}[cos(2\pi(2x-y))] \tag{44}$$
$$= \frac{1}{2}(\delta(u-2, v+1) + \delta(u+2, v-1)) \tag{45}$$

From (42) and using the distributive property of convolution,

$$G(x,y) = (1/4rect(u, v/4)e^{-2\pi ju} * 1/2\delta(u-2, v+1)) \tag{46}$$
$$+(1/4rect(u, v/4)e^{-2\pi ju} * 1/2\delta(u+2, v+1)) \tag{47}$$
$$= 1/8(rect(u-2, \frac{v+1}{4})e^{-2\pi j(u-2)} + rect(u+2, \frac{v-1}{4})e^{-2\pi j(u+2)}) \tag{48}$$

g. $g(x,y) = e^{-2\pi(x^2+9y^2)} * cos(\pi x + 3\pi y)$
Here, we again split the transformation into two.

$$G(x,y) = \frac{1}{6}e^{-\pi(\frac{u^2}{2}+\frac{v^2}{18})}\frac{1}{2}(\delta(u-1/2, v-3/2) + \delta(u+1/2, v+3/2)) \tag{49}$$
$$= \frac{1}{12}(e^{-\pi(1/8+1/8)} + e^{-\pi(1/8+1/8)}) \tag{50}$$
$$= \frac{1}{6}e^{-\pi/4} \tag{51}$$

Question 5

This question requires us to construct a Modified Shepp-Logan phantom image, which is a contrast-enhanced version of the Shepp-Logan image, which is used to be a baseline image for tomography studies. Then, each part requires different tasks.

a. This part requires us to visualize the ideal image and its magnitude spectrum. For code, see Appendix A, hw1.m.

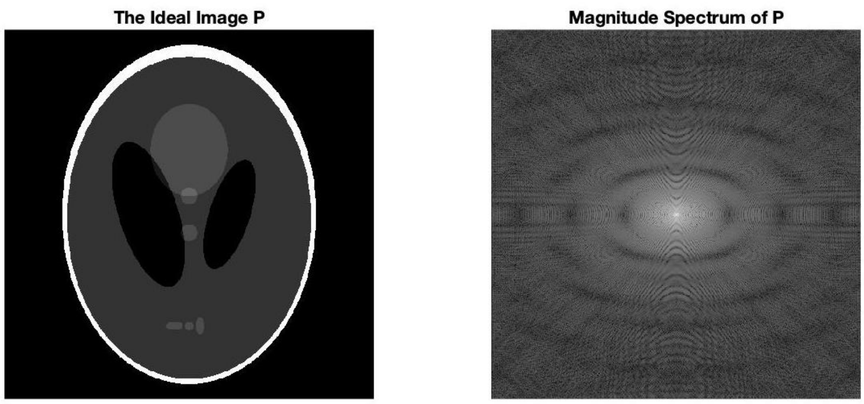

Figure 7: The Ideal Image and its Magnitude Spectrum

b. This part asks us to visualize the given Point Spread Functions (PSFs) and their magnitude spectrum, which are given below. The PSFs are also given as follows.

$$h_1 = sinc^4(x/8, y/2) \tag{52}$$

$$h_2 = \frac{1}{2\pi\sigma^2} e^{-\frac{x^2+y^2}{2\sigma^2}} \tag{53}$$

$$h_3 = rect(x/2, y/6) \tag{54}$$

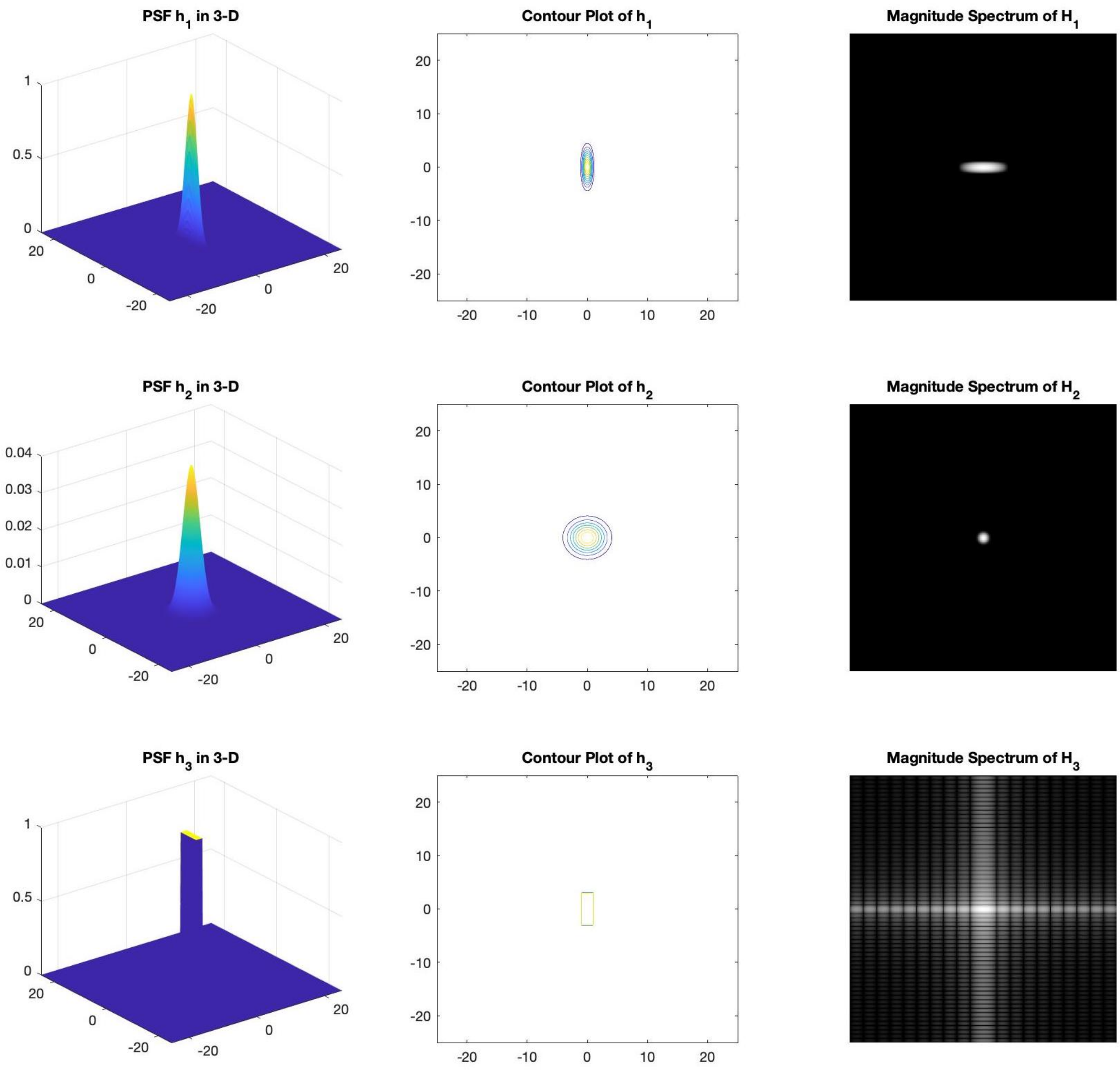

Figure 8: Point Spread Functions of h_1, h_2 and h_3 with their Magnitude Spectra

The first column are the 3-D representations, second gives the contour maps and the third column represents the magnitude spectrum of the PSFs. Here, we have chosen the x and y values as $[-25, 25]$. We noticed that as the range gets bigger, the output images gets closer to the original image since filters' total mass reaches its maximum. We thought that this range is appropriate to capture enough of the functionality with and contrast.

c. This part asks us to find the output images of the ideal images passed through these imaging systems and their magnitude spectra. For this, we can use two different methods. We can either convolve the image and the filter in the temporal domain or we can take the Fourier transforms (since we have already done that in the previous sections), multiply them and take the inverse transform. We have demonstrated both to show that the results are equivalent. The figure is provided below.

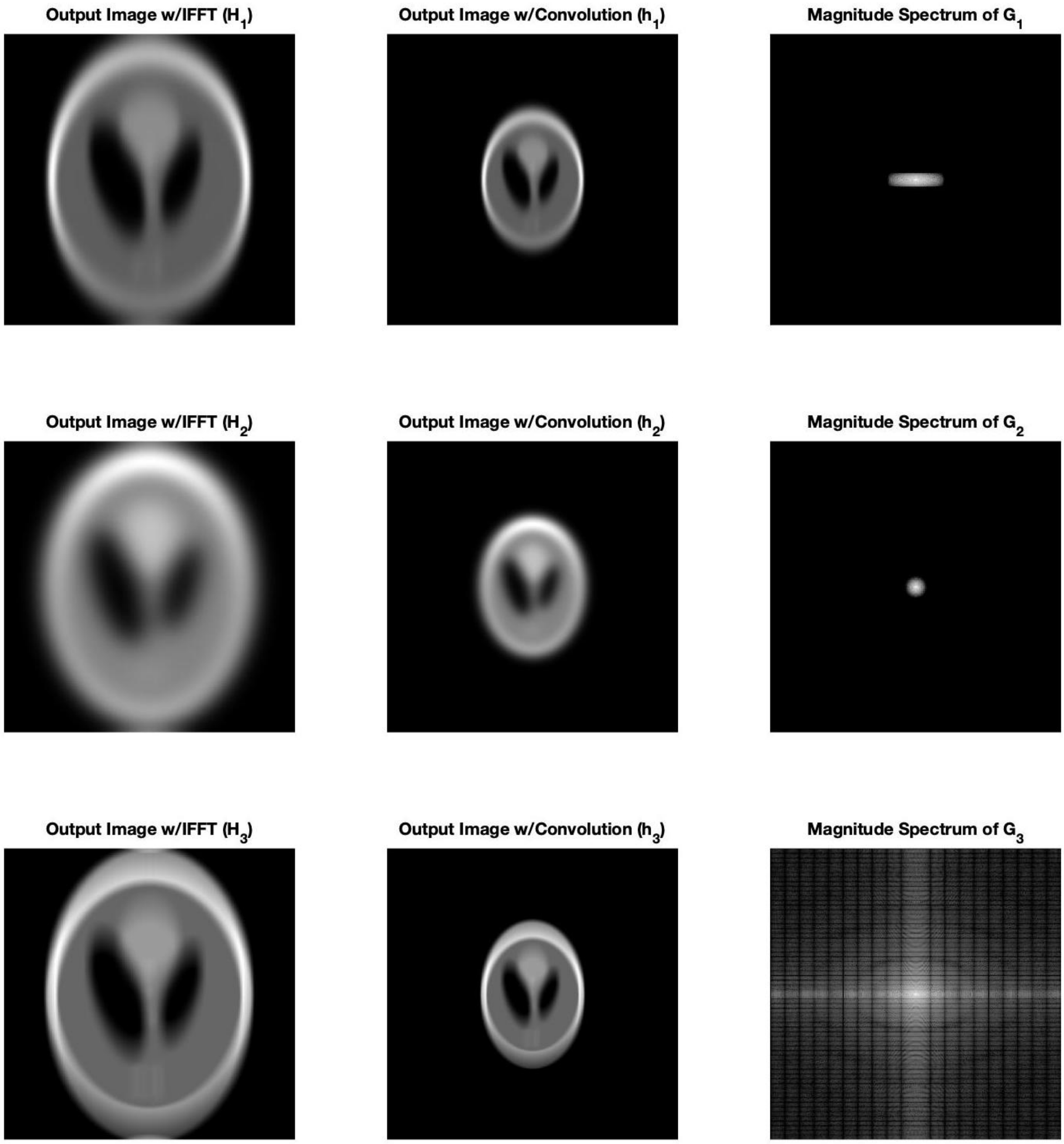

Figure 9: Output Image Reconstructions of g_1, g_2 and g_3 (Both Methods) with their Magnitude Spectra

d. İf we look at the output images, we can clearly state that the exercise reached its purpose. For the first system, we see the output looks like a more blurry but a contrast-enhanced version of the ideal image. The organs are more distinguishable than the background. From the shapes that represent the organs, we see that we can observe the heart and lungs better than the aorta and the vertebra. Here, we see the enhanced contrast on the soft tissues, which resembles the workings of MRI. When we observe the magnitude spectrum, we see that the filter nearly eliminated all high frequency components in either spectral component.

When we observe the second output, which is filtered with a Gaussian filter, is more blurry than the original image, this is reasonable since the Gaussian filter is a filter that is used in noise reduction and blurring. The Gaussian reacts as a smoothing element in the natural images since we usually want to focus on low-frequency elements in the power spectrum. The magnitude spectrum also looks like the filter's spectrum but slightly bigger. If we decrease the filter range further, the contrast decreases even more.

The third system is a rectangular filter, which is similar to the Gaussian is generally used to smooth-out small-scale image features. Here we can see that the edges are smoother than what is in the ideal image, which gives the feeling of topography on the edges rather than a two-dimensional image. When we observe the magnitude spectrum, we see that the sinc function is multiplied with the image. One thing I noticed is that the pass band of this system is substantially higher than the other systems, allowing more high-frequency components to be observed.

MATLAB Code

Code that is written for Question 5 are presented below.

fft2c.m

```
function d = fft2c(im)
% d = fft2c(im)
% fft2c performs a centered fft2
d = fftshift(fft2(ifftshift(im)));
end
```

ifft2c.m

```
function im = ifft2c(d)
% im = ifft2c(d)
% ifft2c performs an centered ifft2
im = fftshift(ifft2(ifftshift(d)));
end
```

rect.m

```
function sig=rect(min,max,num_pts, alph1, alph2)
% 2-D rectangle function
% g(x,y) = rect(alph1*x,alph2*y)
% Takes two separate 1-D rectangle functions and multiplies them
width = 0.5;
base = linspace(min,max,num_pts);
sigx = double(abs(base) <= width./alph1);
sigy = double(abs(base) <= width./alph2);
sig = transpose(sigx) * sigy;
end
```

hw1.m

```
close all; clear all;
%%
P = phantom('Modified Shepp-Logan', 512);
%% Part A
figure();
subplot(1,2,1)
imshow(P)
title("The Ideal Image P")
subplot(1,2,2)
F = fft2c(P);
imshow(log(abs(F)+1),[])
title("Magnitude Spectrum of P")

%% Part B
figure()
%% H-1
subplot(3,3,1)
x=linspace(-25,25,512);
y=x;
h1 = (transpose(sinc(x./8))*sinc(y./2)).^4;
surf(x,y,h1,'LineStyle','none')
title("PSF h_1 in 3-D")
subplot(3,3,2)
```

```
contour(x,y,h1)
title("Contour Plot of h_1")
subplot(3,3,3)
H1 = fft2c(h1);
imshow(log(abs(H1)+1),[])
title("Magnitude Spectrum of H_1")

%% H-2
subplot(3,3,4)
std = 2;
x = linspace(-25,25,512);
y = x;
[X,Y] = meshgrid(x,y);
h2 = (1/(2*pi*std.^2))*exp(-(X.^2+Y.^2)/(2*std.^2));
surf(X,Y,h2,'LineStyle','none')
title("PSF h_2 in 3-D")
subplot(3,3,5)
contour(X,Y,h2)
title("Contour Plot of h_2")
subplot(3,3,6)
H2 = fft2c(h2);
imshow(log(abs(H2)+1),[])
title("Magnitude Spectrum of H_2")

%% H-3
subplot(3,3,7)
min = -25; max = 25; num_pts = 512;
x = linspace(min,max,num_pts);
y = x;
h3 = rect(min,max,num_pts,1/6,1/2);
surf(x,y,h3,'LineStyle','none')
title("PSF h_3 in 3-D")
subplot(3,3,8)
contour(x,y,h3)
title("Contour Plot of h_3")
subplot(3,3,9)
H3 = fft2c(h3);
imshow(log(abs(H3)+1),[])
title("Magnitude Spectrum of H_3")

%% Part C
%% G-1
figure()
subplot(3,3,1)
G1 = F.* H1;
g1 = ifft2c(G1);
imshow(mat2gray(g1))
title("Output Image w/IFFT (H_1)")
subplot(3,3,2)
g1 = conv2(P,h1);
imshow(mat2gray(g1))
title("Output Image w/Convolution (h_1)")
subplot(3,3,3)
imshow(log(abs(G1)+1),[])
title("Magnitude Spectrum of G_1")

subplot(3,3,4)
G2 = F .* H2;
g2 = ifft2c(G2);
imshow(mat2gray(g2))
```

```
title("Output Image w/IFFT (H_2)")
subplot(3,3,5)
g2 = conv2(P,h2);
imshow(mat2gray(g2))
title("Output Image w/Convolution (h_2)")
subplot(3,3,6)
imshow(log(abs(G2)+1),[])
title("Magnitude Spectrum of G_2")

subplot(3,3,7)
G3 = F .* H3;
g3 = ifft2c(G3);
imshow(mat2gray(g3))
title("Output Image w/IFFT (H_3)")
subplot(3,3,8)
g3 = conv2(P,h3);
imshow(mat2gray(g3))
title("Output Image w/Convolution (h_3)")
subplot(3,3,9)
imshow(log(abs(G3)+1),[])
title("Magnitude Spectrum of G_3")
```