[Wk 1]

Statistics
- Descriptive
  - ↳ summary statistics ; numerical data
  - ↳ Plots ; categorical data
- Inferential

- Histogram : numerical data ; binned and count obtained for each bin.

Variable
- quantitative
  - numerical
    - discrete
    - continuous (measured)
  - categorical
    - ↳ ordinal
    - ↳ Nominal (no order)
- qualitative

discrete : $E[x] = \sum x_i \, p(x_i)$

continuous : $E[x] = \int x \, f(x)$

- Variance ; measures spread ; how far you are from the mean.

- Portfolio - collection of financial assets (bonds, funds, etc)
  ↳ consider risk & expected return

E.g.: Portfolio :
1) Bonds: 60% } return = 8%
2) Stock: 40% } return = 12%

weight Σ 100%

weight (how much you're willing to invest)

Expected Return: $\Sigma\, \omega_i r_i$ (r = return)

$E(R_p) = (0.6 \times 0.08) + (0.4 \times 0.12) = 0.096$

- Variance - measures the level of risk in the portfolio.

- 3 scenarios in investment
  - pessimistic scenario / downturn
  - base " / comfort zone
  - optimistic " / good times

* 2008 financial crisis : Read

- **Assets:**

1) Option - a financial contract that gives the right (to sell / or buy) but not obligation.
- 2 types: a) Call option: gives the right to buy (but not obligation) after maturity happens

E.g: Call option:
$t_0 \to 100 \Rightarrow$ Strike price (K) — what's agreed upon

After 3 months $S_t = 120$
* We buy at 100 & sell at 120.

Return = 20

Paid at the onset $\to$ Premiums = 5

Net return = 15

After 3 months $S_t = 80$
* We don't exercise it leads to loss

$$\boxed{\text{Return Call option} = \max(S_t - K, 0)}$$

b) Put option: gives the right to sell but not obligation

E.g.
to sell at: $K = 100$

After 3 months: $S_t = 120$
Buy at 120 & sell at 100 $\to$ loss

$S_t = 80$
Buy at 80 & sell at 100 $\to$ Profit

$$\boxed{\text{Return put option} = \max(K - S_t, 0)}$$

2.) Bonds - are like loans (e.g. government bonds, corporate bonds)
- Coupons $\to$ interest
- Default rate = 0 for gov. bonds.
- Infrastructure bonds is ~~tax~~ free

Q

* Consider the 3 scenario ALWAYS

Portfolio with 2 assets.

Scenario/State	Prob.	R_A	R_B
1	0.3	0.1	0.05
2	0.4	0.06	0.04
3	0.3	0.02	0.07

$$\boxed{* \; E(R) = \sum_{i=1}^{n} P_i R_i}$$

$E(R_A) = (0.3 \times 0.1) + (0.4 \times 0.06) + (0.3 \times 0.02) = 0.06$

$E(R_B) = (0.3 \times 0.05) + (0.4 \times 0.04) + (0.3 \times 0.07) = 0.052$

* Risk / variance : $\sigma^2 = \Sigma P_i [R_i - E(R)]^2$

$\sigma_A^2 = 0.3(0.1-0.06)^2 + 0.4(0.06-0.06)^2 + 0.3(0.02-0.06)^2$
$= 9.6 \times 10^{-4} = 0.00096$

$\sigma_B^2 = 0.3(0.05-0.052)^2 + 0.4(0.04-0.052)^2 + 0.3(0.07-0.052)^2$
$= 1.56 \times 10^{-4} = 0.000156$

Q Call option: $k = 105$ After 3 months: Scenario 1, 2, 3: $S_t = 120, 100, 80$
$P_i = 0.4, 0.4, 0.2$

Return / Pay offs = what you receive after 3 months : 15, 0, 0
$E(R) = (0.4 \times 15) + (0.4 \times 0) + (0.2 \times 0) = 6$

Q Put option: $k = 105$ After 3 months: $S_t = 120, 100, 80$
Prob $= 0.4, 0.4, 0.2$
return $= 0, 5, 25$

[Wk 2]

* Discounting factor $= \frac{1}{(1+i)^n}$ ↳ effective rate
the money in present time

Q $E(R_A) = 0.06$ 70% - bonds
$E(R_B) = 0.052$ 30% - stock

$E(R_p) = \sum_{i=1}^{2} W_i E(R_i) = (0.06 \times 0.7) + (0.052 \times 0.3) = 0.0576 \approx 0.06 = 6\%$

↳ to determine whether to invest or not

* The higher the variance the riskier the asset.

* Covariance - '-ve' the best to invest. (for portfolio)
- gives direction of what is happening.

Q :

S	Prob.	R_A	R_B
1	0.3	0.1	0.05
2	0.4	0.06	0.04
3	0.3	0.02	0.07
		0.06	0.052

* $Var(X) = \Sigma P_i [R_i - E(R)]^2$

* $Cov(X,Y) = \Sigma P_i [R_x - E(R_x)][R_y - E(R_y)]$

$Cov(X,Y) = 0.3[0.1-0.06][0.05-0.052] + 0.4[0.06-0.06][0.04-0.052] +$
$0.3[0.02-0.06][0.07-0.052]$
$= -2.4 \times 10^{-4} = -0.00024$

* 2 correlated variables: check correlation of each variable with target variable. Pick variable with highest correlation with target variable.

-ve ; we invest in the portfolio

* Correlation: $\rho(A,B) = \frac{Cov(A,B)}{\sqrt{Var(A)\,Var(B)}} = \frac{Cov(A,B)}{\sigma_A \sigma_B} = [-1, 1]$

Q $\rho(A,B) = \frac{-0.00024}{\sqrt{0.00096}\sqrt{0.000156}} \simeq -0.6202$

Q

S	Prob.	R_A	R_B
1	0.4	0.4	0.01
2	0.3	0.07	0.8
3	0.3	0.02	0.03
		0.187	0.253

$W_A = 0.6$ $E(R_A) = 0.187$
$W_B = 0.4$ $E(R_B) = 0.253$

a.) $E(R_P)$
b.) Var (A) & var (B)
c.) Cov (A,B)
d.) $\rho(A,B)$

a.) $E(R_P) = \sum W_i E(R_i)$ & $E(R) = \sum P_i R_i$

$E(R_A) = (0.4 \times 0.4) + (0.3 \times 0.07) + (0.3 \times 0.02) = 0.187$
$E(R_B) = (0.4 \times 0.01) + (0.3 \times 0.8) + (0.3 \times 0.03) = 0.253$

$\therefore E(R_P) = (0.6 \times 0.187) + (0.4 \times 0.253) = 0.2134 = \underline{21.34\%}$

Bonds : 5% $\therefore$ The above portfolio is worth investing in.

b.) $Var(A) = \sum P_i [R_A - E(R_A)]^2$
$= 0.4 (0.4 - 0.187]^2 + 0.3 (0.07 - 0.187)^2 + 0.3 (0.02 - 0.187)^2$
$= 0.030621$

$Var(B) = \sum P_i [R_B - E(R_B)]^2$
$= 0.4 (0.01 - 0.253)^2 + 0.3 (0.8 - 0.253)^2 + 0.3 (0.03 - 0.253)^2$
$= 0.128301$ → Asset B riskier than Asset A

c.) $Cov(A,B) = \sum P_i [R_A - E(R_A)][R_B - E(R_B)]$
$= [0.4 (0.4 - 0.187)(0.01 - 0.253)] + [0.3 (0.07 - 0.187)(0.8 - 0.253)] +$
$[0.3 (0.02 - 0.187)(0.03 - 0.253)]$
$= -0.028731$

d.) $\rho(A,B) = \frac{Cov(A,B)}{\sqrt{Var A}\sqrt{Var B}} = \frac{-0.028731}{\sqrt{0.030621}\sqrt{0.128301}} = \underline{-0.458}$

* Options are ~~a~~ contracts ; we don't own them unlike stocks.

Q Call option:

$S_0 = 100$
$K = 120$
$S_t = 140, 100, 80$
Prob = 0.3, 0.5, 0.2

(Call)
* Payoffs = $\max(S_t - K, 0)$
* Pay offs = $\max(K - S_t, 0)$
(Put)

* Hedging ??

1 → Pay offs (Call) = 20, 0, 0
→ " (Put) = 0, 20, 40

Buy at 100 & sell at 140

	(option) Payoffs	Return (stock)	Prob.
1)	20	40	0.3
2)	0	0	0.5
3)	0	-20	0.2
	$E(R) = 6$	$E(R) = 8$	

* +ve correlation → we invest in it

a) var (payoffs)
b) var (returns)
c) cov (p, r)
d) $\rho(P, r)$

Q do the same for put option

a) $Var(P) = \sum P_i (R_P - E(R_P))^2 = [0.3(20-6)^2] + [0.5(0-6)^2] + [0.2(0-6)^2]$
$= 84$

b) $Var(R) = \sum P_i [R_R - E(R_R)]^2 = [0.3(40-8)^2] + [0.5(0-8)^2] + [0.2(-20-8)^2]$
$= 496$

c) $cov(P,R) = \sum P_i [R_P - E(R_P)][R_R - E(R_R)]$
$= 0.3(20-6)(40-8) + 0.5(0-6)(0-8) + 0.2(0-6)(-20-8)$
$= 192$

d) $\rho(P,R) = \dfrac{cov(P,R)}{\sqrt{Var(P)}\sqrt{Var(R)}} = \dfrac{192}{\sqrt{84}\sqrt{496}} = 0.941$

Q Put option:

$S_0 = 100$
$K = 120$
$S_t = 140, 100, 80$
Prob $= 0.3, 0.5, 0.2$

Prob.	Payoffs	Returns
0.3	0	40
0.5	20	0
0.2	40	-20
	18	8

Buy at 140 & sell at 100

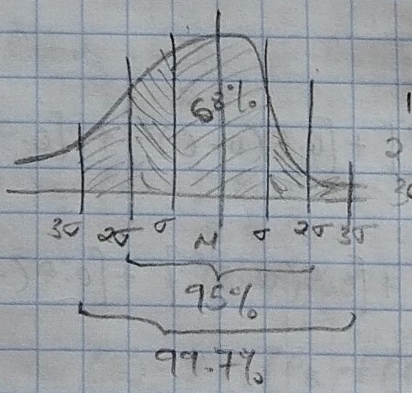
a) $E(P) = (0.3 \times 0) + (0.5 \times 20) + (0.2 \times 40) = 18$
$E(R) = (0.3 \times 40) + (0.5 \times 0) + (0.2 \times -20) = 8$

b) $Var(P) = \Sigma p_i (R_i - E(R_P))^2$
$= 0.3(0-18)^2 + 0.5(20-18)^2 + 0.2(40-18)^2$
$= 196$

$Var(R) = 0.3(40-8)^2 + 0.5(0-8)^2 + 0.2(-20-8)^2$
$= 496$

c) $Cov(P,R) = \Sigma p_i [R_P - E(R_P)][R_R - E(R_R)]$
$= 0.3[0-18][40-8] + 0.5[20-18][0-8] + 0.2[40-18][-20-8]$
$= -304$

d) $\rho(P,R) = \frac{Cov(P,R)}{\sqrt{Var P}\sqrt{Var R}} = \frac{-304}{\sqrt{196}\sqrt{496}} = -0.975$



$1\sigma = 68\%$
$2\sigma = 95\%$
$3\sigma = 99.7\%$

- Simple linear regression: To check the relationship between the independent & dependent variable.
- Logistic regression – when dealing with binary dependent variable

1-) Capital asset pricing model (CAPM) ← Application of regression analysis.

Net Returns $= R_A - R_f$ → dependent variable

independent variable → Net market return $= R_m - R_f$

$y = \alpha + \beta X$
$R_A - R_f = \alpha + B(R_m - R_f)$
volatility

2-) Credit risk (Application of regression analysis) — score card
- Expected Credit loss (ECL) - model 3 parameter
  - prob. of default
  - loss given default; LGD (1 - recovery rate)
  - exposure at default (EAD)

3-) Predicting e.g stock prices (Application of regression analysis).

Probability Distribution. - normal & log normal

* t-distribution used when sample size is small (< 30)
* normal distribution used when sample size is large (> 30)

* Chi-square test used for
  - goodness of fit
  - test of independence

* ANOVA - test of variance

  $H_0: \mu_1 = \mu_2 = \mu_3 = \dots = \mu_k$

  H_A: atleast one mean is different

  Test statistics: f-distribution.

1.) Normal distribution:
- can model returns (since it takes in both +ve & -ve) whereas log-normal models stock prices since prices are never -ve thus log-normal.

  * Log normal distribution considers +ve values only

- $X \sim N(\mu, \sigma^2)$

  $Y = e^X$

  $\ln(Y) = X$

  $\ln(Y) \sim N(\mu, \sigma^2)$

- E.g: $R \sim N(0.02, 0.0001)$ $R \sim N(2\%, 0.01\%)$

  a.) $P(R < 0)$

  b.) $P(R > 3\%)$

  c.) $P(3\% < R < 5\%)$

  a.) Prob. of making a loss; $R < 0$; $P(R < 0)$

$$P\left(\frac{R-\mu}{\sigma} < \frac{0-0.02}{\sqrt{0.0001}}\right) = P(Z < -2) = 0.02275$$

b.) $$P(R > 3\%) = P\left(Z > \frac{0.03-0.02}{\sqrt{0.0001}}\right) = 1 - P(Z < 1) = 1 - 0.84134 = 0.15866$$

c.) $$P(3\% < R < 5\%) = P\left(\frac{0.03-0.02}{\sqrt{0.0001}} < Z < \frac{0.05-0.02}{\sqrt{0.0001}}\right) = P(1 < Z < 3)$$

$$= 0.99865 - 0.84134 = 0.15731$$

$$P(X=x) = \frac{e^{-\lambda}\lambda^{x}}{x!}$$ (Poisson distribution)

$$P(X=x) = \binom{n}{x} p^{x} q^{n-x}$$ (Binomial distribution)

[Wk 3]

* Ind. assign uploaded

2.) Poisson distribution: (discrete)

- On average, the total no. of events over a given period of time.
- PDF: $P(X=x) = \frac{e^{-\lambda}\lambda^{x}}{x!}$ where λ = average.

Q A call center receives an average of 5 calls per minute. What is the prob. that in the next minute they'll receive exactly 7 calls?

$\lambda = 5$ calls/minute $x = 7 = ?$

$$P(X=7) = \frac{e^{-5}5^{7}}{7!} = 0.1044$$

Q.1) A data streaming platform logs an average of 12 errors/hr. What is the prob. of observing no error in an given hr.

Q.2) A fraud detection system detects on average 2 suspicious transactions per 10 mins. What is the prob. that it will detect atleast 3 suspicious transactions ~~and~~, at most 2 suspicious transactions.

Q.1) $\lambda = 12$ errors/hr

$$P(X=0) = \frac{e^{-\lambda}\lambda^{x}}{x!} = \frac{e^{-12}\times 12^{0}}{0!} = 6.1442 \times 10^{-6}$$

Q.2) $\lambda = 2/10$ mins

$$P(X \geq 3) = 1 - P(X=0) - P(X=1) - P(X=2)$$
$$= 1 - \frac{e^{-2}\times 2^{0}}{0!} - \frac{e^{-2}\times 2^{1}}{1!} - \frac{e^{-2}\times 2^{2}}{2!}$$
$$= 1 - 0.1353 - 0.2707 - 0.2707$$
$$= 0.3233$$

$$P(X \leq 2) = P(X=0) + P(X=1) + P(X=2) \text{ OR } 1 - P(X \geq 3)$$
$$= \frac{e^{-2}\times 2^{0}}{0!} + \frac{e^{-2}\times 2^{1}}{1!} + \frac{e^{-2}\times 2^{2}}{2!}$$
$$= 0.1353 + 0.2707 + 0.2707$$
$$= 0.6767$$

OR $1 - 0.3233 = 0.6767$

* Read on beta distribution

3.) Binomial distribution: (discrete)
- Dealing with 2 categorical data; 2 instances (either failure or success).
- PDF: $P(X=x) = {}^{n}C_{x}\, p^{x} q^{n-x}$

Q.1) A machine learning algorithm has a 70% accuracy rate in classifying default rates as either default or non-default. If you test it on 10 ~~different~~ independent accounts, what is the prob. that it correctly classifies exactly 8 accounts.

Q.2) In A/B testing, version A of a website has 15% conversion rate. If 100 visitors are randomly assigned to version A, what is the prob. that between 10 & 20 convert?

Q.3) A company finds that 60% (p) of its customers who sign up for a free trial eventually convert to paying customers. If "12" people sign up for the free trial, what is the prob. that atleast 10 (x) become paying customers?

Q.1) $p = 70\% = 0.7 \quad q = 0.3 \quad n = 10 \quad x = 8$
$P(X=8) = {}^{10}C_{8} \times (0.7)^{8} \times (0.3)^{10-8}$
$= 0.2335$

Q.2) $p = 15\% = 0.15 \quad q = 0.85 \quad n = 100 \quad P(10 \le X \le 20) = ?$ X TO FINISH
$P(10 \le X \le 20) = P(X \le 20) - P(X < 10)$

0 1 2 3 4 5 6 7 8 9 | 10 11 12 13 14 15 16 17 18 19 20

Q.3) $p = 60\% = 0.6 \quad q = 0.4 \quad n = 12 \quad P(X \ge 10) = ?$
$P(X \ge 10) = P(X=10) + P(X=11) + P(X=12)$
$= {}^{12}C_{10}(0.6)^{10}(0.4)^{2} + {}^{12}C_{11}(0.6)^{11}(0.4)^{1} + {}^{12}C_{12}(0.6)^{12}(0.4)^{0}$
$= 0.0639 + 0.0174 + 2.1768 \times 10^{-3}$
$= 0.0835$

* Read on beta distribution & learn on python
* Read on Gauss distribution

Stochastic Processes

- There is uncertainty (non-deterministic) & over time.
- Weiner process: discrete time & discrete state space.
- Tossing a coin twice: sample space $\begin{Bmatrix} HH & TH \\ HT & TT \end{Bmatrix}$ & state space $\{H, T\}$

* Read on Weiner process (Geometric brownian motion)

Q You're analysing the daily prices of stock over 30 day period. let X_t represent the closing stock price on day t.
a.) Possible outcomes - Day 5 has closing stock price \$ 105.30
b.) Sample space - { Positive integers }
c.) State space - { +ve integers } → continuous
Time → discrete { 30 day period ; 1-30 }

Q A bank tracks whether a borrower defaults on their loan each month for one year. Let $X_t = 1$ if the borrower defaults in month t & $X_t = 0$ otherwise.
a.) Give an example of an outcome. 91: 0 0 0 1 0 1 0 0 0 0 0 0 (months 1 2 3 4 5 6 7 8 9 10 11 12)
b.) Sample space { Default (1), No-default (0) }
c.) State space { 1, 0 } discrete
Time → discrete (each month ; 1-12)

Q ~~No of fraudelent accounts. +7 fraudulent ≠0 = non-fraudulent. over a year~~ per m

Q No. of clicks daily. → Random variable.
time = Continuous { no specific interval }
state = discrete

Derivatives - priced
- Def: assets that are not stand-alone (depends on something).

* Call option: Pay-off = max $(\underbrace{S_t}_{\text{derivative}} - k, 0)$

* risk free rate
Geometric Brownian Motion

$$dS_t = \overbrace{\mu S_t dt}^{\text{deterministic}} + \overbrace{\sigma S_t dW_t}^{\text{randomness}}$$

* forward contract - person
* future contract - market

- μ = drift ; growth rate we know
σ = volatility ; there is uncertainty ; flunctuations in price.
W_t = Weiner process (brownian motion) ; models randomness
↳ Expected value = 0

- Properties of Weiner process (W_t, $t \geq 0$)
1 -) $E(W_t) = 0$

2-) Increments are independent
Day 1 Day 5 * x independent of 5
W_1 W_5 independent
$120 \xleftrightarrow{5} 125 \xleftrightarrow{x}$

3-) Increments are stationary.

Day 1 —4 days— Day 5   Day 10 —4 days— Day 14

120 ⟷ 125   110 —— 90

4-) Increments $\sim N(0, s)$: $W_{t+s} - W_t \sim N(0, s)$

$\underbrace{s}$ dependent on the interval not starting points.

$W_2 \xrightarrow{4\text{ days}} W_6$

Q A weiner process with a variance of 4.

a) Find its distribution $W_t \sim N(0,4)$

b) Find the prob. that $W_t > 2$

$$P(W_t > 2) = P\left(\frac{W_t - \mu}{\sigma} > \frac{2-0}{\sqrt{4}}\right) = P(z > 1) = 1 - P(z < 1)$$

$$= 1 - 0.84134 = 0.15866$$

Q Suppose W_t is a weiner process. a) What is the distribution of $W_6 - W_3$:

$W_6 - W_3 \sim N(0,3)$

b) Find the prob. that $W_6 - W_3 < 1$

$P(W_6 - W_3) < 1$

$$P\left[z = \frac{(W_6 - W_3) - \mu}{\sigma}\right] < \frac{1-0}{\sqrt{3}}$$

$$P(z < 0.58) = 0.71904$$

* Read about no-arbitrage property & Martingale property

* Learn : $E(S_t)$ & $Var(S_t)$

[Wk 4]

- Garch - used to predict volatility. (σ)
- The GBM assumes σ is constant.
- μ = drift; growth rate we expect. (expected return)

* Read about no-arbitrage & martingale principle.

- No-arbitrage principle - everything has risk associated to it.
  - μ $\vec{r}$ = risk free rate (to standardize)
  - Real world; P & risk Neutral world = Q

- $dS_t = rS_t\,dt + \sigma S_t\,dW_t$ } Stochastic Differential eqn; SDE

  After integrating: $S_t = S_0 e^{\left[(\mu - \frac{1}{2}\sigma^2)dt + \sigma\, dW_t\right]}$

<u>*</u> Read about Taylor series & Stochastic calculus

- Martingale principle - The past doesn't affect future.
  W_t is said to be martingale if $E(W_s \mid f_t) = W_t$
  * The future doesn't depend on the past but on the present.
  filtration; prior information

- To proof that W_t is a martingale:
  $E[W_t \mid f_s] = W_s \ , \ t > s$

  $W_5 \quad W_3$
  $W_5 = W_3 + (W_5 - W_3)$
  $W_t = W_s + (W_t - W_s)$

  $\therefore = E(W_s + W_t - W_s \mid f_s)$
  $= \underbrace{E(W_s)}_{W_s} + \underbrace{E(W_t - W_s) \mid f_s)}_{0}$
  $\cancel{E(W}$
  $= W_s + 0 = W_s$

- $Var(W_t) = E(W_t^2) - [E(W_t)]^2$

  $E(W_t^2) = t \qquad W_t \sim N(0, t)$

  * $E(\text{constant}) = \text{constant}$
  $E[a + bX] = E(a) + E(bX)$
  $= a + bE(X)$

  * with respect to t so W_s = constant

- Check if $W_t^2 - t$ is a martingale:
  $E(W_t^2 - t \mid f_s) = W_s^2 - s \ ; \ s < t$

  $(a+b)^2 = (a+b)(a+b)$
  $= a^2 + 2ab + b^2$
  $W_s^2 + 2W_s(W_t - W_s) + (W_t - W_s)^2$

  $W_5 = W_3 + (W_5 - W_3)$
  $W_t = \cancel{W_t}\ W_s + (W_t - W_s)$
  $W_t^2 = [W_s + (W_t - W_s)]^2$

  don't open increments

  $W_t^2 = W_s^2 + 2W_s(W_t - W_s) + (W_t - W_s)^2$

  $E[W_t^2] = E[W_s^2 + 2W_s(W_t - W_s) + (W_t - W_s)^2]$

$$E[W_t^2] = E[W_s^2] + 2W_s \overbrace{E(W_t - W_s)}^{0} + E[(W_t - W_s)^2]$$

$$= W_s^2 + 0 + t - s \qquad * \underbrace{Var(W_t - W_s)}_{= t-s} = \underbrace{E[(W_t - W_s)^2]}_{t-s} - \underbrace{[E(W_t - W_s)]^2}_{=0} \quad \therefore$$

$$= W_s^2 + t - s - t$$

$$= W_s^2 - s \quad \text{therefore martingale.}$$

$$dS_t = \mu S_t dt + \sigma S_t dW_t$$

$$S_t = S_0 e^{[(\mu - \frac{1}{2}\sigma^2)dt + \sigma dW_t]}$$

$$E(S_t) = S_0 e^{\mu t}$$

~~$Var(S_t)$~~

$$Var(\ln S_t) = \sigma^2 t$$

stock price

$$\underbrace{Y}_{\text{stock price}} = e^X \sim \log$$

$$\ln Y = X \sim \text{normal}$$

Q Find $E[S_t]$ at $t=3$ given $S_0 = 100$ and $\mu = 8\%$ and $\sigma = 30\%$

$$E(S_t) = S_0 e^{\mu t} = 100 e^{0.08 \times 3} \simeq 127.12$$

Find $Var(\ln S_t) = \sigma^2 t = (0.3)^2 \times 3 = 0.27$

**

- Real world; P : $dS_t = \mu S_t dt + \sigma S_t dW_t^P$
- Risk-neutral world; Q: $dS_t = r S_t dt + \sigma S_t W_t^Q$
- X_t is considered an Ito process iff:

$$dX_t = a(t, X_t)dt + b(t, X_t)dW_t$$

$$dS_t = \mu S_t dt + \sigma S_t dW_t \quad \} \text{ Ito Process}$$

$$S_t = S_0 \exp[(\mu - \frac{1}{2}\sigma^2)dt + \sigma dW_t]$$

- $f(X_t) = f'(x)\, dX_t + \frac{1}{2} f''(x)(dX_t)^2$

$$dS_t = \mu S_t dt + \sigma S_t dW_t$$

$$f(\ln S_t) = f'(S_t)dS_t + \frac{1}{2} f''(S_t)(dS_t)^2$$

Let $y = \ln S_t \quad y' = \frac{1}{S_t} \quad y'' = \frac{-1}{S_t^2}$

$(a+b)^2 = a^2 + 2ab + b^2$

$(\mu S_t dt + \sigma S_t dW_t)^2$
$= \mu^2 S_t^2 \cancel{dt^2} + 2\mu\sigma S_t^2 dt\, dW_t + \sigma^2 S_t^2 (dW_t)^2$
$= \sigma^2 S_t^2 dt$

$(dt)^2 = 0$
~~$(dt\,W_t)$~~
$(dt\, dW_t) = 0$
$(dW_t)^2 = dt$

$$d\ln S_t = \frac{1}{S_t}(\mu S_t dt + \sigma S_t dW_t) + \frac{1}{2}\left(\frac{-1}{S_t^2}\right)(\sigma^2 S_t^2 dt)$$

$$d\ln S_t = \mu dt + \sigma dW_t - \frac{1}{2}\sigma^2 dt$$

$$d\ln S_t = \mu\, dt - \frac{1}{2}\sigma^2 dt + \sigma dW_t$$

$$d\ln S_t = \left(\mu - \frac{1}{2}\sigma^2\right) dt + \sigma dW_t$$

Intergrate: $t=0 \quad t=1$

$$\ln S_t - \ln S_0 = (\mu - \tfrac{1}{2}\sigma^2)dt + \sigma dW_t$$

$$\ln S_t = \ln S_0 + (\mu - \tfrac{1}{2}\sigma^2)dt + \sigma dW_t$$

$$S_t = S_0 \exp\left[(\mu - \tfrac{1}{2}\sigma^2)dt + \sigma dW_t\right]$$

Q. $\underbrace{dX_t}_{\text{Ito process}} = \mu dt + \sigma dW_t$ Use Ito lemma to find $d(X_t^2)$

Ito lemma: $f'(x) dX_t + \frac{1}{2} f''(x)(dX_t)^2$

Let $f(x) = X^2 \quad f'(x) = 2x \quad f''(x) = 2$

$(dX_t)^2 = (\mu dt + \sigma dW_t)^2$
$= \cancel{\mu^2 dt^2}\ \mu^2 \underbrace{(dt)^2}_{=0} + 2\mu\sigma \underbrace{dt\, dW_t}_{=0} + \sigma^2 \underbrace{(dW_t)^2}_{=dt}$
$= \sigma^2 dt$

Substituting:
$$d(x_t^2) = 2x(\mu dt + \sigma dW_t) + \frac{1}{2}(2)(\sigma^2 dt)$$

$$d(x_t^2) = 2x\mu dt + 2x\sigma dW_t + \sigma^2 dt$$
$$d(x_t^2) = \underline{(2x\mu + \sigma^2)dt + 2x\sigma\, dW_t} \Rightarrow \text{Ans!}$$

Q A population follows: $dP_t = rP_t dt + \sigma P_t dW_t$. Use Ito lemma to find the dynamics of $\ln(P_t)$

Ito lemma:

$$d(\ln P_t) = f'(x)\,dP_t + \frac{1}{2} f''(x)(dP_t)^2$$

Let $f(x) = \ln P_t$ so $f'(x) = \frac{1}{P_t}$ $\quad f''(x) = -\frac{1}{P_t^2}$

$$(dP_t)^2 = (rP_t dt + \sigma P_t dW_t)^2 = r^2P_t^2(dt)^2 + 2r\sigma P_t^2\,dt\,dW_t + \sigma^2P_t^2(dW_t)^2$$
$$= \sigma^2 P_t^2\,dt$$

$$d\ln P_t = \frac{1}{P_t}(rP_t dt + \sigma P_t dW_t) + \frac{1}{2}\left[\frac{-1}{P_t^2}\right](\sigma^2P_t^2 dt)$$

$$d\ln P_t = r\,dt + \sigma dW_t - \frac{1}{2}\sigma^2 dt$$

$$\int_0^t d\ln P_t = \ln P_t\Big|_0^t = \ln P_t - \ln P_0$$

$$\ln P_t - \ln P_0 = r\,dt - \frac{1}{2}\sigma^2 dt + \sigma dW_t = \left(r - \frac{1}{2}\sigma^2\right)dt + \sigma dW_t$$

$$\ln P_t = \ln P_0 + \left(r - \frac{1}{2}\sigma^2\right)dt + \sigma dW_t$$

$$P_t = P_0 \exp\left[\left(r - \tfrac{1}{2}\sigma^2\right)dt + \sigma dW_t\right] \Rightarrow \text{Ans}$$

* Read about Black-Scholes Model

Q Let $Y_t = e^{S_t}$ where $dS_t = \mu S_t dt + \sigma S_t dW_t$. Use Ito lemma to find dY_t.

$f(x) = Y_t = e^{S_t}$ $\quad f'(x) = e^{S_t}$ $\quad f''(x) = e^{S_t}$

$$d(Y_t) = f'(x)\,dY_t + \frac{1}{2}f''(x)(dY_t)^2$$

$$(dS_t)^2 = \sigma^2 S_t^2 (dW_t)^2 = \sigma^2 S_t^2\,dt$$

$$= e^{S_t}(\mu S_t dt + \sigma S_t dW_t) + \frac{1}{2}(e^{S_t})(\sigma^2 S_t^2 dt)$$

$$= \left(e^{S_t}\mu S_t + \tfrac{1}{2}e^{S_t}\sigma^2 S_t^2\right)dt + e^{S_t}\sigma S_t\,dW_t \Rightarrow \text{Ans}$$

[Wk 5]

Not tested in exam.

* learn how to derive Black-scholes model (call option & put option).

$$dS_t = \mu S_t dt + \sigma S_t dW_t$$

$$\ln S_t = \underbrace{\ln S_0 + (\mu - \tfrac{1}{2}\sigma^2)dt}_{\text{mean}} + \underbrace{\sigma dW_t}_{\text{std}}$$

$\text{mean} = \ln S_0 + (\mu - \tfrac{1}{2}\sigma^2)dt$

$\text{variance} = \sigma^2 (dW_t)^2 = \sigma^2 dt$

* $(dW_t)^2 = dt$

Black Scholes model:

Assumptions:

1) The stock prices follow a GBM
2) We deal with a frictionless market (no transaction cost)
3) It assumes no-arbitrage
4) It assumes there is no dividends
5) It assumes a constant volatility (σ).

* We are pricing derivatives!

- Call option: a financial contract that gives the right to buy but not obligation.
  - Payoff $= \max(S_t - k, 0)$

$$C_0 = S_0 N(d_1) - k e^{-rT} N(d_2)$$

where $d_1 = \dfrac{\ln\left(\frac{S_0}{k}\right) + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$

$$d_2 = d_1 - \sigma\sqrt{T}$$

→ value of ē call option = expected payoffs × discounted to today

discounting factor for discrete time $= \left(\dfrac{1}{(1+i)}\right)^n$ where i = interest

- e^{-rT} ⇒ discounting factor
- r ⇒ risk free rate
- S_0 ⇒ Stock price at $t=0$
- $N(d_2)$ ⇒ Prob. that under the risk neutral world $S_t > k$
- T ⇒ time at maturity.
- $N(d_1)$ ⇒ shows the sensitivity of the derivative when the underlying asset changes
  - e.g $N(d_1) = 40\%$ then a small change in stock price leads to a 40% increase in ē value of the derivative.

Q Given a call option, find the fair price of the call option today given:

current stock price = 100 $= S_0$
strike price = 110 $= k$
risk free rate = 5% $= r = 0.05$
volatility = 20% $= \sigma = 0.2$
time to maturity = 1 yr $= T$

$$d_1 = \frac{\ln\left(\frac{S_0}{k}\right) + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} = \frac{\ln\left(\frac{100}{110}\right) + \left(0.05 + \frac{0.2^2}{2}\right)1}{0.2\sqrt{1}}$$

$d_1 \approx -0.1266$ $\quad N(d_1) = N(-0.13) = 0.44828$

$d_2 = d_1 - \sigma\sqrt{T} = -0.1266 - 0.2\sqrt{1} = -0.3266$
$N(d_2) = N(-0.33) = 0.37070$

$$C_0 = S_0 N(d_1) - k e^{-rT} N(d_2)$$
$$= (100 \times 0.44828) - (110 \times e^{-0.05 \times 1} \times 0.37070)$$
$$= 6.04$$

$\Rightarrow$ 6.04 = initial premium you pay at the onset

$\Rightarrow N(d_1) \simeq 45\%$: An increase in stock price by one unit then the value of call option increases by 45%.

$\Rightarrow N(d_2) \simeq 37\%$: There is a 37% chance that stock price is greater than strike price. $(S_t > k)$

- Put option:

$$P_0 = k e^{-rT} N(-d_2) - S_0 N(-d_1)$$

* Premium: Expected payoff − premium = Actual expected payoff * only
  - Otherwise use expected payoff for other calculations NOT deducted one.

* Put option: formula different otherwise parameter meaning the same.