

No-Arbitrage, Discounting, and Martingale Principle

Ambachow Kahsay

10/18/2025

1 Introduction

In finance, understanding the relationship between **No-Arbitrage** and the **Martingale Principle** is fundamental. This document explains these concepts step by step, including numeric examples.

2 No-Arbitrage Principle

A market has **no arbitrage** if there is no way to make *risk-free profit*:

- **Arbitrage:** Making profit without risk and without investment.
- **No-Arbitrage:** Market is *fair*, prices adjust so that no guaranteed profit exists.

Example: - Gold price in Nairobi = \$100 - Gold price in Addis Ababa = \$110

Buying in Nairobi and selling in Addis gives a risk-free profit (\$10). Traders exploit this until prices adjust. After adjustment, no free profit exists → *fair market*.

3 Discounting

Future money is worth less today due to the **time value of money**. Discounting adjusts future prices to today's equivalent value.

$$\text{Discounted Price} = \frac{\text{Future Price}}{1 + r} \quad (\text{discrete time})$$

$$\text{Discounted Price} = \text{Future Price} \cdot e^{-rt} \quad (\text{continuous time})$$

Numeric Example: Risk-Free Discounting

Let's assume:

$$r = 0.1, \quad S_0 = 100$$

The future value of today's \$100 after one period:

$$\text{FV of today's } 100 = 100(1 + 0.1) = 110$$

This means that the **future value of the asset** is $S_1 = 110$, which is equivalent to today's \$100 in present value.

To ensure no arbitrage:

$$S_0 = E\left[\frac{S_1}{1+r}\right] \implies 100 = \frac{110}{1.1}$$

$$\text{Discounted FV (because of risk-free)} = \frac{110}{1.1} = 100$$

This shows that *the expected discounted future price equals today's price*, which is the essence of the martingale principle.

4 Martingale Principle

A stochastic process S_t is a **martingale** if:

$$E[S_{t+1} \mid \text{info up to time } t] = S_t$$

In finance, using discounted prices:

$$E\left[\frac{S_{t+1}}{1+r} \mid \text{info at time } t\right] = S_t$$

This means: after adjusting for interest, the *expected future price equals today's price*.

5 Relationship Between No-Arbitrage and Martingale

- **No-Arbitrage:** Market is fair, no free profit exists.
- **Martingale:** Discounted prices evolve so that the expected future value = current value.
- **Connection:** No-arbitrage guarantees that discounted prices follow a martingale.

6 Numeric Example with Probabilities

- Today $S_0 = 100$ - Tomorrow stock can go up to 120 or down to 80 - Risk-free rate $r = 0.1$
Find the *risk-neutral probability* p such that discounted expected price = today's price:

$$S_0 = \frac{E[S_1]}{1+r} = \frac{p \cdot 120 + (1-p) \cdot 80}{1.1}$$

$$100 \cdot 1.1 = 120p + 80(1-p)$$

$$110 = 120p + 80 - 80p = 40p + 80$$

$$p = \frac{30}{40} = 0.75$$

Check:

$$E\left[\frac{S_1}{1.1}\right] = \frac{0.75 \cdot 120 + 0.25 \cdot 80}{1.1} = \frac{100}{1} = 100 = S_0$$

Discounted price is a martingale. Market is fair. No-arbitrage condition is satisfied.

7 Summary

- **No-Arbitrage** = fair market = no risk-free profit.
- **Discounting** adjusts future prices to present value.
- **Martingale Principle** = expected discounted future price = current price.
- **Connection:** No-arbitrage \implies discounted prices are martingales.