

QUANTUM CHROMODYNAMICS

& COLLIDER PHENOMENOLOGY

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Outline

Ø The Beginnings

I Basics of QCD

II QCD in e^+e^- Collisions

- $e^-e^- \rightarrow$ hadrons • higher-order corrections
- infrared singularities & safety • the QCD emission pattern

III Hadron Colliders

IV Jet Physics

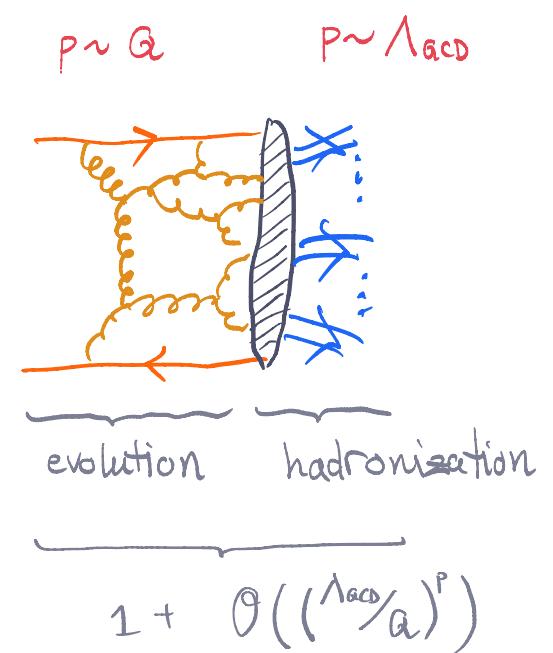
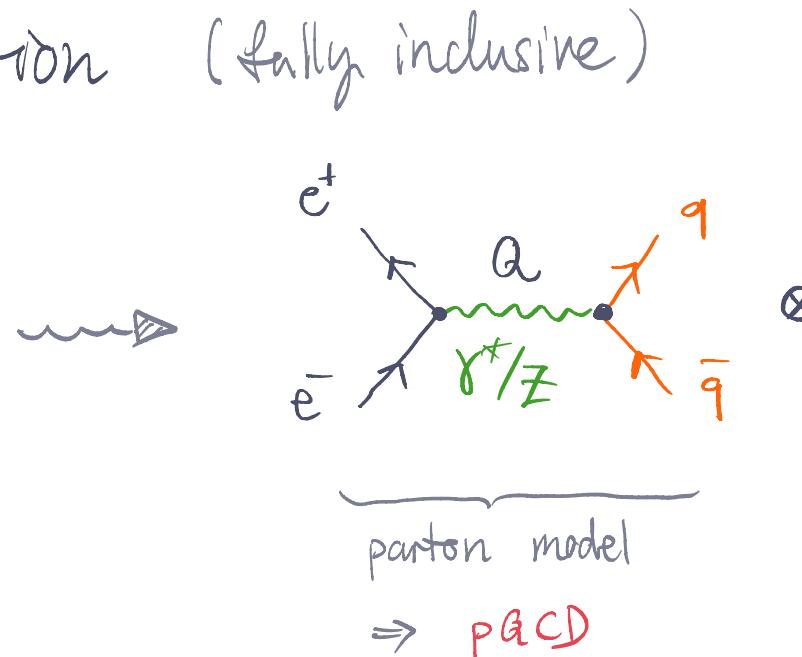
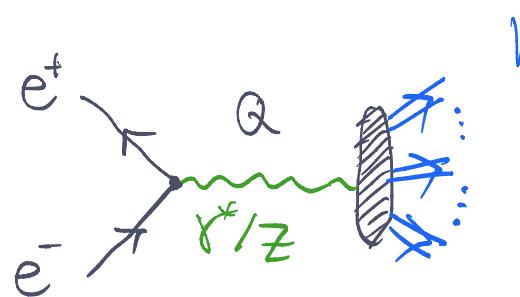
The Parton Model

* asymptotic freedom

↳ at large scales $\alpha \rightarrow$ hadrons \simeq collection of free partons

↳ if not interested in details of hadronic process \Rightarrow use parton picture
 ↑ "sufficiently inclusive"

* total hadronic cross section (fully inclusive)



$e^+ e^- \rightarrow \text{hadrons}$ (check "Extra" of previous lecture)

* for simplicity restrict to photon exchange ($\alpha < M_Z$)

* we're not interested in the angular orientation \rightarrow "average"

$$\sigma(e^+ e^- \rightarrow n \text{ partons}) = \frac{1}{2s} \frac{1}{4} L^{\mu\nu} \frac{1}{s^2} \int d\Omega_n W_{\mu\nu} (\gamma^* \rightarrow n \text{ partons})$$

flux spin leptonic tensor photon propagator hadronic tensor

must be $\sim (-g_{\mu\nu} + \frac{Q_\mu Q_\nu}{Q^2})$ why?

will mainly look at this

$$= \frac{e^2}{4s^2} \frac{2}{3} W_{n\text{-parton}}(Q^2)$$

with $W_{n\text{-parton}}(Q^2) = (-g^{\mu\nu}) \int d\Omega_n W_{\mu\nu} (\gamma^* \rightarrow n \text{ partons})$

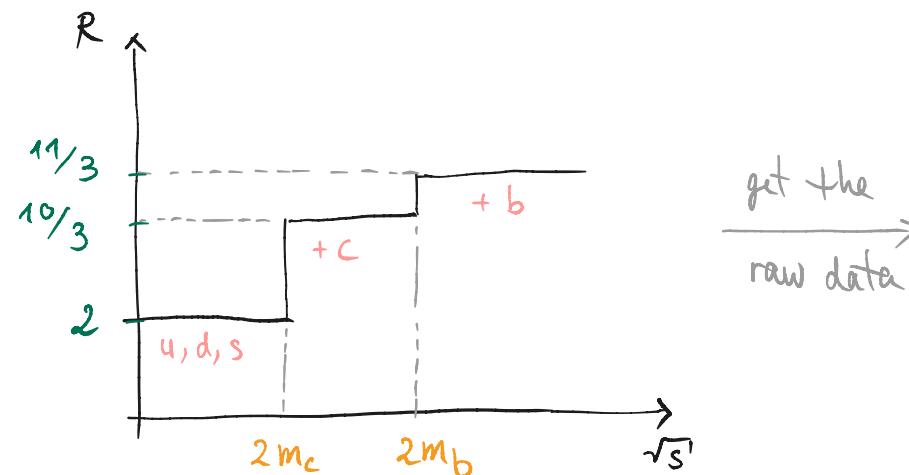
$e^+ e^- \rightarrow \text{hadrons} @ \text{LO}$

* For the LO process, we get $W_{q\bar{q}}^{(0)} = N_c \alpha Q_q^2 2S$

$$\Rightarrow \sigma(e^+ e^- \rightarrow q\bar{q}) = \frac{4}{3} \frac{\pi \alpha^2}{s} N_c Q_q^2$$

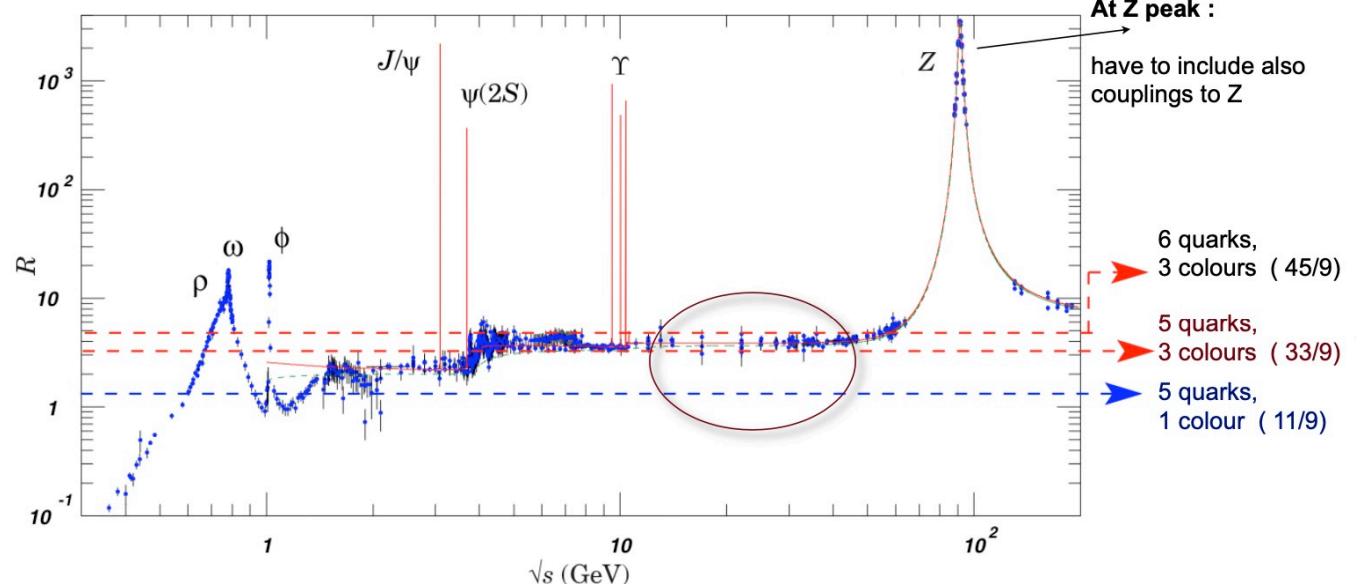
* The R ratio:

$$R = \frac{\sigma(e^+ e^- \rightarrow \text{hadrons})}{\sigma(e^+ e^- \rightarrow \mu^+ \mu^-)}$$



$$= N_c \sum_q Q_q^2 \Theta(\sqrt{s} - 2m_q)$$

production threshold for $q\bar{q}$ pair



<http://pdg.lbl.gov/2019/hadronic-xsections>

QCD at Higher Orders

leading order (LO) predictions in QCD
not sufficient for precision phenomenology

want to include
higher order(s) !
→ diagrams with loops

$$\mathcal{M}_2 = \text{LO diagram} + \text{NLO diagram} + \dots + \text{NNLO diagram} + \dots$$

$\mathcal{O}(\alpha) \leftrightarrow \mathcal{M}_2^{(0)}$ $\mathcal{O}(\alpha_s \alpha) \leftrightarrow \mathcal{M}_2^{(1)}$ $\mathcal{O}(\alpha_s^2 \alpha) \leftrightarrow \mathcal{M}_2^{(2)}$

$$\Rightarrow |\mathcal{M}_2|^2 = |\mathcal{M}_2^{(0)}|^2 + 2 \operatorname{Re} \{ (\mathcal{M}_2^{(0)})^* \mathcal{M}_2^{(1)} \} + |\mathcal{M}_2^{(1)}|^2 + 2 \operatorname{Re} \{ (\mathcal{M}_2^{(0)})^* \mathcal{M}_2^{(2)} \} + \dots$$

$\mathcal{O}(\alpha^2)$

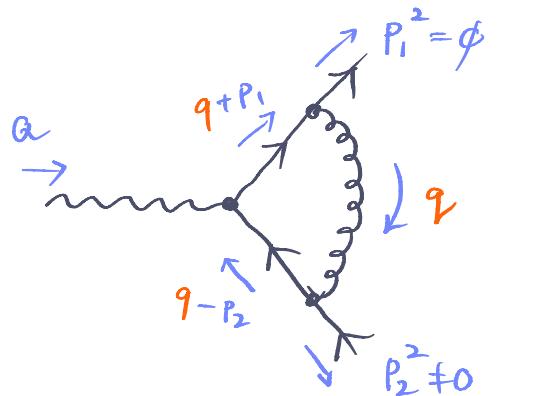
$\mathcal{O}(\alpha_s \alpha^2)$

"virtual corrections"

$\mathcal{O}(\alpha_s^2 \alpha^2)$

Virtual Corrections

- * we already saw how loop integrals can diverge \Rightarrow UV divergence $\frac{1}{\epsilon^n} \in O(\alpha_s^n)$
- * now we encounter a new region that causes us trouble



$$\sim \int \frac{d^4 q}{q^3 d q} \frac{\{1, q, q^2\}}{q^2 (q + p_1)^2 (q - p_2)^2} \underset{q \text{ small}}{\sim} \int \frac{dq}{q} \Rightarrow \log \text{divergence in } q \rightarrow 0$$

$\underbrace{q^2}_{q^2 + 2qp_1} \quad \underbrace{(q - p_2)^2}_{q^2 - 2p_2 q}$

- * These are called infrared (IR) divergences
- \hookrightarrow dimensional regularization also works here (now $\epsilon < 0$)

- * final result

$$\sigma^V = \sigma^{LO} \left\{ -\frac{\alpha_s}{2\pi} C_F \frac{1}{\Gamma(1-\epsilon)} \left(\frac{4\pi\mu^2}{-s-i0} \right)^\epsilon \left[\frac{2}{\epsilon^2} + \frac{3}{\epsilon} + 8 \right] \right\}$$

1-loop result has no UV divergences (why?)
no renormalization!
 \rightarrow what are we missing?

QCD at Higher Orders

Higher-order correction also must account for real emissions

$$\mathcal{M}_3 = \underbrace{\text{diagram with one gluon exchange}}_{\mathcal{O}(\sqrt{\alpha_s} \alpha) \leftrightarrow \mathcal{M}_3^{(0)}} + \text{diagram with two gluon exchanges} + \text{diagram with three gluon exchanges} + \dots$$

$$\mathcal{O}(\alpha_s^{3/2} \alpha) \leftrightarrow \mathcal{M}_3^{(1)}$$

$$\Rightarrow |\mathcal{M}_3|^2 = |\mathcal{M}_3^{(0)}|^2 + 2 \operatorname{Re} \{ (\mathcal{M}_3^{(0)})^* \mathcal{M}_3^{(1)} \} + \dots$$

$$\mathcal{O}(\alpha_s \alpha^2)$$

$$\mathcal{O}(\alpha_s^2 \alpha^2)$$

* at NNLO: also need $\mathcal{M}_4^{(0)}, \mathcal{M}_3^{(1)}, \mathcal{M}_2^{(2)}$

* at N³LO: also need $\mathcal{M}_5^{(0)}, \mathcal{M}_4^{(1)}, \mathcal{M}_3^{(2)}, \mathcal{M}_2^{(3)}$

...

Real Corrections

$$|\mathcal{M}_{q\bar{q}g}^{(3)}|^2 \sim \left| \begin{array}{c} \xrightarrow{Q} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \end{array} \right|^2$$

$$\Rightarrow W_{q\bar{q}g}(Q^2) = \int d\Phi_3 \ e^2 Q^2 g_s^2 8 C_F N_c \frac{s_{13}^2 + s_{23}^2 + 2 s_{12} Q^2}{s_{13} s_{23}}, \quad s_{ij} = 2 p_i \cdot p_j$$

* kinematic variables for real emission

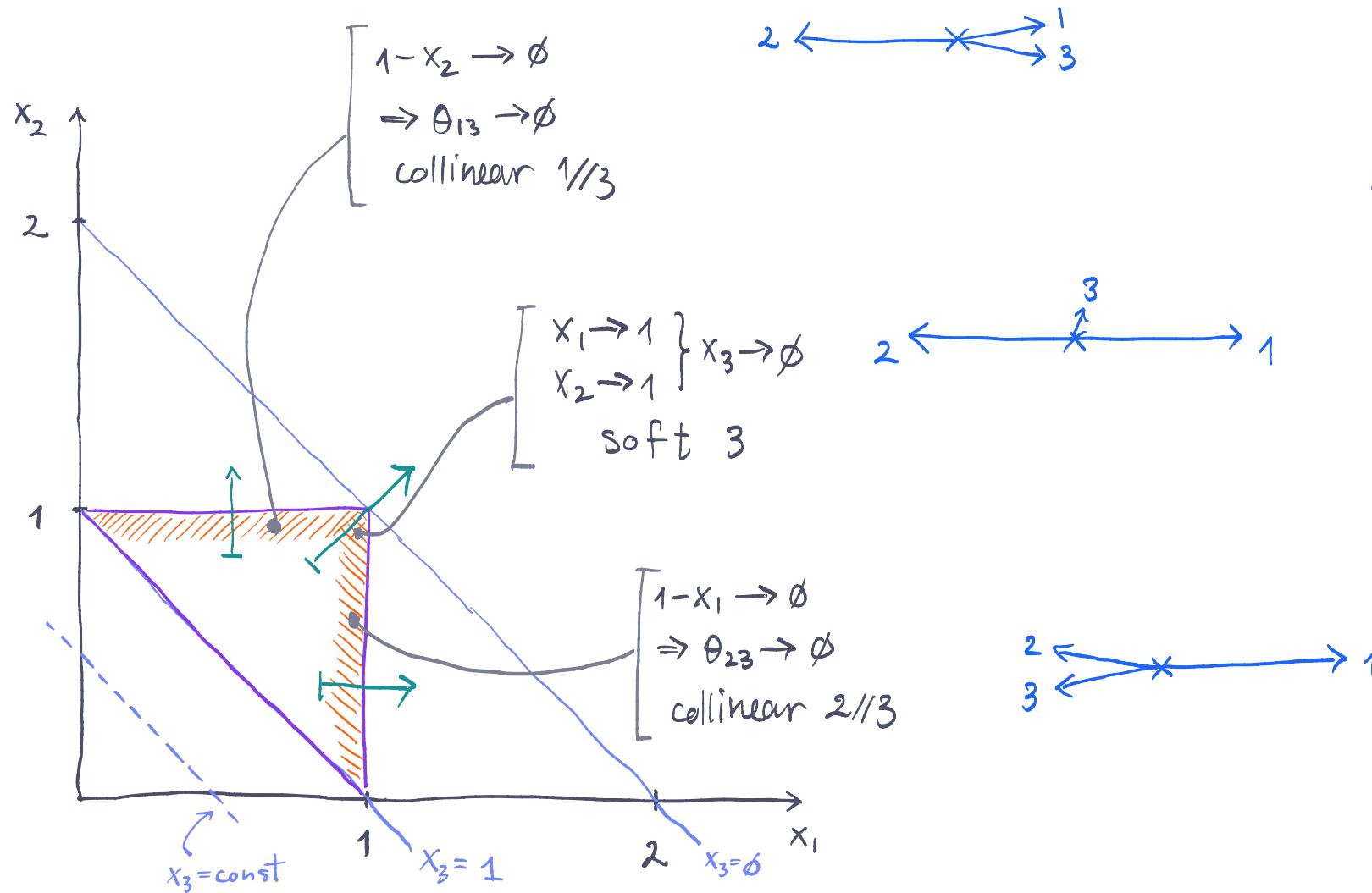
↳ energy fraction: $x_i = \frac{2 p_i \cdot Q}{Q^2}$ c.m. frame $\rightsquigarrow E_i = \frac{g_s}{2} x_i \quad (x_i > 0)$

↳ energy conservation: $x_1 + x_2 + x_3 = \frac{2(\sum p_i) \cdot Q}{Q^2} = 2$

↳ $d\Phi_3(p_1, p_2, p_3; Q) = \frac{s}{2(4\pi)^3} dx_1 dx_2 dx_3 \delta(2 - x_1 - x_2 - x_3)$

The Real Phase Space

* the energy fractions x_i lie within a triangle



angles:

$$2P_1P_3 = (P_1+P_3)^2 = (Q-P_2)^2 = Q^2 - 2P_2Q$$

"

$$2E_1E_3 (1-\cos\theta_{13}) \longleftrightarrow Q^2(1-x_2)$$

Real Corrections

- * using the energy fractions, we get

$$W_{q\bar{q}g}(Q^2) = \frac{s}{2(4\pi)^3} \int_0^1 dx_1 \int_0^1 dx_2 \int_0^1 dx_3 \delta(2-x_1-x_2-x_3) e^2 Q_q^2 g_s^2 8 C_F N_c \frac{x_1^2 + x_2^2}{(1-x_1)(1-x_2)}$$

- * even though no loop integrals, need to regularize

↳ here the divergences appear in $d\mathbb{E}_n$

↳ dimensional regularization : $\frac{d^3 p}{(2\pi)^3 2E} \rightarrow \frac{d^{D-1} p}{(2\pi)^{D-1} 2E}$

$$\delta^{(4)}(p_{in}-p_{out}) \rightarrow \delta^{(D)}(p_{in}-p_{out})$$

- * repeating the entire calculation in $D=4-2\epsilon$ dimensions

$$\sigma^R = \sigma^{LO} \cdot \frac{\alpha_s}{2\pi} C_F \frac{\Gamma^2(1-\epsilon)}{\Gamma(1-3\epsilon)} \left(\frac{4\pi\mu^2}{s}\right)^\epsilon \left[\frac{2}{\epsilon^2} + \frac{3}{\epsilon} + \frac{19}{2} \right]$$

this is terrible!
diverges for $x_{1,2} \rightarrow 1$

$e^+e^- \rightarrow \text{Hadrons} @ \text{NLO}$

* now we finally have all pieces in place

$$\sigma^{\text{NLO}} = \sigma^{\text{LO}} \times \left\{ 1 + \frac{\alpha_s}{2\pi} C_F \frac{\Gamma^2(1-\epsilon)}{\Gamma(1-3\epsilon)} \left(\frac{4\pi\mu^2}{s}\right)^\epsilon \left[\frac{2}{\epsilon^2} + \frac{3}{\epsilon} + \frac{19}{8} \right] \right.$$

expand & simplify

$$- \frac{\alpha_s}{2\pi} C_F \frac{1}{\Gamma(1-\epsilon)} \left(\frac{4\pi\mu^2}{-s-i0}\right)^\epsilon \left[\frac{2}{\epsilon^2} + \frac{3}{\epsilon} + 8 \right] + \mathcal{O}(\epsilon) \left. \right\}$$

↑ ↑ ↓
all divergences cancel

$$= \sigma^{\text{LO}} \left\{ 1 + \frac{\alpha_s}{\pi} \right\}$$

too nice to be just
an accident
→ what is the mechanism?

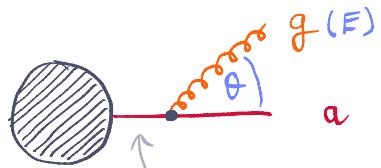
* the Ratio at the Z pole
(including Z-exchange diagram)

$$R^{\text{LO}} = 20.09 \rightarrow R^{\text{NLO}} = 20.84$$

$$R^{\text{exp}} = 20.79 \pm 0.04 \quad (\text{from LEP})$$

The Pattern of IR Singularities

- * IR singularities arise from intermediate propagators going on-shell



$$\sim \frac{1}{(P_a + k)^2} = \frac{1}{2 P_a \cdot k} = \frac{1}{E E_a (1 - \cos \theta)}$$

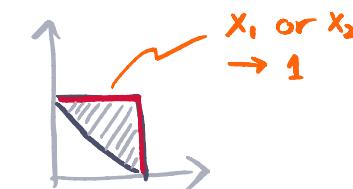
potentially divergent

- soft $E \rightarrow 0$
- collinear $\theta \rightarrow 0$

↳ these correspond to degenerate configurations

→ divergence in $2 \rightarrow 3$ was located @ boundary

→ has correct $2 \rightarrow 2$ kinematics to match the virtuals

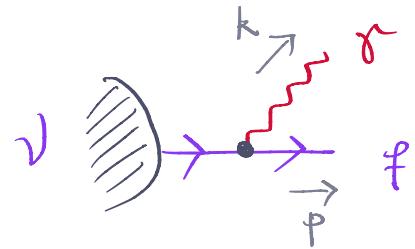


- * the IR singularities **factorize** \leftrightarrow are universal

↳ soft emissions have long wave length \rightarrow cannot resolve hard "blob"

↳ collinear emissions only care about the parent it was emitted from

The Soft Limit in QED



$$= \left[\bar{u}(p) (-ieQ_f \gamma^\mu) \frac{i(p+k)}{(p+k)^2} V(p+k) \right] \cdot \epsilon_\mu^*(k)$$

* k soft $\Rightarrow k \ll p$ & neglect

$$(p+k)^2 = 2p \cdot k$$

$$= \frac{eQ_f}{2p \cdot k} [\bar{u}(p) \gamma^\mu \not{p} V(p)] \epsilon_\mu^*(k)$$

$$\underbrace{-\not{p} \gamma^\mu}_{\cancel{\not{p}}} + 2p^\mu$$

[Dirac algebra $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$]

[Dirac eqn: $\bar{u}(p)\not{p} = 0$]

$$= \frac{eQ_f}{2p \cdot k} 2p^\mu \epsilon_\mu^* [\bar{u}(p) V(p)]$$

the original
before γ -emit.

* soft limit: emission from all legs

$$M_{n+p} \xrightarrow[\text{soft}]{\gamma} \epsilon_\mu^* e J^\mu \cdot M_n \quad \text{with} \quad J^\mu = \sum_{i=1}^n Q_i \frac{p_i^\mu}{(p_i \cdot k)}$$

eikonal current
• indep of spin
• indep of E_i
(only directions)

Factorization of IR Divergences in QCD

* soft limit:

$$M_{n+1}(p_1, \dots, p_n, g(k)) \xrightarrow{k \rightarrow 0} \epsilon_\mu^a(k)^* g_s \hat{J}^{a,\mu} \otimes M_n(p_1, \dots, p_n)$$

gluon current

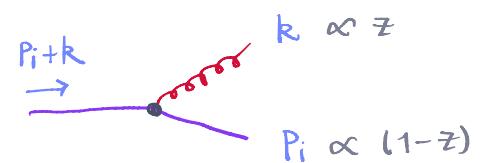
$$\hat{J}^{a,M} = \sum_{i=1}^n \frac{p_i^\mu}{(p_i \cdot k)} \hat{T}_i^a$$

$$\hat{T}_i^a = \begin{cases} t^a & q(\text{in}), \bar{q}(\text{out}) \\ -t^a & q(\text{out}), \bar{q}(\text{in}) \\ T_A^a & g \end{cases}$$

* collinear limit

$$|M_{n+1}(p_1, \dots, p_n, g(k))|^2 \xrightarrow{k \parallel p_i} \frac{g_s^2}{(p_i \cdot k)} P_{gg}(z) |M_n(p_1, \dots, (p_i+k), \dots, p_n)|^2$$

splitting function



* beyond NLO: many more

$$P_{gg}(z) = C_F \frac{1+(1-z)^2}{z}$$

$$P_{gg}(z) = 2 C_A \left[\frac{1-z}{z} + \frac{z}{1-z} + z(1-z) \right]$$

QCD Emission Pattern in the Soft-Collinear Limit

A very useful limit: gluon emission both soft & collinear

↳ start with the collinear & take the gluon soft $z \rightarrow 0$

exercise:
take the soft,
square it & take
a collinear limit
 $\sum_{i,j} \hat{T}_i = -\hat{T}_j$

(colour conservation)

$$\hat{T}_q^2 = C_F, \hat{T}_g^2 = C_A$$

$$|M_{n+g}|^2 \xrightarrow{g \parallel i} \frac{g_s^2}{(p_i \cdot k)} P_{gi}(z) |M_n|^2$$

$\xrightarrow[z \rightarrow 0]$

$$\frac{2 C_x}{z}$$

$$, C_x = \begin{cases} C_F, \bar{i} = q \\ C_A, \bar{i} = g \end{cases}$$

gluons emit almost
twice as much

↳ Also the phase space factorizes

$$[\delta(\sum_i p_i + k - Q) \xrightarrow[\text{soft}]{k} \delta(\sum_i p_i - Q)]$$

$$d\Phi_{n+g} = d\Phi_n \frac{d^3 k}{(2\pi)^3 2E} ;$$

parametrize k w.r.t. p_i

- $d^3 k = E^2 dE d\cos\theta d\varphi$

- $(p_i \cdot k) = E_i E (1 - \cos\theta)$

QCD Emission Pattern in the Soft-Collinear Limit

combining both expressions, the cross section reads

$$d\sigma_{n+g} \sim |M_{n+g}|^2 d\Phi_{n+g} \xrightarrow[\text{collinear}]{\text{soft \&}} d\sigma_n \times d\omega_{i \rightarrow i+g}$$

$$d\omega_{i \rightarrow i+g} = \frac{g_s^2}{E_i E (1 - \cos\theta)} \frac{2}{Z} C_x \frac{1}{(2\pi)^3 2\pi} E^2 dE \cos\theta d\varphi ,$$

$$= 2 \frac{\alpha_s}{\pi} C_x \frac{dE}{E} \frac{d\theta}{\theta} \frac{d\varphi}{2\pi} \xrightarrow[\rightarrow 1]{\text{after azimuthal average}}$$

* two logarithmic divergences

$$\begin{array}{ll} \hookrightarrow \text{soft} & \frac{dz}{z} = \frac{dE}{E} \\ \hookrightarrow \text{collinear} & \frac{d\theta}{\theta} = \frac{1}{2} \frac{d\theta^2}{\theta^2} \end{array}$$

} "double logs": for every α_s , up to two logs from the IR

$$\cos\theta \simeq 1 + \frac{\theta^2}{2} + \dots$$

$$\frac{d\cos\theta}{1 - \cos\theta} \sim 2 \frac{d\theta}{\theta}$$

QCD Emission Pattern in the Soft-Collinear Limit

- * we derived the QCD emission probability

$$d\omega_{i \rightarrow i+g} = 2 \frac{\alpha_s}{\pi} C_x \frac{dE}{E} \frac{d\theta}{\theta}$$

from factorization \Rightarrow allows us to understand general QCD concepts

- * alternatively, we can also use the transverse momentum $k_T \simeq E\theta$ and the pseudorapidity $\eta = -\ln [\tan \frac{\theta}{2}]$ to obtain

$$d\omega_{i \rightarrow i+g} = 2 \frac{\alpha_s}{\pi} C_x \frac{dk_T}{k_T} dy \quad \leftarrow \text{try to derive this}$$

R where is the double log?

Infrared Safety

- * pattern of cancellations dictated by the Kinoshita-Lee-Nanenberg theorem
 - ↳ for sufficiently inclusive observables \Leftrightarrow IR safe observable
- * in general, measurements involve non-trivial fiducial constraints

$$\sigma \sim \int |M_n|^2 d\Phi_n \rightsquigarrow \int |M_n|^2 F_n(p_1, \dots, p_n) d\Phi_n$$

measurement function

- fiducial cuts • jet algorithm
- histogram bins • isolation ...

* an IR safe observable satisfies

↳ soft safety:

$$F_{n+1}(p_1, \dots, p_{n+1}) \xrightarrow{p_i \rightarrow \phi} F_n(p_1, \dots, \cancel{p_i}, \dots, p_{n+1})$$

$$\frac{d\sigma}{d\phi} = \int d\sigma \delta(\phi - \hat{\theta}(\vec{p}))$$

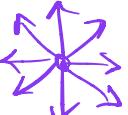
↳ collinear safety

$$F_{n+1}(p_1, \dots, p_{n+1}) \xrightarrow{p_i \parallel p_j} F_n(p_1, \dots, \cancel{p_i}, \dots, \cancel{p_j}, \dots, p_{n+1}, (p_i + p_j))$$

Infrared Safety

* thrust (event shape)

$$T = \max_{\vec{n}} \frac{\sum_i |\vec{P}_i \cdot \vec{n}|}{\sum_j |\vec{P}_j|}$$

- {
- pencil like  $\Rightarrow T \rightarrow 1$
 - mercedes star  $\Rightarrow T \rightarrow \frac{2}{3}$
 - isotropic  $\Rightarrow T \rightarrow \frac{1}{2}$

↳ soft safety: add one soft particle P_{nti} ($P_{nti} \rightarrow \emptyset$)

$$T \xrightarrow{n+1} T' = \max_{\vec{n}} \frac{\sum_{i=1}^n |\vec{P}_i \cdot \vec{n}| + |\vec{P}_{nti} \cdot \vec{n}|}{\sum_{j=1}^n |\vec{P}_j| + |\vec{P}_{nti}|}$$

→ T ✓

Infrared Safety

* thrust (event shape)

$$T = \max_{\vec{n}} \frac{\sum_i |\vec{P}_i \cdot \vec{n}|}{\sum_j |\vec{P}_j|}$$

- \rightsquigarrow
- pencil like  $\Rightarrow T \rightarrow 1$
 - mercedes star  $\Rightarrow T \rightarrow \frac{2}{3}$
 - isotropic  $\Rightarrow T \rightarrow \frac{1}{2}$

\hookrightarrow collinear safety: w.l.o.g split up $P_n^m = \underbrace{z P_n^m}_{P_n^m} + \underbrace{(1-z) P_n^m}_{P_{n+1}^m}$

$$T \xrightarrow{n+1} T' = \max_{\vec{n}} \frac{\sum_{i=1}^{n-1} |\vec{P}_i \cdot \vec{n}| + |\vec{P}_n^m \cdot \vec{n}| + |\vec{P}_{n+1}^m \cdot \vec{n}|}{\sum_{j=1}^{n-1} |\vec{P}_j| + |\vec{P}_n^m| + |\vec{P}_{n+1}^m|} = \frac{\sum_{i=1}^{n-1} |\vec{P}_i \cdot \vec{n}| + \overbrace{z |\vec{P}_n \cdot \vec{n}| + (1-z) |\vec{P}_{n+1} \cdot \vec{n}|}^{|\vec{P}_n \cdot \vec{n}|}}{\sum_{j=1}^{n-1} |\vec{P}_j| + z |\vec{P}_n| + (1-z) |\vec{P}_{n+1}|}$$

$\rightarrow T \quad \checkmark$

Infrared Safety

Which one of these observables are IR safe?

- * number of final-state particles
- * $\sum_{i=1}^n p_{T,i}$
- * $\sum_{i=1}^n (p_{T,i})^2$
- * jets
- * sphericity $S^{\alpha\beta} = \frac{\sum_{i=1}^n p_i^\alpha p_i^\beta}{\sum_{j=1}^n |\vec{p}_j|}$
- * spherocity $S = \left(\frac{4}{\pi}\right)^2 \max_n \left[\frac{\sum_{i=1}^n |\vec{p}_i \times \vec{n}|}{\sum_{j=1}^n |\vec{p}_j|} \right]^2$

Extra

[tɔɪ-nlo]