Quantum Chromodynamics & Collider Phenomenology

Alexander Huss
alexander.huss@cern.ch

2025 CERN Latin-American School of High-Energy Physics

Quantum Chromodynamics – QCD

* what? theory of strong interactions: quarks & gluons

↳ non-Abelian gauge theory $SU(N_c)$

($N_c = 3$; colour d.o.f)

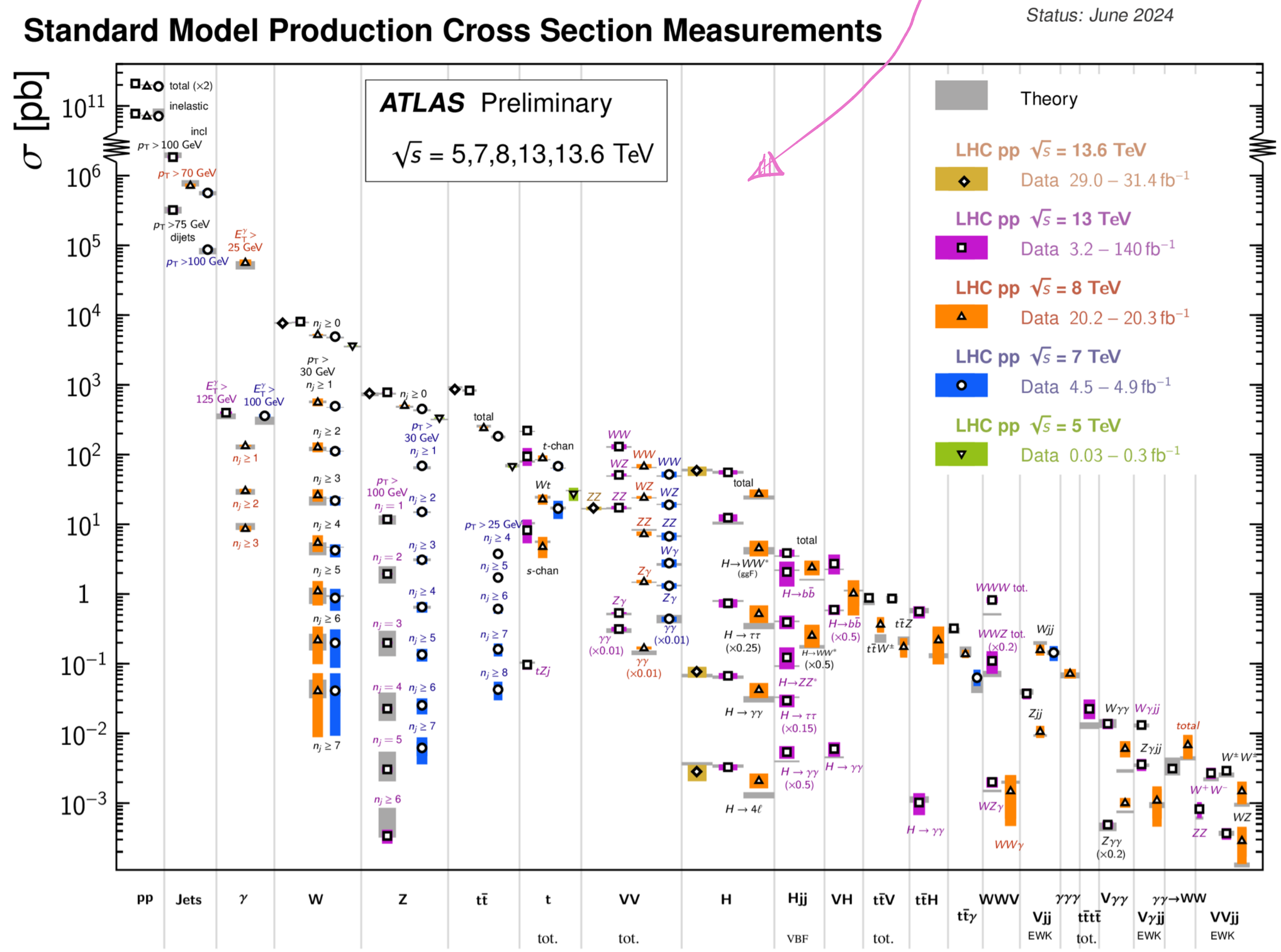
* why? unavoidable in hadron collisions

(LHC collides protons $\simeq$ QCD bound states)

① measure properties & scrutenize the KNOWN

② enhance the discovery reach of the UNKNOWN

"stairway to heaven" plot



> Quantum chromodynamics is conceptually simple.

Frank Wilczek

$$\mathcal{L}_{QCD} = \sum_q \bar{\psi}_q (i \not{D}_\mu - m_q) \psi_q - \frac{1}{4} F^a_{\mu\nu} F^{a,\mu\nu}$$

with

$$D_\mu = \partial_\mu + i g_s A^a_\mu t^a \quad \text{and} \quad F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g_s f^{abc} A^b_\mu A^c_\nu$$

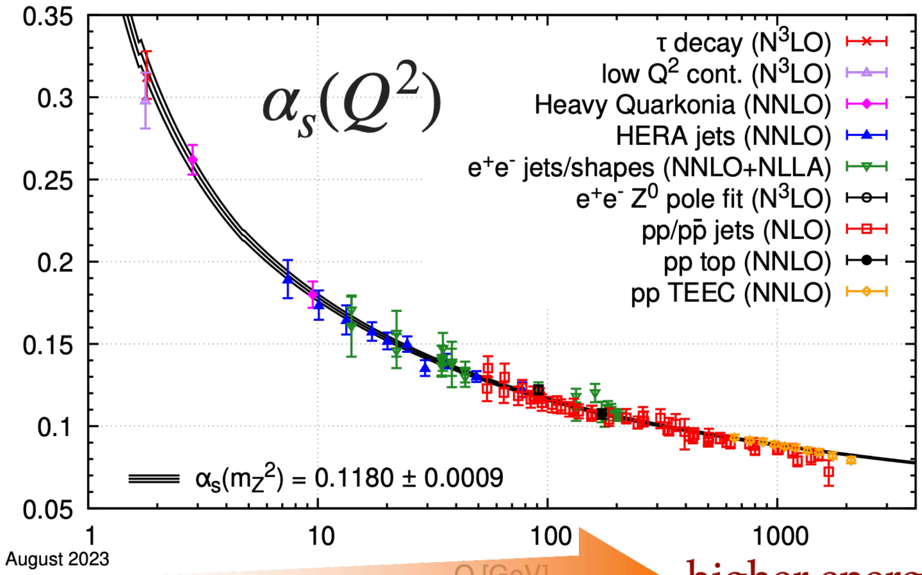
that's it!

"Quantum chromodynamics is conceptually simple. Its realization in nature, however, is usually very complex."

Frank Wilczek

strong interaction ⟷ 

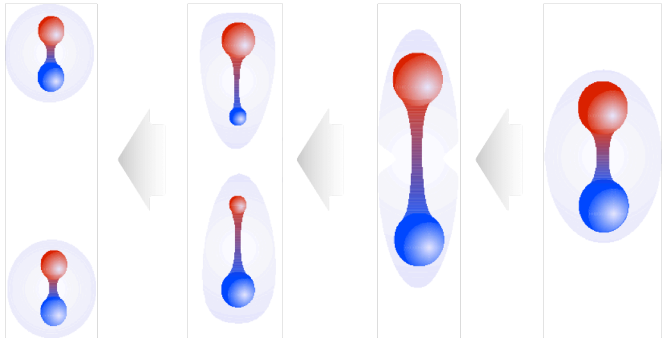
- Quantum Chromodynamics (QCD)



higher energy

larger distance

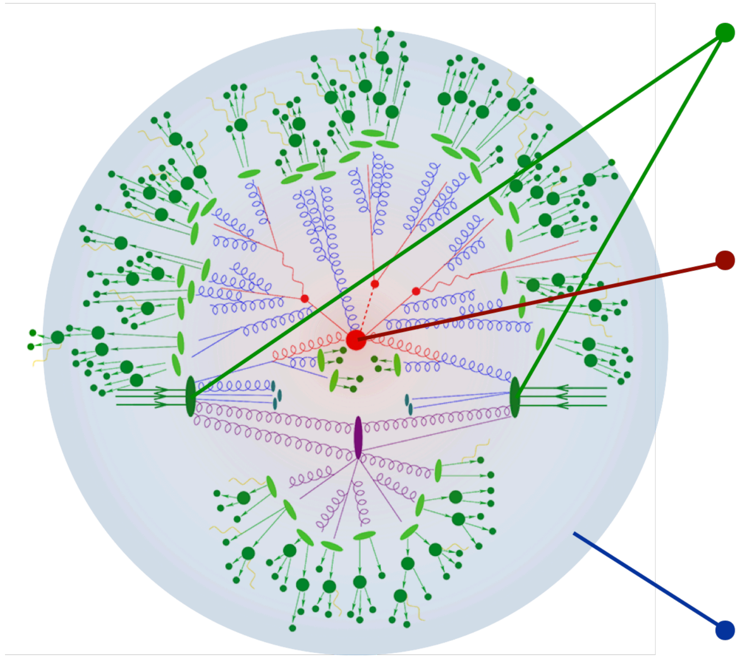
confinement



asymptotic freedom

* no free quarks & gluons
  → spray of hadrons ($\pi^\pm, K^\pm, K^0, p^\pm, n, \dots$)
* colliding objects (p @ LHC) not elementary

> Quantum chromodynamics is conceptually simple. Its realization in nature, however, is usually very complex. But not always.
>
> Frank Wilczek



① factorization

→ relevant physics at disparate scales
(isolates description of proton from rest)

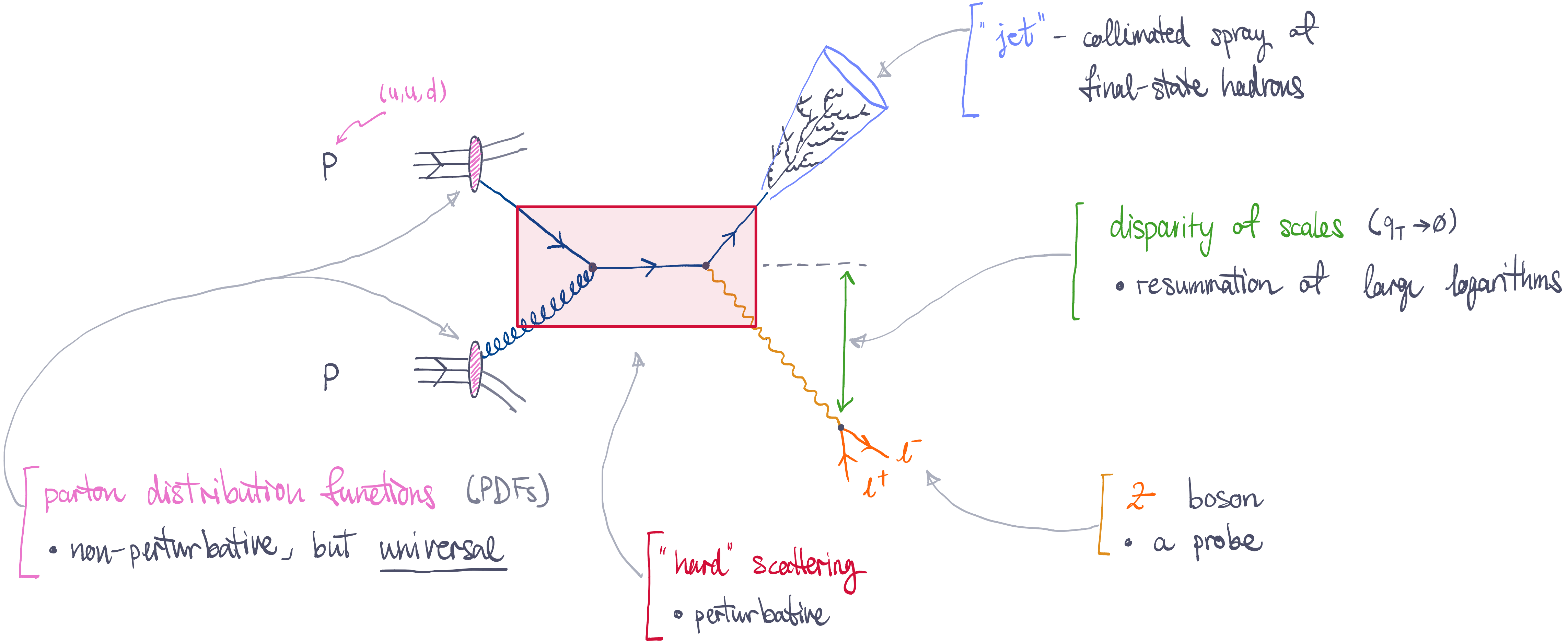
② asymptotic freedom

→ short distance ⟷ perturbation theory

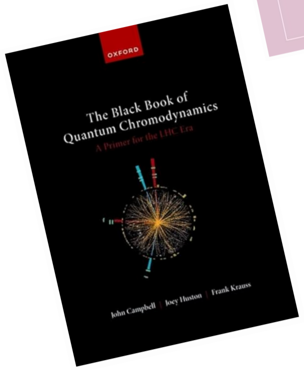
$$\sigma = \sigma_{LO} \left(1 + \alpha_s c^{(1)} + \alpha_s^2 c^{(2)} + \dots \right)$$

Goal of these lectures

* understand, how this picture comes about
* get a feeling of how calculations are done within each part



Some Literature

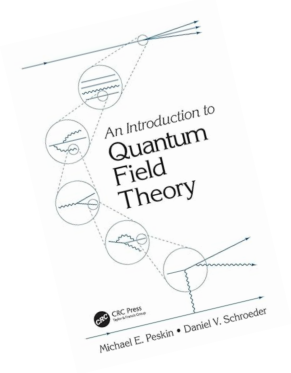


* Quantum Chromodynamics

↪ Ellis, Stirling, Webber aka "The pink book"
"QCD and Collider Physics"

↪ Campbell, Huston, Krauss
"The black book of Quantum Chromodynamics"

available as free download thanks to SCOAP3 foundation



* Quantum Field Theory

↪ Peskin, Schroeder
"An Introduction to Quantum Field Theory"

Conventions & Sources

* Conventions natural units: $[\hbar] = [c] = 1$

↪ $[\text{length}] = [\text{time}] = \text{eV}^{-1}$

$[\text{mass}] = [\text{energy}] = [\text{momentum}] = \text{eV}$

↪ four vectors $x^\mu = (t, x, y, z)^T$, $\partial_\mu = (\partial_t, \vec{\nabla})^T$

$\Rightarrow$ energy-momentum conservation: $(a+b \to 1+2)$

$$\delta^{(4)}(p_1+p_2-(p_a+p_b)) = \delta(E_1+E_2-(E_a+E_b))\,\delta^{(3)}(\vec{p}_1+\vec{p}_2-(\vec{p}_a+\vec{p}_a))$$

↪ scalar product $a \cdot b = a^\mu b_\mu = a^\mu g_{\mu\nu} b^\nu = a^0 b^0 - \vec{a} \cdot \vec{b}$

$$g^{\mu\nu} = \text{diag}(+1, -1, -1, -1)$$

* Notebooks https://github.com/aykhuss/Lectures-CERN-QCD/

Outline

Ø **The Beginnings**

- the eightfold way
- the quark model
- colour quantum number
- confinement

I **Basics of QCD**

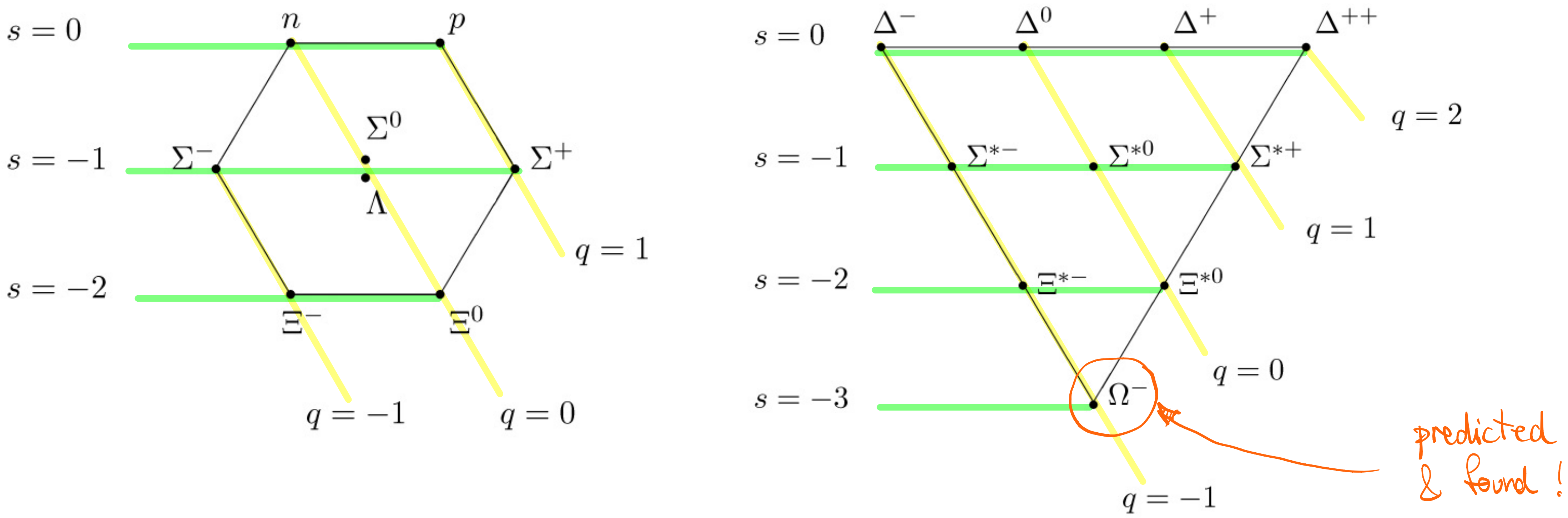
II **QCD in e^+e^- Collisions**

III **Hadron Colliders**

IV **Jet Physics**

The Eightfold Way [Gell-Mann '61]

* the hadron spectrum exhibits a striking pattern (s = "strangeness")



$\hookrightarrow$ What is the reason for this pattern?

The Quark Model [Gell-Mann, Zweig '64]

* proposal: spin-$\frac{1}{2}$ constituents

(quarks: $\boxed{u, d, s}$) SU(3) flavour symmetry

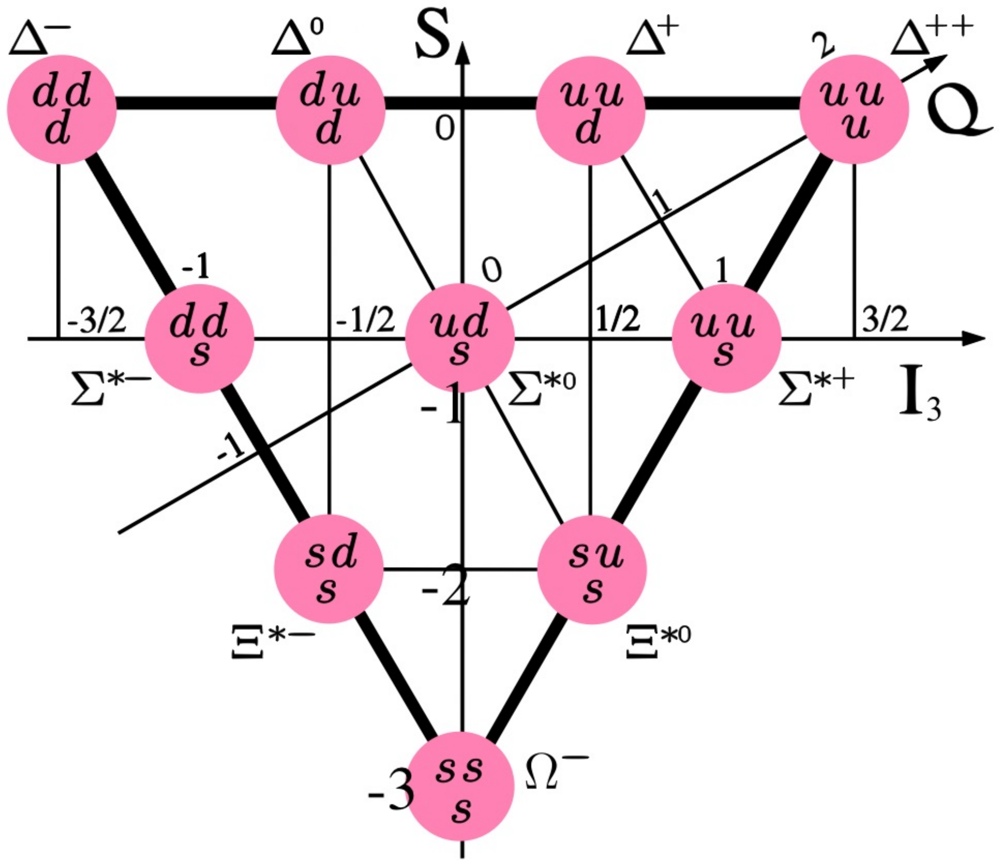
with fractal charges

u $m_u \sim 4$ MeV $(Q = +2/3)$

proton stable

d $m_d \sim 7$ MeV $(Q = -1/3)$

s $m_s \sim 135$ MeV $(Q = -1/3)$



⇒ "Explains" the pattern

but no free quarks can be seen!

The Spin-Statistics Issue

* Δ^{++} is a state with

  - spin $\frac{3}{2}$ $\qquad |\uparrow\uparrow\uparrow\rangle$

  - $3 \times$ up $\qquad |uuu\rangle \qquad (Q = +2)$

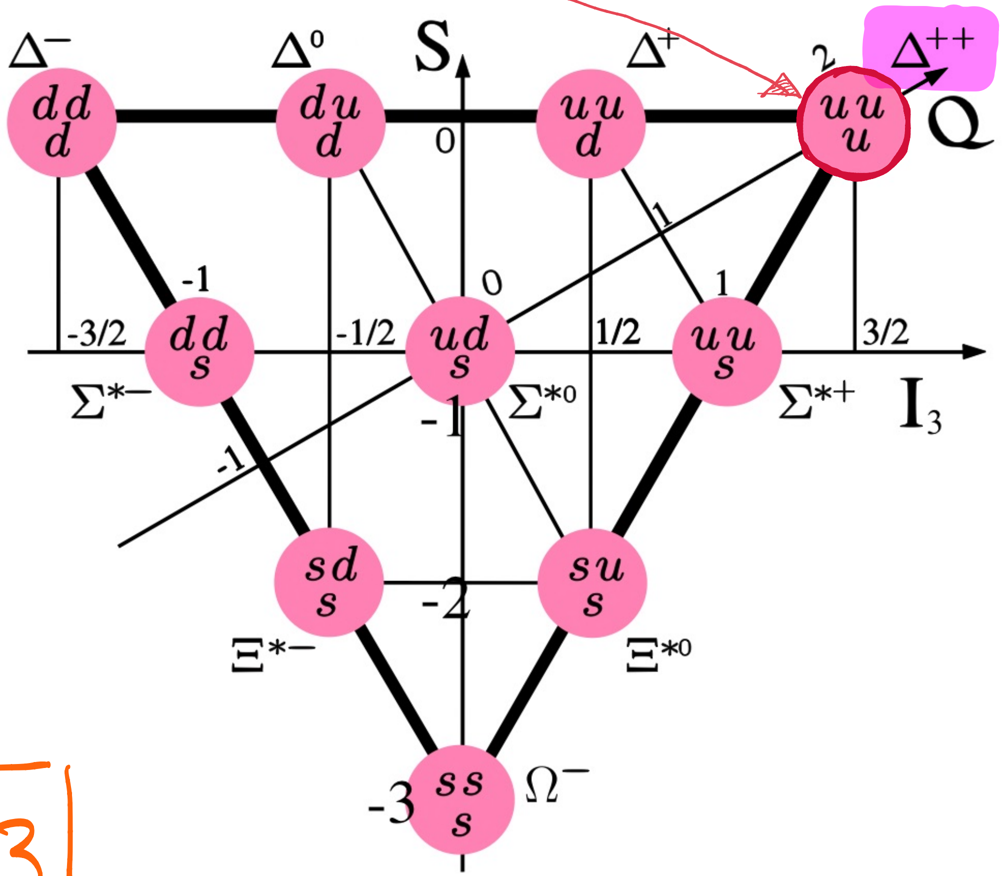
$\lightning$ Pauli's exclusion principle

$\Rightarrow$ Way out: A new quantum number

$u^1 \quad u^2 \quad u^3 \quad ?$ $\Big\}$ COLOUR

$\Delta^{++} \sim \epsilon^{ijk} \; u^i \; u^j \; u^k$

$\Leftrightarrow$ fully anti-symmetric

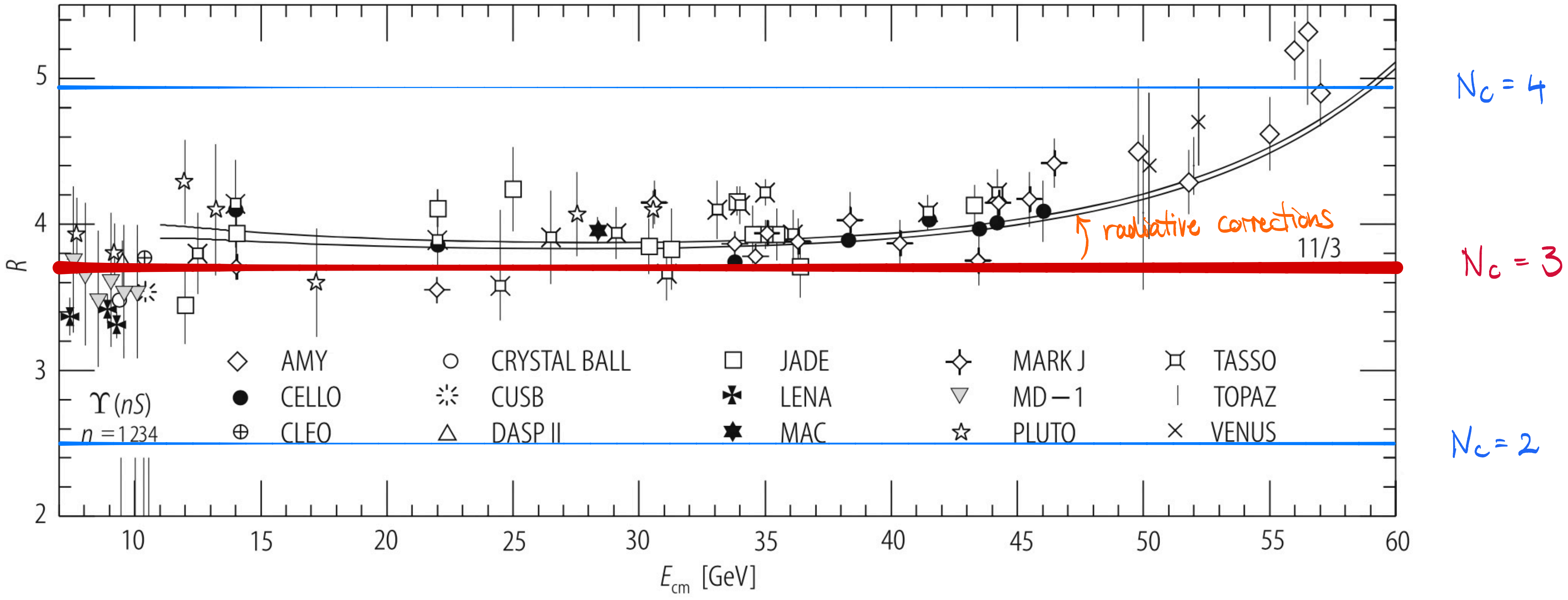


$\boxed{N_c \geq 3}$

Evidence for Colour

* The R-ratio $\quad R \equiv \dfrac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)} \propto N_c \sum_q Q_q^2$

$\underbrace{\text{lowest-order prediction}}$ $\quad R = N_c \, \dfrac{11}{9}$



* pion decay $\quad \Gamma(\pi^0 \to \gamma\gamma) \propto N_c^2$
* ABJ anomaly cancellation
* much much more (BR of W, τ decays ...)

$$\boxed{N_c = 3}$$

QCD & Colour Confinement

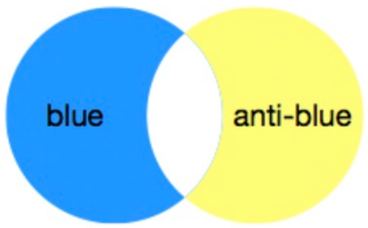
* QCD has an exact $SU(N_c)$ symmetry ← why not $SO(N_c)$?

$$UU^\dagger = U^\dagger U = \mathbb{1}\,, \quad \det(U) = 1$$

* no isolated colour charges $\left(V_{q\bar{q}}(r) \simeq C_F \left[\frac{\alpha_s(r)}{r} + \ldots + \sigma r \right] \right)$

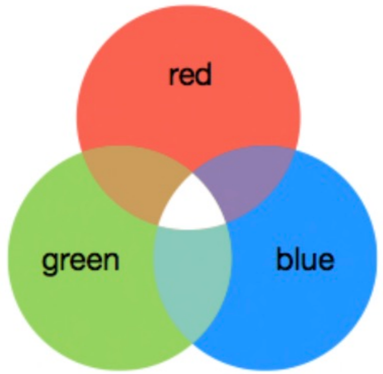
$\Rightarrow$ only colour singlet particles $\rightsquigarrow$ hadrons have integer electric charge

① Mesons (bosons: $\pi, \rho, \ldots$)



$$\bar{q}^i q^i \longrightarrow \underbrace{\underbrace{U^*_{ij}}\,\bar{q}^j\,\underbrace{U_{ik}}}_{(U^\dagger)_{ji}\,U_{ik} = \delta_{jk}}\,q^k = \bar{q}^i q^i$$

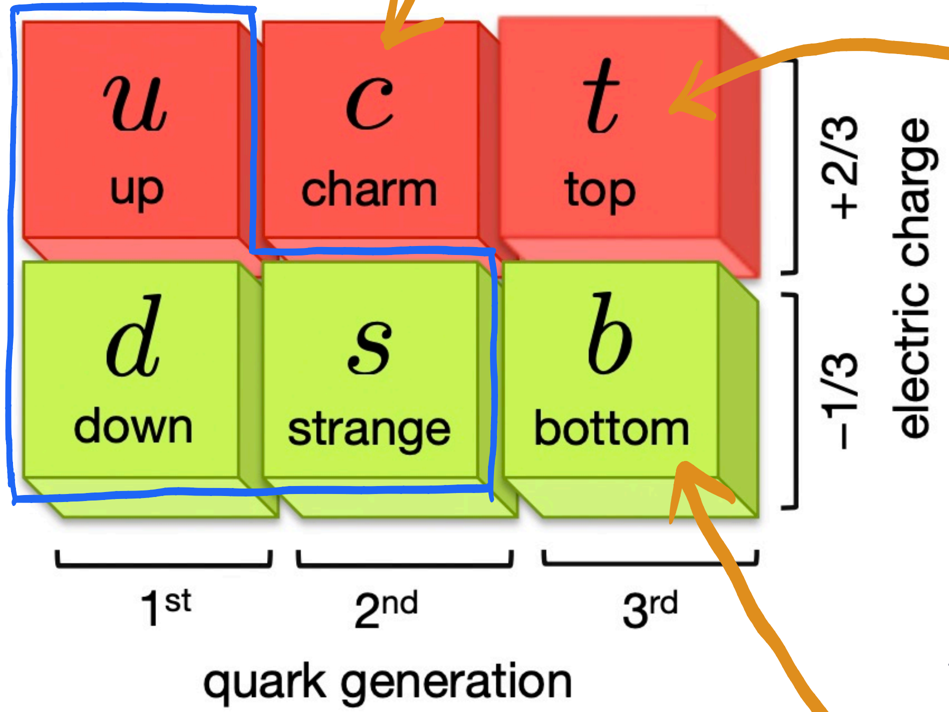
② Baryons (fermions: $p, n, \ldots$)



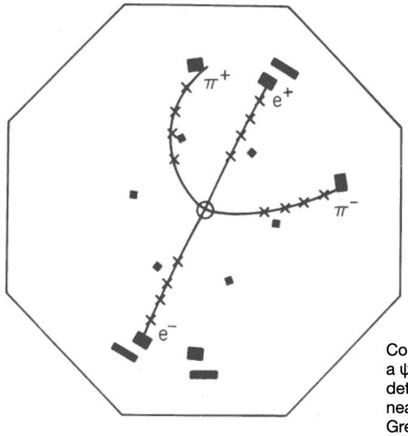
$$\epsilon_{ijk}\, q^i q^j q^k \longrightarrow \underbrace{\epsilon_{ijk}\, U_{ii'} U_{jj'} U_{kk'}}_{\det(U)\,\epsilon_{i'j'k'}}\, q^{i'} q^{j'} q^{k'} = \epsilon_{ijk}\, q^i q^j q^k$$

The Rest of the Family

The eight fold way



[postulated '70
GIM mechanism] → discovery '74



Computer reconstruction of a ψ′ decay in the Mark I detector at SLAC, making a near-perfect image of the Greek letter ψ

$J/\psi \,\hat{=}\, (c\bar{c})$ "charmonium"

['87 $m_t > 50$ GeV (B- oszillation)
<'94 $m_t \in [145, 185]$ GeV (EW precision)
discovery '95 $m_t = 173$ GeV]

[postulated '73
CP violation
Kobayashi & Maskawa] → discovery '77 (Υ)

Outline

0 The Beginnings

I Basics of QCD

- the QCD Lagrangian
- pQCD & Feynman rules
- the $SU(N_c)$ colour algebra
- renormalization & running coupling

II QCD in e^+e^- Collisions

III Hadron Colliders

IV Jet Physics

The QCD Lagrangian

* idea: use information gathered from observations (Part $\emptyset$)
  & use gauge invariance as the construction principle

* follow QED from $U(1)$

  ① free theory of electrons $\mathcal{L}_{\text{fermion}}(\psi, \partial_\mu \psi)$ $\rightarrow$ has a global $U(1)$ symmetry

  ② promote to a local (gauge) symmetry

  $\Rightarrow$ minimal substitution $\partial_\mu \longmapsto D_\mu = \partial_\mu + ieQ_f A_\mu^a$

  ③ add dynamics for photons: $\mathcal{L}_{\text{Maxwell}} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$

  (field strength tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A^\mu$)

  $\Rightarrow$ $\mathcal{L}_{QED} = \mathcal{L}_{\text{fermion}}(\psi, D_\mu \psi) + \mathcal{L}_{\text{Maxwell}}$

* Let us follow the same steps to arrive at $\mathcal{L}_{QCD}$

The QCD Lagrangian

① * quarks are spin-$\frac{1}{2}$ fermions & exist in 6 "flavours" $(q = u, d, s, c, b, t) \Rightarrow \psi_q$

* quarks have an additional colour d.o.f $\Rightarrow \psi_q = (\psi_q^r, \psi_q^g, \psi_q^b)^T$

↪ Dirac Lagrangian $\mathcal{L}_q(\psi_q, \partial_\mu \psi_q) = \sum_q \bar{\psi}_q^i (i\not{\partial} - m_q)\delta^{ij}\psi_q^j = \sum_q \bar{\psi}_q (i\not{\partial} - m_q)\psi_q$

* This Lagrangian has a global SU(3) symmetry $\psi_q^i \longrightarrow U_{ij}\psi_q^j$

$U = \exp[-i g_s \theta^a t^a]$; $U \in SU(3)$ $(N_c^2 - 1) = 8$ parameters (θ^a) / generators (t^a)

* one explicit representation: $t^a = \lambda^a/2$ (λ^a: Gell-Mann matrices)

$$\lambda^1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda^2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda^4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$\lambda^5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \quad \lambda^6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad \lambda^7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \quad \lambda^8 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

normalization

$\mathrm{tr}[t^a t^b] = T_R\, \delta^{ab}$

$T_R = \frac{1}{2}$

The QCD Lagrangian

② * $SU(N_c)$ is an exact symmetry ⤳ "gauge" it $(\theta^a \longrightarrow \theta^a(x))$

⇒ For $\mathcal{L}_q$ to remain invariant, need to substitute

$$\partial_\mu \longmapsto D_\mu = \partial_\mu + i g_s \underbrace{A_\mu^a t^a}_{\mathcal{A}_\mu}$$

"covariant derivative"

$\mathcal{A}_\mu$ transforms as $\mathcal{A}_\mu \mapsto U \mathcal{A}_\mu U^\dagger + \frac{i}{g_s} (\partial_\mu U) U^\dagger$

* forced us to introduce 8 new spin-1 fields, the gluons: A_μ^a $(a = 1, \dots, 8)$

③ * add dynamics for the gluon fields:

structure constant

$[t^a, t^b] = i f^{abc} t^c$

$$\mathcal{L}_A = -\frac{1}{4} F_{\mu\nu}^a F^{a,\mu\nu} \qquad \text{with} \qquad F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a \underbrace{- g_s f^{abc} A_\mu^b A_\nu^c}_{\text{not present in QED}}$$

* gluon self-interaction
* no mass term allowed (not gauge invariant)

The QCD Lagrangian

* Putting it together ($D_\mu = \partial_\mu + i g_s A_\mu^a t^a$, $F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a - g_s f^{abc} A_\mu^b A_\nu^c$)

$$\mathcal{L}_{QCD} = \sum_q \bar{\psi}_q (i \not{D} - m_q) \psi_q - \frac{1}{4} F_{\mu\nu}^a F^{a,\mu\nu}$$ fully gauge invariant by construction

* gauge symmetry introduces redundancy $\rightsquigarrow$ $\mathcal{L}_{fix}$ for reduction to physical d.o.f.

[$A_{\mu=0,1,2,3}$ (4) v.s. massless spin-1 (2)]

↪ covariant gauge $\mathcal{L}_{fix} = -\frac{1}{2\xi} (\partial^\mu A_\mu^a)^2$

$\Rightarrow$ introduces new ghost fields to cancel unphysical d.o.f.

* $\xi = 1$: Feynman gauge

↪ axial gauges $\mathcal{L}_{fix} = -\frac{1}{2\xi} (n^\mu A_\mu^a)^2$ (n^μ arbitrary $\leftrightarrow$ not covariant)

* $n^2 = 0$: "light-cone" gauge
* $\xi \to 0$: "axial" gauge is ghost free $\Rightarrow$ physical modes only

Quantization & Perturbation Theory

* Quantization in the path-integral formalism:

$$\mathcal{M}_{\Phi_1 \cdots \Phi_n} \longleftrightarrow \int \mathcal{D}[\Phi]\, \Phi_1(x_1) \cdots \Phi_n(x_n)\, e^{i \int d^4x\, \mathcal{L}_{QCD}(\Phi, \partial_\mu \Phi)}$$

↳ $\Phi \in \{ \psi_q, A_\mu^a \}$

all information contained

⇝ In principle, could compute <u>anything</u> from it but extremely difficult to solve (starting point for lattice QCD)

* solve approximately using perturbation theory α_s^n ↔ systematically improvable

⇒ diagrammatic representation in form of Feynman diagrams

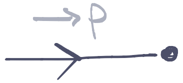
① terms in $\mathcal{L}_{QCD}$ with $\leq 2\,\Phi$: free theory ⇝ external states & propagators

② terms in $\mathcal{L}_{QCD}$ with $> 2\,\Phi$: interactions ⇝ vertices

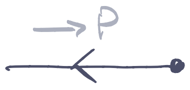
Feynman Rules of QCD

* external legs

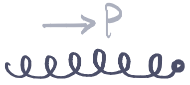
incoming

ψ_q^i 

$u^i(p)$

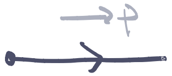
$\bar{\psi}_q^i$ 

$\bar{v}^i(p)$

A_μ^a 

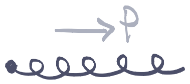
$\varepsilon_\mu^a(p)$

outgoing



$\bar{\psi}_q^i$ $\bar{u}^i(p)$

ψ_q^i $v^i(p)$

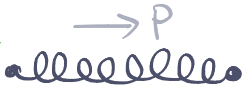


A_μ^a $\varepsilon_\mu^a(p)^*$

* propagators

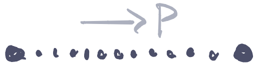
$\bar{\psi}_q^i$ 

ψ_q^j $\quad \dfrac{i\delta^{ij}(\not{p}+m_q)}{p^2-m^2}$

A_μ^a 

A_ν^b $\quad \dfrac{-i\delta^{ab}}{p^2}\left[g_{\mu\nu}-(1-\xi)\dfrac{p_\mu p_\nu}{p^2}\right]$

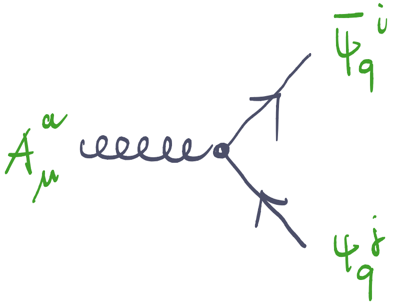
covariant gauge

$\bar{u}^a$ 

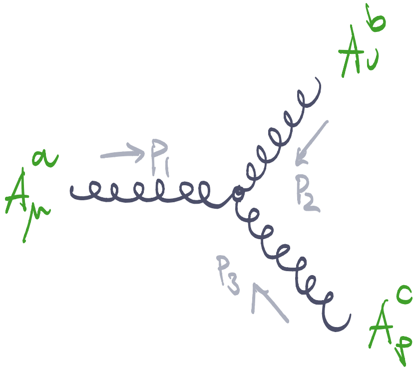
u^b $\quad \dfrac{i\delta^{ab}}{p^2}$ (the Fadeev-Popov ghost)

Feynman Rules of QCD

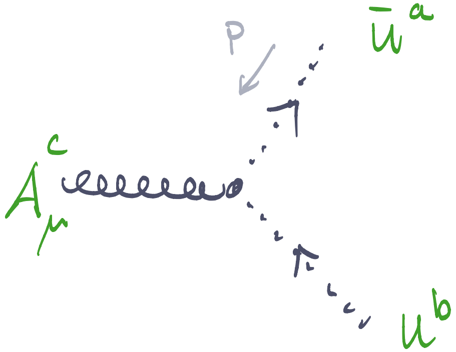
* Vertices



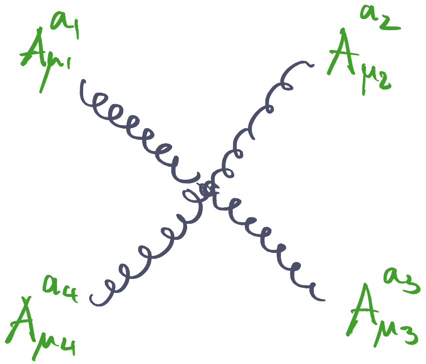
$$-i g_s (t^a)_{ij} \gamma_\mu$$



$$-g_s f^{abc} \left[g_{\mu\nu} (p_1 - p_2)_\rho + g_{\nu\rho} (p_2 - p_3)_\mu + g_{\rho\mu} (p_3 - p_1)_\nu \right]$$



$$g_s f^{abc} p_\mu$$

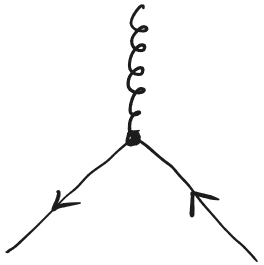


$$-i g_s^2 \left[f^{a_1 a_2 b} f^{a_3 a_4 b} \left(g^{\mu_1\mu_3} g^{\mu_2\mu_4} - g^{\mu_1\mu_4} g^{\mu_2\mu_3} \right) + \text{perms} \right]$$

Colour Algebra

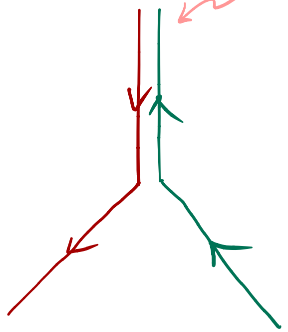
* main distinction from QED: colour $\longleftrightarrow$ non-Abelian group $(U_1 U_2 \neq U_2 U_1)$

* pictorial representation of quark-gluon interaction



$\bar{\psi}^i \quad t^a_{ij} \quad \psi^j$

$$\underbrace{(1,0,0)}_{\bar{\psi}^r} \underbrace{\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{t^1} \underbrace{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}}_{\psi^g} \quad \longleftrightarrow$$



gluons carry colour & anti-colour

↪ "repaint" the quarks

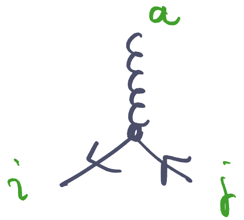
Graphical Representation of $SU(N_c)$

* propagators ⟷ identity matrix

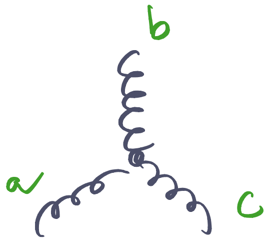
δ^{ij}

δ^{ab}

* vertices ⟷ generators

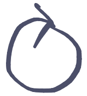


$(t^a)_{ij}$



$i f^{abc}$ ⟵ ordering matters

* loops ⟷ traces



$\mathrm{Tr}[\mathbb{1}_F] = \delta^{ii} = N_c$

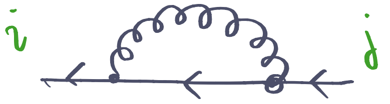
$\mathrm{Tr}[\mathbb{1}_A] = \delta^{aa} = N_c^2 - 1$

$\mathrm{Tr}[t^a] = \phi$

$\mathrm{Tr}[T^a_{(ad)}] = \phi$ (f^{abc} anti-symm.)

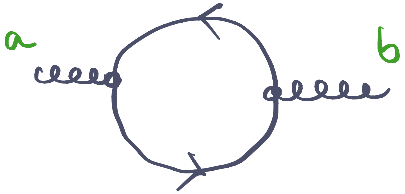
Graphical Representation of SU(N_c)

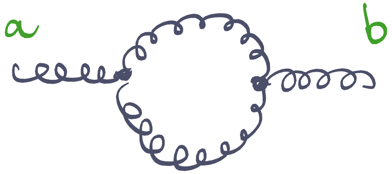
* normalization & quadratic Casimirs

 $= \sum_a t^a t^a = C_F \, \delta^{ij} = C_F$ 

$$C_F = T_R \, \frac{N_c^2 - 1}{N_c} = \frac{4}{3}$$

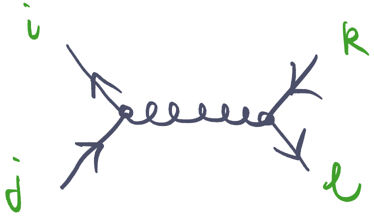
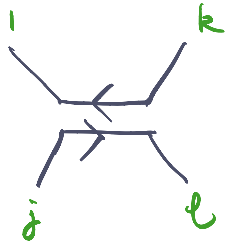
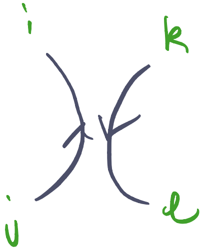
show this as an exercise

 $= \mathrm{Tr}[t^a t^b] = T_R \, \delta^{ab} = T_R$ 

 $= \sum_c T^c_{(ad)} \, T^c_{(ad)} = C_A \, \delta^{ab} = C_A$ 

$$C_A = 2 \, T_R \, N_c = 3$$

* Fiertz identity

 $= T_R \Big[$  $- \dfrac{1}{N_c}$  $\Big]$

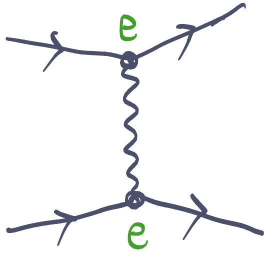
Renormalization & the Running Coupling

* all parameters, i.e. also α_s, must be determined experimentally
  ↳ renormalization: measurable quantities expressed in terms of measurable quantities

* QED ⇝ the fine structure constant $\alpha = \frac{e^2}{4\pi}$

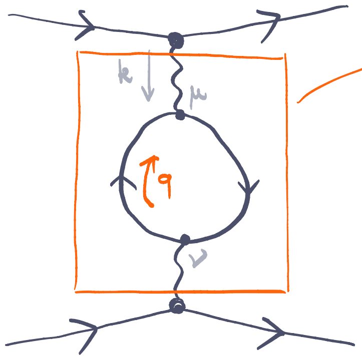
  EM interaction ⟷ exchange of photon

  Let's imagine, we measure it from the scattering of 2 charged particles



$\mathcal{M}^{(0)} \sim \alpha$

exchange affected by vacuum polarization (higher order corr.)



$\mathcal{M}^{(1)} \sim \alpha^2$

$$\int \frac{d^4q}{(2\pi)^4} \frac{\mathrm{Tr}\left[\gamma^\mu(\not{q}+m)\gamma^\nu(\not{q}+\not{k}+m)\right]}{(q^2-m^2)\left[(q+k)^2-m^2\right]}$$

$$\underset{\text{large}}{\overset{q}{\sim}} \int dq\, q^3 \frac{\{1, q, q^2\}}{q^4}$$

at least $\int \frac{dq}{q}$ in the UV

logarithmic divergent ?!

Intermezzo: Dimensional Regularization

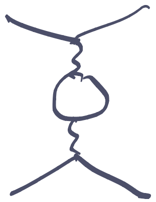
* modify theory (e.g. $\int_0^\infty d^4q \to \int_{|q|<\Lambda} d^4q$) to render expression finite w/ parameter (Λ)
  - ↳ original (divergent) expression by taking limit of parameter ($\Lambda \mapsto \infty$)
* de facto standard: dimensional regularization using $D = 4 - 2\epsilon$ as dimension
  - ↳ preserves Lorentz- & gauge-invariance; works also for the infrared
  - ↳ divergence as poles in $\frac{1}{\epsilon^n}$ $(\epsilon \to \emptyset)$
  - ↳ momentum integrals: $\int \frac{d^4q}{(2\pi)^4} \to \int \frac{d^Dq}{(2\pi)^D}$ (impose integral axioms)
  - ↳ retain dimensionless coupling $g \to \mu^{\frac{4-D}{2}} g = \mu^{\epsilon} g$

why this power?

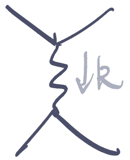
need to introduce a scale

Renormalization & the Running Coupling

* with DimReg, we obtain the following result

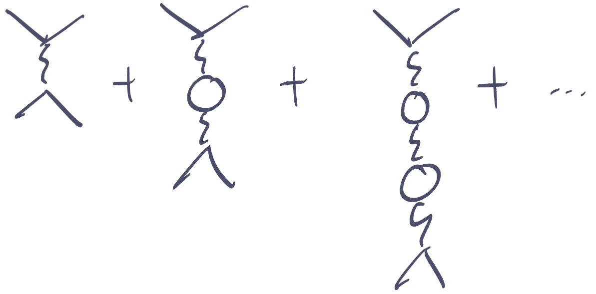


$$= \left[-\Pi^{\gamma\gamma}(k^2)\right] \cdot$$



with $\Pi^{\gamma\gamma}(k^2) = \frac{\alpha}{3\pi}\left[\frac{1}{\epsilon} - \gamma_E + \ln(4\pi) + \ln\left(\frac{\mu^2}{k^2}\right) + \frac{5}{3}\right] + \mathcal{O}(\epsilon)$

* we can anticipate even more bubble insertions:



$$= \mathcal{M}^{(0)}\left[1 + \left[-\Pi^{\gamma\gamma}\right] + \left[-\Pi^{\gamma\gamma}\right]^2 + \dots\right]$$

$$= \mathcal{M}^{(0)} \sum_{n=0}^{\infty}\left[-\Pi^{\gamma\gamma}\right]^n = \mathcal{M}^{(0)} \frac{1}{1+\Pi^{\gamma\gamma}}$$

Dyson sum

Renormalization & the Running Coupling

* Let's make the coupling dependence explicit and label it the bare coupling α^{bare} in anticipation of the renormalization step.

$$\mathcal{M}^{(0)} = \alpha^{bare}\,\bar{\mathcal{M}}^{(0)}\,, \qquad \Pi^{\gamma\gamma}(k^2) = \alpha^{bare}\,\bar{\Pi}^{\gamma\gamma}(k^2)$$

↪ we can introduce an effective coupling $\alpha^{eff}(k^2)$ that incorporates all bubble insertions

$$\mathcal{M}^{\Sigma} = \bar{\mathcal{M}}^{(0)} \underbrace{\frac{\alpha^{bare}}{1+\alpha^{bare}\,\bar{\Pi}^{\gamma\gamma}(k^2)}}_{\equiv\,\alpha^{eff}(k^2)} = \mathcal{M}^{(0)}\Big|_{\alpha^{bare}\to\,\alpha^{eff}(k^2)}$$

Renormalization & the Running Coupling

* Now, in the final step, we will renormalize the coupling.

  $\Leftrightarrow$ we perform a measurement of α $\leftrightsquigarrow$ defines the renormalization scheme

* e.g. let's define α as measured at the Z pole

$$\alpha^{ren} \equiv \alpha^{eff}(Q^2 = M_Z^2) = \frac{\alpha^{bare}}{1 + \alpha^{bare}\,\underbrace{\overline{\Pi}^{\gamma\gamma}(M_Z^2)}} \simeq 1/129$$

this is a finite measured value!

this has divergences!

what if we had picked the Thompson limit? ($Q^2 \to \emptyset$)

* We can rewrite all $\alpha_s^{eff}(Q^2)$ in terms of the measured value α^{ren}

$$\frac{1}{\alpha^{eff}(Q^2)} = \frac{1}{\alpha_{ren}}\left[\overline{\Pi}^{\gamma\gamma}(Q^2) - \overline{\Pi}^{\gamma\gamma}(M_Z^2)\right]$$

something magical has happened!

Renormalization & the Running Coupling

* by expressing measurable quantities $\alpha^{eff}(Q^2)$ in terms of measured quantities α^{ren}

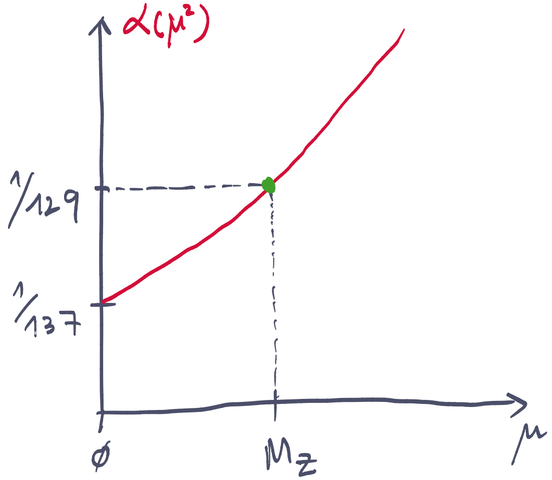
  $\Leftrightarrow$ all divergences have disappeared Renormalization

$$\left[\bar{\Pi}^{\gamma\gamma}(Q^2) - \bar{\Pi}^{\gamma\gamma}(M_Z^2)\right] = \frac{1}{3\pi}\left[\ln\left(\mu^2/Q^2\right) - \ln\left(\mu^2/M_Z^2\right)\right] = -\frac{1}{3\pi}\ln\left(Q^2/M_Z^2\right)$$

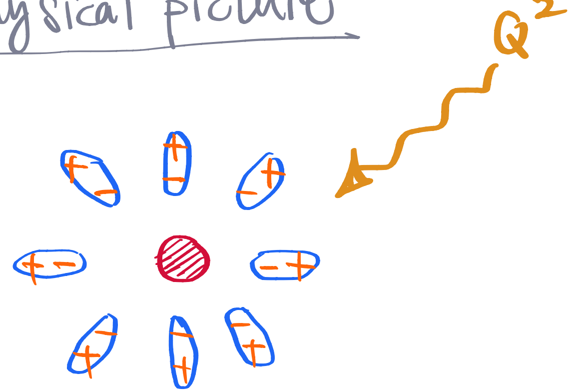
$\cancel{\frac{1}{\epsilon}}$ $\cancel{\frac{1}{\epsilon}}$

* what we get is the running QED coupling

$$\alpha(\mu^2) = \frac{\alpha(\mu_0^2)}{1 - \frac{\alpha(\mu_0^2)}{3\pi}\ln\left(\mu^2/\mu_0^2\right)}$$



physical picture



Q^2 small $\longleftrightarrow$ large λ : screening

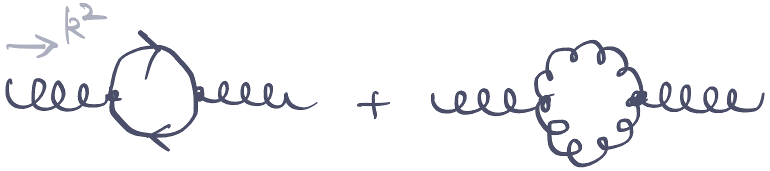
Q^2 large $\longleftrightarrow$ small λ : "see" more of Q

Renormalization & the Running Coupling

* QCD: more subtle because gluons carry colour & gauge cancellations

$\Leftrightarrow$ exploit gauge freedom: light-cone (axial) gauge $\Rightarrow$ only 

* now we have two diagrams



$$= \Pi^{gg}(k^2) = -\frac{\alpha_s^{\text{bare}}}{2\pi}\left[\beta_0\left(\frac{1}{\epsilon}-\gamma_E+\ln(4\pi)+\ln\left(\frac{\mu^2}{k^2}\right)\right)+h+\mathcal{O}(\epsilon)\right]$$

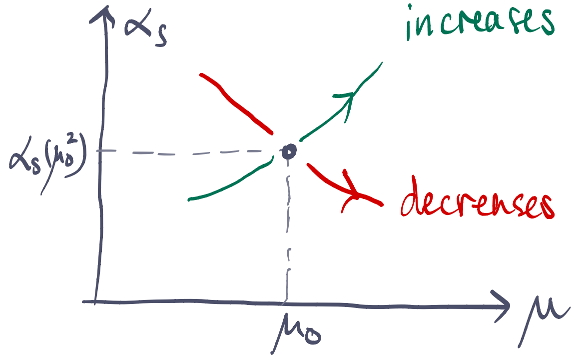
constant (no k^2-dep)

$$\text{with} \quad \beta_0 = \frac{11}{6}C_A - \frac{N_f}{3}$$

flavours

* following same steps as in QED: running of $\alpha_s(\mu^2)$ (at lowest order)

$$\alpha_s(\mu^2) = \frac{\alpha_s(\mu_0^2)}{1+\beta_0\frac{\alpha_s(\mu_0^2)}{2\pi}\ln\left(\mu^2/\mu_0^2\right)}$$



increases if $\beta_0 < 0$

decreases if $\beta_0 > 0$

[for $N_f \leq 16$]

asymptotic freedom

Renormalization & the Running Coupling

* asymptotic freedom: $\alpha_s(\mu^2) \to 0$ for $\mu^2 \to \infty$
  - $\hookrightarrow$ feature of non-Abelian gauge theories
    - $\Rightarrow$ at high scales: hadrons $\hat{=}$ collection of free partons & pQCD applicable
* scale choice μ is arbitrary $\rightsquigarrow$ mass scale of QCD Λ_{QCD}

$$\Lambda_{QCD} \simeq \mu_0 \exp\left\{-\frac{1}{2\beta_0 \frac{\alpha_s(\mu_0^2)}{2\pi}}\right\} \simeq 200\,\text{MeV}$$

← signals breakdown of perturbative treatment

$$\Rightarrow \quad \alpha_s(\mu^2) = \frac{1}{\frac{\beta_0}{2\pi} \ln\left(\mu^2/\Lambda_{QCD}^2\right)}$$

* Why does no one talk about Λ_{QED}?

Renormalization & the Running Coupling

* cross section for a scale choice μ_0

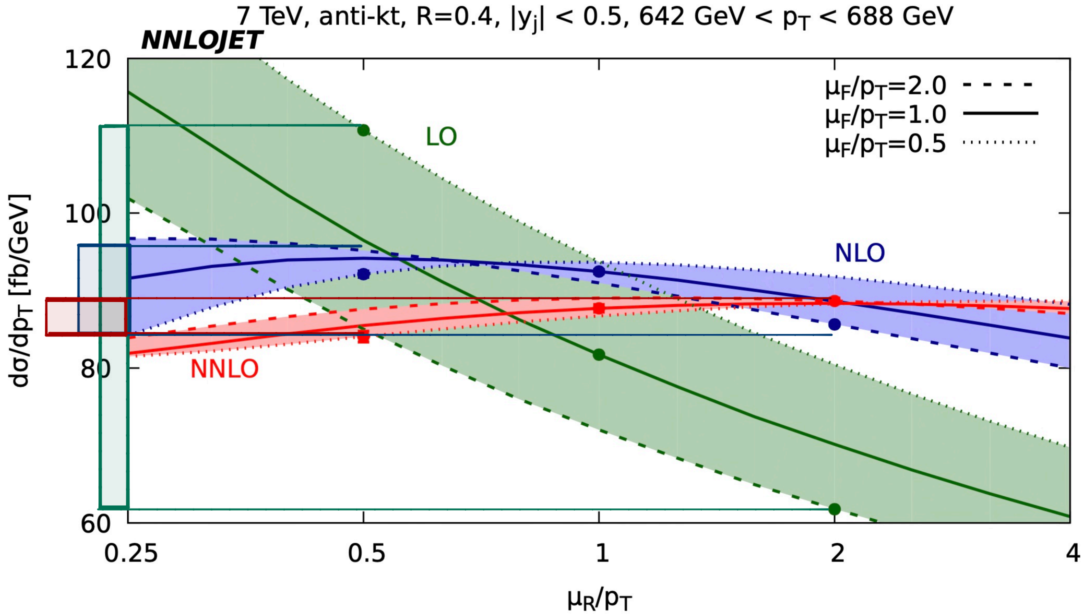
$$\sigma(\alpha_s(\mu_0), \mu_0) = \left(\frac{\alpha_s(\mu_0)}{2\pi}\right)^n \underbrace{\sigma^{(0)}}_{\text{LO}} + \left(\frac{\alpha_s(\mu_0)}{2\pi}\right)^{n+1} \underbrace{\sigma^{(1)}}_{\text{NLO}} + \dots$$

scale variation @ lower orders generates higher-order terms

$\Rightarrow$ prediction for scale dependence

$$\sigma(\alpha_s(\mu), \mu) = \left(\frac{\alpha_s(\mu)}{2\pi}\right)^n \sigma^{(0)} + \left(\frac{\alpha_s(\mu)}{2\pi}\right)^{n+1} \Big[\sigma^{(1)} + \underbrace{n\,\beta_0 \ln\left(\frac{\mu^2}{\mu_0^2}\right) \sigma^{(0)}}_{} \Big] + \dots$$

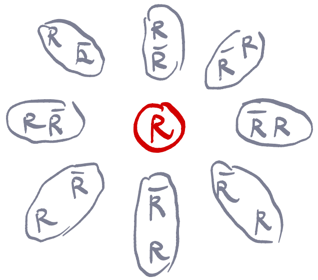
$\hookrightarrow$ handle on missing terms from truncation of the series



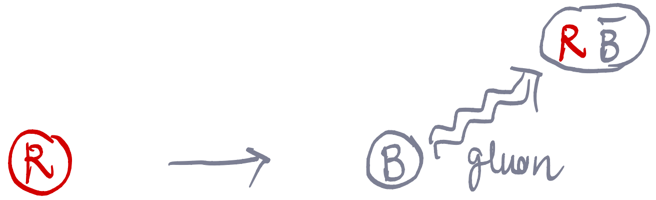
Renormalization & the Running Coupling

physical picture:

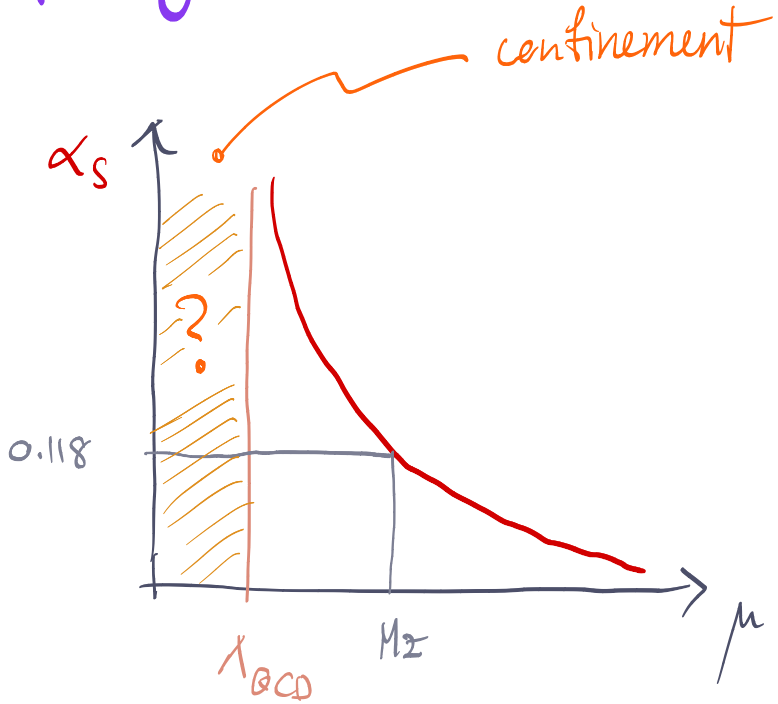
① screening



② anti-screening



net charge: still R
but gluon spreads
the charge



⟹ the closer I look (high Q^2 → small λ)

→ chance of finding R → $\emptyset$

(this effect dominates over the screening)

Renormalization & the Running Coupling

* more formally:

$$\alpha_s^{\text{bare}} = \mu^{2\varepsilon} Z_{\alpha_s} \alpha_s^{\text{ren}} \quad \& \quad \frac{d\alpha_s^{\text{bare}}}{d\mu} \overset{!}{=} 0$$

$\Rightarrow$ beta function

$$\beta(\alpha_s) = \frac{1}{\alpha_s} \frac{d\alpha_s}{d\ln\mu^2} = -\sum_{n=1}^{\infty} \beta_{n-1} \left(\frac{\alpha_s}{2\pi}\right)^n$$

* known to 5 loops

$$\beta_0 = 11 - \frac{2}{3} n_f , \qquad \beta_1 = 102 - \frac{38}{3} n_f ,$$

$$\beta_2 = \frac{2857}{2} - \frac{5033}{18} n_f + \frac{325}{54} n_f^2 ,$$

$$\beta_3 = \frac{149753}{6} + 3564\,\zeta_3 + n_f \left(-\frac{1078361}{162} - \frac{6508}{27}\zeta_3\right) + n_f^2 \left(\frac{50065}{162} + \frac{6472}{81}\zeta_3\right) + \frac{1093}{729} n_f^3$$

$$\begin{aligned}\beta_4 = {} & \frac{8157455}{16} + \frac{621885}{2}\zeta_3 - \frac{88209}{2}\zeta_4 - 288090\,\zeta_5 \\ & + n_f \left(-\frac{336460813}{1944} - \frac{4811164}{81}\zeta_3 + \frac{33935}{6}\zeta_4 + \frac{1358995}{27}\zeta_5\right) \\ & + n_f^2 \left(\frac{25960913}{1944} + \frac{698531}{81}\zeta_3 - \frac{10526}{9}\zeta_4 - \frac{381760}{81}\zeta_5\right) \\ & + n_f^3 \left(-\frac{630559}{5832} - \frac{48722}{243}\zeta_3 + \frac{1618}{27}\zeta_4 + \frac{460}{9}\zeta_5\right) + n_f^4 \left(\frac{1205}{2916} - \frac{152}{81}\zeta_3\right)\end{aligned}$$

* running

$\hookrightarrow$ τ decay

$M_\tau \sim 2\,\text{GeV}$

3x from running

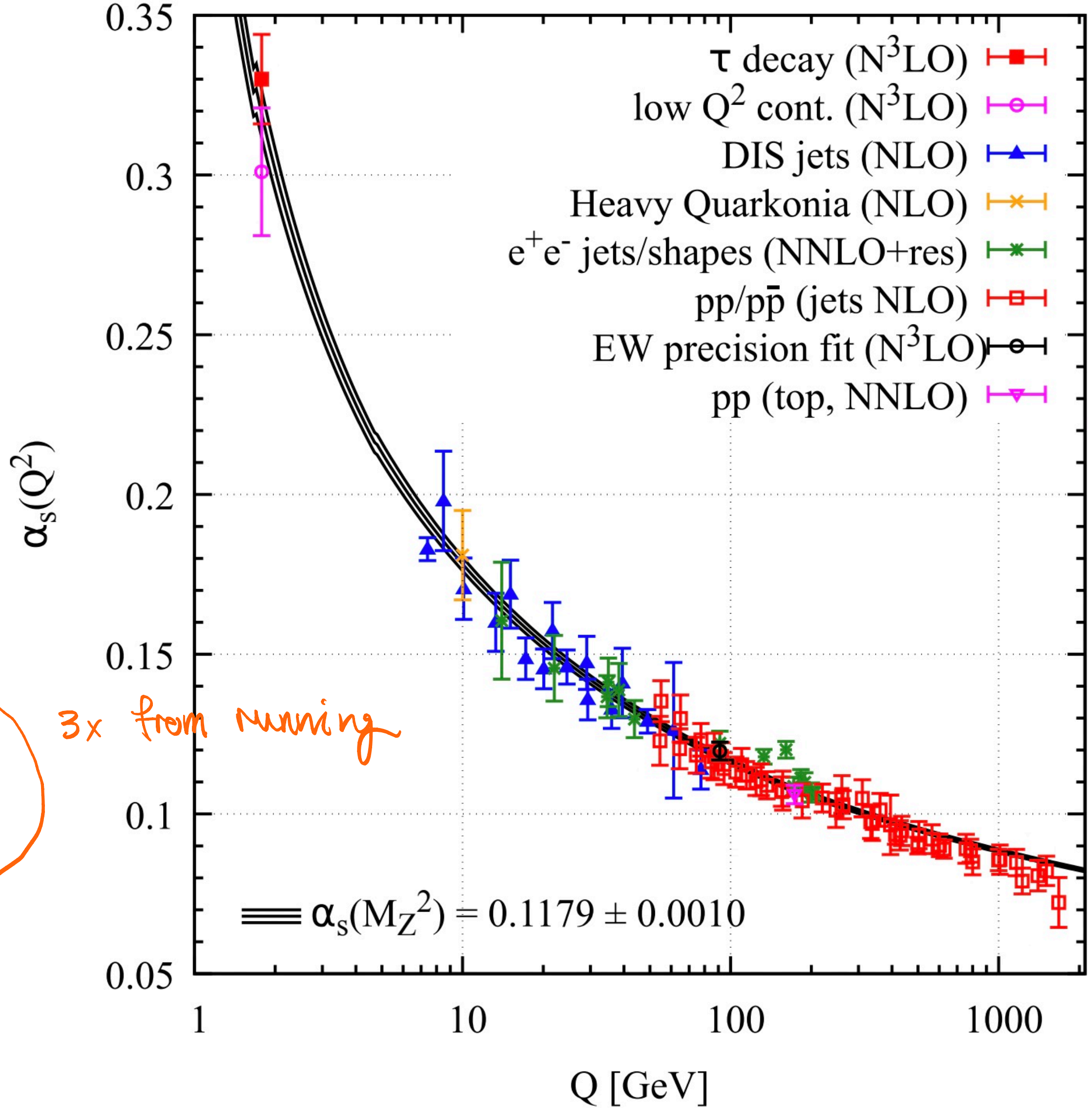
$\hookrightarrow$ LEP

$M_Z \sim 100\,\text{GeV}$

$\hookrightarrow$ LHC

$Q \sim 1\,\text{TeV}$

$\alpha_s(M_Z^2) = 0.1179 \pm 0.0010\,.$



Determinations of α_s

* τ decays & low Q^2 $\longrightarrow$ running helps!

  $\alpha_s(M_Z^2) = 0.1173\,(17)$ $\qquad$ $(\alpha_s(m_\tau^2) = 0.314\,(14))$

  $\delta' = [\alpha_s(M_Z^2)/\alpha_s(Q^2)]^2\,\delta$

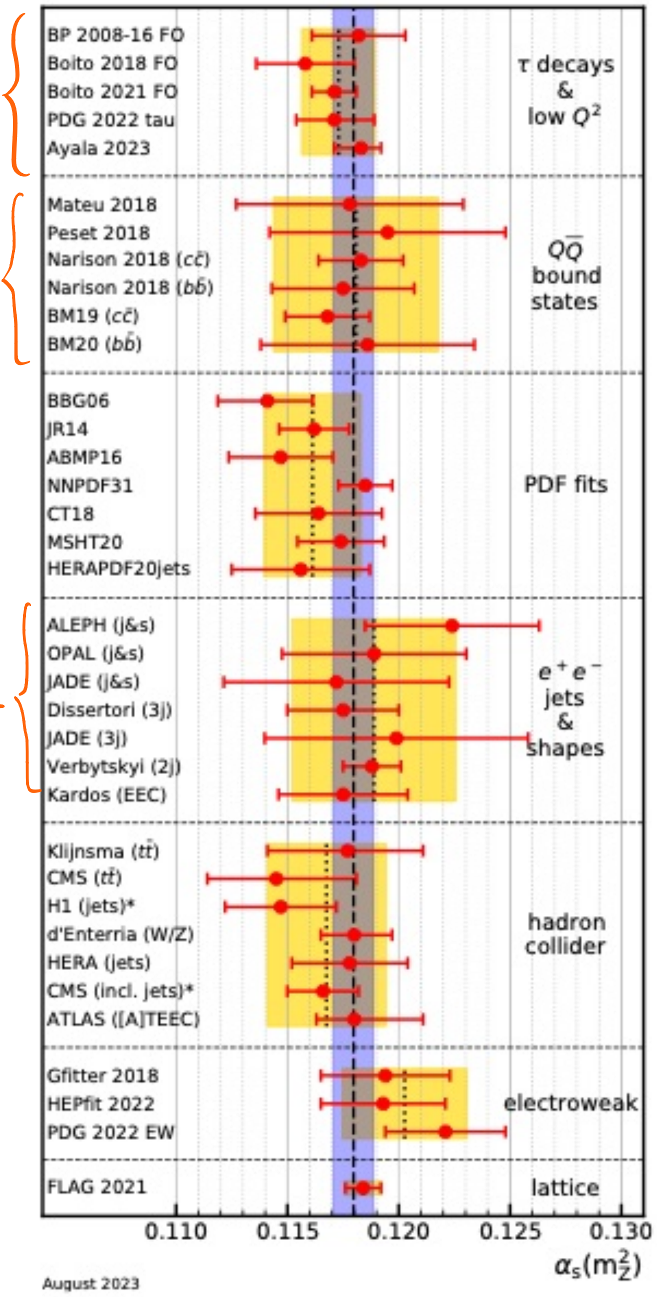
* quarkonia decays $\longrightarrow$ α_s sensitivity $\Gamma(\Upsilon \to had) \sim \alpha_s^3$

  $\alpha_s(M_Z^2) = 0.1181\,(37)$

* hadronic final states in e^+e^-
  - $\hookrightarrow$ $e^+e^- \to$ hadrons $\longrightarrow$ weak α_s sensitivity but $NP \sim (\Lambda_{QCD}/Q)^4$
  - $\hookrightarrow$ event shapes $\longrightarrow$ LO α_s sensitivity but $NP \sim (\Lambda_{QCD}/Q)$

  $\alpha_s(M_Z^2) = 0.1189\,(37)$

* World average $\quad \alpha_s(M_Z^2) = 0.1180\,(9)$



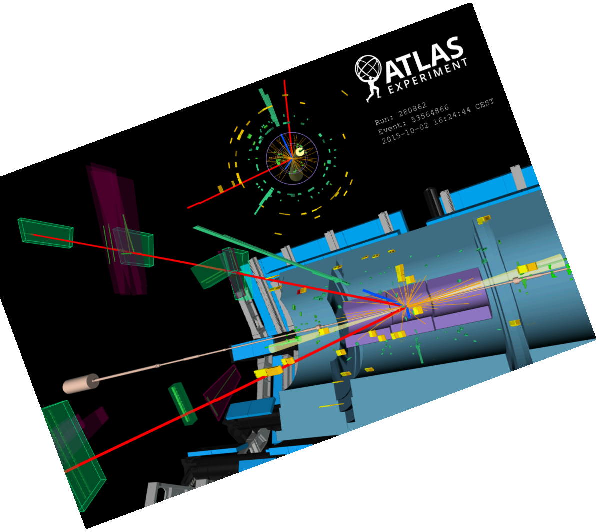
Recap

* QCD: non-Abelian gauge theory with spin-$\frac{1}{2}$ matter quark fields
  - ↳ describes the pattern of hadrons we observe (mesons: $q\bar{q}$, baryons qqq)
  - ↳ analogous construction as QED
* non-Abelian group $SU(N_c)$ ⇝ gluons also carry colour (self interactions)
  - ↳ much more complex algebra
  - ↳ diagrammatic representation, e.g.  $= C_F$ 
* renormalization & asymptotic freedom
  - ↳ UV divergences inside loops → requires regularization: DimReg $D = 4 - 2\epsilon$
  - ↳ renormalization → measurable quantities in terms of measured parameters

    ⇒ all UV divergences cancel!
  - ↳ the running coupling $\alpha_s(\mu^2)$

Extra

From Lagrangian Densities to Event Rates



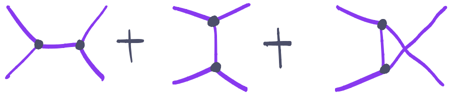
Event Rates

$$N = L \underbrace{\sigma}$$

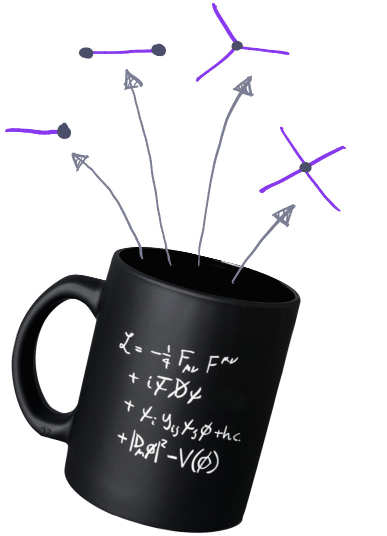
↳ **Cross Sections**

$$d\sigma_{2\to n} = \frac{1}{F} \underbrace{\langle |\mathcal{M}|^2 \rangle} \, d\Phi_n$$

↳ **Scattering Amplitudes**

 + ...

Feynman diagrams & rules $\longleftrightarrow$ $\mathcal{L}$

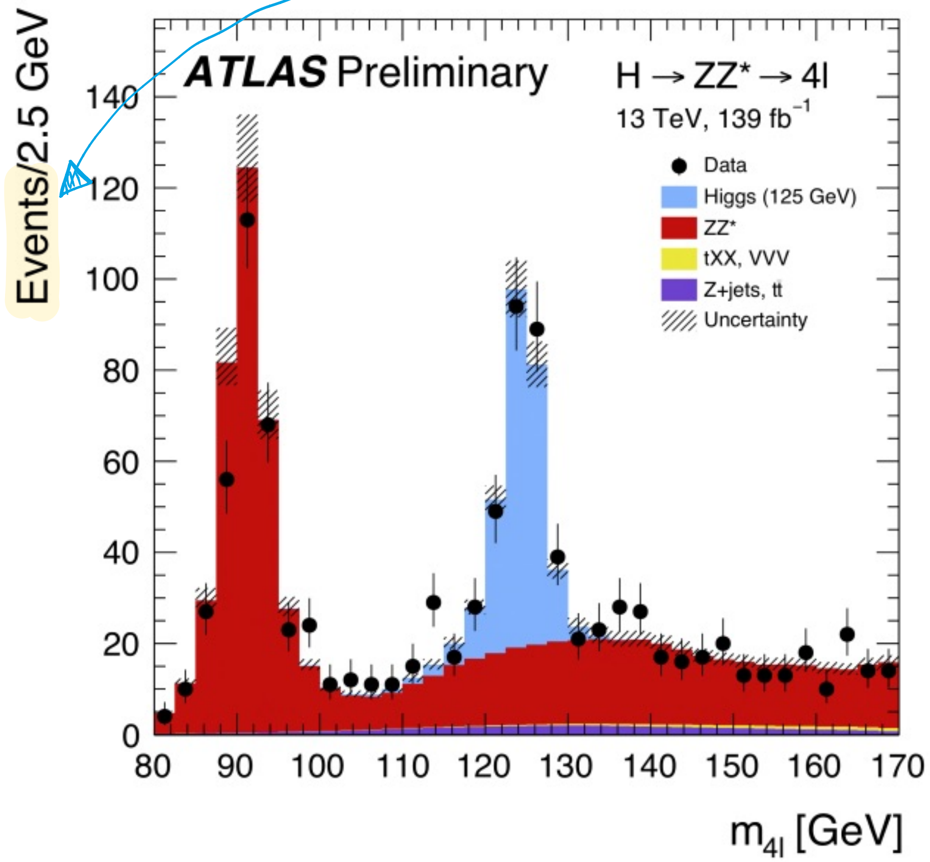


* another important case: Decay Rates $(\tau = 1/\Gamma)$

$$d\Gamma_{1\to n} = \frac{1}{2M} \langle |\mathcal{M}|^2 \rangle \, d\Phi_n$$

Event Rates

We ultimately measure # Events for a specific process: $a + b \to 1 + 2 + \dots + n$



Luminosity ~ # collisions

cross section

$$dN = \mathcal{L}\, d\sigma$$

* σ_H (13 TeV) $\approx 50\,\mathrm{pb}$

  $\int_{\text{Run2}} dt\, \mathcal{L} \approx 150\,\mathrm{fb}^{-1}$

  ~ 7 million Higgs bosons produced!

* σ_Z (13 TeV) $\simeq 50\,\mathrm{nb}$

  $\sigma_{W^\pm}$ (13 TeV) $\simeq 200\,\mathrm{nb}$

  $\mathcal{L}$ (instantaneous) $\simeq 0.02\,\mathrm{pb}^{-1}\mathrm{s}^{-1}$

  ~ 1000 Z's

  ~ 4000 $W^\pm$'s

  every second!

Calculating Cross Sections

Fermi's Golden Rule $\quad a+b \to 1+2+\dots+n$

$$d\sigma = \underbrace{\frac{1}{F}}_{\text{flux}} \underbrace{\langle |\mathcal{M}|^2 \rangle}_{|\text{amplitude}|^2} \underbrace{d\Phi}_{\substack{\text{phase space}\\ \text{(LIPS)}}}$$

$$= \frac{1}{4(p_a \cdot p_b)} = \frac{1}{2E_{cm}^2}$$

$$= \frac{1}{n_a^{d.o.f.}\, n_b^{d.o.f.}} \sum_{d.o.f.} |\mathcal{M}|^2$$

(degrees of freedom
spin, colour)

$$d\Phi_n(p_1,\dots,p_n;\, p_a+p_b)$$
$$= \prod_{i=1}^{n} \frac{d^4 p_i}{(2\pi)^4}\, (2\pi)\,\delta(p_i^2 - m_i^2)\, \Theta(p_i^0)$$
$$(2\pi)^4\, \delta^{(4)}(p_1 + \dots + p_n - (p_a+p_b))$$

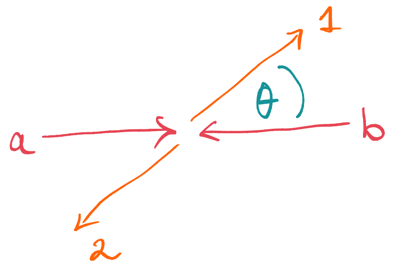
"on shell"

$E_i > 0$

energy-momentum conservation

Special case $\quad a+b \to 1+2$

$$d\Phi_2 = \frac{d\cos\theta}{16\pi} \quad \text{(massless)}$$



Exercise: Lepton Collider

Consider the process $e^+ e^- \to \mu^+ \mu^-$

at lowest order (tree level). There are two diagrams

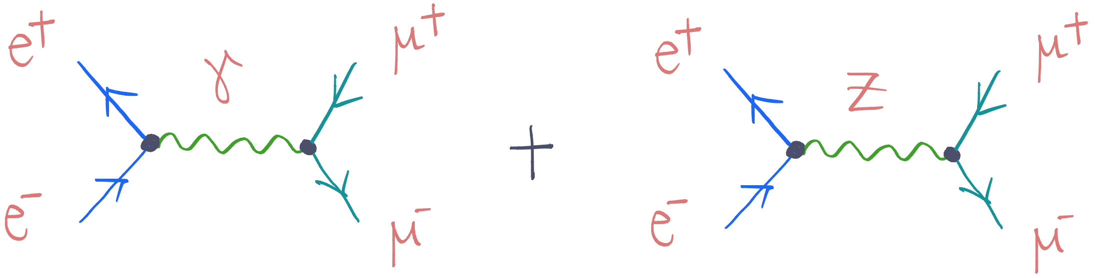
What are they? [demo: FeynGame]

Exercise: Lepton Collider

[demo: $e^+e^- \to \mu^+\mu^-$]

Consider the process $e^+ e^- \to \mu^+ \mu^-$

at lowest order (tree level). There are two diagrams



$$= i\mathcal{M}^{\text{tree}}_{e^+e^-\to\mu^+\mu^-}$$

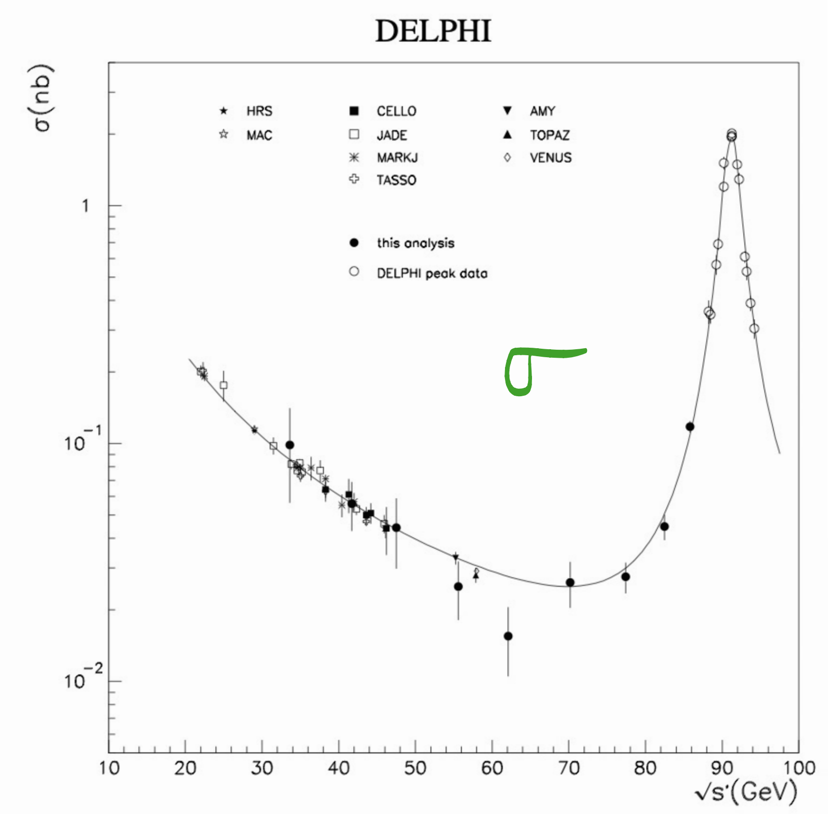
⇒ Inserting into Fermi's golden rule $[s = E_{CM}^2 \; ; \; p_a \cdot p_1 = p_a^\mu \, p_{1,\mu} = E_{cm}^2 (1-\cos\theta)]$

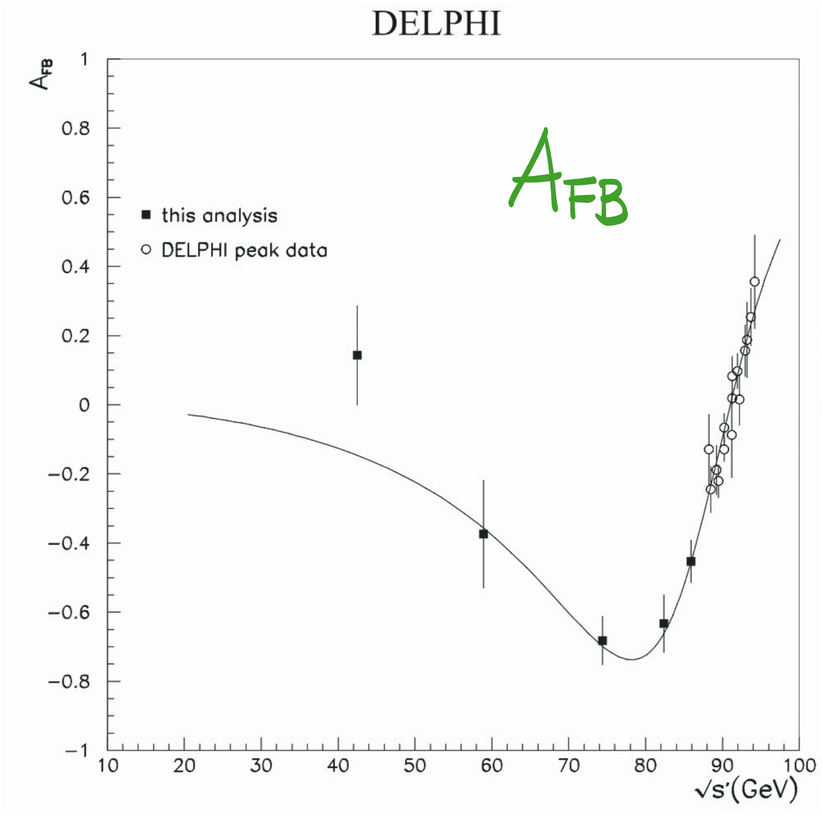
$$\frac{d\sigma}{d\cos\theta} = \frac{\alpha^2 \pi}{2s}\left[(1+\cos^2\theta)\, G_1(s) + 2\cos\theta\, G_2(s)\right]$$

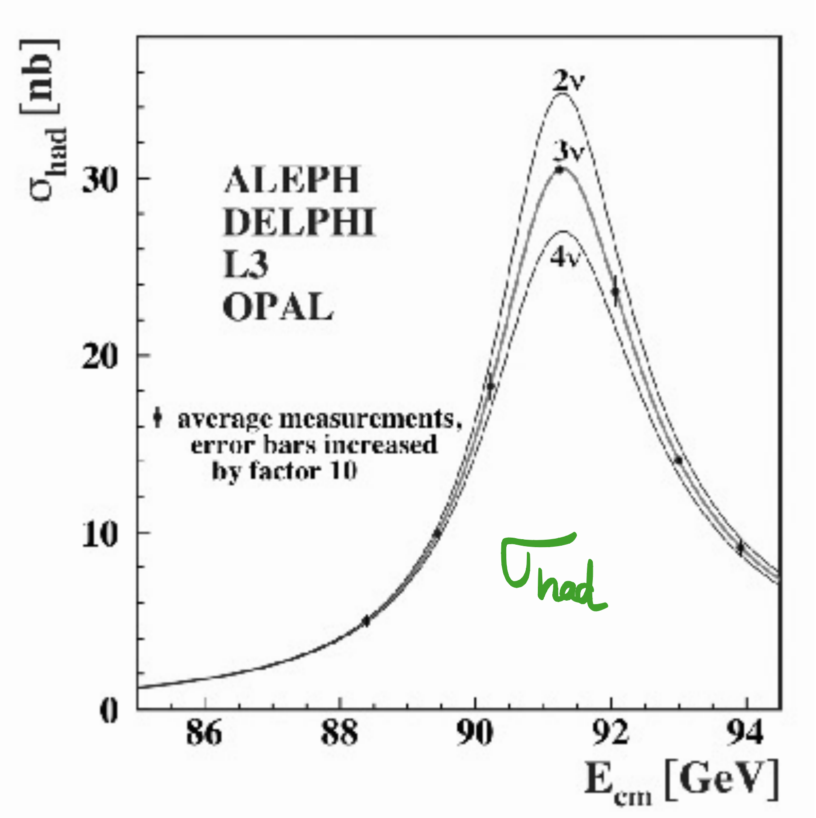
$$G_1(s) = 1 + 2v_\ell^2 \,\mathrm{Re}\left\{\frac{s}{s-M_Z^2+iM_Z\Gamma_Z}\right\} + (v_\ell^2+a_\ell^2)\left|\frac{s}{s-M_Z^2+iM_Z\Gamma_Z}\right|^2$$

$$G_2(s) = 0 + 2a_\ell^2 \,\mathrm{Re}\left\{\frac{s}{s-M_Z^2+iM_Z\Gamma_Z}\right\} + 4v_\ell^2 \cdot a_\ell^2\left|\frac{s}{s-M_Z^2+iM_Z\Gamma_Z}\right|^2$$

"Comparison" to Data







* In principle, you now can use the predictions to fit M_Z & $\sin^2\theta_W$ from the data (at leading order)

* σ_{had} is the hadronic cross section: @ LO: $e^+e^- \to q\bar{q}$

  $\Rightarrow$ what changes compared to $\mu^+\mu^-$?