Quantum Chromodynamics & Collider Phenomenology

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Outline

Ø The Beginnings

I Basics of QCD

II QCD in e^+e^- Collisions

- $e^+e^- \to$ hadrons
- higher-order corrections
- infrared singularities & safety
- the QCD emission pattern

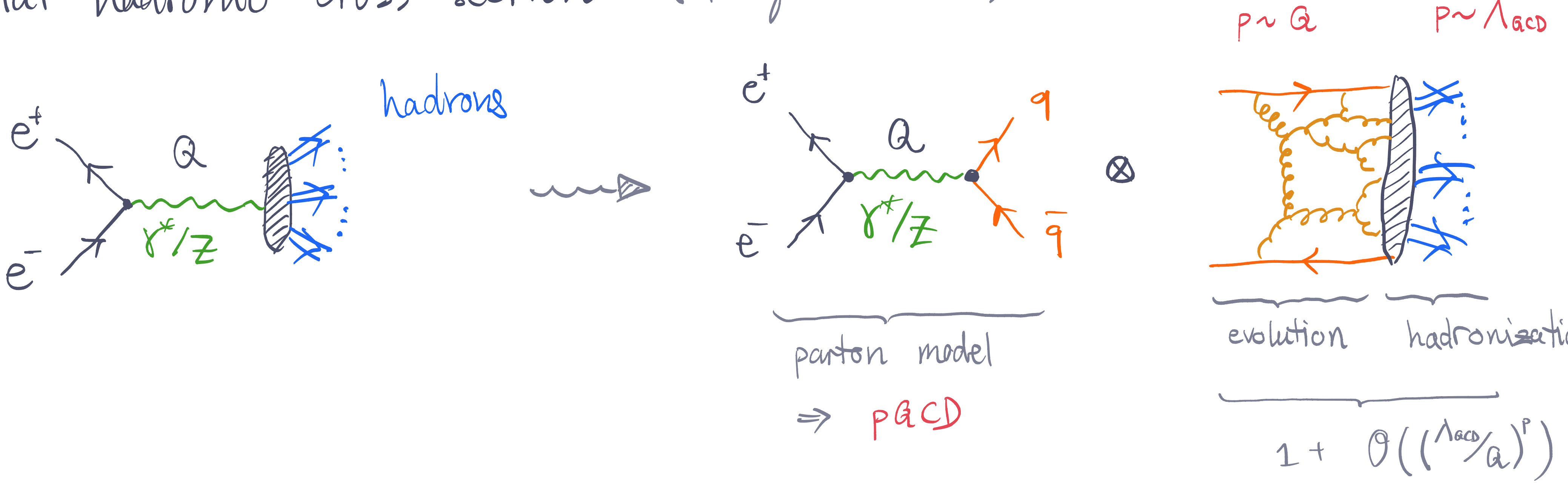
III Hadron Colliders

IV Jet Physics

The Parton Model

* asymptotic freedom
  - ↳ at large scales Q ⟶ hadrons ≃ collection of free partons
  - ↳ if not interested in details of hadronic process ⇒ use parton picture
    ↖ "sufficiently inclusive"

* total hadronic cross section (fully inclusive)



$e^+e^- \to$ hadrons (check "Extra" of previous lecture)

* for simplicity restrict to photon exchange $(Q < M_Z)$
* we're not interested in the angular orientation $\to$ "average"

hadronic tensor

$$\sigma(e^+e^- \to n \text{ partons}) = \underbrace{\frac{1}{2s}}_{\text{flux}} \underbrace{\frac{1}{4}}_{\text{spin avg}} \underbrace{L^{\mu\nu}}_{\text{lepton tensor}} \underbrace{\frac{1}{s^2}}_{\text{photon propagator}} \underbrace{\int d\Phi_n \, W_{\mu\nu}(\gamma^* \to n \text{ partons})}_{\text{must be} \sim \left(-g_{\mu\nu} + \frac{Q_\mu Q_\nu}{Q^2}\right)}$$

why?

$$= \frac{e^2}{4s^2} \frac{2}{3} W_{n\text{-parton}}(Q^2)$$

will mainly look at this

with $W_{n\text{-parton}}(Q^2) \equiv (-g^{\mu\nu}) \int d\Phi_n \, W_{\mu\nu}(\gamma^* \to n \text{ partons})$

$e^+e^- \to$ hadrons @ LO

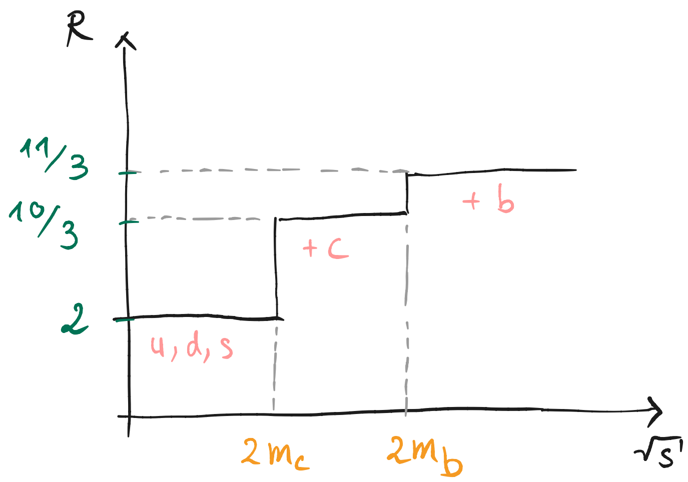
* For the LO process, we get $W^{(0)}_{q\bar{q}} = N_c \alpha Q_q^2 \, 2S$

$$\Rightarrow \quad \sigma(e^+e^- \to q\bar{q}) = \frac{4}{3}\frac{\pi\alpha^2}{s} \underline{N_c} \, \underline{Q_q^2}$$

* The R ratio:

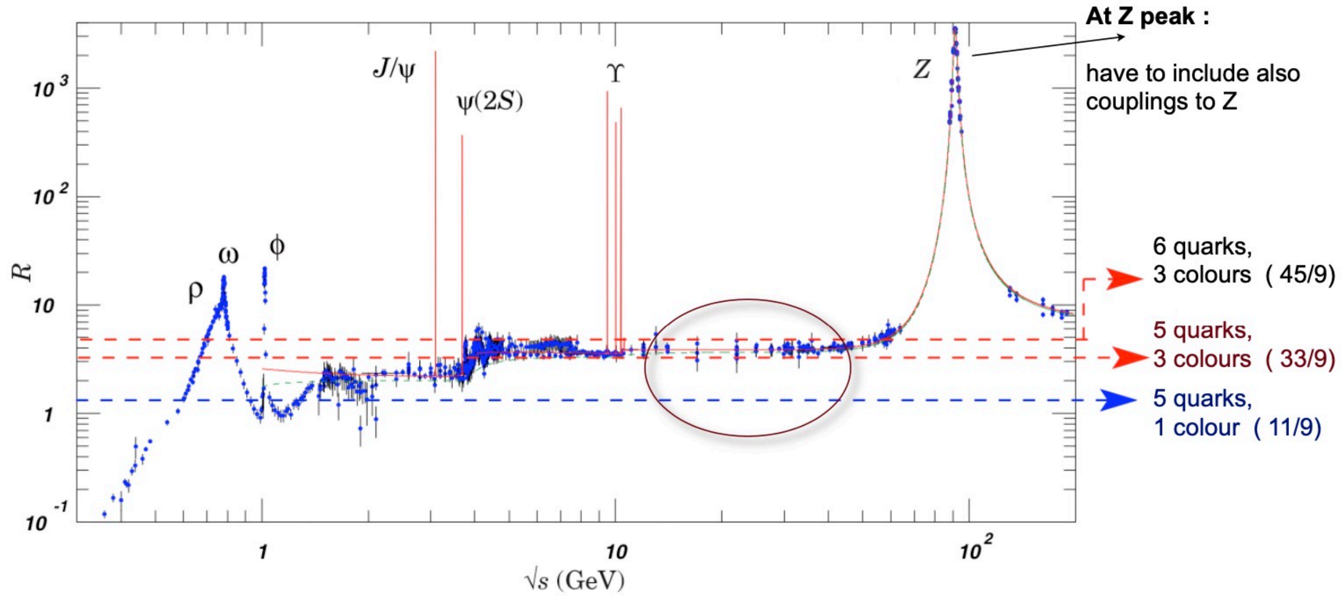
$$R = \frac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)} = N_c \sum_q Q_q^2 \; \Theta(\sqrt{s} - 2m_q)$$

production threshold for $q\bar{q}$ pair



get the raw data $\longrightarrow$

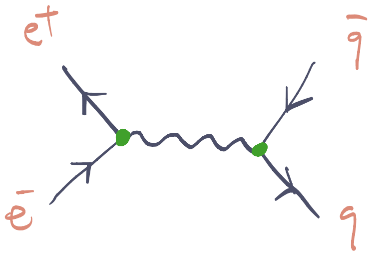
http://pdg.lbl.gov/2019/hadronic-xsections

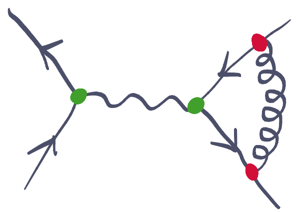
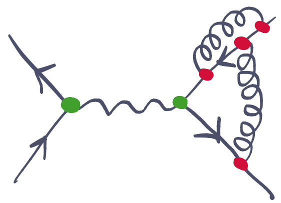


QCD at Higher Orders

leading order (LO) predictions in QCD
not sufficient for precision phenomenology

want to include
higher order(s) !
↪ diagrams with <u>loops</u>

$\mathcal{M}_2 =$ 

$+$  $+ \dots +$  $+ \dots$

$\mathcal{O}(\alpha) \leftrightarrow \mathcal{M}_2^{(0)}$ $\qquad$ $\mathcal{O}(\alpha_s \alpha) \leftrightarrow \mathcal{M}_2^{(1)}$ $\qquad$ $\mathcal{O}(\alpha_s^2 \alpha) \leftrightarrow \mathcal{M}_2^{(2)}$

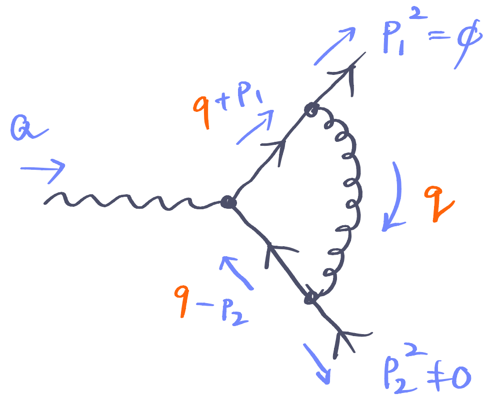
$$\Rightarrow |\mathcal{M}_2|^2 = |\mathcal{M}_2^{(0)}|^2 + 2\,\mathrm{Re}\left\{ (\mathcal{M}_2^{(0)})^* \mathcal{M}_2^{(1)} \right\} + \underbrace{|\mathcal{M}_2^{(1)}|^2 + 2\,\mathrm{Re}\left\{ (\mathcal{M}_2^{(0)})^* \mathcal{M}_2^{(2)} \right\}}_{\mathcal{O}(\alpha_s^2 \alpha^2)} + \dots$$

$\mathcal{O}(\alpha^2)$ $\qquad$ $\mathcal{O}(\alpha_s \alpha^2)$

"virtual corrections"

Virtual Corrections

* we already saw how loop integrals can diverge $\Rightarrow$ UV divergence $\frac{1}{\epsilon^n}$ @ $\mathcal{O}(\alpha_s^n)$
* now we encounter a new region that causes us trouble



$$\sim \int \underbrace{d^4q}_{q^3 dq} \frac{\{1, q, q^2\}}{q^2 \underbrace{(q+p_1)^2}_{q^2+2qp_1} \underbrace{(q-p_2)^2}_{q^2-2p_2 q}} \overset{q \text{ small}}{\sim} \int_{\emptyset} \frac{dq}{q}$$

$\Rightarrow$ log divergence in $q \to \emptyset$

* These are called infrared (IR) divergences
  $\hookrightarrow$ dimensional regularization also works here (now $\epsilon < \emptyset$)
* final result

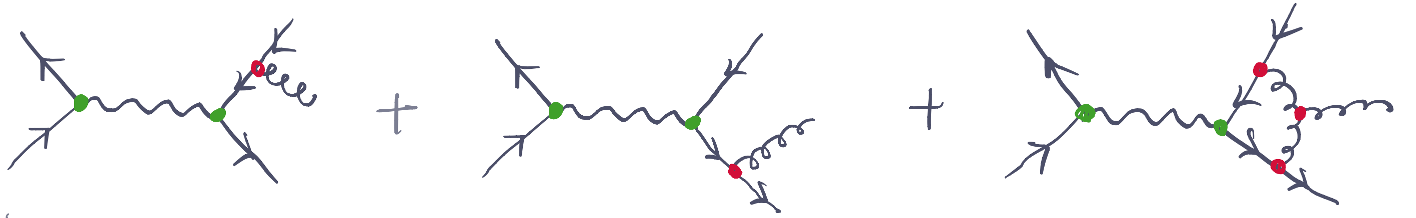
$$\sigma^V = \sigma^{LO} \left\{ -\frac{\alpha_s}{2\pi} C_F \frac{1}{\Gamma(1-\epsilon)} \left(\frac{4\pi\mu^2}{-s-i\emptyset} \right)^{\epsilon} \left[\frac{2}{\epsilon^2} + \frac{3}{\epsilon} + 8 \right] \right\}$$

1-loop result has no UV divergences (why?)
no renormalization!
$\rightarrow$ what are we missing?

QCD at Higher Orders

higher-order correction also must account for real emissions

$\mathcal{M}_3 =$



$+ \dots$

$\mathcal{O}(\sqrt{\alpha_s}\,\alpha) \leftrightarrow \mathcal{M}_3^{(0)}$ $\qquad$ $\mathcal{O}(\alpha_s^{3/2}\,\alpha) \leftrightarrow \mathcal{M}_3^{(1)}$

$$\Rightarrow |\mathcal{M}_3|^2 = \underbrace{|\mathcal{M}_3^{(0)}|^2}_{\mathcal{O}(\alpha_s\alpha^2)} + \underbrace{2\,\mathrm{Re}\{(\mathcal{M}_3^{(0)})^* \mathcal{M}_3^{(1)}\}}_{\mathcal{O}(\alpha_s^2\alpha^2)} + \dots$$

* at NNLO: also need $\mathcal{M}_4^{(0)}, \mathcal{M}_3^{(1)}, \mathcal{M}_2^{(2)}$
* at N^{3}LO: also need $\mathcal{M}_5^{(0)}, \mathcal{M}_4^{(1)}, \mathcal{M}_3^{(2)}, \mathcal{M}_2^{(3)}$

...

Real Corrections

$\left|\mathcal{M}^{(3)}_{q\bar{q}g}\right|^2 \sim \left|\right.$  $\left.\right|^2$

$$\Rightarrow W_{q\bar{q}g}(Q^2) = \int d\Phi_3 \; e^2 Q_q^2 \, g_s^2 \, 8 \, C_F N_c \, \frac{s_{13}^2 + s_{23}^2 + 2 s_{12} Q^2}{s_{13} \, s_{23}} \, , \qquad s_{ij} \equiv 2\, p_i \cdot p_j$$

* kinematic variables for real emission

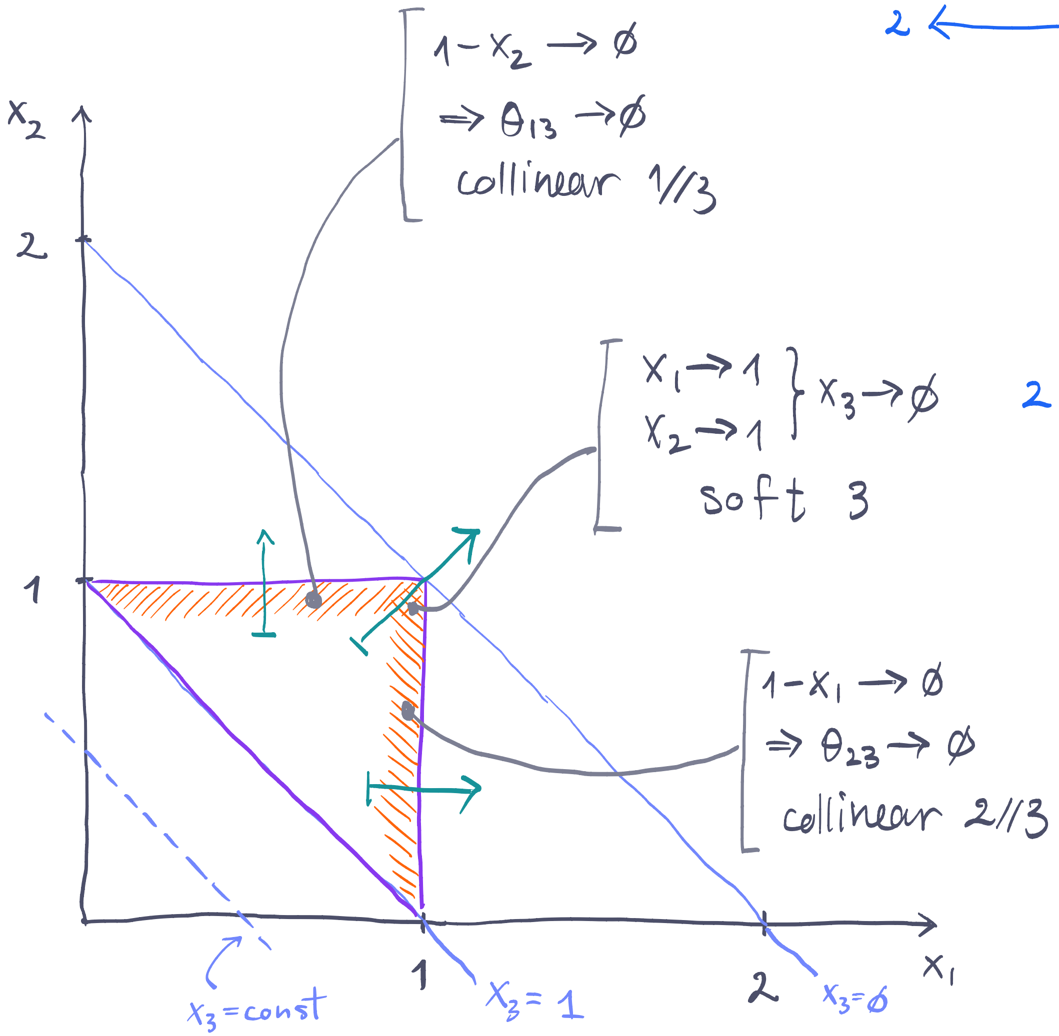
↳ energy fraction: $x_i = \dfrac{2 p_i Q}{Q^2}$ $\xrightarrow{\text{c.m. frame}}$ $E_i = \dfrac{\sqrt{s}}{2} x_i \quad (x_i > 0)$

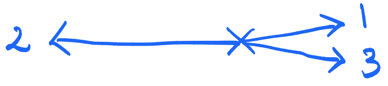
↳ energy conservation: $x_1 + x_2 + x_3 = \dfrac{2(\Sigma p_i) Q}{Q^2} = 2$

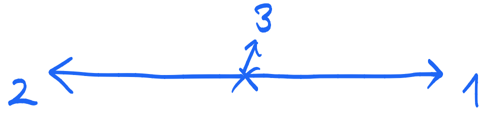
↳ $d\Phi_3(p_1, p_2, p_3; Q) = \dfrac{s}{2(4\pi)^3} \, dx_1 \, dx_2 \, dx_3 \;\; \delta(2 - x_1 - x_2 - x_3)$

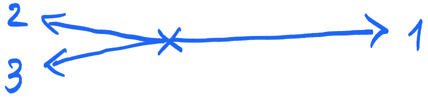
The Real Phase Space

* the energy fractions x_i lie within a triangle









angles:

$$2p_1p_3 = (p_1+p_3)^2 = (Q-p_2)^2 = Q^2 - 2p_2Q$$

$$\shortparallel \qquad\qquad\qquad\qquad \shortparallel$$

$$2E_1E_3(1-\cos\theta_{13}) \longleftrightarrow Q^2(1-x_2)$$

Real Corrections

* using the energy fractions, we get

$$W_{q\bar{q}g}(Q^2) = \frac{s}{2(4\pi)^3} \int_0^1 dx_1 \int_0^1 dx_2 \int_0^1 dx_3 \; \delta(2-x_1-x_2-x_3) \; e^2 Q_q^2 g_s^2 \, 8 \, C_F N_c \frac{x_1^2 + x_2^2}{(1-x_1)(1-x_2)}$$

this is terrible! diverges for $x_{1/2} \to 1$

* even though no loop integrals, need to regularize
  - here the divergences appear in $d\Phi_n$
  - dimensional regularization: $\frac{d^3 p}{(2\pi)^3 2E} \to \frac{d^{D-1} p}{(2\pi)^{D-1} 2E}$

$$\delta^{(4)}(P_{in}-P_{out}) \to \delta^{(D)}(P_{in}-P_{out})$$

* repeating the entire calculation in $D = 4 - 2\epsilon$ dimensions

$$\sigma^R = \sigma^{LO} \cdot \frac{\alpha_s}{2\pi} C_F \frac{\Gamma^2(1-\epsilon)}{\Gamma(1-3\epsilon)} \left(\frac{4\pi\mu^2}{s}\right)^\epsilon \left[\frac{2}{\epsilon^2} + \frac{3}{\epsilon} + \frac{19}{2}\right]$$

$e^+e^- \to$ Hadrons @ NLO

* now we finally have all pieces in place

$$\sigma^{NLO} = \sigma^{LO} \times \Big\{ 1 + \frac{\alpha_s}{2\pi} C_F \frac{\Gamma^2(1-\epsilon)}{\Gamma(1-3\epsilon)} \left(\frac{4\pi\mu^2}{s}\right)^{\epsilon} \left[\frac{2}{\epsilon^2} + \frac{3}{\epsilon} + \frac{19}{8}\right]$$

all divergences cancel

$$- \frac{\alpha_s}{2\pi} C_F \frac{1}{\Gamma(1-\epsilon)} \left(\frac{4\pi\mu^2}{-s-i0}\right)^{\epsilon} \left[\frac{2}{\epsilon^2} + \frac{3}{\epsilon} + 8\right] + \mathcal{O}(\epsilon) \Big\}$$

expand & simplify

$$= \sigma^{LO} \left\{ 1 + \frac{\alpha_s}{\pi} \right\}$$

too nice to be just an accident
→ what is the mechanism?

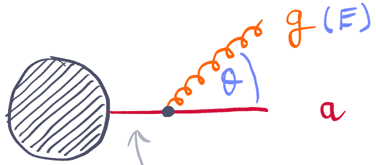
* the Ratio at the Z pole
(including Z-exchange diagram)

$$R^{LO} = 20.09 \quad \to \quad R^{NLO} = 20.84$$

$$R^{exp} = 20.79 \pm 0.04 \quad \text{(from LEP)}$$

The Pattern of IR Singularities

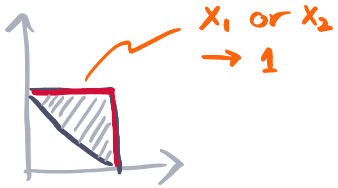
* IR singularities arise from intermediate propagators going on-shell



$$\sim \frac{1}{(p_a+k)^2} = \frac{1}{2 p_a \cdot k} = \frac{1}{E \, E_a \, (1-\cos\theta)}$$

potentially divergent

- soft $E \to 0$
- collinear $\theta \to 0$



↳ these correspond to degenerate configurations

⇒ divergence in $2\to3$ was located @ boundary

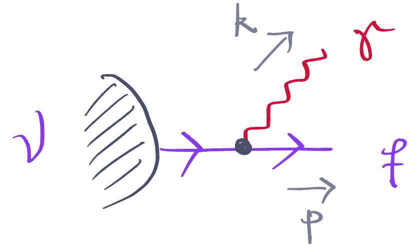
⇒ has correct $2\to2$ kinematics to match the virtuals

* the IR singularities factorize ⟷ are universal

↳ soft emissions have long wave length ⇝ cannot resolve hard "blob"

↳ collinear emissions only care about the parent it was emitted from

The Soft Limit in QED



$$= \left[\bar{u}(p) \left(-ieQ_f\gamma^\mu \right) \frac{i(\not{p}+\not{k})}{(p+k)^2} V(p+k) \right] \cdot \varepsilon^*_\mu(k)$$

$*$ k soft $\Rightarrow$ $k \ll p$ & neglect $\qquad (p+k)^2 = 2p\cdot k$

$$= \frac{eQ_f}{2p\cdot k} \left[\bar{u}(p) \underbrace{\gamma^\mu \not{p}}_{= -\cancel{\not{p}\gamma^\mu} + 2p^\mu} V(p) \right] \varepsilon^*_\mu(k)$$

$\cancel{\not{p}\gamma^\mu} \to \emptyset$

[Dirac algebra $\{\gamma^\mu,\gamma^\nu\} = 2g^{\mu\nu}$]

[Dirac eqn: $\bar{u}(p)\not{p} = \emptyset$]

$$= \frac{eQ_f}{2p\cdot k} 2p^\mu \varepsilon^*_\mu \left[\bar{u}(p) V(p) \right]$$

the original before γ-emit.

$*$ soft limit: emission from all legs

$$\mathcal{M}_{n+\gamma} \xrightarrow[\text{soft}]{\gamma} \varepsilon^*_\mu \, e\, J^\mu \cdot \mathcal{M}_n \quad \text{with} \quad J^\mu = \sum_{i=1}^n Q_i \frac{p_i^\mu}{(p_i\cdot k)}$$

eikonal current
- indep of spin
- indep of E_i
(only directions)

Factorization of IR Divergences in QCD

* soft limit:

$$\mathcal{M}_{n+1}(p_1,\dots,p_n,g(k)) \xrightarrow{k\to 0} \varepsilon^{a}_{\mu}(k)^{*}\, g_s\, \hat{J}^{a,\mu} \otimes \mathcal{M}_n(p_1,\dots,p_n)$$

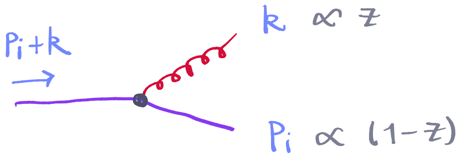
eikonal current $\quad \hat{J}^{a,\mu} = \sum_{i=1}^{n} \frac{p_i^{\mu}}{(p_i\cdot k)} \hat{T}_i^{a}$

$$\hat{T}_i^{a} = \begin{cases} t^a & q(in), \bar{q}(out) \\ -t^a & q(out), \bar{q}(in) \\ T_A^a & g \end{cases}$$

* collinear limit

$$|\mathcal{M}_{n+1}(p_1,\dots,p_n,g(k))|^2 \xrightarrow{k\parallel p_i} \frac{g_s^2}{(p_i\cdot k)} P_{gi}(z)\, |\mathcal{M}_n(p_1,\dots,(p_i+k),\dots p_n)|^2$$

Splitting function



$$P_{gq}(z) = C_F\,\frac{1+(1-z)^2}{z}$$

$$P_{gg}(z) = 2\,C_A\left[\frac{1-z}{z} + \frac{z}{1-z} + z(1-z)\right]$$

* beyond NLO: many more

QCD Emission Pattern in the Soft-Collinear Limit

A very useful limit: gluon emission both soft & collinear

exercise: take the soft, square it & take a collinear limit

$\sum_{i \neq j} \hat{T}_i = -\hat{T}_j$

(colour conservation)

$\hat{T}_q^2 = C_F$, $\hat{T}_g^2 = C_A$

$\Leftrightarrow$ start with the collinear & take the gluon soft $z \to \phi$

$$|\mathcal{M}_{n+g}|^2 \xrightarrow{g // i} \frac{g_s^2}{(p_i \cdot k)} \underbrace{P_{gi}(z)}_{\xrightarrow[z \to \phi]{} \frac{2 C_x}{z}} |\mathcal{M}_n|^2$$

$$\frac{2 C_x}{z}, \quad C_x = \begin{cases} C_F, & i = q \\ C_A, & i = g \end{cases}$$

gluons emit almost twice as much

$\Leftrightarrow$ Also the phase space factorizes $\left[\delta\left(\Sigma_i p_i + k - Q\right) \xrightarrow[\text{soft}]{k} \delta\left(\Sigma_i p_i - Q\right) \right]$

$$d\Phi_{n+g} = d\Phi_n \frac{d^3 k}{(2\pi)^3 \, 2E} \quad ;$$

parametrize k w.r.t. p_i

- $d^3 k = E^2 \, dE \, d\cos\theta \, d\varphi$
- $(p_i \cdot k) = E_i E (1 - \cos\theta)$

QCD Emission Pattern in the Soft-Collinear Limit

combining both expressions, the cross section reads

$$d\sigma_{n+g} \sim |\mathcal{M}_{n+g}|^2 \, d\Phi_{n+g} \xrightarrow[\text{collinear}]{\text{soft \&}} d\sigma_n \times d\omega_{i\to i+g}$$

$$d\omega_{i\to i+g} = \frac{g_s^2}{E_i E(1-\cos\theta)} \frac{2}{z} C_X \frac{1}{(2\pi)^3 2E} E^2 dE \, d\cos\theta \, d\varphi \, , \qquad \frac{d\cos\theta}{1-\cos\theta} \sim 2\frac{d\theta}{\theta}$$

$\cos\theta \simeq 1+\frac{\theta^2}{2}+\ldots$

$$= 2\frac{\alpha_s}{\pi} C_X \frac{dE}{E} \frac{d\theta}{\theta} \underbrace{\frac{d\varphi}{2\pi}}_{\to 1 \text{ after azimuthal average}}$$

* two logarithmic divergences

- soft $\frac{dz}{z} = \frac{dE}{E}$
- collinear $\frac{d\theta}{\theta} = \frac{1}{2}\frac{d\theta^2}{\theta^2}$

"double logs": for every α_s, up to two logs from the IR

QCD Emission Pattern in the Soft-Collinear Limit

* we derived the QCD emission probability

$$d\omega_{i\to i+g} = 2\,\frac{\alpha_s}{\pi}\,C_x\,\frac{dE}{E}\,\frac{d\theta}{\theta}$$

from factorization $\rightsquigarrow$ allows us to understand general QCD concepts

* alternatively, we can also use the transverse momentum $k_T \simeq E\theta$ and the pseudorapidity $\eta = -\ln\left[\tan\frac{\theta}{2}\right]$ to obtain

$$d\omega_{i\to i+g} = 2\,\frac{\alpha_s}{\pi}\,C_x\,\frac{dk_T}{k_T}\,d\eta$$

try to derive this

where is the double log?

Infrared Safety

* pattern of cancellations dictated by the Kinoshita-Lee-Nauenberg theorem
  - ⇾ for sufficiently inclusive observables ⇝ IR safe observable
* in general, measurements involve non-trivial fiducial constraints

$$\sigma \sim \int |\mathcal{M}_n|^2 \, d\Phi_n \quad \rightsquigarrow \quad \int |\mathcal{M}_n|^2 \, \mathcal{F}_n(p_1, \dots, p_n) \, d\Phi_n$$

measurement function
- fiducial cuts
- jet algorithm
- histogram bins
- isolation ...

$$\frac{d\sigma}{d\theta} = \int d\sigma \, \delta(\theta - \hat{\theta}(\Phi))$$

* an IR safe observable satisfies
  - ⇾ soft safety:

$$\mathcal{F}_{n+1}(p_1, \dots, p_{n+1}) \xrightarrow{p_i \to 0} \mathcal{F}_n(p_1, \dots, \cancel{p_i}, \dots, p_{n+1})$$

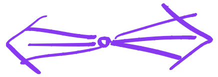
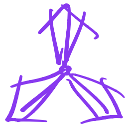
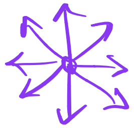
  - ⇾ collinear safety

$$\mathcal{F}_{n+1}(p_1, \dots p_{n+1}) \xrightarrow{p_i /\!/ p_j} \mathcal{F}_n(p_1, \dots, \cancel{p_i}, \dots, \cancel{p_j}, \dots, p_{n+1}, (p_i + p_j))$$

Infrared Safety

* thrust (event shape)

$$T = \max_{\vec{n}} \frac{\sum_i |\vec{p}_i \cdot \vec{n}|}{\sum_j |\vec{p}_j|}$$

- pencil like  $\Rightarrow T \to 1$
- mercedes star  $\Rightarrow T \to 2/3$
- isotropic  $\Rightarrow T \to 1/2$

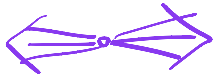
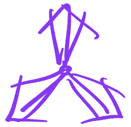
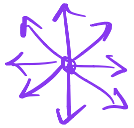
soft safety: add one soft particle p_{n+1} ($p_{n+1} \to \emptyset$)

$$T \xrightarrow{n+1} T' = \max_{\vec{n}} \frac{\sum_{i=1}^{n} |\vec{p}_i \vec{n}| + \overbrace{|\vec{p}_{n+1} \cdot \vec{n}|}^{\to \phi}}{\sum_{j=1}^{n} |\vec{p}_j| + \underbrace{|\vec{p}_{n+1}|}_{\to \phi}} \to T \quad \checkmark$$

Infrared Safety

* thrust (event shape)

$$T = \max_{\vec{n}} \frac{\sum_i |\vec{p}_i \cdot \vec{n}|}{\sum_j |\vec{p}_j|}$$

- pencil like  $\Rightarrow T \to 1$
- mercedes star  $\Rightarrow T \to 2/3$
- isotropic  $\Rightarrow T \to 1/2$

collinear safety: w.l.o.g. split up $p_n^\mu = \underbrace{z\, p_n^\mu}_{p_n'^\mu} + \underbrace{(1-z)\, p_n^\mu}_{p_{n+1}'^\mu}$

$$T \xrightarrow{n+1} T' = \max_{\vec{n}} \frac{\sum_{i=1}^{n-1} |\vec{p}_i \cdot \vec{n}| + |\vec{p}_n{}' \cdot \vec{n}| + |\vec{p}_{n+1}{}' \cdot \vec{n}|}{\sum_{j=1}^{n-1} |\vec{p}_j| + |\vec{p}_n{}'| + |\vec{p}_{n+1}{}'|} = \frac{\sum_{i=1}^{n-1} |\vec{p}_i \cdot \vec{n}| + \overbrace{z\,|\vec{p}_n \cdot \vec{n}| + (1-z)\,|\vec{p}_n \cdot \vec{n}|}^{|\vec{p}_n \cdot \vec{n}|}}{\sum_{j=1}^{n-1} |\vec{p}_j| + \underbrace{z\,|\vec{p}_n| + (1-z)\,|\vec{p}_n|}_{|\vec{p}_n|}}$$

$\to T$ ✓

Infrared Safety

Which one of these observables are IR safe?

* number of final-state particles
* $\sum_{i=1}^{n} p_{T,i}$
* $\sum_{i=1}^{n} (p_{T,i})^2$
* jets
* sphericity $S^{\alpha\beta} = \dfrac{\sum_{i=1}^{n} p_i^{\alpha} p_i^{\beta}}{\sum_{j=1}^{n} |\vec{p}_j|}$
* spherocity $S = \left(\frac{4}{\pi}\right)^2 \max_{\vec{n}} \left[\dfrac{\sum_{i=1}^{n} |\vec{p}_i \times \vec{n}|}{\sum_{j=1}^{n} |\vec{p}_j|} \right]^2$

Extra

[demo: toy NLO]