Parton Showers

Alexander Huss

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Contents

1 Introduction

We will investigate the gluon-emission probability off quarks and gluons and use that to implement a **very** simplified parton shower (single final-state leg, only primary branching, double-log approximation, only k_T and not proper kinematics, ...).

2 Emission probability and the Sudakov form factor

In the double-log approximation (soft *and* collinear emission), we have seen in the lecture that the emission probability is given as

$$\mathrm{d}\omega_{X\to X+g} = \frac{2\alpha_s C_X}{\pi}\,\frac{\mathrm{d}\theta}{\theta}\,\frac{\mathrm{d}E}{E}\,, \tag{1}$$

where E denotes the energy of the emitted gluon and θ the angle w.r.t. the parent particle. We denote the emitting particle by "X" and C_X is the associated colour factor. For quarks, $C_X = C_F = \frac{4}{3}$ and for gluons $C_X = C_A = 3$.

For any parton shower, we need to choose an evolution variable w.r.t. which we want to generate ordered emissions (angle θ, transverse momentum k_T, virtuality q^2, ...). We will perform a slight change of variables, $k_T \propto E\theta$ (transverse momentum w.r.t. the

parent parton) and $z \propto E$ (energy fraction of the emitted gluon) and integrate out the momentum fraction with the k_T constraint to obtain the emission probability

$$\mathrm{d}\mathcal{P}_{\text{emit}} = \frac{2\alpha_s C_X}{\pi}\,\frac{\mathrm{d}k_T}{k_T}\int_{k_T/k_T^{\max}}^{1}\frac{\mathrm{d}z}{z} = \frac{2\alpha_s C_X}{\pi}\,\frac{\mathrm{d}k_T}{k_T}\ln\left(\frac{k_T^{\max}}{k_T}\right), \tag{2}$$

where $k_T^{\max}$ denotes the upper bound on the transverse momentum that the emission can carry and is of the order of the hard scale.

We will further integrate over k_T with a lower cut-off k_T^{cut}, below which we consider the emission to be unresolved, and get for the probability of a single resolved emission

$$\mathcal{P}_{\text{emit}}(k_T > k_T^{\text{cut}}) = \frac{\alpha_s C_X}{\pi}\,\ln^2\left(\frac{k_T^{\max}}{k_T^{\text{cut}}}\right). \tag{3}$$

Note that this fixed-order result comes with a serious problem: For a power of α_s, we get two powers of a potentially large logarithm (the so-called "double logarithms" that appear frequently in higher-order calculations), a pattern that will continue to higher orders. For some representative values ($\alpha_s \sim 0.1$, $k_T^{\text{cut}} \sim \Lambda_{\text{QCD}} \sim 0.2\,\text{GeV}$, $k_T^{\max} \sim 100\,\text{GeV}$), we quickly realize that the large logarithm compensates the small value of the coupling, giving rise to a non-converging expansion. In such situations, where we are sensitive to large logarithms, we need to re-arrange the perturbative expansion in such a way to "re-sum" these large logarithms to all orders.

To accomplish this, we define the so-called Sudakov form factor $\Delta(t, t')$, which is the probability for *no resolved emissions* to happen between the evolution $t \to t'$, where we introduced an "evolution time" $t \equiv \ln(k_T^{\max}/k_T)$. The Sudakov form factor is *multiplicative*, i.e. obeys $\Delta(t, t'') = \Delta(t, t')\Delta(t', t'')$, and satisfies a differential equation reminiscent of that of radiative decay ($\mathcal{P}_{\text{no-emit}} = 1 - \mathcal{P}_{\text{emit}}$):

$$\frac{\mathrm{d}\Delta(t_0, t)}{\mathrm{d}t} = \Delta(t, t')\,\frac{\mathrm{d}\mathcal{P}_{\text{no-emit}}}{\mathrm{d}t'} = -\Delta(t, t')\,\frac{\mathrm{d}\mathcal{P}_{\text{emit}}}{\mathrm{d}t'}, \tag{4}$$

which has a simple solution

$$\begin{aligned}\Delta(t, t') &= \Delta(t')/\Delta(t)\,,\\ \Delta(t) \equiv \Delta(0, t) &= \exp\left\{-\frac{\alpha_s C_X}{2\pi}\,t^2\right\} = \exp\left\{-\frac{\alpha_s C_X}{2\pi}\,\ln^2\left(\frac{k_T^{\max}}{k_T}\right)\right\},\end{aligned} \tag{5}$$

which now has the large logarithm in the exponent. This solution therefore accomplishes exactly what we wanted: sum up the problematic logarithms to all orders, and in doing so, tame the otherwise divergent behaviour ($k_T \to 0$). It turns out that we can use the Sudakov form factor to sample successive emissions (it is a Markovian process), which we discuss in the next section together with a simple implementation.

3 Implementation

3.1 Interlude: Sampling

Assume we have a uniform random number generator at our disposal sampling values in the range $r \in [0, 1]$. We wish to generate samples x_i drawn from a probability distribution

$p(x)$. In the so-called *inversion method*, we use the fact that the *cumulant* distribution, $P(x) = \int_{-\infty}^{x} \mathrm{d}x' p(x')$, is a strictly monotonic function with values in $P(x) \in [0, 1]$ (it is a probability). We can thus obtain a sample x_i by drawing r_i uniformly from $[0, 1]$ and then *inverting* the relation $r_i = P(x_i)$ for $\mathrm{x_i}$. The sample x_i generated this way, follows the probability distribution $p(x)$.

Often times, the cumulant of the distribution is not easy to invert. In such cases, one can also use the *rejection method* (or "hit-and-miss") by finding a simpler distribution $\tilde{p}(x)$ that bounds $p(x)$ from above (in the simplest case the bound is just a constant). If we can draw samples from $\tilde{p}(x)$, all we need to do is to correct for the difference with respect to $p(x)$. This can be accomplished by drawing another uniform random number $s_i \in [0, 1]$ and only accepting the point $\tilde{x}_i$ generated with $\tilde{p}(x)$ with the probability $p(\tilde{x}_i)/\tilde{p}(\tilde{x}_i)$.

3.2 Our Toy Shower

With the Sudakov form factor in Eq. (5) at hand, we can easily iterate the sampling of emissions using the *inversion method* described above given that the cumulant that corresponds to the probability for the *next emission* is precisely the Sudakov. The steps of the shower are:

1. set $k_T = k_T^{\max}$
2. draw a uniform random number r in the range [0, 1]
3. solve $r = \Delta(k_T, k_T')$ for k_T', which is the new emission scale.
4. if $k_T' < k_T^{\mathrm{cut}}$, no resolvable emission can be generated: Terminate loop.
5. "generate" the emission at k_T', set $k_T = k_T'$ and go back to step 2.

The shower cut-off k_T^{cut} is typically set to $\mathcal{O}(1\,\mathrm{GeV})$ and represents the scale at which the perturbative shower description breaks down and the generated emissions are handed over to the hadronization model to simulate the non-perturbative hadronization stage of the simulation.

```python
#!/usr/bin/env python

import math
import random
import sys
import numpy as np

random.seed(42)
alphas = 0.118


def generate_emissions(kt_max: float, kt_cut: float, CX: float) -> list[float]:
    emissions = list()
    fac = CX * alphas / math.pi  # save common factor
    sudakov = 1.  # initialize to the starting scale
    while True:
```

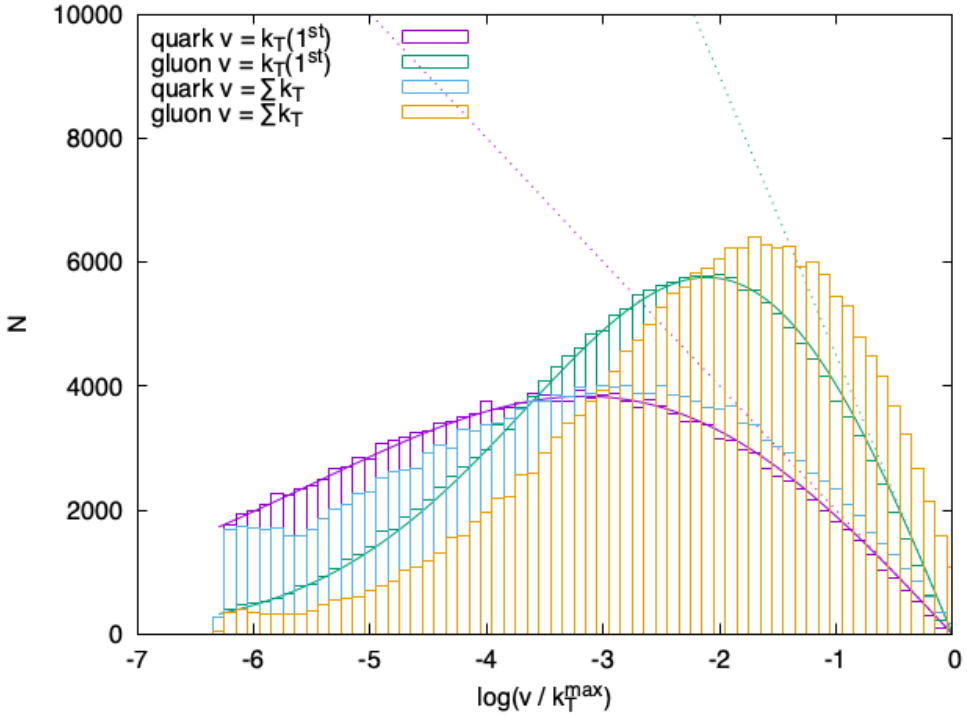
```
        sudakov *= random.uniform(0., 1.)
        #> invert `r = sudakov(kt, kt_new)`
        L2 = -math.log(sudakov) / fac
        kt = kt_max * math.exp(-math.sqrt(L2))
        if kt <= kt_cut:
            break
        emissions.append(kt)
    return emissions


if __name__ == "__main__":
    if len(sys.argv) < 3:
        raise RuntimeError("I expect at least two arguments:  kt_max [g|q]")
    kt_max = float(sys.argv[1])  # the hard scale
    kt_cut = 1.  # shower cutoff
    if sys.argv[2].lower() == "q":
        CX = 4. / 3.
    elif sys.argv[2].lower() == "g":
        CX = 3.
    else:
        raise RuntimeError("unrecognised parton: {}".format(sys.argv[2]))
    if len(sys.argv) >= 4:
        alphas = float(sys.argv[3])
    if len(sys.argv) >= 5:
        nevents = int(sys.argv[4])
    else:
        nevents = 1000
    for i in range(nevents):
        print("#event {} [{} {} {} {} {}]".format(i, kt_max, sys.argv[2], CX,
                                                  alphas, nevents))
        emissions = generate_emissions(kt_max, kt_cut, CX)
        if len(emissions) > 0:
            print("#summary {} {} {} {} {}".format(
                len(emissions), sum(emissions),
                math.log(sum(emissions) / kt_max), emissions[0],
                math.log(emissions[0] / kt_max)))
```

4 Playground

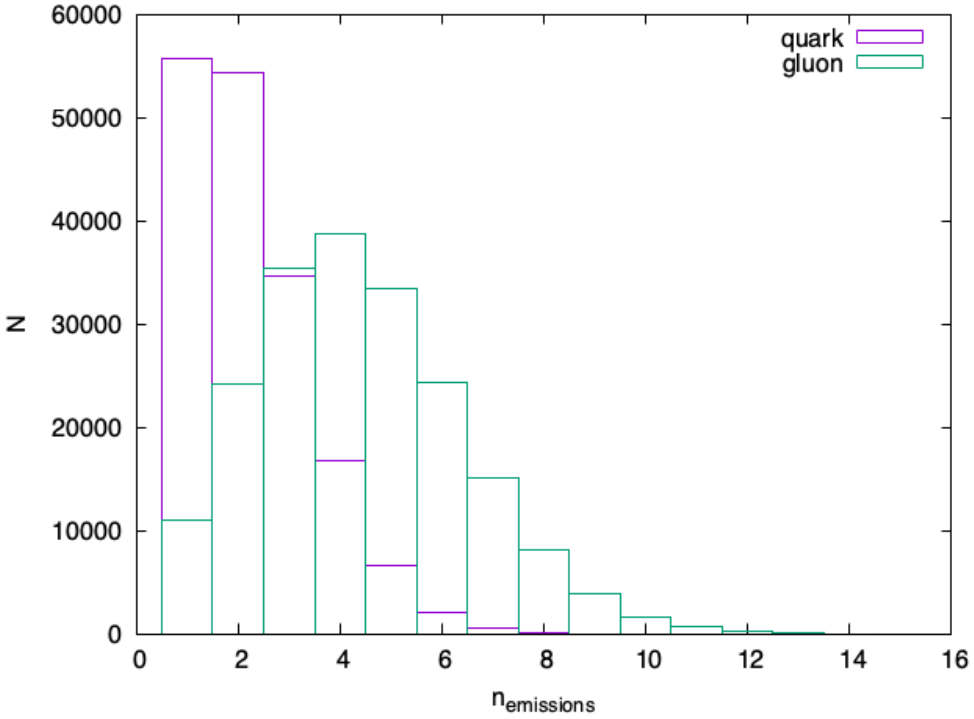
Let us use the implementation to generate some “events”

```
#> N, sum, log(sum), first, log(first)
python main.py 500 g 0.118 200000 | awk '$1~/summary/{print $2,$3,$4,$5,$6}' > data_g.dat
python main.py 500 q 0.118 200000 | awk '$1~/summary/{print $2,$3,$4,$5,$6}' > data_q.dat
```

First we plot the transverse momentum of the generated emissions (only first, sum of all).



We can see that the all-order description damps the divergent behaviour of a pure fixed-order prediction for $k_T \to 0$. We show separately the first emission alone, which can be compared to the analytic expression in Eq. (4) given by the solid line and is in excellent agreement with the shower. In addition, we include a dotted line that corresponds to the fixed-order expression that is logarithmically divergent. Given $C_A > C_F$, we also see how a gluon generates more emissions than quarks. This property can be exploited to try and discriminate between "quark jets" and "gluon jets".



- To increase the amount of emissions, try out setting the strong coupling to $\alpha_s = 0.5$. How does the picture change?