

QUANTUM CHROMODYNAMICS

& COLLIDER PHENOMENOLOGY

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Outline

Ø The Beginnings

I Basics of QCD

II QCD in e^+e^- Collisions

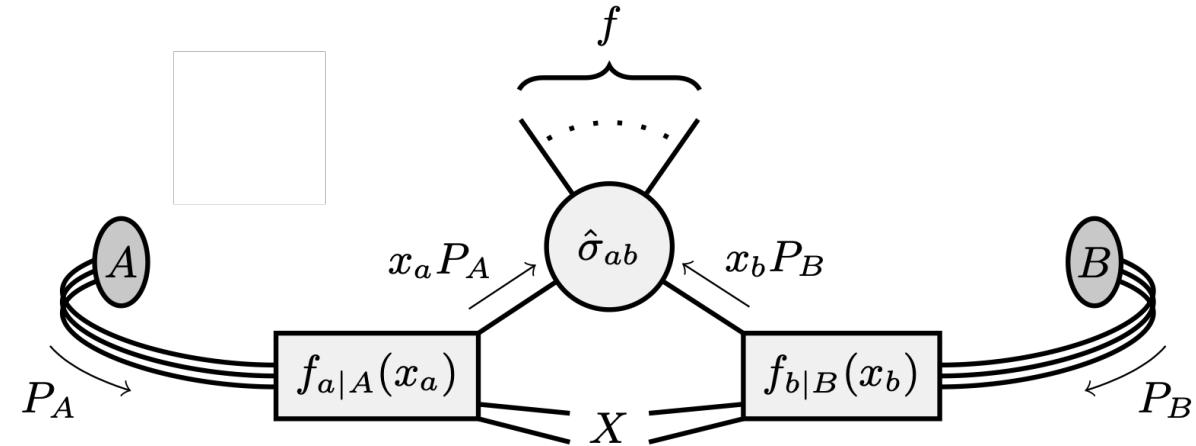
III Hadron Colliders

- parton distribution functions
- Drell Yan process
- DGLAP evolution
- resummation

IV Jet Physics

Hadron Colliders: The Parton Model

- * time scale of interactions binding the proton $\sim \gamma/m_p = T_{\text{proton}}$
- * hard interaction $Q \gg m_p$
 ↳ time scales $\sim \gamma_a \ll T_{\text{proton}}$
 ⇒ no time for struck parton to "negotiate" a response with others
- * asymptotic freedom: at high energies
 ↳ collection of free partons
 ⇒ incoherent sum of possible partonic cross sections
 ↳ sufficient to consider single parton



$$\begin{aligned}
 d\sigma_{A+B \rightarrow f} & (P_A, P_B) \\
 &= \sum_{a,b} \int_0^1 dx_a \int_0^1 dx_b \ f_a(x_a) f_b(x_b) d\hat{\sigma}_{a+b \rightarrow f} (x_a P_A, x_b P_B)
 \end{aligned}$$

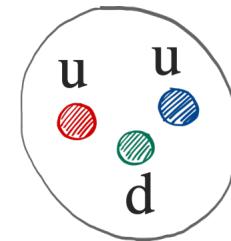
momentum fraction

parton distribution function (PDF)

$\hat{\sigma}$ ≈ number density for parton a w/ momentum fraction $[x_a, x_a dx_a]$ of parent hadron A

Parton Distribution Functions

* just free quarks? ($p \simeq (uud)$)



- How would the PDFs look like for such a proton?

PDF: number density of a parton flavour within the proton
to carry the momentum fraction x .

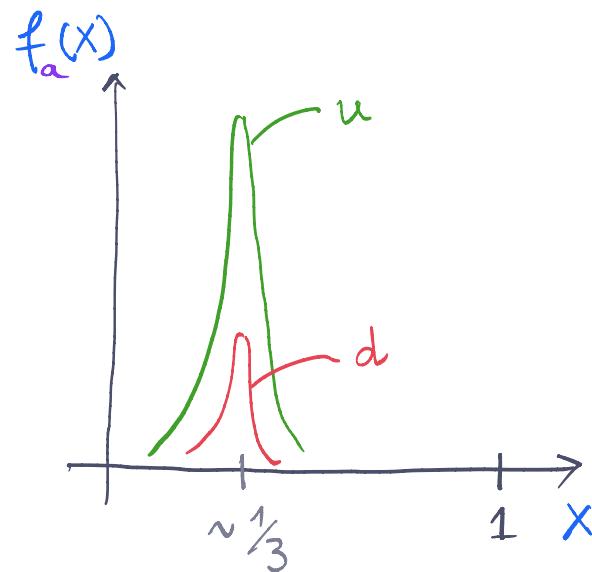
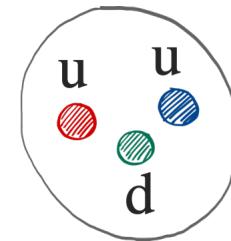
- what is the value of

$$\int_0^1 dx [f_u(x) - f_{\bar{u}}(x)]$$

$$\int_0^1 dx [f_d(x) - f_{\bar{d}}(x)]$$

Parton Distribution Functions

* just free quarks? ($p \simeq (uud)$)



$$f_u(x) \sim 2 \delta(x - \frac{1}{3})$$

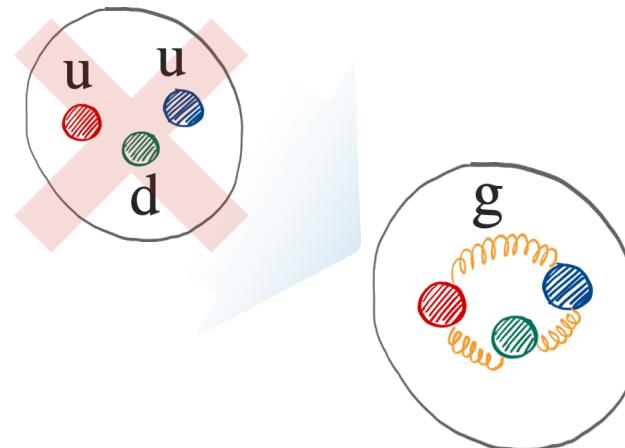
$$f_d(x) \sim 1 \delta(x - \frac{1}{3})$$

$$f_{\text{etc}}(x) \sim \emptyset$$

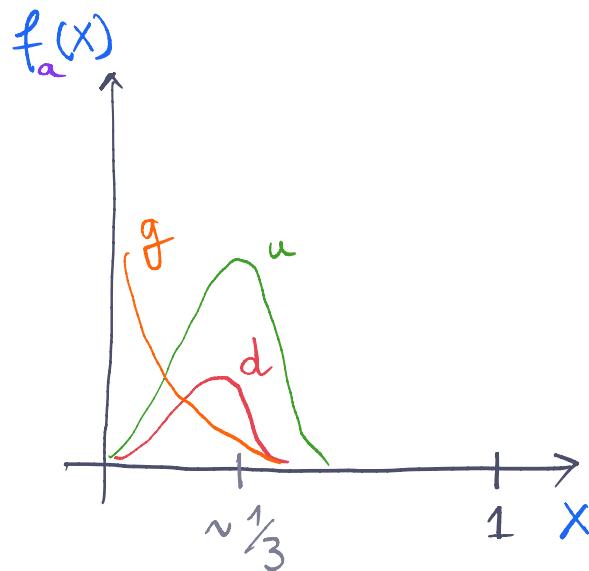
(+ some smearing)

Parton Distribution Functions

* just free quarks? ($p \simeq (uud)$)



* bound by gluons?



naive parton model

\leftrightarrow composition of point particles

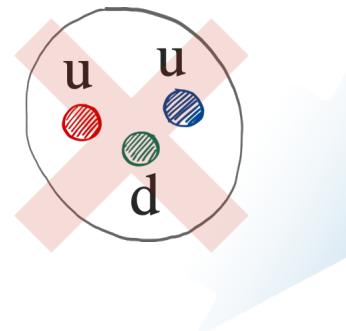
\mapsto zoom in ($Q^2 \uparrow$) \leftrightarrow same composition: scaling

PDFs independent on scale, at which it is probed
(as long as $Q^2 \gg m_p^2$)

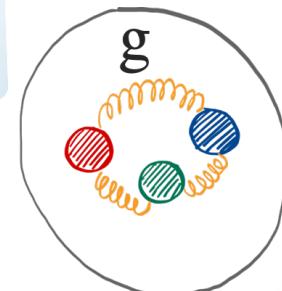
Parton Distribution Functions

[demo: PDFs]

* just free quarks? ($p \simeq (uud)$)

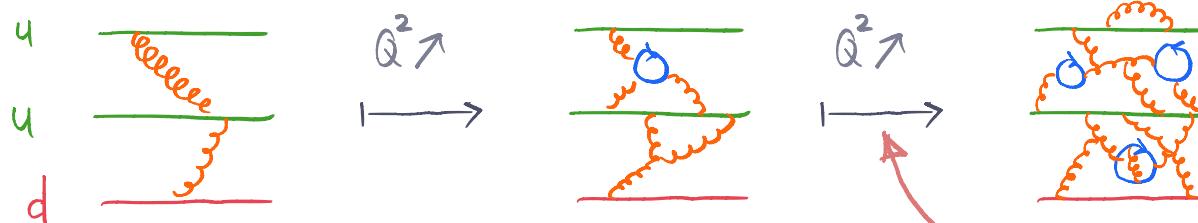


* bound by gluons?



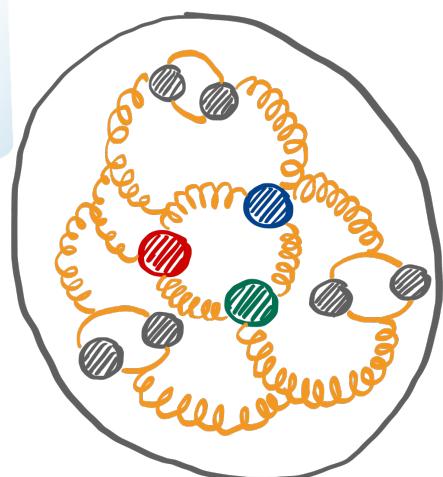
* QCD - improved parton model

↳ quantum fluctuations \rightarrow more g & $(g\bar{g})$ as we "zoom in" ($Q^2 \uparrow$)



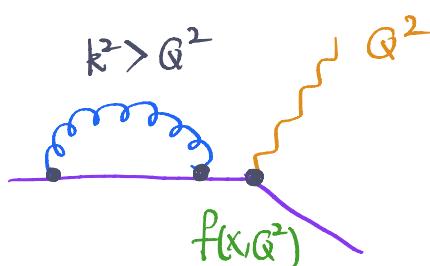
\Rightarrow predominantly shifts partons from high- x to low- x

evolution is perturbatively calculable!
(test of QCD)



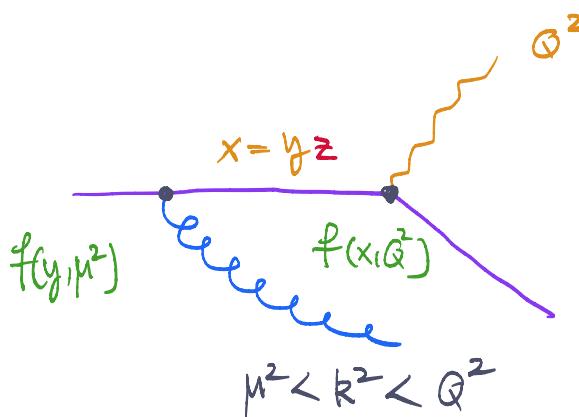
PDF Evolution

- * interaction of high-energetic ($Q^2 \gg m_p^2$) probe γ with a parton in the proton
 ↳ the probe "sees" a distribution $f(x, Q^2)$ for momentum fraction x
- * any virtual corrections @ $k^2 > Q^2$ cannot be resolved



(hard exchange among partons)
 Suppressed $\sum \sum_{k^2 > Q^2} \sim \frac{1}{\alpha^2}$

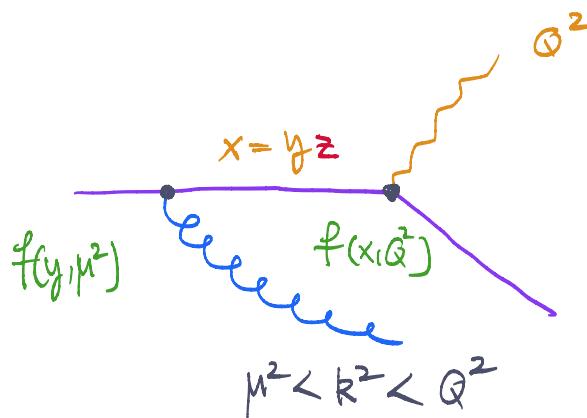
- * let's see what happens when Q^2 resolves a loop & how PDFs are affected



$$f(x, Q^2) = f(x, \mu^2) + \text{LO: no emission / evolution} \rightarrow \text{scaling}$$

$$+ \int_x^1 dy f(y, \mu^2) \int_0^1 dz \delta(x - yz) \int_{\mu^2}^{Q^2} dk^2 P(z, k^2) \Phi(x_s)$$

PDF Evolution



$$f(x, Q^2) = \underbrace{f(x, \mu^2)}_{\text{independent of } \mu} + \int_x^1 dy f(y, \mu^2) \int_0^1 dz \delta(x - yz) \int_{\mu^2}^{Q^2} dk^2 P(z, k^2)$$

independent of μ
(arbitrary reference)

one can show that \checkmark

$$P(z, \mu^2) = \frac{\alpha_s}{2\pi} \frac{1}{\mu^2} P(z)$$

calculable in
QCD

$$\frac{d f(x, Q^2)}{d \mu^2} = \phi \Rightarrow \frac{d f(x, \mu^2)}{d \mu^2} = \int_x^1 \frac{dz}{z} P(z, \mu^2) f\left(\frac{x}{z}, \mu^2\right) + (\text{higher orders})$$

\Rightarrow DGLAP evolution equations [Altarelli, Parisi; Gribov, Lipatov; Dokshitzer '77]

$$\frac{d f(x, \mu^2)}{d \ln \mu^2} = \int_x^1 \frac{dz}{z} \frac{\alpha_s}{2\pi} P(z) f\left(\frac{x}{z}, \mu^2\right)$$

PDF Evolution

* more than one flavour

$$\begin{array}{c} a \xrightarrow{\quad} b \\ \text{---} \quad \diagdown \\ z \quad (1-z) \end{array} \simeq P_{ba}(z)$$

↳ coupled integro-differential equations

$$\frac{d f_a(x/\mu^2)}{d \ln \mu^2} = \int_x^1 \frac{dz}{z} \frac{\alpha_s}{2\pi} P_{ab}(z) f_b(\frac{x}{z}/\mu^2)$$

* we cannot compute PDFs, but predict the evolution in p^{QCD}

* universality of P_{ab}

↳ measure in one process, use as input for another

Splitting Functions

$$P_{gg} \rightarrow \text{eeee} \quad C_F \left(\frac{1+z^2}{1-z} \right)_+ \xleftrightarrow{z \leftrightarrow (1-z)} P_{gq} \rightarrow \text{eeee} \quad C_F \frac{1+(1-z)^2}{z}$$

$$P_{qg} \cancel{\rightarrow \text{eeee}} \quad T_R [z^2 + (1-z)^2]$$

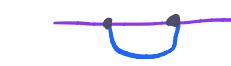
$$P_{qg} \cancel{\rightarrow \text{eeee}} \quad z \rightarrow (1-z)$$

$$2 C_A \left[\frac{z}{(1-z)_+} + \frac{1-z}{z} + z(1-z) \right] + \delta(1-z) \left(\frac{11N_c - 2N_f}{6} \right)$$

* the plus distribution $(\dots)_+$
 ↳ regulates the soft-gluon pole \rightsquigarrow why no need to regulate $z=\phi$?

$$\int_0^1 dz \left[f(z) \right]_+ g(z) \equiv \int_0^1 dz f(z) [g(z) - g(1)]$$

$$\text{show: } \left(\frac{1+z^2}{1-z} \right)_+ = \frac{3}{2} \delta(1-z) + 2z \left(\frac{1}{1-z} \right)_+ + (1-z)$$

↑ think of it as the divergence from 

PDF Evolution

* PDFs are universal

↳ pick a common "starting scale"

$$\mu_0^2 = \mathcal{O}(1 \text{ GeV}^2) \leftrightarrow m_p, m_c$$

↳ parametrise functions

$$x f(x, \mu_0^2) = A x^B (1-x)^C [1 + D x + E x^2] - A' x^{B'} (1-x)^{C'} \quad (\text{HERAPDF})$$

- Neural Network (NNPDF)

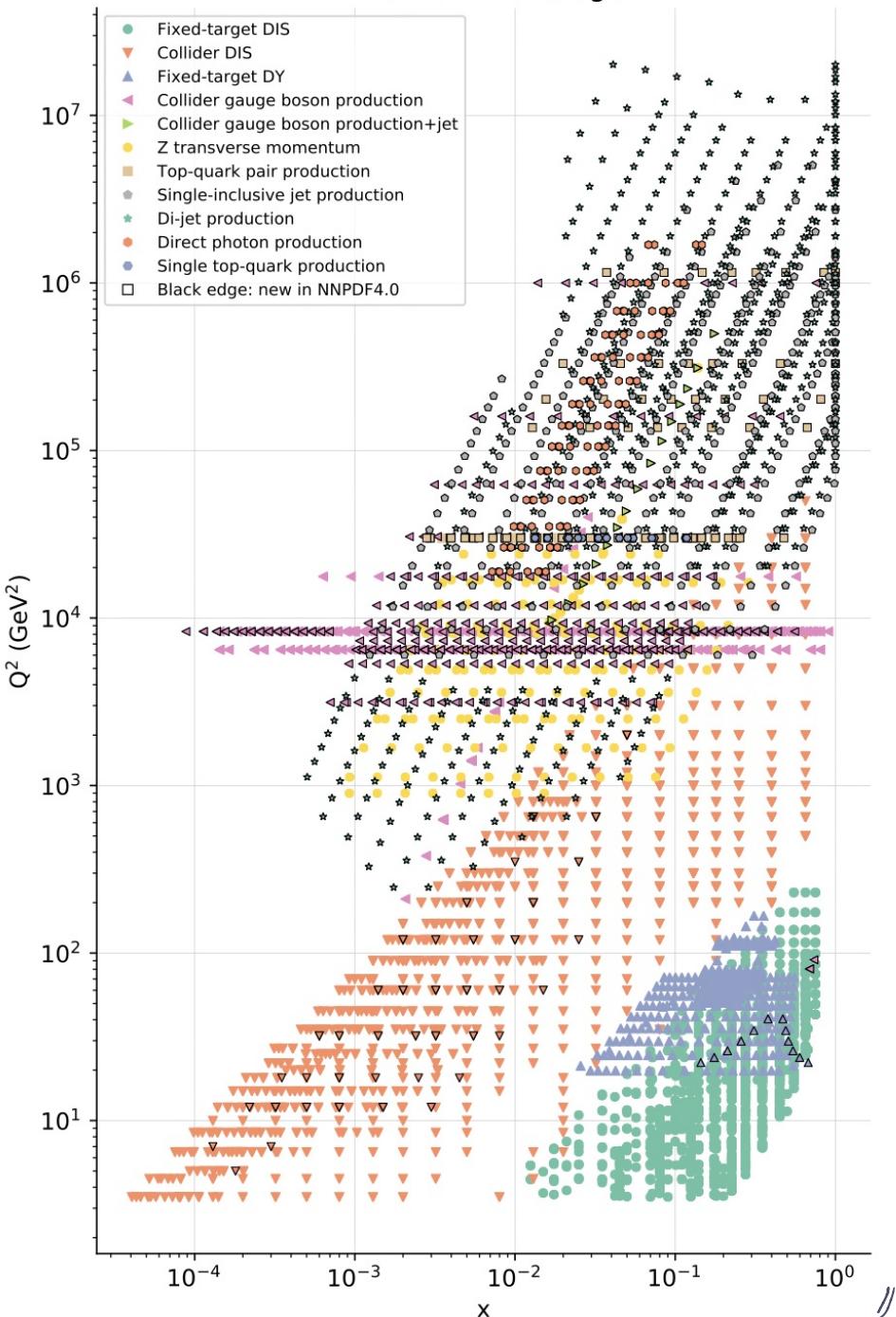
- ...

↳ fit the data

⇒ use evolution for predictions

[notebook: pdf]

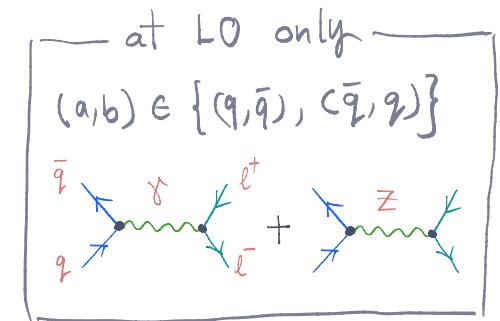
Kinematic coverage



The Drell-Yan Process $P + P \rightarrow l + \bar{l}$

[demo: Drell-Yan]

$$d\sigma_{DY}(P_A, P_B) = \sum_{a,b} \int_0^1 dx_a \int_0^1 dx_b f_a(x_a) f_b(x_b) d\hat{\Gamma}_{a+b \rightarrow l^+ l^-}(x_a P_A, x_b P_B)$$



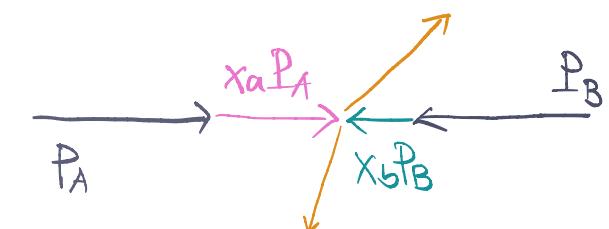
* Integrate out $Z \rightarrow l^+ l^-$ decay

* Observables of intermediate gauge boson $q^\mu = (P_1 + P_2)^\mu = (P_A + P_B)^\mu$

$$M_{ll} = \sqrt{q^2} \quad ; \quad Y_{ll} = \frac{1}{2} \ln \left(\frac{q^0 + q^3}{q^0 - q^3} \right)$$

rapidity: $Y \mapsto Y + \frac{1}{2} \ln(\frac{x_a}{x_b})$

$$\Rightarrow \frac{d^2\sigma_{DY}}{dM_{ll} dY_{ll}} = f_a(x_a) f_b(x_b) \frac{2 M_{ll}}{E_{cm}^2} \hat{\Gamma}_{a+b \rightarrow l^+ l^-} \Bigg|_{x_a x_b = \frac{M_{ll}}{E_{cm}} e^{\pm Y_{ll}}}$$



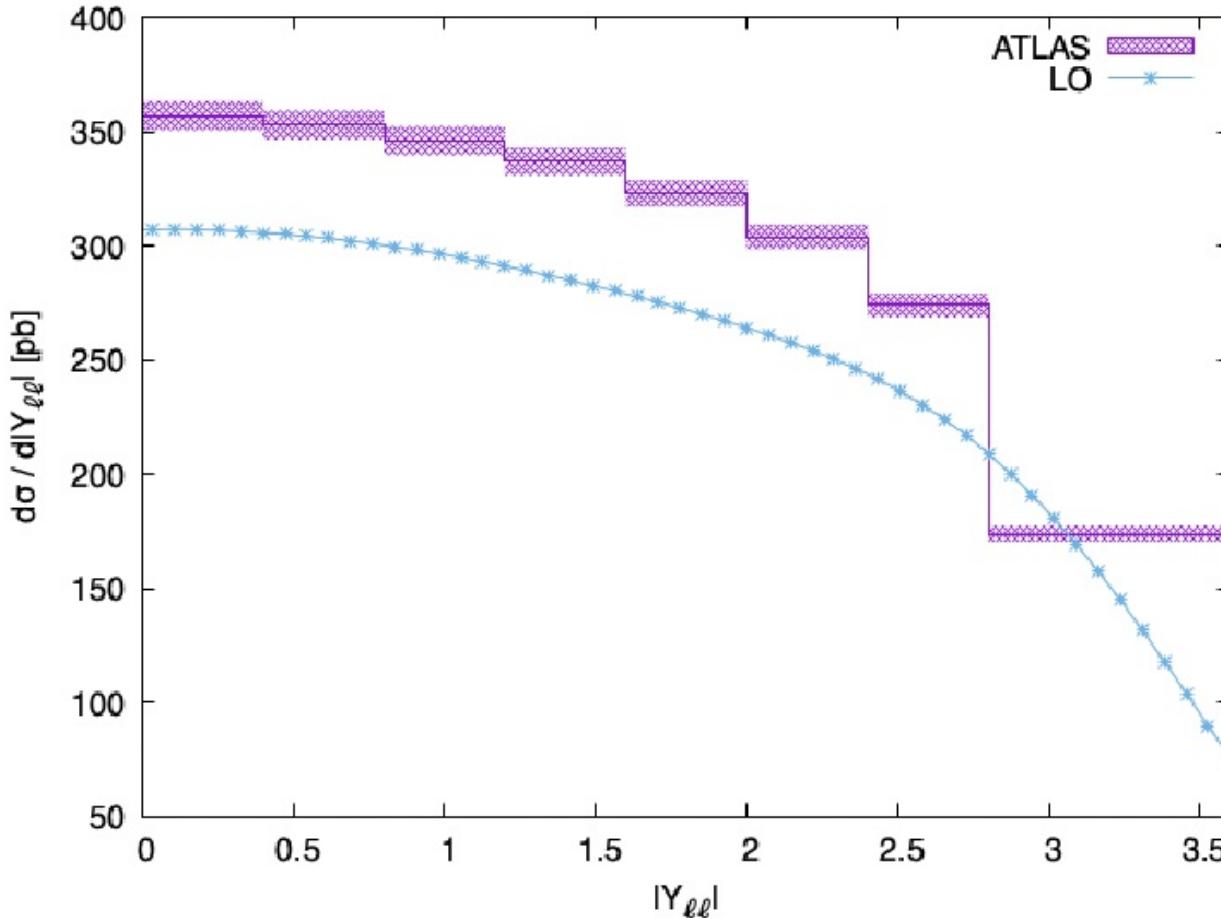
DY @ LO

ATLAS

$$\sigma_Z = 1055.3 \pm 0.7 \text{ (stat.)} \pm 2.2 \text{ (syst.)} \pm 19.0 \text{ (lumi.) pb}$$

LO :

#total 897.1018242299992 pb



* right ballpark

$$L_S \sim 0.1$$

↳ $O(10\%)$ anticipated

↳ often much worse!

(extreme: $gg \rightarrow H$ 100%)

@ LO only $q\bar{q}$ annihilation

↳ 50% of protons are g !

* no error estimate here

↳ no quantitative comp.

would scale variation help much?

⇒ need at least NLO!

Drell Yan @ NLO

- * the virtual & real corrections

$$M_{2 \rightarrow 2}^v = \text{diagram with loop} + \dots$$

$$M_{2 \rightarrow 3}^R = \text{diagram with loop} + \dots ; \quad \text{diagram with loop} + \dots$$

no UV divergences; why?
 ⇒ no renorm. needed

@ NLO, new partonic
 channels open up:
 $q\bar{q}, g\bar{q}, \bar{q}g, g\bar{q}\bar{q}$

IR divergences all
 over the place!
 ⇒ need to do everything
 in $D=4-2\epsilon$

- * Let's first rewrite the differential cross section

$$(\xi_{ab} = \frac{M}{\sqrt{s}} e^{iY})$$

$$\frac{d^2\sigma}{dM dY} = \sum_{q,b} \frac{1}{n_a n_b} \underbrace{\frac{M}{\xi_a \xi_b s^2}}_{\text{arg. fact.}} \int_{\xi_a}^1 \frac{dz_a}{z_a} \int_{\xi_b}^1 \frac{dz_b}{z_b} f_a(\xi_a/z_a) f_b(\xi_b/z_b) F_{ab}(Q^2, z_a, z_b)$$

with a perturbative expansion for the formfactor $F_{ab} = F_{ab}^{(0)} + F_{ab}^{(1)} + \dots$

Drell Yan @ NLO

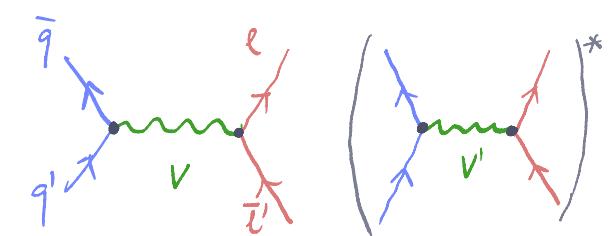
* at LO, we only have $q\bar{q}$ & $\bar{q}q$ channels (can work in $D=4$)

$$F_{q\bar{q}}^{(0)}(Q^2, z_a, z_b) = \delta(1-z_a) \delta(1-z_b) \sum_{V,V'} L^{VV'}(Q^2) \cdot H^{VV'}(Q^2) \quad (V,V' \in \{\gamma, Z, W^\pm\})$$

$$L^{VV'} = \frac{2}{3} \frac{\alpha}{Q^2} (V_\nu V'_\nu + a_V a_{V'}) \left(\frac{Q^2}{Q^2 - M_{V'}^2} \right) \left(\frac{Q^2}{Q^2 - M_V^2} \right)^*$$

$$H^{VV'} = 16\pi \alpha N_c (V_q V'_{q'} + a_q a_{q'}) Q^2$$

$$F_{q\bar{q}}^{(0)}(Q^2, z_a, z_b) = F_{q\bar{q}}^{(0)}(Q^2, z_b, z_a) = F_{q\bar{q}}^{(0)}(Q^2, z_a, z_b)$$



Drell Yan @ NLO

- * at NLO, we have 6 channels, 2 distinct ones: $(q\bar{q} \& \bar{q}\bar{q})$, $(qg \& \bar{q}g) \xleftrightarrow{z_a \leftrightarrow z_b} (gg \& g\bar{q})$
- * corrections w.r.t. LO formfactor:

$$F_{q\bar{q}}^{(1)} = \delta_{q\bar{q}}^{\text{NLO}}(Q^2, z_a, z_b) F_{q\bar{q}}^{(0)} \quad \& \quad F_{qg}^{(1)} = \delta_{qg}^{\text{NLO}}(Q^2, z_a, z_b) F_{q\bar{q}}^{(0)}$$

↳ let's just look at the divergent pieces

$$\delta_{q\bar{q}}^R(Q^2, z_a, z_b) \Big|_{\text{poles}} = \frac{\alpha_s}{\pi} C_F c_\epsilon \left(\frac{\mu^2}{Q^2}\right)^\epsilon \left\{ \begin{array}{l} \delta(1-z_a) \delta(1-z_b) \left[\frac{1}{\epsilon^2} \right] \\ - \delta(1-z_a) \frac{1}{\epsilon} \left[\frac{1-z_b}{2} + z_b \left(\frac{1}{1-z_b} \right)_+ \right] \\ - \delta(1-z_b) \frac{1}{\epsilon} \left[\frac{1-z_a}{2} + z_a \left(\frac{1}{1-z_a} \right)_+ \right] + \dots \end{array} \right\}$$

$$\delta_{q\bar{q}}^V(Q^2, z_a, z_b) \Big|_{\text{poles}} = \frac{\alpha_s}{\pi} C_F c_\epsilon \left(\frac{\mu^2}{Q^2}\right)^\epsilon \left\{ \delta(1-z_a) \delta(1-z_b) \left[-\frac{1}{\epsilon^2} - \frac{3}{2} \frac{1}{\epsilon} \right] + \dots \right\}$$

↳ $\frac{1}{\epsilon^2}$ cancels between "real" & "virtual" but left-over $\frac{1}{\epsilon}$ poles ?!

$$c_\epsilon \equiv \frac{(4\pi)^\epsilon}{\Gamma(1-\epsilon)}$$

Dress You @ NLO

- * Let's take $V+R$ and massage the expressions

$$\delta_{q\bar{q}}^{V+R} = -\frac{\alpha_s}{2\pi} C_F c_E \left(\frac{\mu^2}{Q^2}\right)^E \frac{1}{E} \left\{ \delta(1-z_a) \left(\frac{1+z_b^2}{1-z_b}\right)_+ + \delta(1-z_b) \left(\frac{1+z_a^2}{1-z_a}\right)_+ \right\}$$

$$= -\frac{\alpha_s}{2\pi} c_E \left(\frac{\mu^2}{Q^2}\right)^E \frac{1}{E} \left\{ \delta(1-z_a) P_{qg}(z_b) + \delta(1-z_b) P_{gq}(z_a) \right\}$$

$$\delta_{q\bar{q}}^{V+R} = \delta_{qg}^R = -\frac{\alpha_s}{2\pi} C_F c_E \left(\frac{\mu^2}{Q^2}\right)^E \frac{1}{E} \delta(1-z_a) [z_b^2 + (1-z_b)^2]$$

$\frac{N_c^2-1}{N_c} T_R$

$$= -\frac{\alpha_s}{2\pi} c_E \left(\frac{\mu^2}{Q^2}\right)^E \frac{1}{E} \underbrace{\frac{N_c^2-1}{N_c}}_{\text{changes the average factor}} \delta(1-z_a) P_{qg}(z_b)$$

changes the average factor $\frac{1}{n_g} \rightarrow \frac{1}{n_q}$

- * left-over singularities \rightsquigarrow collinear (universal!)



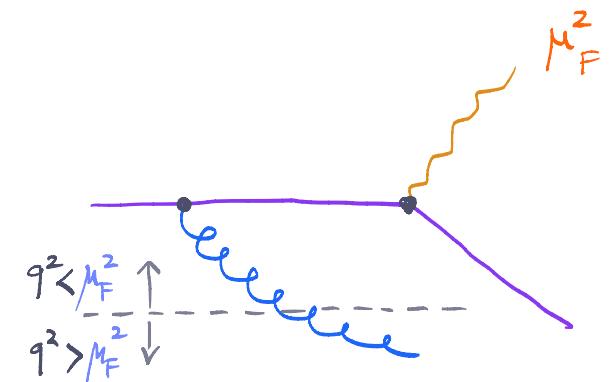
this should give you
a hint at what
we were missing!

The NLO PDF

$$\frac{1}{\epsilon} \left(\frac{\mu^2}{Q^2} \right) = \frac{1}{\epsilon} + \ln \left(\frac{\mu^2}{Q^2} \right) = \underbrace{\frac{1}{\epsilon}}_{q^2 < \mu_F^2} + \ln \left(\frac{\mu^2}{\mu_F^2} \right) + \underbrace{\ln \left(\frac{\mu_F^2}{Q^2} \right)}_{q^2 > \mu_F^2}$$

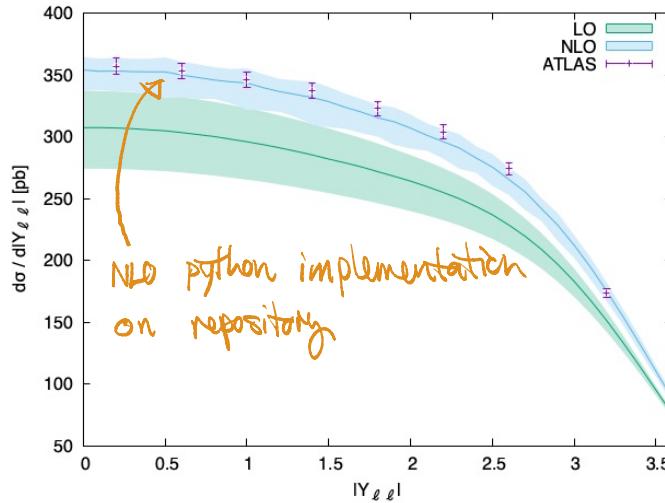
*the NLO PDF (renormalization)

$$f_a^{(0)}(x) \rightarrow f_a^{(0)}(x) + \frac{x_s}{2\pi} \frac{(4\pi)^{\epsilon}}{T(r-\epsilon)} \left(\frac{\mu^2}{\mu_F^2}\right)^{\epsilon} \frac{1}{\epsilon} \int_x^1 \frac{dz}{z} P_{ab}(z) f_b^{(0)}\left(\frac{x}{z}\right)$$



* with this, all divergences are gone! [repository]

Drell Yan @ NLO



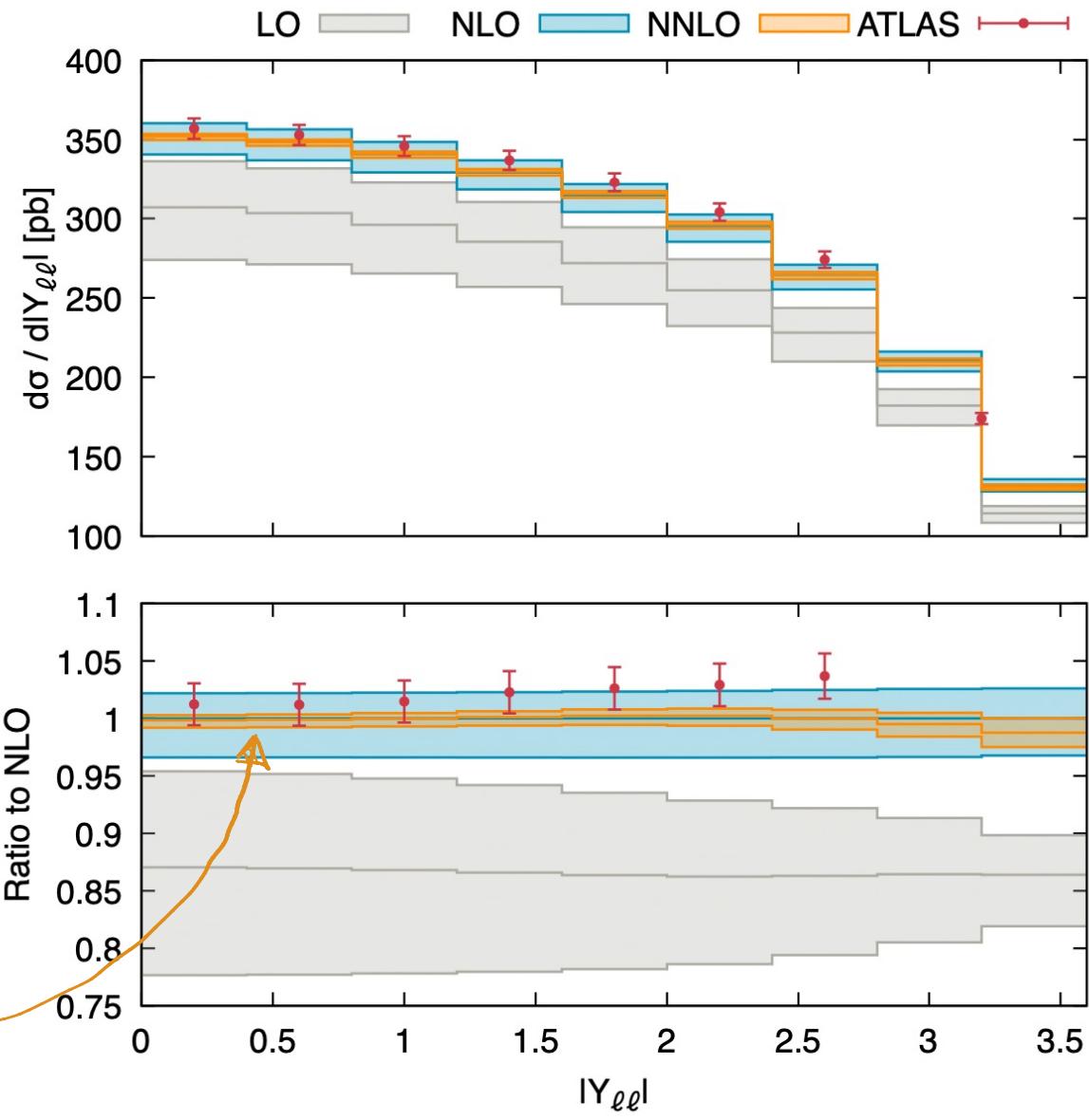
* NLO not enough: $\Delta_{\text{TH}} \sim 2x \Delta_{\text{EXP}}$

→ need NNLO

no longer something you want

to do in python

→ runcard for NNLOJET on repo



Accuracy of the Calculation v.s. Observables

using the Drell-Yan @ NLO calculation derived here,
what is the formal perturbative accuracy of those observables?

* σ^{fid}

* $\frac{d\sigma}{dY}$

* $\frac{d\sigma}{dP_{T,z}}$

* $\frac{d\sigma}{d\eta_{e^+}}$

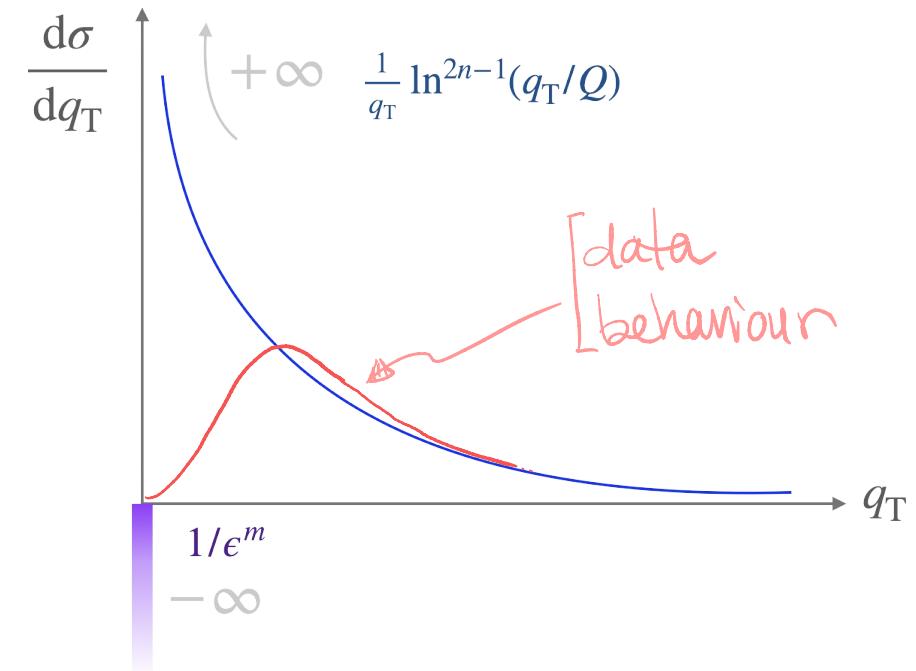
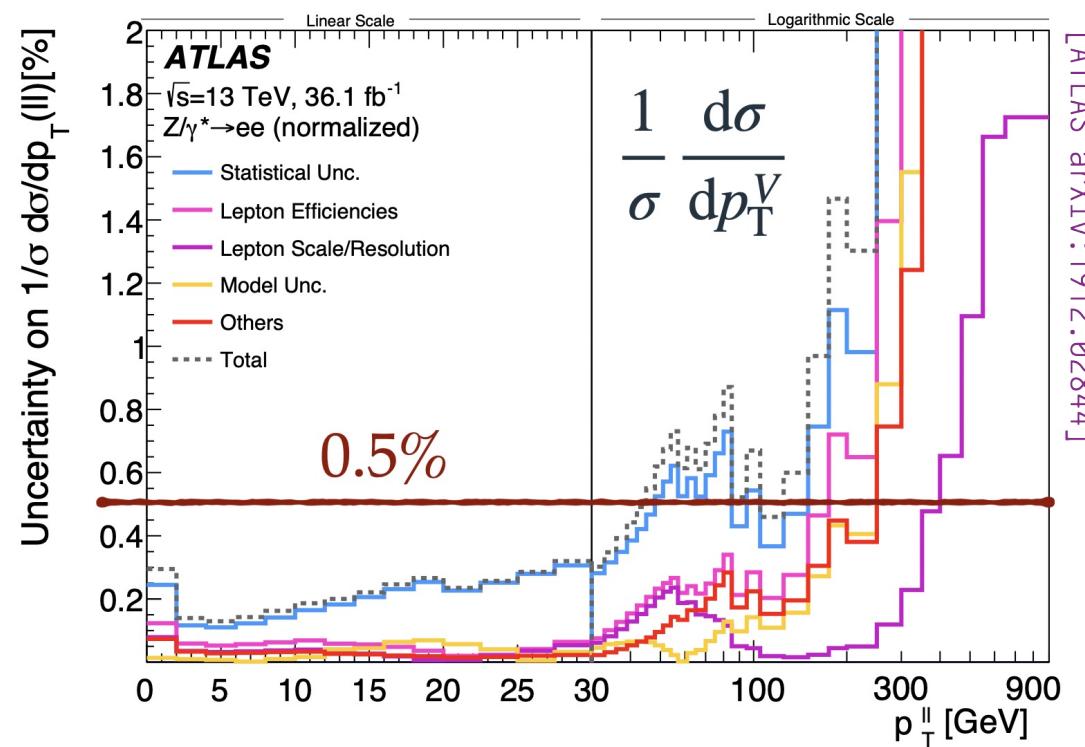
* $\frac{d\sigma}{dP_{T,e^+}}$

Drell-Yan @ small q_T

q_T distribution enters many important measurements: M_W , α_S , modeling, ...

* extraordinary experimental precision

* fixed-order: completely off



Drell-Yan @ small q_T

q_T probes small scales in a high-energy process ($Q \sim M_Z$)

\Rightarrow sensitivity to large logarithms $\ln(k_T/Q)$

* recall $d\omega_{x \rightarrow x+g}$ emission pattern in the soft-collinear limit

$$d\omega_{z \rightarrow z+g} = \frac{4\alpha_s}{\pi} C_F \ln(k_T/Q) \frac{dk_T}{k_T} \quad (\text{2x from } q \& \bar{q} \text{ leg})$$

* easiest to look at the cumulant $\Sigma(q_T)$ (prob for $P_{T,z} < q_T$)

$$\Sigma(q_T) = \frac{1}{\sigma} \int_0^{q_T} \frac{dk_T}{dp_{T,z}} dp_{T,z} = 1 + \frac{4\alpha_s}{\pi} C_F \int \frac{dk_T}{k_T} \ln(k_T/Q) [\Theta(k_T - q_T) - 1]$$

$P_{T,z} \equiv \phi$
@ LO

$Z \& \text{emission}$
 $\text{back-to-back @ NLO}$

$P_{T,z} \equiv k_T$

"virtual" corrections
via unitarity

$$= 1 - \frac{4\alpha_s}{\pi} C_F \int_{q_T}^Q \frac{dk_T}{k_T} \ln(k_T/Q) = 1 - \frac{2\alpha_s}{\pi} C_F \ln^2(Q/q_T)$$

can run the perturbative expansion!
(@ $q_T \approx 5 \text{ GeV}$)
 $\alpha_s L^2 \approx 1$

Drell-Yan @ small q_T

the pattern of logarithmic divergences follows from soft and collinear factorization ($L = \ln \frac{Q}{q_T}$)

| | | | | | | |
|---------------------------|---------------------|-----------------------|------------------|----------------|--------------|--|
| $\mathcal{O}(\alpha_s)$ | $\alpha_s L^2$ | $\alpha_s L$ | α_s | | | |
| $\mathcal{O}(\alpha_s^2)$ | $\alpha_s^2 L^4$ | $\alpha_s^2 L^3$ | $\alpha_s^2 L^2$ | $\alpha_s^2 L$ | α_s^2 | |
| $\mathcal{O}(\alpha_s^3)$ | $\alpha_s^3 L^6$ | $\alpha_s^3 L^5$ | | ... | α_s^3 | |
| : | | | | | | |
| $\mathcal{O}(\alpha_s^n)$ | $\alpha_s^n L^{2n}$ | $\alpha_s^n L^{2n-1}$ | | ... | α_s^n | |

double-logarithmic (DL)
tower of terms

\Rightarrow need to account for
arbitrary # of emissions
in DL ($\alpha_s L^2 \sim 1$) counting

- * Convergence can be restored by reorganizing the perturbative series across orders: **resummation**

q_T Resummation

Let's first consider the virtual & real corrections separately with δ to regulate

$$\text{Diagram} = \sum^{(I)V}(q_T) = \frac{4\alpha_s}{\pi} C_F \int_{\delta}^Q \frac{dk_T}{k_T} \ln\left(\frac{Q}{k_T}\right) [-1] = -\frac{4\alpha_s}{\pi} C_F \frac{1}{2} \ln^2\left(\frac{Q}{\delta}\right)$$

$$\text{Diagram} = \sum^{(I)R}(q_T) = \frac{4\alpha_s}{\pi} C_F \int_{\delta}^Q \frac{dk_T}{k_T} \ln\left(\frac{Q}{k_T}\right) \Theta(k_T < q_T) = \dots$$

$$= \frac{4\alpha_s}{\pi} C_F \left[\frac{1}{2} \ln^2\left(\frac{q_T}{\delta}\right) + \ln\left(\frac{Q}{q_T}\right) \ln\left(\frac{q_T}{\delta}\right) \right]$$

$$\sum^{(I)}(q_T) = \sum^{(I)V}(q_T) + \sum^{(I)R}(q_T) = \frac{4\alpha_s}{\pi} C_F \frac{1}{2} \ln^2\left(\frac{Q}{q_T}\right) \text{ or same as before}$$

q_T Resummation

arbitrary # of real emissions. For DL can assume strong ordering

$k_{T,1} \gg k_{T,2} \gg \dots \gg k_{T,n}$ \longleftrightarrow the configuration with highest logarithmic enhancement

$$= \sum^R (q_T)$$

$$= \left[\frac{4\alpha_s}{\pi} C_F \right]^n \int \limits_{\delta}^{k_{T,n}} \frac{dk_{T,1}}{k_{T,1}} \ln \left(\frac{Q}{k_{T,1}} \right) \int \limits_{\delta}^{k_{T,2}} \frac{dk_{T,2}}{k_{T,2}} \ln \left(\frac{Q}{k_{T,2}} \right) \dots \int \limits_{\delta}^{k_{T,n}} \frac{dk_{T,n}}{k_{T,n}} \ln \left(\frac{Q}{k_{T,n}} \right) \Theta(k_{T,1} < q_T)$$

1st emission sets the $p_{T,z}$

$$= \frac{1}{n!} \left[\frac{4\alpha_s}{\pi} C_F \right]^n \left[\int \limits_{\delta}^{q_T} \frac{dk_T}{k_T} \ln \left(\frac{Q}{k_T} \right) \right]^n \Rightarrow \boxed{\sum^{\sum_n R^n} = \exp \left(-\frac{E}{\epsilon} \right)}$$

q_T Resummation

analogously: arbitrary # of virtual amissions.

\Rightarrow all amissions to all orders

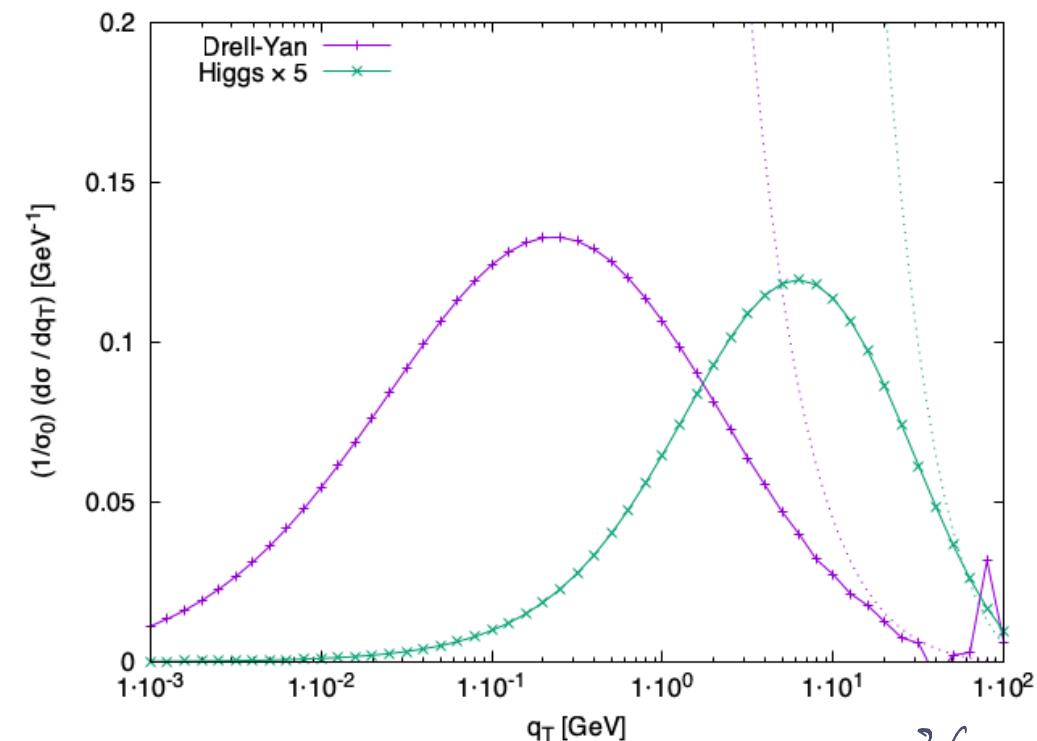
$$\sum^{\text{DL}}(q_T) = \sum^{\Sigma_n V^n} \times \sum^{\Sigma_n R^n} = \exp\left(-\frac{\epsilon}{\text{f}} + \text{cloud}\right)$$

$$= \exp\left(-\frac{2\alpha_s}{\pi} C_F \ln^2\left(\frac{Q}{q_T}\right)\right)$$



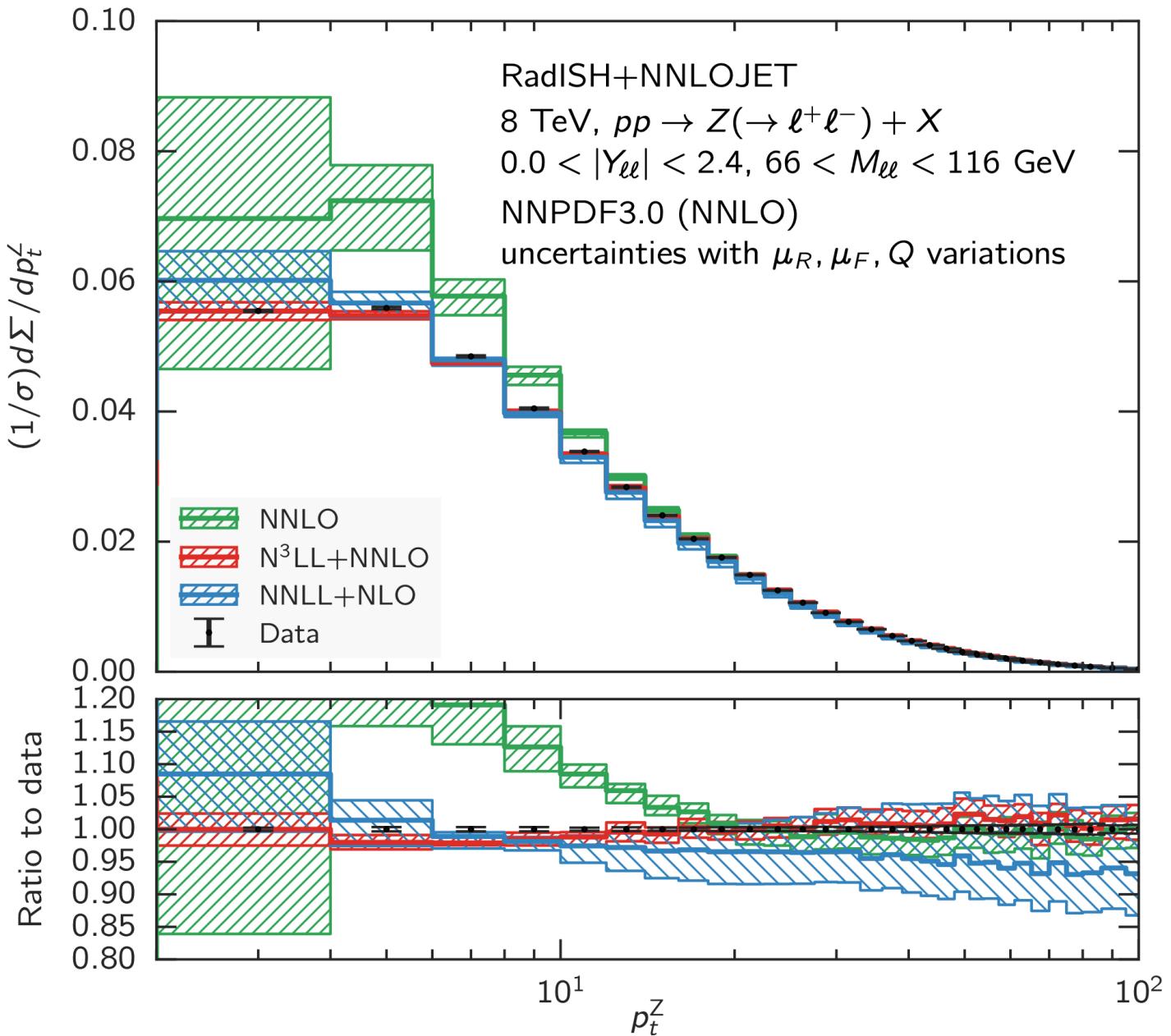
Sudakov formfactor

$$\boxed{\sum^{\Sigma_n V^n} = \exp\left(\text{cloud}\right)}$$



q_T Resummation

- * DL resummation can be extended to include additional tower of logarithms
 - ↳ drop strong ordering
 - ↳ b-space for $\delta^{(2)}(\vec{P}_{T,z} + \sum_i \vec{k}_{T,i})$
 - ↳ hard-collinear limit, α_s running, DGLAP evolution h.o. corrections to ingredients
 - ↳ state of the art: N^3LL



Extra

PDF evolution

