

Exercise 1:

Remind yourself that

$$\not{p} = p^\mu \gamma_\mu$$

$$\{\gamma^\mu, \gamma^\nu\} = 2 g^{\mu\nu}$$

$$\frac{1}{\not{p}-m} = \frac{\not{p}+m}{p^2-m^2}$$

$$\not{p} \not{p} = p^2 \mathbb{1}$$

$$\frac{1}{\not{p}-m} = \frac{\not{p}+m}{p^2-m^2}$$

$$\gamma^\mu \gamma_\mu = 4 \cdot \mathbb{1}$$

$$\gamma^\mu \gamma^\nu \gamma_\mu = -2 \gamma^\nu$$

$$\text{Tr}(\gamma^\mu \gamma^\nu) = 4 g^{\mu\nu}$$

Exercise 2:

Show that translational invariance of $G^{\vec{P}_1, \dots, \vec{P}_n}(x_1, \dots, x_n)$ implies momentum conservation:

$$\tilde{G}^{\vec{P}_1, \dots, \vec{P}_n}(p_1, \dots, p_n) = (2\pi)^4 \delta^{(4)}(p_1 + \dots + p_n) f(p_1, \dots, p_{n-1})$$

$$\tilde{f}(p) = \int d^4x e^{-ipx} f(x), \quad f(x) = \int \frac{d^4p}{(2\pi)^4} e^{+ipx} \tilde{f}(p)$$

Exercise 3

- consider the Lagrangian of the free photon field (+ gauge fixing term)

$$\mathcal{L}_{A,0} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2\zeta} (\partial_\mu A^\mu)^2 \quad (F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu)$$

$$\Rightarrow \text{e.o.m.} \quad [g^{\mu\nu} \square - (1 - \frac{1}{\zeta}) \partial^\mu \partial^\nu] A_\nu(x) = 0$$

The Green's function is defined via

$$[g_{\mu\nu} \square - (1 - \frac{1}{\zeta}) \partial_\mu \partial_\nu] \Delta_F^{\nu\rho}(x) \stackrel{!}{=} g_\mu^\rho \delta(x)$$

Derive $\Delta_F^{\mu\nu}$ in momentum space using

$$\Delta_F^{\mu\nu}(x) = \int \frac{d^4 p}{(2\pi)^4} e^{ipx} \Delta_F^{\mu\nu}(p)$$

Exercise 4

(a) draw all graphs to the 2-pt function $G^{\phi\phi}$
up to $\mathcal{O}(g^2)$ in $Z_{\text{int}} = \frac{g}{3!} \phi^3$. (

determine the symmetry factors.

- connected graphs $G_{\text{con}}^{\Phi_1 \dots \Phi_n}(x_1, \dots x_n) := G^{\Phi_1 \dots \Phi_n}(x_1, \dots x_n) \Big| \text{only connected graphs}$
 $\hookrightarrow G_{\text{con}}^{\Phi_1 \dots \Phi_n} = \frac{\delta}{i \delta J_1} \dots \frac{\delta}{i \delta J_n} Z_{\text{con}}[\{J\}] \Big|_{J=0} \quad (Z_{\text{con}} = \ln(Z))$

(b) draw all connected graphs to the 2-pt function $G^{\phi \dots \phi}$
up to $\mathcal{O}(\lambda^2)$ in $Z_{\text{int}} = -\frac{\lambda}{4!} \phi^4$. (

determine the symmetry factors.

Exercise 5

- Consider a scalar theory with an arbitrary interaction (\mathcal{L}_{int})

Let $i \sum_{(p^2)}^{\phi\phi} \equiv \text{---} \xrightarrow{\text{P}} \textcircled{1\text{PI}} \text{---}$ be the sum of all 1PI graphs
"self energy" \hookrightarrow not allowed $\text{---} \textcircled{O} \text{---}$

- Give a graphical representation for the 2-pt function $G^{\phi\phi}(p_1 - p)$ to all orders using $i \sum_{(p^2)}^{\phi\phi}$
- resum the diagrammatic series ("Dyson series" $\sum_{n=0}^{\infty} a^n = \frac{1}{1-a}$)
- give an expression for the 2-pt vertex function $\Gamma^{\phi\phi}(p_1 - p)$

Exercise 6

- the generating functional $\Gamma[\{\Phi\}]$ for vertex functions is the Legendre-transformation of $Z_{\text{con}}[\{J\}]$

$$\Gamma^{\Phi_1 \dots \Phi_n}(x_1, \dots, x_n) = \frac{\delta}{\delta \Phi_1(x_1)} \dots \frac{\delta}{\delta \Phi_n(x_n)} \Gamma[\{\Phi\}]$$

→ one can show that at tree-graph level \rightsquigarrow Appendix

$$\Gamma^{(0)}[\{\Phi\}] = i \int d^4x \mathcal{L}(\{\Phi(x)\})$$

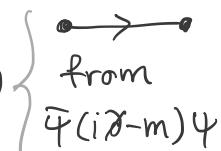
⇒ very easy derivation of Feynman rules ($\Gamma^{(0)} \Phi_1 \dots \Phi_n$)

- ① collect terms in $i\mathcal{L}$ that contain the fields $\Phi_1 \dots \Phi_n$
- ② replace all derivatives by "-i" the incoming momentum ($\partial_\mu \rightarrow -i p_\mu$)
- ③ sum over all permutations of indices & momenta of identical external fields
- ④ drop all fields.

derive the Feynman rules for

(a) $\mathcal{L}_g = ig A^\mu (B \partial_\mu C - C \partial_\mu B)$

(b) $\mathcal{L}_g = g A^\mu A_\mu CD$

(c) 

Exercise 7

- using the tree-level identity

$$k^\mu \cdot \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} = eQ \left[\begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} - \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} \right]$$

\vec{k}
 \vec{P}
 $\vec{P+k}$

show the transversality of the photon self-energy

$$k^\mu \cdot \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} \xrightarrow{\text{1PI}} = k^\mu \sum_{\mu\nu}^{AA}(k, -k) = 0$$

at 1-loop and 2-loop order.

- similarly, one can show

$$k^\mu \left[\begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} + \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} + \dots + \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} \right] = 0$$

\Rightarrow Ward Identity for the photon propagator:
 $- \frac{1}{i} k^2 k^\mu G_{\mu\nu}^{AA}(k, -k) = i k^\mu$

(no h.o. corrections to the longitudinal part)

$$k^\mu \cdot \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} = eQ \left[\begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} - \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} \right]$$

\vec{q}
 $\vec{q-k}$
 \vec{p}
 $\vec{p+k}$

Exercise 8

(a) determine a formula for the superficial degree of divergence

$$\omega(G) = \begin{cases} 0 \rightarrow \text{log. div} \\ 1 \rightarrow \text{lin. div.} \\ 2 \rightarrow \text{quad. div.} \\ \vdots \end{cases} \quad (\text{we're interested in 1PI graphs})$$

↑
graph

① write $\omega(G)$ in terms of L (# loops), I_A (# internal photon prop.), I_4 (# internal fermion prop.).

② "Euler's loop equation": count the # of momentum-conserv. constraints to write L in terms of I_A, I_4 , and V (# vert.)

③ inspect the vertex $(A_\mu \bar{\psi} \psi)$ and find relations

$$V = X(I_4, E_4) \quad \& \quad V = Y(I_A, E_A)$$

with $E_{4/A}$: # of external ψ/A fields

④ show that $\omega(G)$ only depends on $E_{4/A}$

cont.
 ↗

Exercise 8

(b) find all superficially divergent vertex functions in QED

Exercise 9

- compute the D-dimensional solid angle $\Omega_D = \int d\Omega_D$

↪ the 1-dim Gaussian integral:

$$\sqrt{\pi} = \int_{-\infty}^{+\infty} dx e^{-x^2}$$

↪ Gamma function:

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt$$

Exercise 10

- expand

$$A_0(m) = -m^2 \left(\frac{m^2}{4\pi\mu^2} \right)^{\frac{D-4}{2}} \Gamma\left(\frac{2-D}{2}\right)$$
$$= -m^2 \left(\frac{m^2}{4\pi\mu^2} \right)^{-\epsilon} \Gamma(-1+\epsilon)$$

in ϵ ($D \rightarrow 4$) up to finite terms

- Γ has poles for $z=0, -1, -2, \dots$

- $\Gamma(\epsilon) = \frac{1}{\epsilon} - \gamma_E + \theta(\epsilon)$

- $\Gamma(z+1) = z \Gamma(z)$

Exercise 11

- derive the Feynman parametrisation: $\frac{1}{A_1 A_2} = \int_0^1 dx \frac{1}{[A_1(1-x) + A_2 x]^2}$

① convince yourself that $\frac{1}{A} = \int_0^\infty dt e^{-At}$ & write

$$\frac{1}{A_1 A_2} = \int_0^\infty dt_1 dt_2 e^{-A_1 t_1 - A_2 t_2} \quad (\text{Schwinger parametrisation})$$

② hint: insert $1 = \int_0^\infty d\lambda \delta(\lambda - t_1 - t_2)$

-
- generalization to higher powers:

$$\frac{1}{A^\nu} = \frac{1}{\Gamma(\nu)} \int_0^\infty dt t^{\nu-1} e^{-At} \quad (\text{by differentiation})$$

- Cheng-Wu Theorem

only a subset $S \subset \{1, \dots, N\}$ in the delta: $\delta(1 - \sum_{i=1}^N x_i) \rightsquigarrow \delta(1 - \sum_{i \in S} x_i)$
for $j \notin S$: $\int_0^\infty dx_j$

Exercise 12

- compute $B_0(\mathbb{P}^2, 0, m)$ up to $\Theta(\epsilon)$

imagine: 

↪ result: $1 - \ln\left(\frac{m^2}{\mu^2}\right) + 2 + \frac{m^2 - p^2}{p^2} \ln\left(\frac{m^2 - p^2 - i0}{m^2}\right) + \Theta(\epsilon)$

Exercise 13

- solve the system of linear equations for B_{00} & B_{11} in terms of A_0, B_0, B_1 ,

$$D B_{00} + p^2 B_{11} = A_0(m_1) + m_0^2 B_0$$

$$B_{00} + p^2 B_{11} = \frac{1}{2} A_0(m_1) - \frac{1}{2} (p^2 - m_1^2 + m_0^2) B_1$$

- give the divergent part of B_{00} & B_{11}

$$A_0(m) \Big|_{\text{div}} = \frac{m^2}{\epsilon}, \quad B_0(p^2, m_0, m_1) \Big|_{\text{div}} = \frac{1}{\epsilon}, \quad B_1(p^2, m_0, m_1) = -\frac{1}{2\epsilon}$$

Exercise 14

- the photon self energy is given by

$$\Sigma_{\mu\nu}^{AA}(k) = \frac{\alpha}{2\pi} Q^2 \left\{ g_{\mu\nu} \left[k^2 B_0 - 2A_0 + 4B_{00} \right] + 4k_\mu k_\nu \left[B_{11} + B_1 \right] \right\}$$

apply the reduction to scalar integrals:

$$B_1(k^2, m_1, m_1) = -\frac{1}{2} B_0$$

$$B_{00}(k^2, m_1, m_1) = \frac{1}{6} \left[A_0(m_1) + 2m^2 B_0 + k^2 B_1 + 2m^2 - \frac{k^2}{3} \right] + O(\epsilon)$$

$$B_{11}(k^2, m_1, m_1) = \frac{1}{6k^2} \left[2A_0(m_1) - 2m^2 B_0 - 4k^2 B_1 - 2m^2 + \frac{k^2}{3} \right] + O(\epsilon)$$

and identify the transversal & longitudinal parts in the decomposition

$$\Sigma_{\mu\nu}^{AA}(k) = \left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \right) \Sigma_T^{AA}(k^2) + \frac{k_\mu k_\nu}{k^2} \Sigma_L^{AA}(k^2)$$

what is the behaviour of $\Sigma_T^{AA}(k^2)$ for $k^2 \rightarrow 0$?
what physical consequence does this have?

Exercise 15

- the electron self energy is given by:

$$i \bar{\Sigma}^{\bar{4}4}(k) = \rightarrow \textcircled{1PI} \rightarrow = \begin{array}{c} k \\ \rightarrow \\ \text{---} \\ \text{---} \\ q \end{array} + \dots \quad (\text{Feynman gauge: } \xi = 1)$$

$$\Rightarrow \bar{\Sigma}^{\bar{4}4}(k) = -\frac{\alpha}{4\pi} Q^2 \frac{(2\pi\mu)^{4-D}}{-\pi^2} \int d^D q \frac{\gamma_\mu (q+m) \gamma^\mu}{(q^2-m^2)(q-k)^2}$$

and can be decomposed into its vector & scalar part as

$$\bar{\Sigma}^{\bar{4}4}(k) = k \bar{\Sigma}_v^{\bar{4}4}(k^2) + m \bar{\Sigma}_s^{\bar{4}4}(k^2)$$

determine $\bar{\Sigma}_v^{\bar{4}4}$ & $\bar{\Sigma}_s^{\bar{4}4}$ in terms of scalar integrals.

- simplify the Dirac structure in the numerator
 - perform a tensor reduction of $B_\mu(p, m_0, m_1) = p_\mu B_1(p, m_0, m_1)$
- $$B_1(p^2, m_0, m_1) = \frac{1}{2p^2} [A_0(m_0) - A_0(m_1) - (p^2 - m_1^2 + m_0^2) B_0(p^2, m_0, m_1)]$$
- get all finite terms using $A_0(m)|_{\text{div}} = \frac{m^2}{e}$, $B_0|_{\text{div}} = \frac{1}{e}$

Exercise 16

- explicitly calculate the divergent part of the vertex correction.
- ① for the UV behaviour ($q \rightarrow \infty$) external momenta are irrelevant

$$\Delta_\mu(p', p) \Big|_{\substack{\text{uv} \\ \text{div}}} = \Delta_\mu(0, 0) \Big|_{\substack{\text{uv} \\ \text{div}}} = \frac{\alpha}{4\pi} Q^2 \frac{(2\pi\mu)^{4-D}}{i\pi^2} \int d^D q \quad \frac{\gamma_\alpha q^\alpha \gamma_\mu q^\mu \gamma_\nu q^\nu}{[q^2]^3} \Big|_{\substack{\text{uv} \\ \text{div}}}$$

also neglect masses

- ② what is the possible Lorentz-structure of $\Delta_\mu(0, 0)$?
 ↳ how can we extract the scalar coefficient of this decomposition?
- ③ determine the UV-divergent part

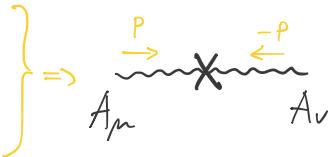
$$\frac{(2\pi\mu)^{4-D}}{i\pi^2} \int d^D q \frac{1}{(q^2)^2} \Big|_{\substack{\text{uv} \\ \text{div}}} = B_0(0, 0, 0) \Big|_{\substack{\text{uv} \\ \text{div}}} = \frac{1}{\epsilon}$$



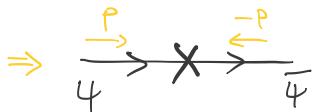
Exercise 17

- derive the counterterm Feynman rules for

$$\mathcal{L}_{ct} = -\delta Z_A \frac{1}{2} (\partial_\mu A_\nu) (\partial^\mu A^\nu) + \left[\delta Z_A \left(1 - \frac{1}{3}\right) + \delta Z_\xi \frac{1}{3} \right] \frac{1}{2} (\partial_\mu A_\nu) (\partial^\nu A^\mu)$$



$$+ \delta Z_4 \bar{\psi} (i\cancel{\partial} - m) \psi - \delta m \bar{\psi} \psi$$



$$- (\delta Z_e + \delta Z_4 + \frac{1}{2} \delta Z_A) e Q \bar{\psi} \gamma^\mu \psi$$



Exercise 18

- field strength renormalization ($\tilde{\Phi}_i = \sqrt{z_{\Phi_i}} \Phi_i$) is not needed in S-Matrix elements / physical observables.
They are useful to make n-pt functions finite.
how are the n-pt functions with & without z_{Φ_i} related?

$$G^{\tilde{\Phi}_1^\circ \dots \tilde{\Phi}_n^\circ} \quad \text{vs.} \quad G^{\tilde{\Phi}_1 \dots \tilde{\Phi}_n}$$

- what about truncated n-pt functions?

$$G_{\text{trunc}}^{\tilde{\Phi}_1^\circ \dots \tilde{\Phi}_n^\circ} \quad \text{vs.} \quad G_{\text{trunc}}^{\tilde{\Phi}_1 \dots \tilde{\Phi}_n}$$

cont.
G ↗

Exercise 18:

- The LSZ reduction formula is given in terms of bare fields:

$$i\mathcal{M}^{n \rightarrow m}(p_1, \dots, p_n, p'_1, \dots, p'_m) = \prod_{i=1}^n f_{in}^{\Xi_i}(p_i) \sqrt{R_{\Xi_i}} \prod_{j=1}^m f_{out}^{\Xi'_j}(p'_j) \sqrt{R_{\Xi'_j}}$$

$$\times G_{\text{trunc}}^{\Xi_1 \dots \Xi_n \Xi'_1 \dots \Xi'_m}(p_1, \dots, p_n, -p'_1, \dots, -p'_m) \quad \Big| \text{on-shell}$$

what is the formula in terms of $G_{\text{trunc}}^{\Xi_1 \dots \Xi_m'}$? (convenient choice?
meaning?)

- Ξ_i are only relevant for external legs. Show that f_{ZA} cancels between these diagrams



Exercise 19

- determine the electron WF renormalization from

$$\lim_{p^2 \rightarrow m^2} \left(\frac{i}{p-m} \hat{\Gamma}^{44}(-p, p) \right) u(p) \stackrel{!}{=} -u(p)$$

$$* (p-m) u(p) = 0$$

$$* \hat{\Gamma}^{44}(-p, p) = i(p-m) + i \hat{\Sigma}_l^{44}(p)$$

$$* \hat{\Sigma}_l^{44}(p) = p \hat{\Sigma}_V^{44}(p^2) + m \hat{\Sigma}_S^{44}(p^2)$$

$$* \hat{\Sigma}_V^{44}(p^2) = \hat{\Sigma}_V^{44}(p^2) + \delta Z_4, \quad \hat{\Sigma}_S^{44}(p^2) = \hat{\Sigma}_S^{44}(p^2) - \delta Z_4 - \frac{\delta m}{m}$$

$$* \frac{\delta m}{m} = \hat{\Sigma}_V^{44}(m^2) + \hat{\Sigma}_S^{44}(m^2)$$

Exercise 20

- determine the charge renormalization from the condition

$$0 \stackrel{!}{=} \bar{u}(p) \hat{\Delta}_\mu(p, p) u(p)$$

$$\textcircled{1} \quad \hat{\Delta}_\mu(p, p) = \Delta_\mu(p, p) + \gamma_\mu (\delta z_e + \delta z_4 + \frac{1}{2} \delta z_A)$$

use the Ward identity and $\sum^{\bar{4}4}(p) = p \sum_V^{\bar{4}4}(p^2) + m \sum_S^{\bar{4}4}(p^2)$

$$\Delta_\mu(p, p) = \frac{\partial}{\partial p^\mu} \sum^{\bar{4}4}(p)$$

\textcircled{2} Show the special Gordon identity $\bar{u}(p) \gamma_\mu u(p) = \frac{p_\mu}{m} \bar{u}(p) u(p)$ and use it to bracket out $\bar{u}(p) \gamma_\mu u(p)$

\textcircled{3} Use $\delta z_4 = - \sum_V^{\bar{4}4}(m^2) - 2m^2 \left[\sum_V^{\bar{4}4}(m^2) + \sum_S^{\bar{4}4}(m^2) \right]$ to arrive at $\delta z_e = -\frac{1}{2} \delta z_A$

Exercise 21

- determine the transformation property of $A_\mu = A_\mu^a T^a$ from

$$D_\mu \rightarrow D'_\mu = U D_\mu U^\dagger \quad ; \quad D_\mu = \partial_\mu + ig A_\mu^a T^a = \partial_\mu + ig A'_\mu$$
$$U = U(\theta) = \exp \{-ig \theta^a T^a\}$$

- show that for infinitesimal gauge transformations $\delta\theta$ the gauge fields transform as

$$\delta A_\mu^a = g c^{abc} \delta \theta^b A_\mu^c + \partial_\mu (\delta \theta^a)$$

$$[T^a, T^b] = i c^{abc} T^c$$

Exercise 22

- Let $y_1, \dots, y_N, y_1^*, \dots, y_N^*$ be the $2N$ generators of a Grassmann alg.

$$\hookrightarrow \{y_i, y_j\} = \{y_i, y_j^*\} = \{y_i^*, y_j^*\} = 0$$

$$\Rightarrow (y_i)^2 = (y_i^*)^2 = 0 \quad \Rightarrow \text{only function } f(y) = a + b y \in \mathbb{C}$$

\hookrightarrow derivatives:

$$\frac{\partial}{\partial y_i} y_k = \delta_{ik} = \frac{\partial}{\partial y_i^*} y_k^*, \quad \frac{\partial}{\partial y_i} y_k^* = 0 = \frac{\partial}{\partial y_i^*} y_k$$

\hookrightarrow "integration" equiv. to derivative

$$\int dy_i = 0 = \int dy_k^*, \quad \int dy_k y_i = \delta_{ik} = \int dy_k^* y_i, \quad \int dy_i y_k^* = \int dy_k^* y_i = 0$$

- show the substitution rule: ($A = N \times N$ regular matrix)

$$z_i = A_{ij} y_j \Rightarrow dz_i = (\tilde{A})_{ji} dz_j$$

- show $\int dz_1 \dots dz_N f(z) = \int dy_1 \dots dy_N f(z(y)) [\det A]^{-1}$

- Finally, show: $\int dy_1 \dots \int dy_N \int dy_N^* \dots \int dy_1^* \exp \{y_i^* A_{ij} y_j\} = \det A$

Exercise 23

- derive the Ward identity for $k^\mu G_\mu^{+4\bar{4}}(k, p, \bar{p})$ using BRS symmetry

$$\delta_{\text{BRS}} A_\mu = \delta \bar{\lambda} [\partial_\mu u]$$

$$\delta_{\text{BRS}} 4 = \delta \bar{\lambda} [-ieQ u 4], \quad \delta_{\text{BRS}} \bar{4} = \delta \bar{\lambda} [+ieQ u \bar{4}]$$

$$\delta_{\text{BRS}} \bar{u} = \delta \bar{\lambda} \left[-\frac{1}{\xi} \partial^\mu A_\mu \right], \quad \delta_{\text{BRS}} u = 0$$

① consider $0 \stackrel{!}{=} \delta_{\text{BRS}} \langle T \bar{u}(x) 4(y) \bar{4}(z) \rangle$

② exploit the fact that the FP ghosts decouple:

$$\langle T u(y_1) \bar{u}(y_2) \underline{\Phi}_1(x_1) \dots \underline{\Phi}_n(x_n) \rangle = \langle T u(y_1) \bar{u}(y_2) \rangle \langle T \underline{\Phi}_1(x_1) \dots \underline{\Phi}_n(x_n) \rangle$$

$$= i \Delta_F(y_1 - y_2) = \underbrace{\int \frac{d^4 k}{(2\pi)^4} e^{-ik(y_1-y_2)}}_{\frac{i}{k^2}}$$

③ go into momentum space using

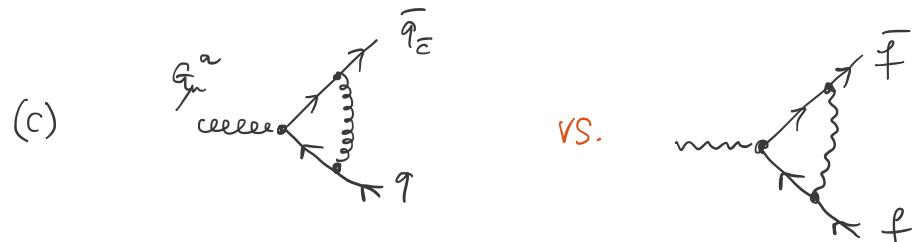
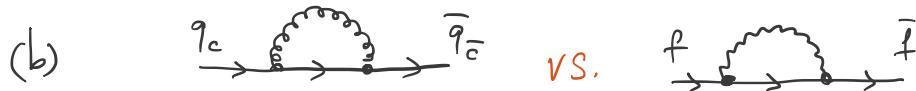
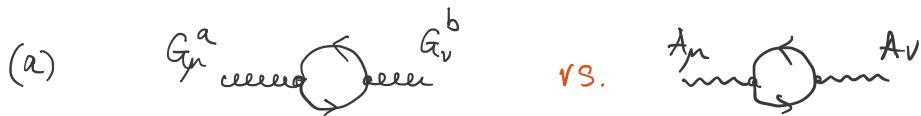
$$\int d^4 x \int d^4 y \int d^4 z \ e^{-ikx - ipy - ip'z} (\dots)$$

Exercise 24

- compare the vertices in QCD vs QED



to derive conversion rules (looking at the colour structure) between:



Exercise 25

- calculate the 1-loop beta function

① consider $g_{s,0} = \mu^{\epsilon} g_s z_j$ with $z_g = 1 + 8z_g$, $\delta z_g = \frac{\alpha_s}{4\pi} \Delta \left(\frac{2}{3} N_f T_F - \frac{11}{6} G_A \right)$

- ② use the fact that the bare coupling is scale-indep ($\mu \frac{d\alpha_{s,0}}{d\mu} = 0$) to obtain an equation for

$$\mu \frac{d\alpha_s}{d\mu} = \dots$$

- ③ take the limit $D \rightarrow 4$ ($\epsilon \rightarrow 0$)

Exercise 26

- consider a diagram with an outgoing quark:

$$\text{V}_q \rightarrow q \quad = M_q^c(p) = \bar{u}(p) \text{V}_q^c(p)$$

- and let's attach a gluon

$$\text{V}_q \rightarrow q \quad = M_{qg}^{ca}(p, k)$$

- show:

$$M_{qg}^{ca}(p, k) = \frac{g_s T_{cc}^a}{2(p \cdot k)} \bar{u}(p) \gamma^\mu(p+k) \text{V}_q^c(p+k) \epsilon_\mu^*(k)$$

cont.

Exercise 26 : soft limit

- use $k = \xi q$ with $\xi \rightarrow 0$ and show

$$M_{gg}^{ca}(p, k) \xrightarrow{\xi \rightarrow 0} \frac{1}{\xi} \int_S J_{cc'}^{a/\mu}(q) \epsilon_n^*(k) \times M_q^{c'}(p)$$

- give $J_{cc'}^{a/\mu}(q)$

cont.

Exercise 26: collinear limit

- use the Sudakov decomposition

$$p^\mu = z q^\mu + k_\perp^\mu - \frac{k_\perp^2}{z} \frac{n^\mu}{2(q \cdot n)}$$

$$n^2 = 0$$

$$k_\perp^2 < 0$$

defines how we approach $p \parallel k$

$$k^\mu = (1-z) q^\mu - k_\perp^\mu - \frac{k_\perp^2}{(1-z)} \frac{n^\mu}{2(q \cdot n)}$$

$$k_\perp \cdot q = k_\perp \cdot n = 0$$

- and derive (use D-dimensional Dirac Algebra)

$$\sum_{\text{rel.}} \sum_{\text{col.}} |M_{qg}(p, k)|^2 \xrightarrow{p \parallel k} \frac{1}{(p \cdot k)} 4\pi\alpha_s C_F \left[\frac{1+z^2}{1-z} - \epsilon(1-z) \right] * |M_g(p+k)|^2$$

Exercise 27

- compute the sq. ME for on-shell W^+ production:

$$\langle |M_{u\bar{d} \rightarrow W^+}|^2 \rangle$$

$\langle \dots \rangle \stackrel{\wedge}{=} \text{average/sum over}$
 initial-/final-state d.o.f (col., hel.)

$$W^+ = i \frac{e}{\sqrt{2} s_W} \gamma^\mu \omega_-$$

$$\omega_- = \frac{1}{2}(1 - \gamma^5)$$

$$\sum_{\text{pol.}} \epsilon_\mu^*(p_W) \epsilon_\nu(p_W) = -g_{\mu\nu} + \frac{p_{W,\mu} p_{W,\nu}}{M_W^2}$$

- calculate the partonic cross section $\hat{\sigma}_{u\bar{d} \rightarrow W^+}(p, \bar{p})$
- calculate the differential hadronic cross section

$$\frac{d\sigma_{pp \rightarrow W^+}(p_1, p_2)}{dY_W}$$

rapidity: $Y_W = \frac{1}{2} \ln \left(\frac{E_W + P_W^2}{E_W - P_W^2} \right)$

Exercise 28

- compute the 1-loop corrections:

$$\mu^{2\epsilon} \int \frac{d^D q}{(2\pi)^D} \frac{\gamma^\nu (q-p) \gamma^\mu (q+p) \gamma_\nu}{q^2 (q+p)^2 (q-p)^2}$$

- combine the propagators & shift loop momentum
- simplify the numerator:
 - linear in $q \mapsto 0$
 - $q^\mu q^\nu \sim g^{\mu\nu} \Rightarrow q^\mu q^\nu \mapsto g^{\mu\nu} \frac{q^2}{D}$
 - Dirac Eqn. $\bar{v}_u(\vec{p}) \vec{\gamma} = 0 = \vec{\gamma} u_u(p)$
- identify $I_{2,3}(-xy\hat{s})$
- expand in ϵ

$$\frac{1}{abc} = 2 \int_0^1 dx \int_0^{1-x} dy \frac{1}{[a(1-x-y) + bx + cy]^3}$$

$$I_n(A) = \int d^D q \frac{1}{[q^2 - A]^n}$$

Exercise 29

- determine ξ_i for the mapping $p + \bar{p} \rightarrow p_w + k \Rightarrow \xi_1 p + \xi_2 \bar{p} \rightarrow \tilde{p}_w$
so that $p_w^2 = \tilde{p}_w^2$ & $y_w = \tilde{y}_w$

- insert into $d\Phi_2(p_w, k; p + \bar{p})$ some "1"'s

$$1 = \int_0^1 dz_1 \int_0^1 dz_2 \delta(z_1 - \xi_1) \delta(z_2 - \xi_2)$$

$$1 = \int \frac{d^D \tilde{p}_w}{(2\pi)^D} (2\pi)^D \delta^{(D)}(\tilde{p}_w - (z_1 p + z_2 \bar{p}))$$

to get $d\Phi_2(p_w, k; p + \bar{p}) = \frac{dz_1}{z_1} \frac{dz_2}{z_2} d\Phi(\tilde{p}_w; z_1 p + z_2 \bar{p}) \underbrace{[dk]}_{\otimes} z_1 z_2 \delta(z_1 - \xi_1) \delta(z_2 - \xi_2)$

- integrate out \otimes

$$\hookrightarrow d^D k = dk^0 dk^z \underbrace{d^{D-2} k_L}_{\otimes}$$

$$dk_L^2 \frac{1}{2} \left(k_L^2 \right)^{\frac{D-4}{2}} d\Omega_{D-2}$$

$$\hookrightarrow \text{variable change: } k^z = \frac{1}{2\pi} \left[\frac{S_{uz}}{x_1} + \frac{S_{\bar{u}\bar{g}}}{x_2} \right] \Rightarrow \begin{cases} S_{uz} = S_{u\bar{u}} \frac{z_1(1-z_2^2)}{z_1+z_2} \\ S_{\bar{u}\bar{g}} = S_{u\bar{u}} \frac{z_2(1-z_1^2)}{z_1+z_2} \end{cases}$$