Radiative Corrections & Renormalization

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The Plan

∅ Motivation & Conventions

I. QFT in a Nutshell

II. QED & Renormalization

III. Yang-Mills Theory & QCD

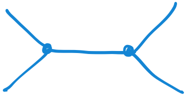
IV. Infrared Singularities

Ø. Motivation & Conventions

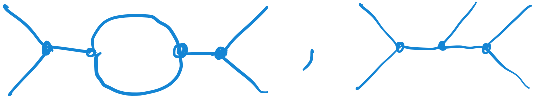
Motivation

- radiative corrections $\leftrightarrow$ perturbative evaluation of a QFT

  $\hookrightarrow$ diagrammatic representation: Feynman diagrams

  Born-level  , ...

  "radiative corrections"

  higher order  , ...

  $\hookrightarrow$ ultraviolet (UV) divergences

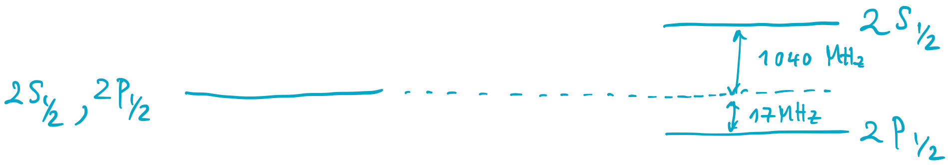
  $\Rightarrow$ renormalization

  $\hookrightarrow$ infrared (IR) divergences

  $\Rightarrow$ compensation between virtual & real-emission

Motivation: Measurable effects

- Lamb-Shift: (H - atom)



(QED corrections)

- anomalous magnetic moment (e, μ)

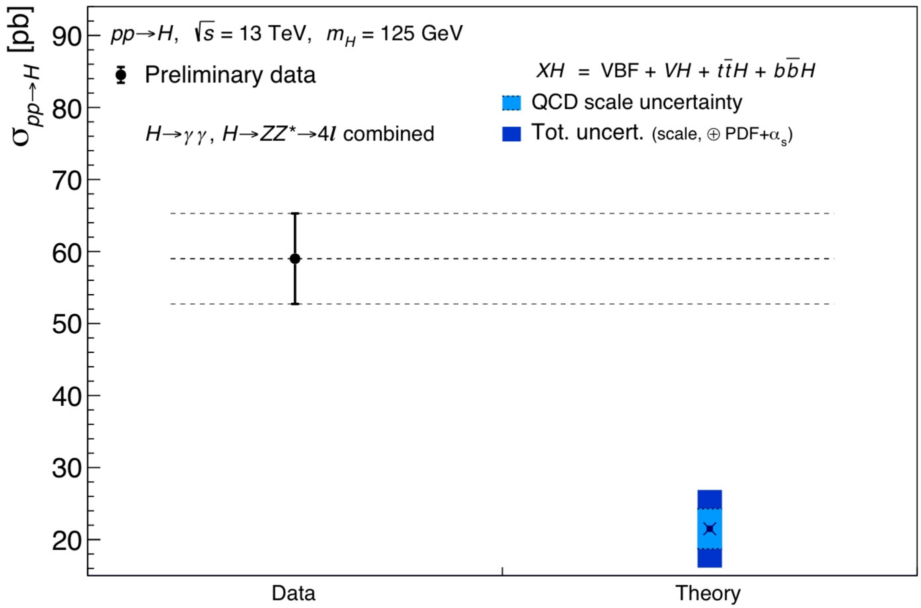
$$g_\mu = 2 \left[1 + \frac{\alpha}{2\pi} + \dots \right]$$

Dirac — Schwinger

- electroweak precision tests   Input parameters ⇝ observables
  ↪ fit Standard Model (SM) to Data: good agreement
  ↪ predict m_t, constrain M_H

Predict & Compare

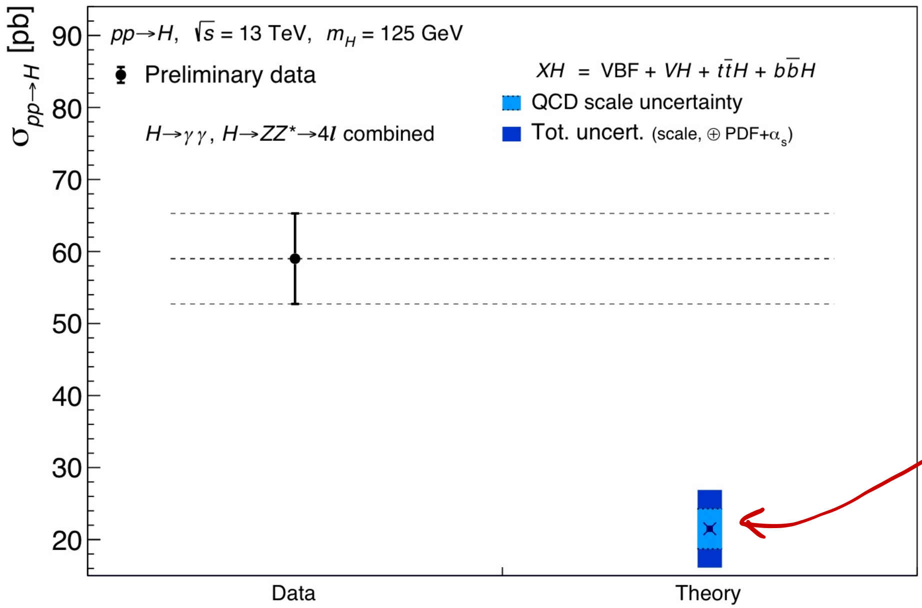
- Higgs production @ LHC:



- $3.8\,\sigma$ deviation !!!
- Standard Model fails?!

Predict & Compare

- Higgs production @ LHC:

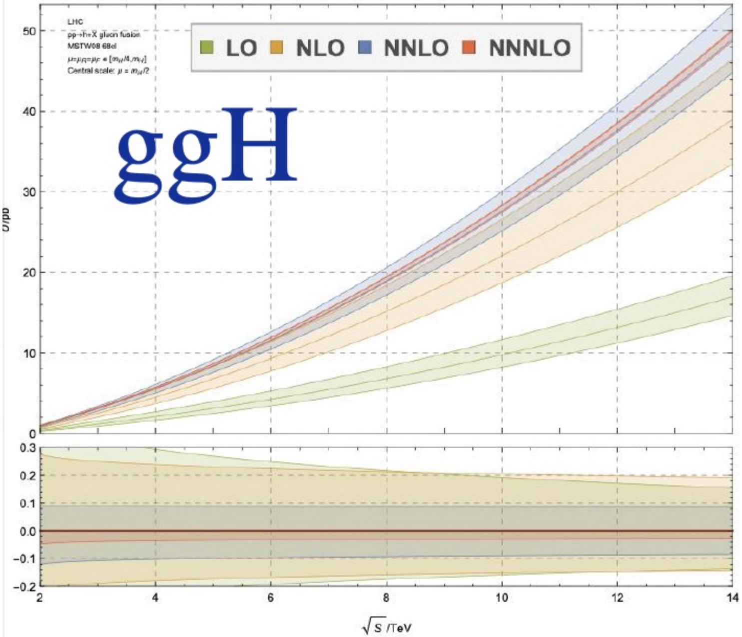


- ~~3.8 σ deviation !!!~~
- ~~Standard Model fails?!~~

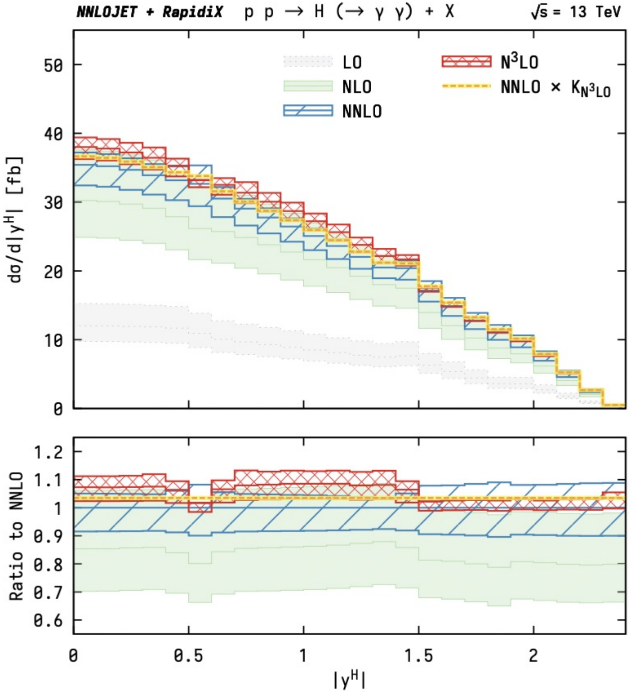
LEADING ORDER

Higgs Production @ N^3LO

[Anastasiou, Duhr, Dulat, Herzog, Mistlberger '15]



[Chen, Gehrmann, Glover, AH, Mistlberger, Pelloni '21]



Conventions & Notation

one should avoid this...

- natural units: $\hbar = c = 1$
- metric tensor $g_{\mu\nu} = g^{\mu\nu} = \mathrm{diag}(1,-1,-1,-1)$

contravariant vector: $x^\mu = (x^0, \vec{x}) = (t, x, y, z)$

covariant vector: $x_\mu = g_{\mu\nu} x^\nu = (x^0, -\vec{x})$

derivative: $\partial_\mu \equiv \frac{\partial}{\partial x^\mu} = \left(\frac{\partial}{\partial t}, \vec{\nabla}\right)$

scalar product: $x \cdot y = x^\mu y_\mu = x^\mu y^\nu g_{\mu\nu} = x^0 y^0 - \vec{x} \cdot \vec{y}$

$$\Box \equiv \partial^2 = \partial^\mu \partial_\mu = \partial_t^2 - \Delta$$

Levi-Civita:
$$\varepsilon^{\mu\nu\rho\sigma} = -\varepsilon_{\mu\nu\rho\sigma} = \begin{cases} +1, & \{\mu,\nu,\rho,\sigma\} \text{ even perm } \{0,1,2,3\} \\ -1, & \text{— " — odd — " —} \\ 0, & \text{else} \end{cases}$$

Conventions & Notation

- Dirac algebra (4 - dimensional)

$$\{\gamma^\mu,\gamma^\nu\} = \gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu = 2g^{\mu\nu}\,\mathbb{1} \qquad (\mathrm{Tr}(\mathbb{1})=4)$$

$$\gamma^5 = -i\,\gamma_0\gamma_1\gamma_2\gamma_3 = +i\,\gamma^0\gamma^1\gamma^2\gamma^3 = -\frac{i}{4}\,\varepsilon_{\mu\nu\rho\sigma}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma$$

$$\{\gamma^5,\gamma^\mu\} = 0\,,\quad (\gamma^5)^2 = \mathbb{1}$$

$$\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu,\gamma^\nu] = -\sigma^{\nu\mu}$$

Dirac - slash: $\not{a} = \gamma^\mu a_\mu$

- Dirac spinors: $(\bar{\psi} = \psi^\dagger\gamma^0)$

$$(\not{p}-m)\,u(p) = 0\,, \qquad (\not{p}+m)\,v(p) = 0$$

$$\bar{u}(p)\,(\not{p}-m) = 0\,, \qquad \bar{v}(p)\,(\not{p}+m) = 0$$

Ex 1

I. QFT in a Nutshell

Overview

- starting point: (local) Lagrange density $\mathcal{L}$ $\leftrightarrow$ defines our theory

  $\hookrightarrow$ ϕ^4-theory: $\mathcal{L}_{\phi^4} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2} m^2 \phi^2 - \frac{\lambda}{4!} \phi^4$

  $\hookrightarrow$ QED: $\mathcal{L}_{QED} = \bar{\psi}(i\not{\partial} - m)\psi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - eQ A_\mu \bar{\psi} \gamma^\mu \psi$

  $e > 0$

  $\hookrightarrow$ construction via symmetry considerations: $SU(3)_c \times SU(2)_L \times U(1)_Y$

  $\Rightarrow$ equation of motion (e.o.m)

$$\partial_\mu \frac{\delta \mathcal{L}}{\delta(\partial_\mu \Phi)} - \frac{\delta \mathcal{L}}{\delta \Phi} = 0$$

I write: $\boxed{\Phi \in \{\phi, \psi, A^\mu, \ldots\}}$

- fundamental objects: n-point correlation functions

$$G^{\Phi_1 \ldots \Phi_n}(x_1, \ldots, x_n)$$

Ex 2

The Road Map

$G^{\Phi_1 \ldots \Phi_n}$

full n-point corr. function

Feynman diagrams

$d\sigma$, $d\Gamma$

cross sections

decay rates

...

LSZ reduction

pert. theory

diag. representation

$\mathcal{L}(\{\Phi_i\})$

theory.

S-Matrix, $\mathcal{M}_{fi}$

scattering ampl.

$G_0^{\Phi_1 \Phi_2}$ & interaction vertices

free 2-point function

$\hat{=}$ "propagator"

Free Field theory

- "free" part of $\mathcal{L} \leftrightarrow$ bilinear in the fields:

$\hookrightarrow$ $$\mathcal{L}_{\phi,0} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2} m^2 \phi^2$$

$\hookrightarrow$ $$\mathcal{L}_{\psi,0} = \bar{\psi}(i\partial\!\!\!/ - m)\psi$$

$\hookrightarrow$ $$\mathcal{L}_{A,0} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \quad ; \; F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

- n-point correlation function

time ordering: $\Phi_{i_1}(x_{i_1}) \dots \Phi_{i_n}(x_{i_n})$
for $x_{i}^{0} > \dots > x_{i_n}^{0}$

$$G_0^{\Phi_1 \dots \Phi_n}(x_1, \dots, x_n) = \langle 0 | T \, \Phi_1(x_1) \dots \Phi_n(x_n) | 0 \rangle$$ (canonical quant.)

$$= \frac{1}{N_0} \int \mathcal{D}[\{\Phi\}] \, \Phi_1(x_1) \dots \Phi_n(x_n) \, \exp\left(i \, S_0[\{\Phi\}]\right)$$

(path integral quant.)

Free Scalar Theory

- $\mathcal{L}_{\phi,0} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2} m^2 \phi^2$ $\Rightarrow$ e.o.m: $(\Box + m^2)\phi = 0$
- Green's function: $(\Box + m^2)\Delta_F(x) = -\delta(x)$

$$\Delta_F(x) = \int \frac{d^4p}{(2\pi)^4} e^{-ipx} \frac{1}{p^2 - m^2 + i0}$$

(Feynman prop.)

time ordering.

- n-point function:

$$G_0^{\phi\dots\phi}(x_1,\dots x_n) = \langle 0 | T\, \phi(x_1)\dots\phi(x_n) | 0 \rangle = \frac{1}{N_0}\int \mathcal{D}[\phi]\, \phi(x_1)\dots\phi(x_n)\, e^{iS_0[\phi]}$$

- generating functional:

$$Z_{\phi,0}[J] = \frac{1}{N_0}\int \mathcal{D}[\phi] \exp\left\{ i \int d^4x \left[\mathcal{L}_{\phi,0} + J(x)\phi(x) \right] \right\}$$

$$\Rightarrow G_0^{\phi\dots\phi}(x_1,\dots,x_n) = \frac{\delta}{i\delta J(x_1)} \cdots \frac{\delta}{i\delta J(x_n)} Z_{\phi,0}[J] \Big|_{J=0}$$

Free Scalar Theory

- generating functional:

P.I.

$$Z_{\phi,0}[J] = \frac{1}{N_0} \int \mathcal{D}[\phi] \exp\left\{ i \int d^4x \left[-\frac{1}{2} \phi (\Box + m^2) \phi + J\phi \right] \right\}$$

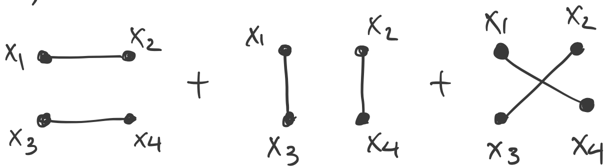
... "complete the square" $\left(\phi = \phi' - \int d^4y \, \Delta_F(x-y) J(y) \right)$

... absorb $\int \mathcal{D}[\phi']$ into normalisation $\left(Z_{\phi,0}[0] \overset{!}{=} 1 \right)$

$$= \exp\left\{ \int d^4x \int d^4y \, \frac{1}{2} \, iJ(x) \, i\Delta_F(x-y) \, iJ(y) \right\}$$

$$\Rightarrow \quad G_0^{\phi\phi}(x_1, x_2) = \frac{\delta}{i\,\delta J(x_1)} \frac{\delta}{i\,\delta J(x_2)} Z_{\phi,0}[J] \Big|_{J=0} = i\Delta_F(x_1 - x_2) =$$

$$G_0^{\phi\phi\phi\phi}(x_1, x_2, x_3, x_4) = \ldots =$$



(no $G^{\overbrace{\phi\ldots\phi}^{\text{uneven}}}$)

Free Dirac Theory

- $\mathcal{L}_{\psi,0} = \bar{\psi}(i\not{\partial}-m)\psi$ $\Rightarrow$ e.o.m. $(i\not{\partial}-m)\psi = 0$
- Green's function: $(i\not{\partial}-m)S_F(x) = \delta(x)$

$$S_F(x) = \int \frac{d^4p}{(2\pi)^4}\, e^{-ipx}\, \frac{1}{\not{p}-m+i0}$$

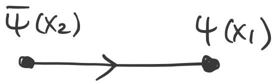
- generating functional:

Grassmann-valued fields.

$$Z_{\psi,0}[\eta,\bar{\eta}] = \frac{1}{N_0}\int \mathcal{D}[\psi]\,\mathcal{D}[\bar{\psi}]\, \exp\left\{ i\int d^4x \left[\bar{\psi}(i\not{\partial}-m)\psi - \bar{\psi}\eta + \bar{\eta}\psi \right]\right\}$$

$$= \dots = \exp\left\{ -\int d^4x \int d^4y\; i\bar{\eta}(x)\; iS_F(x-y)\; i\eta(y)\right\}$$

$$\Rightarrow \quad G_0^{\psi\bar{\psi}}(x_1,x_2) = \langle 0|\, T\, \psi(x_1)\,\bar{\psi}(x_2)\,|0\rangle = \begin{cases} t_1 > t_2: \text{ create } f \text{ @ } x_2 \longrightarrow \text{destroy @ } x_1 \\ t_2 > t_1: \text{ create } \bar{f} \text{ @ } x_1 \longrightarrow \text{destroy @ } x_2 \end{cases}$$

$=$ 

$\Leftrightarrow$ definite fermion-number flow $\hat{=}$ arrow

$$= \frac{\delta}{i\,\delta\bar{\eta}(x_1)}\,\frac{\delta}{i\,\delta\eta(x_2)}\, Z_{\psi,0}[\eta,\bar{\eta}]\Big|_{\eta=\bar{\eta}=0}$$

$$= i\,S_F(x_1-x_2)$$

Free Photon Field

- $\mathcal{L}_{A,0} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$ $\left(F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \right)$ $\Rightarrow$ e.o.m. $\left[g^{\mu\nu} \Box - \partial^\mu \partial^\nu \right] A_\nu = 0$
- Green's function: $\left(g^{\mu\nu} \Box - \partial^\mu \partial^\nu \right) \Delta_{F,\nu\rho}(x) = g^\mu{}_\rho \, \delta(x)$ $\lightning$ $\partial_\mu [\dots]$

$\hookrightarrow$ origin gauge symmetry: $A_\mu(x) \longrightarrow A_\mu(x) + \partial_\mu \Lambda(x)$
(unphysical d.o.f)

$$\left(g^{\mu\nu} \Box - \partial^\mu \partial^\nu \right) \partial_\mu \Lambda(x) \equiv 0 \quad \hat{=} \quad \text{eigenvalue zero}$$

$\hookrightarrow$ Green's function $\Delta_{F,\nu\rho}$ is "inverse" of the differential-operator $\left[g^{\mu\nu} \Box - \partial^\mu \partial^\nu \right]$

eigenvalues zero $\Rightarrow$ not invertible $\Rightarrow$ $\nexists \; \Delta_{F,\nu\rho}$

Ex 3

Free Photon Field (with covariant gauge fixing)

- $\mathcal{L}_{A,0} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2\xi} (\partial \cdot A)^2 \quad \Rightarrow \text{ e.o.m. } \left[g^{\mu\nu} \Box - \left(1 - \frac{1}{\xi}\right) \partial^\mu \partial^\nu \right] A_\nu = 0$

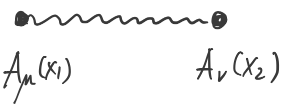
- Green's function: $\left[g^{\mu\nu} \Box - \left(1 - \frac{1}{\xi}\right) \partial^\mu \partial^\nu \right] \Delta_{F,\nu\rho}(x) = g^\mu{}_\rho \, \delta(x)$

$$\Delta_F^{\mu\nu}(x) = \int \frac{d^4p}{(2\pi)^4} \, e^{-ipx} \, \frac{-1}{p^2} \left[g^{\mu\nu} - (1-\xi) \frac{p^\mu p^\nu}{p^2} \right]$$

- generating functional:

$$Z_{A,0}[J_\mu] = \exp\left\{ \frac{1}{2} \int d^4x \int d^4y \; iJ_\mu(x) \; i\Delta_F^{\mu\nu}(x-y) \; iJ_\nu(x) \right\}$$

$$\Rightarrow G^{AA}_{\mu\nu,0}(x_1, x_2) = \frac{\delta}{i\delta J^\mu(x_1)} \frac{\delta}{i\delta J^\nu(x_2)} Z_{A,0}[J^\mu] \Big|_{J^\mu = 0}$$

= 

$= i\Delta_{F,\mu\nu}(x_1 - x_2)$

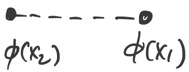
Summary: Free Field Theory

- exact solutions for $Z_0[J]$ $\Rightarrow$ arbitrary n-point functions
- general procedure: $\mathcal{L}_0 = \Phi^{(t)}\, \mathbb{D}\, \Phi$ $\Rightarrow$ e.o.m. $\mathbb{D}\Phi = 0$

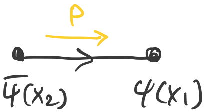
  $\hookrightarrow$ solve using Green's functions: $\mathbb{D}\, \Delta_F(x) \sim \delta(x)$

  $\hookrightarrow$ perform path integral: $Z_0[J] = \exp\left\{ \pm \int d^4x \int d^4y \; iJ(x)\, i\Delta_F(x-y)\, iJ(y) \right\}$

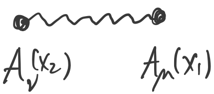
  $\Rightarrow$ basic building block: 2-point functions ("PROPAGATORS") $G_0^{\Phi\Phi}(x_1,x_2) = i\Delta_F(x-y)$

- scalar 

$= i\Delta_F(x_1-x_2)$

- fermion 

$= i S_F(x_1-x_2)$

- photon 

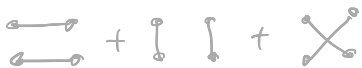
$= i\Delta_F^{\mu\nu}(x_1-x_2)$

$\xrightarrow{\text{momentum space}}$

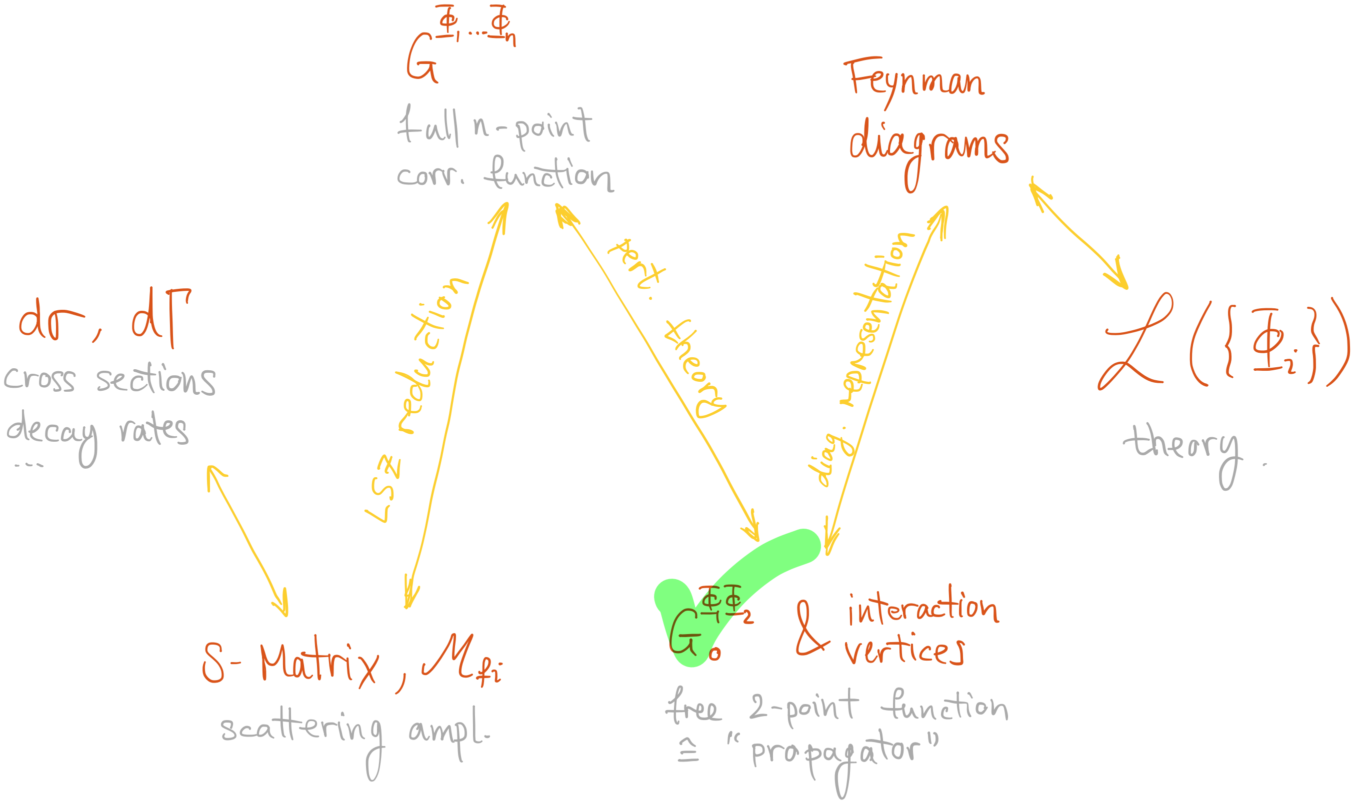
$$\frac{i}{p^2-m^2}$$

$$\frac{i}{\not{p}-m}$$

$$\frac{-i}{p^2}\left[g^{\mu\nu} - (1-\xi)\frac{p^\mu p^\nu}{p^2}\right]$$

- so far: not very exciting: $G^{\phi\phi\phi\phi} =$ 

The Road Map



$G^{\Phi_1 \dots \Phi_n}$
full n-point corr. function
Feynman diagrams
$\mathcal{L}(\{\Phi_i\})$
theory.
$d\sigma$, $d\Gamma$
cross sections
decay rates
...
LSZ reduction
pert. theory
diag. representation
S-Matrix, $\mathcal{M}_{fi}$
scattering ampl.
$G_0^{\Phi_1\Phi_2}$ & interaction vertices
free 2-point function
≙ "propagator"

Interacting Field Theory

- separate interactions from free part: $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{int}$

  $\hookrightarrow \mathcal{L}_{\phi^4} = -\frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \frac{1}{2}m^2\phi^2 - \frac{\lambda^4}{4!}\phi^4$

  $\hookrightarrow \mathcal{L}_{QED} = \bar{\psi}(i\not{\partial} - m)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - eQA_\mu\bar{\psi}\gamma^\mu\psi$

- n-point function

vacuum of the interacting theory $(\neq |0\rangle)$

$$G^{\Phi_1 \dots \Phi_n}(x_1, \dots, x_n) = \langle\Omega| T\, \Phi_1(x_1)\dots\Phi_n(x_n)|\Omega\rangle \qquad \text{(canonical quant.)}$$

$$= \frac{1}{N}\int \mathcal{D}[\{\Phi\}]\, \Phi_1(x_1)\dots\Phi_n(x_n)\, \exp\left\{i\, S[\{\Phi\}]\right\} \qquad \text{(path-integral quant.)}$$

Perturbation Theory

- generating functional for n-point functions

$$Z[\{J\}] = \frac{1}{N'}\int \mathcal{D}[\{\Phi\}]\ \exp\left\{ i\int d^4x \left[\mathcal{L}(\{\Phi(x)\}) + \sum_i J_i(x)\,\Phi_i(x) \right]\right\}$$

$$= \frac{1}{N'}\int \mathcal{D}[\{\Phi\}]\ \exp\left\{ i\int d^4y\ \mathcal{L}_{int}(\{\Phi_i(y)\})\right\} \exp\left\{ i\int d^4x \left[\mathcal{L}_0(\{\Phi_i(x)\}) + \sum_i J_i(x)\,\Phi_i(x) \right]\right\}$$

$$= \frac{1}{N}\ \underbrace{\exp\left\{ i\int d^4y\ \mathcal{L}_{int}\left(\left\{\Phi_i(y) \to \frac{\delta}{i\,\delta J_i(y)}\right\}\right)\right\}}_{\text{series expansion } \hat{=} \text{ perturbative expansion in } \mathcal{L}_{int}}\quad Z_0[\{J_i\}]$$

- $Z[\{J=0\}] \overset{!}{=} 1$

$\longleftrightarrow$ no "vacuum bubbles" 

Example : ϕ^3 Theory

$\mathcal{L}_{int} = \frac{g}{3!}\phi^3$

permutation over vertices $(y_1 \leftrightarrow y_2)$

$$Z[J] = \frac{1}{N}\left\{1 + \frac{ig}{3!}\int d^4y \left(\frac{\delta}{i\delta J(y)}\right)^3 + \frac{1}{2}\left(\frac{ig}{3!}\right)^2 \int d^4y_1\, d^4y_2 \left(\frac{\delta}{i\delta J(y_1)}\right)^3\left(\frac{\delta}{i\delta J(y_2)}\right)^3 + \dots\right\}$$

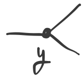
$$\times \left\{1 + \int d^4x \int d^4x' \, \frac{1}{2}\, iJ(x)\; i\Delta_F(x-x')\; iJ(x') + \dots\right\}$$

compensates:

$$\left(\frac{\delta}{i\delta J(y)}\right)^3 J(x_1)J(x_2)J(x_3)$$

$$= 3!\, \delta(x_1-y)\,\delta(x_2-y)\,\delta(x_3-y)$$

- rules for $G^{\phi\dots\phi} = \frac{\delta}{i\delta J}\cdots\frac{\delta}{i\delta J} Z[J]\Big|_{J=0}$ at $\mathcal{O}(g^N)$:

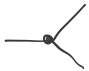
① propagator  $= i\Delta_F(x-x')$, vertex:  $= ig\int d^4y$

② draw all graphs with N vertices and n external legs

(drop all "vacuum bubbles" $\longleftrightarrow$ $Z[0] = 1$)

③ multiply by symmetry factor $\frac{1}{S_G}$; S_G: # of permutations leaving graph inv.

- momentum space : (all momenta incoming)

①  $= \frac{i}{p^2 - m^2 + i0}$;  $= ig$

④ momentum cons. @ each vertex ; undetermined $\to \int \frac{d^4p}{(2\pi)^4}$

EX4

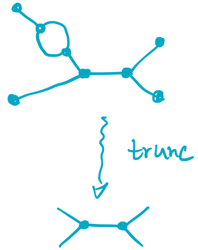
Vertex function

- we saw $G \longrightarrow G_{con}$ (connected) $\longrightarrow$ EX 4

- truncated ("amputated") correlation function

$G_{trunc} = G_{con}\Big|$ no external propagators / propagator corrections

(do not draw ext. field points)



$$G_{trunc}^{\Phi_1 \cdots \Phi_n}(p_1, \dots, p_n) = G_{con}^{\Phi_1 \cdots \Phi_n}(p_1, \dots, p_n) \cdot \prod_{j=1}^{n} \left[G^{\Phi_j \Phi_j^{(+)}}(p_j, -p_j) \right]^{-1}$$

$$\Rightarrow G_{trunc}^{\Phi \Phi^{(+)}} = \left[G^{\Phi \Phi^{(+)}} \right]^{-1}$$

- vertex functions $\longleftarrow$ elementary building blocks

EX 5 & 6

$$\Gamma^{\Phi_1 \Phi_2}(p, -p) := - G_{trunc}^{\Phi_1 \Phi_2}(p, -p) = - \left[G^{\Phi_1 \Phi_2}(p, -p) \right]^{-1}$$

$$\Gamma^{\Phi_1 \cdots \Phi_n}(p_1, \dots, p_n) := G_{trunc}^{\Phi_1 \cdots \Phi_n}(p_1, \dots p_n) \Big|_{\text{only 1PI}}$$

"1PI": 1-particle irred.

no disconnected subgraphs by cutting <u>one internal</u> line

Källén-Lehmann spectral representation

interaction: ϕ also creates multi-particle states!

$$G^{\phi\phi}(x_1,x_2) = \langle \Omega | T\phi(x_1)\phi(x_2) | \Omega \rangle$$

$$= \dots = \int_0^\infty \frac{d\mu^2}{2\pi} \rho(\mu^2) \int \frac{d^4p}{(2\pi)^4} \frac{i\, e^{-ip(x_1-x_2)}}{p^2-\mu^2+i0}$$

- momentum space

$$G^{\phi\phi}(p,-p) = \frac{i\, R_\phi}{p^2-m^2+i0} + \int_{\sim 4m^2}^{\infty} \frac{d\mu^2}{2\pi} \rho(\mu^2) \frac{i}{p^2-\mu^2+i0}$$

$\hookrightarrow$ R_ϕ : residuum $\hat{=}\ |\langle p|\phi|\Omega\rangle|^2$

(prob. for 1-particle state)

$\hookrightarrow$ m : physical mass (in general $\neq$ param. in $\mathcal{L}$)

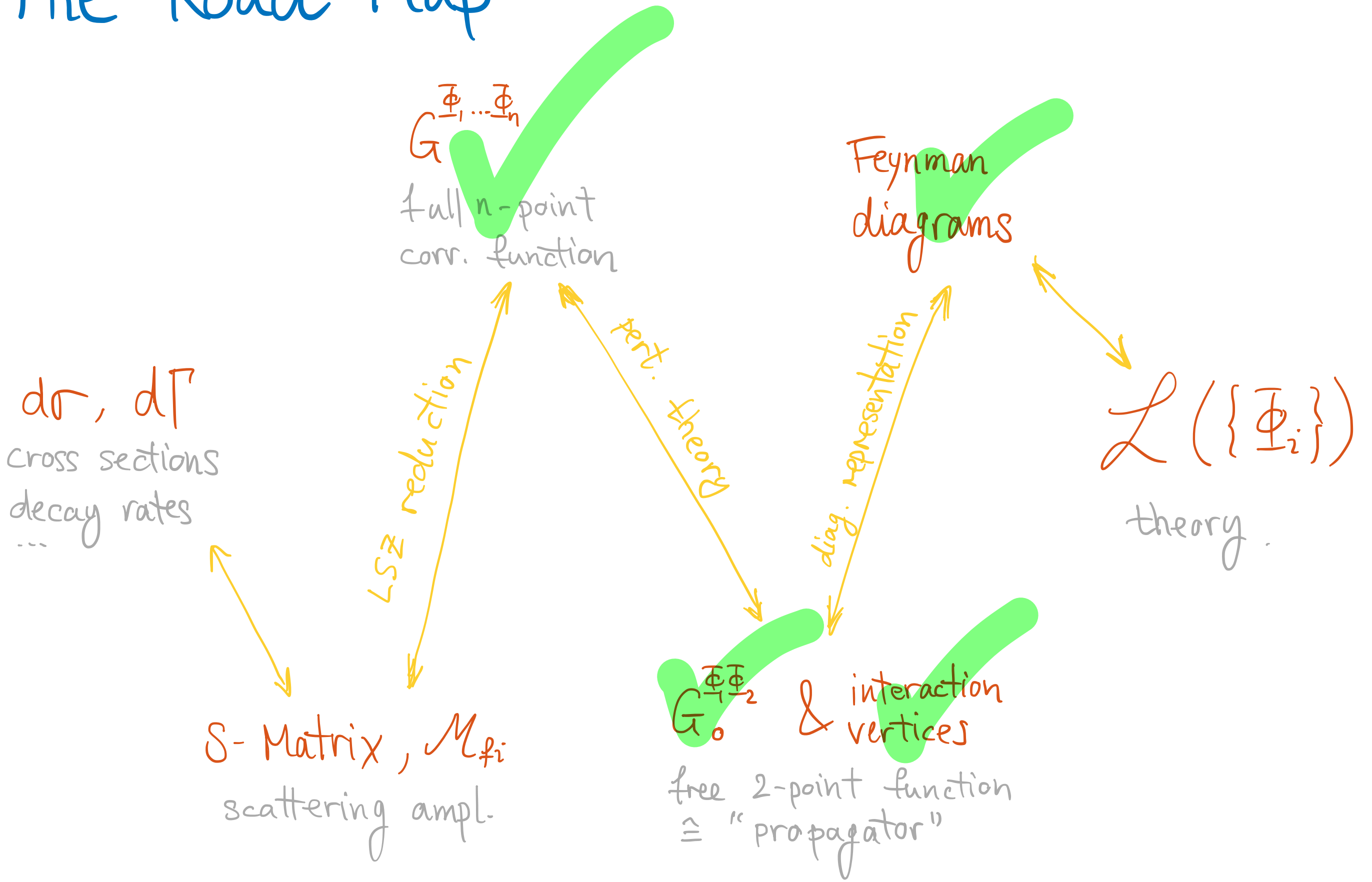
$\rightsquigarrow$ free fields: $R_\phi = 1$, $m = m_0$

Done with n-pt functions

- G : all diags (no vacuum bubbles)
- G_{con} : only connected
- G_{trunc} : "amputate" all external legs
- Γ : only 1PI graphs of G_{trunc} $\left(\Gamma^{\phi\phi} = -[G^{\phi\phi}]^{-1} \right)$

Feynman diagrams & rules to compute them

The Road Map


$G^{\Phi_1 \ldots \Phi_n}$
full n-point corr. function
Feynman diagrams
$d\sigma$, $d\Gamma$
cross sections
decay rates
...
LSZ reduction
pert. theory
diag. representation
$\mathcal{L}(\{\Phi_i\})$
theory
S-Matrix, $\mathcal{M}_{fi}$
scattering ampl.
$G_0^{\Phi_1 \Phi_2}$ & interaction vertices
free 2-point function
$\hat{=}$ "propagator"

The S Matrix & LSZ reduction

- central object in scattering & decays: S matrix

  $\hookrightarrow$ time evolution operator from $t \to -\infty$ ("in") to $t \to +\infty$ ("out")

$$S_{fi} = \langle f | S | i \rangle = \langle f | \mathbb{1} + iT | i \rangle$$

scattering amplitude

$$= \langle f | i \rangle + (2\pi)^4 \delta^4 (p_i - p_f) \, i\mathcal{M}_{fi}$$

- Lehmann Symanzik Zimmermann (LSZ) reduction formula:

$$i\mathcal{M}^{n\to m}(p_1, \ldots, p_n, p_1', \ldots p_m') = \prod_{i=1}^{n} f_{in}^{\Phi_i}(p_i) \sqrt{R_{\Phi_i}} \prod_{j=1}^{m} f^{\Phi_j'}(p_j') \sqrt{R_{\Phi_j'}}$$

$$\times \left. G_{trunc}^{\Phi_1 \ldots \Phi_n \Phi_1' \ldots \Phi_m'}(p_1, \ldots p_n, -p_1', \ldots, -p_m') \right|_{\text{on-shell}}$$

$\hookrightarrow$ truncate ext. legs & replace by 1-particle wave functions

$$f^{\phi}_{in/out} = 1 \;,\quad f^{\psi}_{in/out} = u(p), v(p) \;,\quad f^{\bar\psi}_{in/out} = \bar v(p), \bar u(p) \;,\quad f^{A_\mu}_{in/out} = \varepsilon_\mu^{(*)}(p)$$

$\hookrightarrow$ go on-shell: $p_i^2 = m_i^2 \;,\; p_j'^2 = m_j'^2$

What we were after all this time

- differential cross section $\quad p_a + p_b \longrightarrow p_1 + \dots p_n \quad (2 \to n)$

$$d\sigma = \underbrace{\frac{1}{4\sqrt{(p_a \cdot p_b)^2 - m_a^2 m_b^2}}}_{\text{flux}} \underbrace{|\mathcal{M}_{fi}|^2}_{\text{sq. ME}} \underbrace{\frac{1}{S_{\{n\}}}}_{\text{symmetry factor}} \underbrace{d\Phi_n(p_1, \dots, p_n; p_a + p_b)}_{\text{LIPS}}$$

$S_{\{n\}} = n_1! \cdot \dots$

- decay width $\quad p_a \longrightarrow p_1 + \dots p_n$

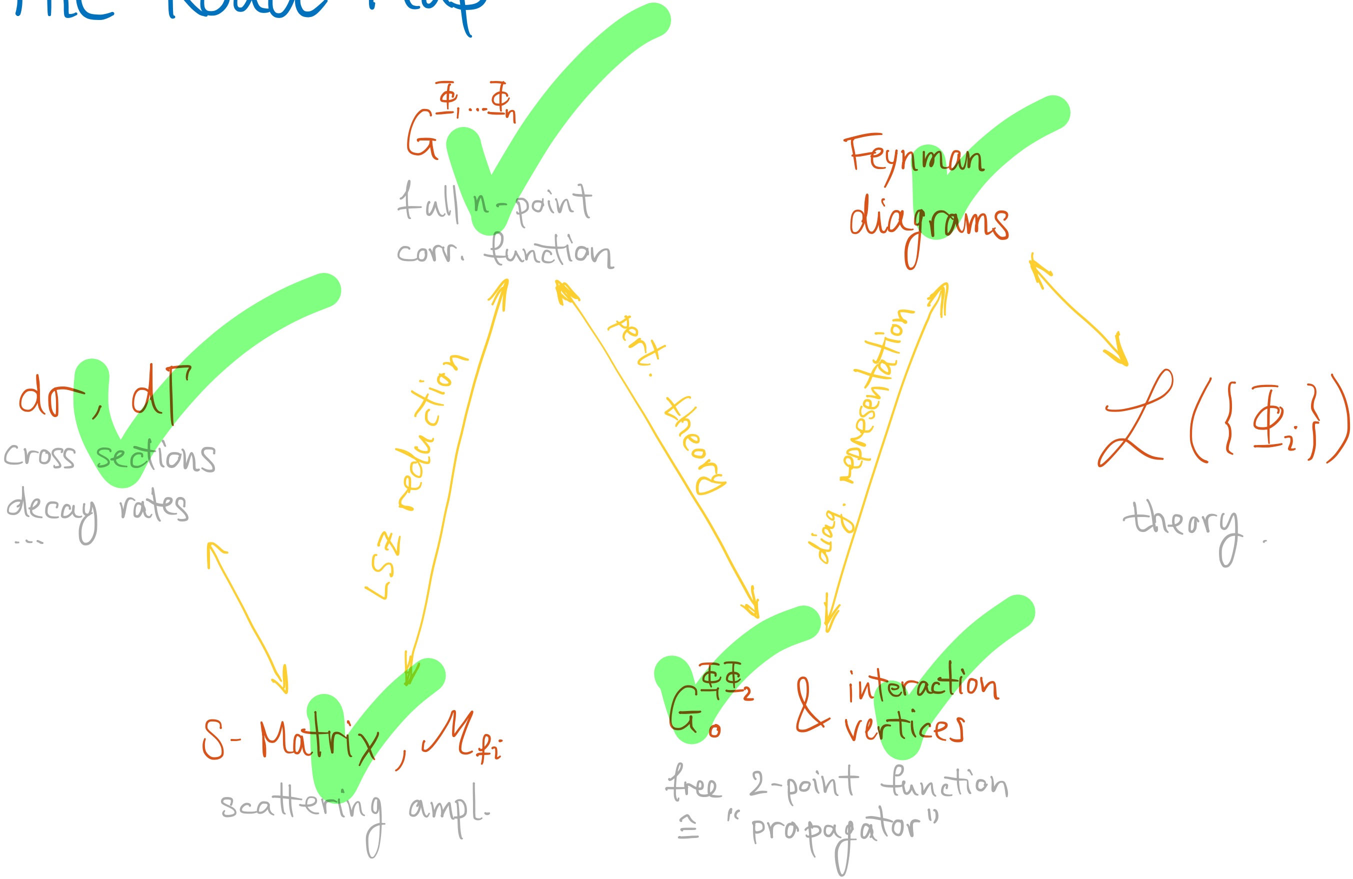
$$d\Gamma = \frac{1}{2E_{p_a}} |\mathcal{M}_{fi}|^2 \frac{1}{S_{\{n\}}} d\Phi_n(p_1, \dots p_n; p_a)$$

- LIPS

$$d\Phi_n(p_1, \dots p_n; Q) = \prod_{i=1}^{n} \left[\frac{d^4 p_i}{(2\pi)^3} \cdot \overbrace{\delta^+(p_i^2 - m_i^2)}^{\delta(p_i^2 - m_i^2)\,\Theta(p^0)} \right] \overbrace{(2\pi)^4 \delta\left(Q - \sum_{i=1}^{n} p_i\right)}^{\text{over-all 4-momentum conservation}}$$

$$= \prod_{i=1}^{n} \left[\frac{d^3 p_i}{(2\pi)^3\, 2E_i} \right] (2\pi)^4 \delta\left(Q - \sum p_i\right)$$

The Road Map



$G^{\Phi_1 \ldots \Phi_n}$
full n-point corr. function
Feynman diagrams
$d\sigma$, $d\Gamma$
cross sections
decay rates
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pert. theory
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theory.
S-Matrix, M_{fi}
scattering ampl.
$G_0^{\Phi_1 \Phi_2}$ & interaction vertices
free 2-point function
≙ "propagator"

Appendix: Vertex Functions @ Tree-Graph Level

- reintroduce $\hbar$:

$$Z[J] = \frac{1}{N} \int \mathcal{D}[\Phi] \exp\left\{ \frac{i}{\hbar} \int d^4x \left[\mathcal{L} + \hbar J(x) \Phi(x) \right] \right\}$$

$$= \frac{1}{N} \exp\left\{ \frac{i}{\hbar} \int d^4y \, \mathcal{L}_{int}\left(\frac{\delta}{i\delta J(y)} \right) \right\} \exp\left\{ \frac{\hbar}{2} \int d^4x \, d^4x' \; iJ(x) \; i\Delta_F(x-x') \; iJ(x') \right\}$$

$\Rightarrow$ each vertex (V): $\hbar^{-1}$, each propagator (I): $\hbar^{+1}$

$\Leftrightarrow$ for vertex graphs (1PI) $\quad L = I - V + 1$

$\Rightarrow$ $\hbar$ expansion $\hat{=}$ Loop expansion : $\Gamma = \sum_L \hbar^L \Gamma^{(L)}$

- method of stationary phase:

$\Rightarrow$ classical $(\hbar \to 0)$ solution Φ_{cl} from e.o.m. : $\dfrac{\delta \mathcal{L}}{\delta \Phi} = \partial_\mu \dfrac{\delta \mathcal{L}}{\delta(\partial_\mu \Phi)}$

0: tree-graph level

$$\Rightarrow Z^{(0)}[J] = \frac{1}{N} \exp\left\{ \frac{i}{\hbar} \int d^4x \, \mathcal{L}(\Phi_{cl}, \partial_\mu \Phi_{cl}) \right\} \exp\left\{ i \int d^4y \, J(y) \, \Phi_{cl}(y) \right\}$$

$$\Rightarrow Z^{(0)}_{con}[J] = \ln\left(Z[J] \right) = \text{const.} + \frac{i}{\hbar} \int d^4x \, \mathcal{L}(\Phi_{cl}, \partial_\mu \Phi_{cl}) + i \int d^4y \, J(y) \, \Phi_{cl}(y)$$

conjugate field: $\dfrac{\delta Z^{(0)}_{con}[J]}{i\delta J(x)} = \Phi_{cl}(x)$

$\Rightarrow$ Legendre-transf : $\Gamma^{(0)}[\Phi_{cl}] = -i \int d^4x \, \Phi_{cl}(x) J(x) + Z^{(0)}_{con}[J] = \dfrac{i}{\hbar} \int d^4x \, \mathcal{L}(\Phi_{cl}, \partial_\mu \Phi_{cl})$

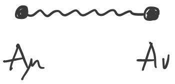
II. QED & Renormalization

The QED Feynman Rules

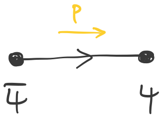
$$\mathcal{L}_{QED} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\psi}\,(i\not{\partial} - Q\, e \not{A} - m\,)\psi \;\; - \frac{1}{2\xi}(\partial\cdot A)^2$$

let's take this for granted ... ⇝ part III

- propagators:

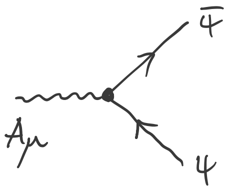


$$= \frac{-i}{k^2+i0}\left[g^{\mu\nu} - (1-\xi)\frac{k^\mu k^\nu}{k^2} \right]$$



$$= \frac{i}{\not{k} - m + i0}$$

- vertex:



$$= -ie\,Q_f\,\gamma^\mu$$

- external legs:

incoming $\xrightarrow{k}$

$= \varepsilon_\mu(k,\lambda)$

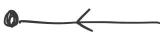
$= u(k,\sigma)$

$= \bar{v}(k,\sigma)$

outgoing $\xrightarrow{k}$

$= \varepsilon^*_\mu(k,\lambda)$

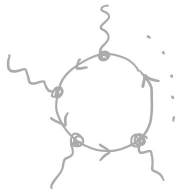
$= \bar{u}(k,\sigma)$



$= v(k,\sigma)$

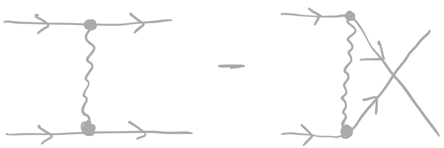
More Rules ...

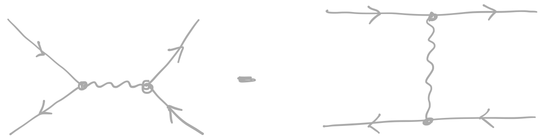
- momentum conservation @ each vertex
- undetermined (loop) momenta $\rightsquigarrow \int \frac{d^4p}{(2\pi)^4}$
- traverse fermion lines in opposite direction to the arrow
- symmetry factors $1/S_G$
- (-1) for each closed fermion loop



- relative (-1) between diagrams related by exchange of external fermion legs

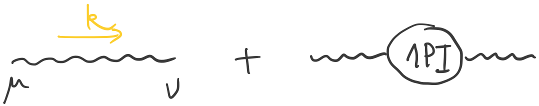
$e^- e^- \to e^- e^-$ or Bhabha





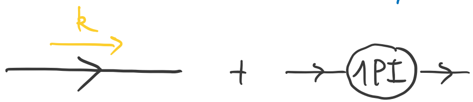
QED Vertex Functions

- photon 2-pt:



$$\Gamma^{AA}_{\mu\nu}(k,-k) = -\left[G^{AA}_{\mu\nu}(k,-k)\right]^{-1} = -i\left[g_{\mu\nu}k^2 - \left(1-\frac{1}{\xi}\right)k_\mu k_\nu\right] - i\,\Sigma^{AA}_{\mu\nu}(k)$$

- electron 2-pt:

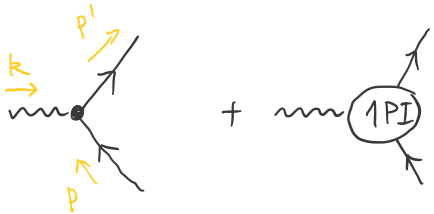


"self energies"

$$\Gamma^{\bar\psi\psi}(-k,k) = -\left[G^{\psi\bar\psi}(k,-k)\right]^{-1} = i(\not{k}-m) + i\,\Sigma^{\bar\psi\psi}(k)$$

- electron-photon vertex:

"vertex corrections"



$$\Gamma^{A\bar\psi\psi}_{\mu}(k,-p',p) = -ieQ\gamma_\mu - ieQ\,\Lambda_\mu(p',p)$$

The Ward-Takahashi Identity

- identities between correlation functions due to electromagnetic $U(1)$ symmetry

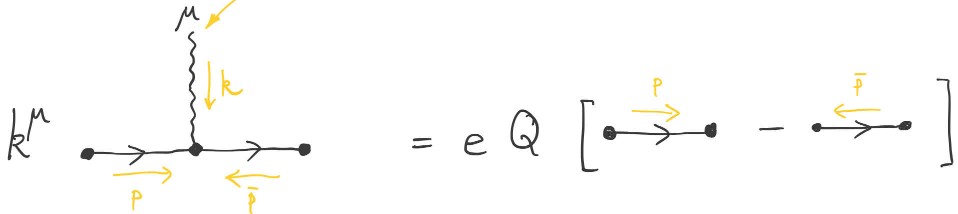
- Ward identity for the electron-photon vertex:

$$k^\mu \; G^{\psi\bar\psi}(\bar p,-\bar p)\;\Gamma^{A\bar\psi\psi}_\mu(k,\bar p,p)\;G^{\psi\bar\psi}(-p,p) = eQ\left[G^{\psi\bar\psi}(-p,p) - G^{\psi\bar\psi}(\bar p,-\bar p)\right]$$

↪ tree level:

no dot: truncated

EX 7



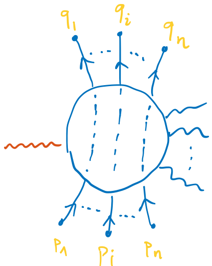
(not using the vertex $\Gamma^{A\bar\psi\psi}_\mu$ but the Green's function

$$\frac{i}{\xi}k^2 k^\mu\, G^{A\bar\psi\psi}_\mu(k,\bar p,p) = eQ\left[G^{\psi\bar\psi}(-p,p) - G^{\psi\bar\psi}(\bar p,-\bar p)\right]\Big)$$

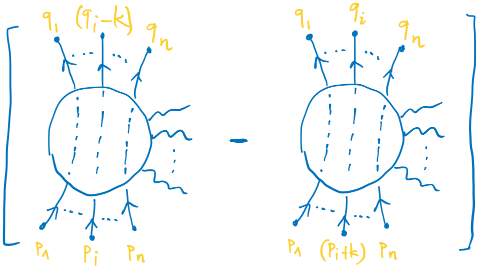
The Ward-Takahashi Identity

- general amplitude with an external photon:

$$\mathcal{A} = \varepsilon_\mu^{(*)}(k)\,\mathcal{A}^\mu =$$



$$\Rightarrow k_\mu \mathcal{A}^\mu = eQ \sum_{i=1}^{n}$$

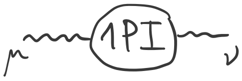


- for scattering amplitudes (LSZ $\longleftrightarrow$ truncation / on-shell residue)

$$\mathcal{M} = \varepsilon_\mu^{(*)}(k)\,\mathcal{M}^\mu \quad \Rightarrow \quad k_\mu \mathcal{M}^\mu = 0$$

1-Loop Corrections: Photon Self Energy

$$-i\Sigma^{AA}_{\mu\nu}(k) =$$



$$=$$

$$+ \dots$$

$$\Rightarrow \Sigma^{AA}_{\mu\nu}(k) = i(-1)\int\frac{d^4q}{(2\pi)^4}\,\mathrm{Tr}\left\{(-ieQ\gamma_\mu)\frac{i}{q-m}(-ieQ\gamma_\nu)\frac{i}{\not{q}+\not{k}-m}\right\}$$

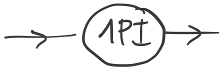
$$= -ie^2Q^2\int\frac{d^4q}{(2\pi)^4}\,\frac{\mathrm{Tr}\left[\gamma_\mu(\not{q}+m)\gamma_\nu(\not{q}+\not{k}+m)\right]}{(q^2-m^2)\left((q+k)^2-m^2\right)}$$

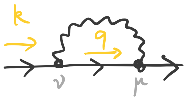
↳ UV behaviour $(q \to \infty)$

$$\int d^4q \sim \int dq\, q^3 \; ; \quad (q^2-m^2),\ \left((q+k)^2-m^2\right) \sim q^2$$

$\Rightarrow \Sigma^{AA}_{\mu\nu} \sim \int dq \left\{\frac{1}{q}, 1, q\right\}$ ⇒ quadratically divergent!

1-Loop Corrections : Elektron Self Energy

$i\Sigma_1^{\bar{\psi}\psi}(k)$ = 

= 

$+ \ldots$ (Feynman gauge: $\xi = 1$)

$$\Rightarrow \Sigma_1^{\bar{\psi}\psi}(k) = (-i)\int \frac{d^4q}{(2\pi)^4}\,(-ieQ\gamma_\mu)\,\frac{i}{\not{q}-m}\,(-ieQ\gamma_\nu)\cdot\frac{-i\,g^{\mu\nu}}{(q-k)^2-\lambda^2}$$

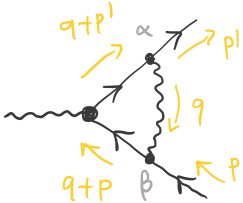
$$= ie^2Q^2\int\frac{d^4q}{(2\pi)^4}\,\frac{\gamma_\mu(\not{q}+m)\gamma^\mu}{(q^2-m^2)\left((q-k)^2-\lambda^2\right)}$$

$\hookrightarrow$ UV behaviour ($q \to \infty$)

$\Rightarrow \Sigma_1^{\bar{\psi}\psi} \sim \int dq \left\{\frac{1}{q}, 1\right\} \Rightarrow$ linearly divergent!

1-Loop Corrections: Electron-Photon Vertex

(Feynman gauge: $\xi = 1$)

$$-ieQ\,\Lambda_\mu(p',p) =$$  $$=$$  $$+ \dots$$

$$\Rightarrow \Lambda_\mu(p',p) = \frac{i}{eQ} \int \frac{d^4q}{(2\pi)^4} \frac{-i g^{\alpha\beta}}{q^2-\lambda^2} (-ieQ\gamma_\alpha) \frac{i}{\not{q}+\not{p}'-m} (-ieQ\gamma_\mu) \frac{i}{\not{q}+\not{p}-m} (-ieQ\gamma_\beta)$$

$$= -ie^2Q^2 \int \frac{d^4q}{(2\pi)^4} \frac{\gamma_\alpha(\not{q}+\not{p}'+m)\gamma_\mu(\not{q}+\not{p}-m)\gamma^\alpha}{(q^2-\lambda^2)\left((q+p')^2-m^2\right)\left((q+p)^2-m^2\right)}$$

$\hookrightarrow$ UV behaviour ($q \rightarrow \infty$)

EX 8

$$\Rightarrow \Lambda_\mu(p',p) \sim \int dq \left\{ \frac{1}{q^3}, \frac{1}{q^2}, \frac{1}{q} \right\} \Rightarrow \text{logarithmically divergent!}$$

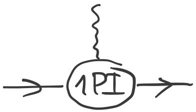
Superficially Divergent Vertex Functions

- EX 8 $\Rightarrow$ $\omega(G) = 4 - \frac{3}{2} E_4 - E_A$

$\omega = 3$ $\longrightarrow$ $= 0$ (quant. number of vacuum)

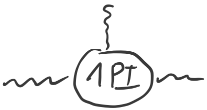
$\omega = 2$

$\omega = 1$

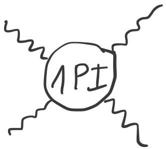


$\omega = 0$

$\Bigg\}$ actually divergent ones



$\omega = 1$ $\longrightarrow$ $= 0$ (Furry's theorem)



$\omega = 0$ $\longrightarrow$ $=$ finite (Ward Id.) $\rightsquigarrow$ Appendix

(gauge symmetry essential in renormalizability!)

The General Idea

Problem: radiative corrections give infinities (UV)

① regularization: tame the infinities by modifying the theory such that calculations become formally well-defined.

$\leftrightarrow$ regularization parameter δ for $\delta \to \delta_0$: recover div. expressions

② renormalization: how do we know what λ_0 (bare param) is?

$\hookrightarrow$ we have to <u>measure</u> it

$\hookrightarrow$ prescription / observable how to measure it: scheme

$\hookrightarrow$ once measurable quantities are re-expressed in terms of other measurable quantities $\Rightarrow$ δ-dependence cancels!

(independent of how it was regularized)

Regularization Schemes

- momentum cut-off: $\int d^4q \longrightarrow \int_{|q^0|,|\vec{q}|<\Lambda} d^4q$ $\quad (\delta = \Lambda\,,\ \delta_0 = \infty)$

- "Pauli-Villars": $\dfrac{1}{q^2-m^2} \longrightarrow \dfrac{1}{q^2-m^2} - \dfrac{1}{q^2-M^2}$ $\quad (\delta = M,\ \delta_0 = \infty)$

- "Momentum subtraction": ex. $\Gamma(p_1^2, \ldots) \longrightarrow \Gamma(p_1^2, \ldots) - \Gamma(p_1^2 = q_1^2, \ldots)$

- put it on the lattice: $\int d^4x \longrightarrow \sum_{x_i} \Delta_i$ $\quad (\delta = a\,,\ \delta_0 = 0)$

  $x_i \leftarrow$ lattice spacing

- dimensional regularization: $\delta = D\,,\ \delta_0 = 4$

  $\oplus$ Lorentz- & gauge-invariance, IR regularization, simple

Dimensional Regularization (Dim Reg)

$$\int \frac{d^4q}{(2\pi)^4} \longrightarrow \int \frac{d^Dq}{(2\pi)^D} \qquad (D = 4-2\epsilon)$$

- defined as a set of axioms that integrals should satisfy:
  - ↪ shifts of loop momenta <u>always</u> allowed:

$$\int d^Dq\, f(k+p) \equiv \int d^Dq\, f(q)$$

  - ↪ analog of D-dim. rotational symmetry / Lorentz transformations

$$\int d^Dq\, f(\Lambda q) \equiv \int d^Dq\, f(q)$$

Scale-less integrals vanish: $\int d^Dq\, (q^2)^\alpha \equiv 0$

  - ↪ scaling property $\int d^Dq\, f(a\cdot q) \equiv a^{-D} \int d^Dq\, f(q)$
  - ↪ linearity: $\int d^Dq\, [f(q)+g(q)] \equiv \int d^Dq\, f(q) + \int d^Dq\, g(q)$
  - ↪ differentiation & integration commute:

$$\frac{\partial}{\partial p_\mu} \int d^Dq\, f(p,q) \equiv \int d^Dq\, \frac{\partial}{\partial p_\mu} f(p,q)$$

More on Dim Reg

- metric $\delta^\mu_\mu = g^\mu_\mu = D$
- Dirac matrices : $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} \Rightarrow \gamma^\mu \gamma_\mu = g^\mu_\mu = D$

  $\gamma^\mu \gamma_\nu \gamma_\mu = (2-D)\gamma_\nu$, ...

  can define $\mathrm{Tr}(\mathbb{1}) = 4$
- no natural extension of γ^5 in D dimensions

  $\mathrm{Tr}[\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma^5] \sim \varepsilon^{\mu\nu\rho\sigma}$ $\rightsquigarrow$ need to adapt γ^5 algebra
- field / coupling dimensions ($[S]=0 \Rightarrow [\mathcal{L}]=D$)

  $[\psi] = \frac{D-1}{2}$, $[\phi] = [A_\mu] = \frac{D-2}{2}$

  $\Rightarrow [e] = \frac{4-D}{2}$; keep $[e]=0 \Rightarrow e \to \mu^{\frac{4-D}{2}} e$ (often absorbed into loop integration)

  scale of Dim Reg !
- $\omega(G)$ in D-dim $\rightsquigarrow$ EX 8

  $$\omega(G) = D - \frac{D-1}{2} E_\psi - \frac{D-2}{2} E_A + V\left(\frac{D-4}{2}\right)$$

1-Loop Integrals — The Roadmap

generally confronted with

tensor integrals $T^N_{\mu_1 \dots \mu_M}$

$$\int d^D q \, \frac{q_{\mu_1} \cdots q_{\mu_M}}{\prod_{i=1}^{N} D_i}$$

$$\left(D_i = (q+p_i)^2 - m_i^2 \right)$$

Passarino-Veltman reduction

scalar integrals S^N

$$\int d^D q \, \frac{1}{\prod_{i=1}^{N} D_i}$$

Feynman parameters

merge propagators

to isolate loop integral

$$\int \left(\prod_{i=1}^{N} dx_i \right) \frac{\delta(1 - \sum x_i)}{\left[\sum_i D_i x_i \right]^N}$$

tadpole

generic integral

perform $\int d^D q$

$$I_N(A) := \int d^D q \left[q^2 - A + i0 \right]^{-N}$$

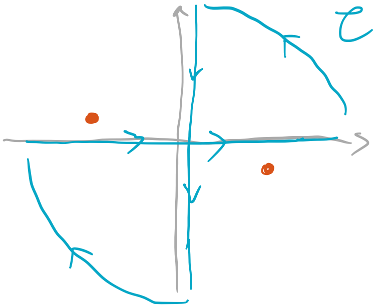
Our First Integral

$$I_n(A) := \int d^Dq \frac{1}{(q^2 - A + i0)^n} \qquad (D < 2n \,,\; A > 0)$$

convergence

① Wick-rotation in the complex q^0 plane

$\hookrightarrow$ poles @ $q^0 = \pm\sqrt{\vec{q}^{\,2} + A^2 - i0} = \pm\sqrt{\vec{q}^{\,2} + A^2} \mp i0$



$$\oint_{\mathcal{C}} dq^0 \, (q^2 - A^2 + i0)^{-n} = 0$$

Eucledian coordinates: $q_E^0 = i\, q^0$

$$\Rightarrow \int_{-\infty}^{+\infty} dq^0 \, (\ldots)^{-n} = \int_{-i\infty}^{i\infty} dq^0 \, (\ldots)^{-n} = i \int_{-\infty}^{\infty} dq_E^0 \, (\ldots)^{-n}$$

$$\left(q^2 = (q^0)^2 - \vec{q}^{\,2} = -(q_E^0)^2 - \vec{q}^{\,2} = -\, q_E^2 \right)$$

② Integration in polar coordinates:

$$\int d^Dq_E = \int d\Omega_D \int_0^\infty dq_E \, (q_E)^{D-1} = \int d\Omega_D \int_0^\infty dq_E^2 \, \frac{1}{2} \left(q_E^2 \right)^{\frac{D}{2} - 1}$$

Ex 9

Result for $I_n(A)$

$$I_n(A) \overset{(y = q_E^2)}{=} i(-1)^n \frac{\pi^{D/2}}{\Gamma(D/2)} (A - i0)^{\frac{D}{2}-n} \underbrace{\int_0^\infty dy \; y^{\frac{D}{2}-1} (1+y)^{-n}}_{B\left(\frac{D}{2}, n-\frac{D}{2}\right)}$$

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$$

$$= i(-1)^n \pi^{D/2} \frac{\Gamma(n - D/2)}{\Gamma(n)} (A - i0)^{\frac{D}{2}-n}$$

$\hookrightarrow$ analytic continuation in D & A possible!

- properties of the Gamma function

$\hookrightarrow$ $\Gamma(z)$ is meromorphic: Poles @ $z = -n$, $n \in \mathbb{N}_0$

$\hookrightarrow$ $\Gamma(z+1) = z\,\Gamma(z)$

$\hookrightarrow$ $\Gamma(n+1) = n!$ $\forall n \in \mathbb{N}$

$\hookrightarrow$ $\Gamma(\epsilon) = \frac{1}{\epsilon} - \gamma_E + \mathcal{O}(\epsilon)$ with $\gamma_E = 0.57\ldots$ (Euler const)

UV divergences as $\frac{1}{\epsilon}$ poles

1-Loop Integrals — The Roadmap

generally confronted with

tensor integrals $T^N_{\mu_1 \cdots \mu_M}$

$$\int d^D q \, \frac{q_{\mu_1} \cdots q_{\mu_M}}{\prod_{i=1}^{N} D_i}$$

$$\left(D_i = (q+p_i)^2 - m_i^2 \right)$$

Passarino-Veltman reduction

scalar integrals S^N

$$\int d^D q \, \frac{1}{\prod_{i=1}^{N} D_i}$$

Feynman parameters

merge propagators

to isolate loop integral

$$\int \left(\prod_{i=1}^{N} dx_i \right) \frac{\delta(1-\sum x_i)}{\left[\sum_i D_i x_i\right]^N}$$

tadpole

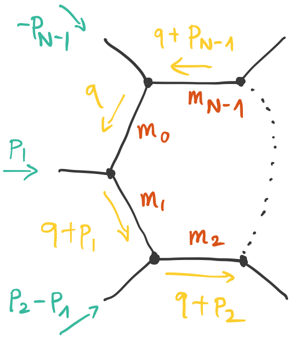
generic integral

perform $\int d^D q$

$$I_N(A) := \int d^D q \, \left[q^2 - A + i0 \right]^{-N}$$

Scalar Integrals

$$S^N(p_1,\dots,p_{N-1},m_0,\dots,m_{N-1}) := \frac{(2\pi\mu)^{4-D}}{i\pi^2}\int d^Dq \prod_{n=0}^{N-1}\frac{1}{(q+p_n)^2-m_n^2+i0} \qquad (p_0=0)$$



Notation: $S^1 = A_0$ (tadpole)

$S^2 = B_0$ (bubble)

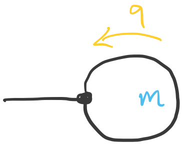
$S^3 = C_0$ (triangle)

$\vdots$ $\quad\vdots$

- all scalar integrals can be brought into the form of $I_n(A)$

  ↪ D-dimensional momentum integration ✓

1-pt Function

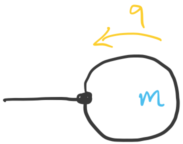


$$A_0(m) = \frac{(2\pi\mu)^{4-D}}{i\pi^2} \underbrace{\int d^Dq \left(q^2 - m^2 + i0 \right)^{-1}}_{I_1(m^2)}$$

$$= -m^2 \left(\frac{m^2}{4\pi\mu^2} \right)^{\frac{D-4}{2}} \Gamma\left(\frac{2-D}{2} \right)$$

EX10

1-pt Function



$$A_0(m) = \frac{(2\pi\mu)^{4-D}}{i\pi^2} \underbrace{\int d^D q \, (q^2 - m^2 + i0)^{-1}}_{I_1(m^2)}$$

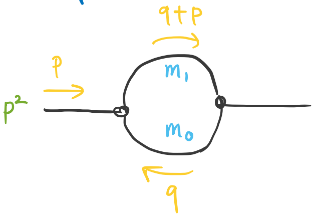
$$= -m^2 \left(\frac{m^2}{4\pi\mu^2}\right)^{\frac{D-4}{2}} \Gamma\left(\frac{2-D}{2}\right)$$

$$= m^2 \Big[\underbrace{\frac{1}{\epsilon} - \gamma_E + \ln(4\pi)}_{=:\Delta} - \ln\left(\frac{m^2}{\mu^2}\right) + 1 \Big] + \mathcal{O}(\epsilon)$$

(divergent constant) $\rightsquigarrow$ $\overline{\text{MS}}$ scheme

- note: $A_0(m)\big|_{\text{div}} = \dfrac{m^2}{\epsilon}$

2-pt Function



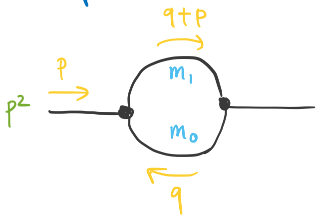
$$B_0(p, m_0, m_1) = \frac{(2\pi\mu)^{4-D}}{i\pi^2} \int d^D q \, \underbrace{(q^2-m_0^2+i0)^{-1}\left((q+p)^2-m_1^2+i0\right)^{-1}}_{\frac{1}{a}\cdot\frac{1}{b}}$$

- to reduce S^N to $I_n(A)$ we use Feynman parameters

$$\frac{1}{A_1\cdot\ldots\cdot A_N} = \Gamma(N)\int_0^1 \prod_{i=1}^{N} dx_i \, \frac{\delta\left(1-\sum_{i=1}^{N} x_i\right)}{\left[\sum_{i=1}^{N} x_i A_i\right]^N}$$

EX 11

2-pt Function



$$B_0(p,m_0,m_1) = \frac{(2\pi\mu)^{4-D}}{i\pi^2}\int d^Dq\ \underbrace{(q^2-m_0^2+i0)^{-1}\left((q+p)^2-m_1^2+i0\right)^{-1}}$$

(Feynman param)

$$= \int_0^1 dx \left\{ (q^2-m_0^2+i0)(1-x) + \left[(q+p)^2-m_1^2+i0\right]x \right\}^{-2}$$

$$= \int_0^1 dx \left\{ \underbrace{(q+xp)^2}_{q'} \underbrace{- x^2p^2 + x(p^2-m_1^2+m_0^2) - m_0^2 + i0}_{=:-A} \right\}^{-2}$$

$$\Rightarrow B_0(p,m_0,m_1) = \frac{(2\pi\mu)^{4-D}}{i\pi}\int_0^1 dx\, I_2(A) = (4\pi\mu^2)^{\frac{4-D}{2}}\,\Gamma\left(\frac{4-D}{2}\right)\int_0^1 dx\left[x^2p^2 + x(p^2-m_1^2+m_0^2)+m_0^2-i0\right]^{\frac{D-4}{2}}$$

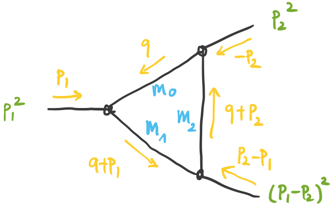
$$= \Delta - \int_0^1 dx\ \ln\left[\frac{x^2p^2 - x(p^2-m_1^2+m_0^2)+m_0^2-i0}{\mu^2}\right] + \mathcal{O}(\epsilon)$$

- note: $B_0(p,m_0,m_1)\big|_{div} = \frac{1}{\epsilon}$
- alternative notation: $B_0(p^2,m_0,m_1)$
- symmetry: $B_0(p^2,m_0,m_1) = B_0(p^2,m_1,m_0)$

EX 12

3-pt Function

finite in $D=4$ dimensions!



$$C_0(p_1,p_2,m_0,m_1,m_2) = \frac{1}{i\pi^2}\underbrace{\int d^4q\, \left[q^2-m_0^2+i0\right]^{-1}\left[(q+p_1)^2-m_1^2+i0\right]^{-1}\left[(q+p_2)^2-m_2^2+i0\right]^{-1}}$$

(Feynman param) $\displaystyle \frac{1}{abc} = 2\int_0^1 dx \int_0^{1-x} dy \left[a(1-x-y)+bx+cy\right]^{-3}$

... complete the square in q, shift momentum $q \to q + \ldots$, identify $I_3(A)$ ...

$$\Rightarrow C_0(p_1,p_2,m_0,m_2,m_3) = -\int_0^1 dx \int_0^{1-x} dy \left[x^2p_1^2 + y^2p_2^2 + 2xy\, p_1p_2 - x(p_1 - m_1^2+m_0^2) - y(p_2^2-m_2^2+m_0^2) + m_0^2 + i0\right]^{-1}$$

- steps: - linearize in x or y: $x \to x + \alpha y$ to kill all y^2 terms $\rightsquigarrow$ y-integration
  - decompose quad. forms into lin. factors $\rightsquigarrow$ $\displaystyle \int dx \frac{\ln(ax+b)}{cx+d}$ $\rightsquigarrow$ $\ln$, $\ln^2$, Li_2
- note: $C_0(p_1,p_2,m_0,m_1,m_2)\Big|_{div} = 0$
- alternative notation: $C_0(p_1^2, (p_1-p_2)^2, p_2^2, m_0, m_1, m_2)$

1-Loop Integrals — The Roadmap

generally confronted with

tensor integrals $T^N_{\mu_1 \dots \mu_M}$

$$\int d^D q \, \frac{q_{\mu_1} \cdots q_{\mu_M}}{\prod_{i=1}^{N} D_i}$$

$$\left(D_i = (q+p_i)^2 - m_i^2 \right)$$

Passarino-Veltman reduction

scalar integrals S^N

$$\int d^D q \, \frac{1}{\prod_{i=1}^{N} D_i}$$

Feynman parameters

merge propagators

to isolate loop integral

$$\int \left(\prod_{i=1}^{N} dx_i \right) \frac{\delta(1-\sum x_i)}{\left[\sum_i D_i x_i \right]^N}$$

tadpole

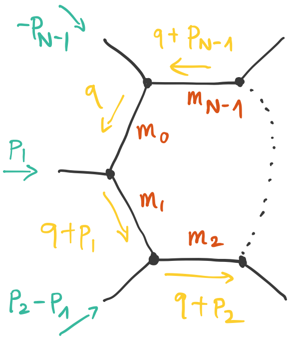
generic integral

perform $\int d^D q$

$$I_N(A) := \int d^D q \, \left[q^2 - A + i0 \right]^{-N}$$

Tensor Integrals

$$T^{N}_{\mu_1\cdots\mu_M}(p_1,\dots,p_{N-1},m_0,\dots,m_{N-1}) := \frac{(2\pi\mu)^{4-D}}{i\pi^2}\int d^Dq\; q_{\mu_1}\cdots q_{\mu_M}\prod_{n=0}^{N-1}\frac{1}{(q+p_n)^2-m_n^2+i0} \qquad (p_0=0)$$



Notation:

$$T^1 = S^1 = A_0$$

$$T^2 = S^2 = B_0, \quad T^2_\mu = B_\mu, \quad T^2_{\mu\nu} = B_{\mu\nu}$$

$$T^3 = S^3 = C_0, \quad T^3_\mu = C_\mu, \quad T^3_{\mu\nu} = C_{\mu\nu}$$

$$\vdots \qquad \vdots$$

- symmetric in $\mu_1\dots\mu_M$, Lorentz-covariance

$\Rightarrow$ decomposition into symmetric tensors of degree M

example:

scalar!

$$C_{\mu\nu}(p_1,p_2,m_0,m_1,m_2) = g_{\mu\nu}C_{00} + p_{1,\mu}p_{1,\mu}C_{11} + p_{2,\mu}p_{2,\mu}C_{22} + (p_{1,\mu}p_{2,\nu}+p_{2,\mu}p_{1,\nu})C_{12}$$

Passarino – Veltman Reduction

- algebraic reduction of tensor coefficients $B_1, B_{11}, B_{00}, C_1, \dots$
  to scalar integrals $A_0, B_0, C_0, \dots$
- recursive algorithm: $\left(T^N_{\mu_1 \dots \mu_M}\right)$

  ① contraction of integral representation & tensor decomposition with external momenta p_i^μ and the metric $g^{\mu\mu}$

  ↪ in the integrand:

  - $$p_i^\mu q_\mu = \frac{1}{2}\left[(q+p_i)^2 - q^2 - p_i^2\right] = \underbrace{\frac{1}{2}\overbrace{\left[(q+p_i)-m_i^2\right]}^{\text{kill prop. } i} - \frac{1}{2}\overbrace{\left[q^2-m_0^2\right]}^{\text{kill prop. } 0}}_{(N-1)\text{-pt integrals}} - \underbrace{\frac{1}{2}\left(p_i^2-m_i^2+m_0^2\right)}_{\text{rank }(M-1)}$$
  - $$g^{\mu\nu} q_\mu q_\nu = \underbrace{\overbrace{\left[q^2-m_0^2\right]}^{\text{kill prop. } 0}}_{(N-1)\text{-pt}} + \underbrace{m_0^2}_{\text{rank }(M-2)}$$

  ↪ in the tensor decomposition ⇝ linear combination of tensor coefficients

  ② solve the system of linear equations for $T^N_{ij\dots}$

2-pt Function

- short-hand notation $\quad \left\langle \ldots \right\rangle_q \equiv \frac{(2\pi\mu)^{D-4}}{i\pi^2} \int d^D q \, (\ldots)$

- rank 1: $\quad B_\mu(p, m_0, m_1) = \left\langle \frac{q_\mu}{(q^2-m_0^2)\left[(q+p)^2-m_1^2\right]} \right\rangle_q = p_\mu B_1(p^2, m_0, m_1)$

$\hookrightarrow$ contraction with p^μ

$$\Rightarrow p^2 B_1 = \left\langle \frac{p\cdot q}{(q^2-m_0^2)\left[(q+p)^2-m_1^2\right]} \right\rangle_q$$

$$= \left\langle \frac{\frac{1}{2}\left[(q+p)^2-m_1^2\right] - \frac{1}{2}\left[q^2-m_0^2\right] - \frac{1}{2}\left(p^2-m_1^2+m_0^2\right)}{(q^2-m_0^2)\left[(q+p)^2-m_1^2\right]} \right\rangle_q$$

$$= \frac{1}{2}\left\langle \frac{1}{q^2-m_0^2} \right\rangle_q - \frac{1}{2}\left\langle \frac{1}{(q+p)^2-m_1^2} \right\rangle_q - \frac{1}{2}\left\langle \frac{p^2-m_1^2+m_0^2}{(q^2-m_0^2)\left[(q+p)^2-m_1^2\right]} \right\rangle_q$$

$$= \frac{1}{2} A_0(m_0) - \frac{1}{2} A_0(m_1) - \frac{1}{2}\left(p^2-m_1^2+m_0^2\right) B_0$$

$$\Rightarrow B_1 = \frac{1}{2p^2}\left[A_0(m_0) - A_0(m_1) - \left(p^2-m_1^2+m_0^2\right) B_0\right]$$

$\hookrightarrow$ note: $\quad B_1(p^2, m_0, m_1)\Big|_{div} = \frac{1}{2p^2}\left[\frac{m_0^2}{\epsilon} - \frac{m_1^2}{\epsilon} - \left(p^2-m_1^2+m_0^2\right)\frac{1}{\epsilon}\right] = -\frac{1}{2\epsilon}$

2-pt Function

- rank 2: $$B_{\mu\nu}(p,m_0,m_1) = \left\langle \frac{q_\mu q_\nu}{(q^2-m_0^2)\left[(q+p)^2-m_1^2\right]} \right\rangle_q = g_{\mu\nu} B_{00} + p_\mu p_\nu B_{11}$$

$\hookrightarrow$ contraction with $g^{\mu\nu}$

$$\Rightarrow D\, B_{00} + p^2 B_{11} = \left\langle \frac{q^2-m_0^2+m_0^2}{(q^2-m_0^2)\left[(q+p)^2-m_1^2\right]} \right\rangle_q = A_0(m_1) + m_0^2 B_0$$

$\hookrightarrow$ contraction with p^μ

$$\Rightarrow p_\nu \left(B_{00} + p^2 B_{11} \right) = \left\langle \frac{q_\nu \left\{ \frac{1}{2}\left[(q+p)^2-m_1^2\right] - \frac{1}{2}\left[q^2-m_0^2\right] - \frac{1}{2}(p^2-m_1^2+m_0^2) \right\}}{(q^2-m_0^2)\left[(q+p)^2-m_1^2\right]} \right\rangle_q$$

$$= \frac{1}{2} \underbrace{\left\langle \frac{q_\nu}{q^2-m_0^2} \right\rangle_q}_{=0} - \frac{1}{2} \underbrace{\left\langle \frac{q_\nu}{(q+p)^2-m_1^2} \right\rangle_q}_{\left\langle \frac{q'_\nu - p_\nu}{(q')^2-m_1^2} \right\rangle_{q'} = -p_\nu A_0(m_1)} - \frac{1}{2}(p^2-m_1^2+m_0^2) \underbrace{B_\nu}_{p_\nu B_1}$$

$$= p_\nu \left[\frac{1}{2} A_0(m_1) - \frac{1}{2}(p^2-m_1^2+m_0^2) B_1 \right]$$

EX13

3-pt Function

- rank 1: $C_\mu(p_1, p_2, m_0, m_1, m_3) = p_{1,\mu} C_1 + p_{2,\mu} C_2$

↪ contraction with p_1^μ

$$\Rightarrow p_1^2 C_1 + p_1 \cdot p_2 C_2 = \frac{1}{2} B_0(p_2^2, m_0, m_2) - \frac{1}{2} B_0((p_1-p_2)^2, m_1, m_2) - \frac{1}{2}(p_1^2 - m_1^2 + m_0^2) C_0$$

↪ contraction with p_2^μ: as above, $1 \leftrightarrow 2$

$f_i = p_i^2 - m_i^2 + m_0$

$$\Rightarrow \begin{pmatrix} p_1^2 & p_1 \cdot p_2 \\ p_2 \cdot p_1 & p_2^2 \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} B_0(p_2^2, m_0, m_2) - \frac{1}{2} B_0((p_1-p_2)^2, m_1, m_2) - \frac{1}{2} f_1 C_0 \\ \frac{1}{2} B_0(p_1^2, m_0, m_1) - \frac{1}{2} B_0((p_1-p_2)^2, m_1, m_2) - \frac{1}{2} f_2 C_0 \end{pmatrix}$$

$$\Rightarrow C_1, C_2 = \ldots$$

↪ note: $C_1\big|_{div} = C_2\big|_{div} = 0$

1-Loop Integrals — The Roadmap

generally confronted with

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$$\left(D_i = (q+p_i)^2 - m_i^2\right)$$

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tadpole

generic integral

perform $\int d^Dq$

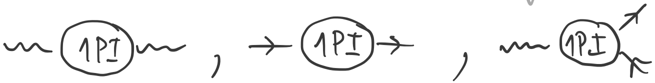
$$I_N(A) := \int d^Dq \, \left[q^2 - A + i0\right]^{-N}$$

A Bird's Eye View

① QED Ward identities

$$k^\mu \Sigma^{AA}_{\mu\nu}(k) = 0 \qquad\qquad k^\mu \Lambda_\mu(p',p) = \Sigma^{\bar\psi\psi}(p') - \Sigma^{\bar\psi\psi}(p)$$

② Study UV divergent behaviour (superficial degree of divergence)

our problem childs: 

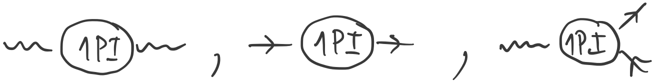
(

⇝ Appendix)

REGULARIZATION

③ Let's get a handle on infinities

dimensional regularization & 1-loop integrals

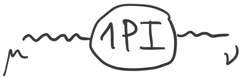
④ compute  in DimReg

RENORMALIZATION

⑤ renormalize the theory

⑥ the on-shell scheme

1-Loop Corrections: Photon Self Energy

$-i\Sigma^{AA}_{\mu\nu}(k) =$  $=$  $+ \mathcal{O}(\alpha^2)$

$$\Rightarrow \Sigma^{AA}_{\mu\nu}(k) = -ie^2 Q^2 \mu^{4-D} \int \frac{d^D q}{(2\pi)^D} \frac{\mathrm{Tr}[\gamma_\mu(\not{q}+m)\gamma_\nu(\not{q}+\not{k}+m)]}{(q^2-m^2)((q+k)^2-m^2)}$$

(Dim Reg)

- trace in the numerator:

$\mathrm{Tr}[\gamma_\mu(\not{q}+m)\gamma_\nu(\not{q}+\not{k}+m)]$ (only even # of γ's)

$$= \underbrace{\mathrm{Tr}[\gamma_\mu \not{q} \gamma_\nu (\not{q}+\not{k})]}_{4[-g_{\mu\nu}\, q\cdot(q+k) + 2q_\mu q_\nu + q_\mu k_\nu + k_\mu q_\nu]} + m^2 \underbrace{\mathrm{Tr}[\gamma_\mu\gamma_\nu]}_{4 g_{\mu\nu}}$$

$\left(\mathrm{Tr}[\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma] = 4(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho})\right)$

$$= -2g_{\mu\nu}\{\underbrace{[q^2-m^2]}_{\text{kills a propagator}} + \underbrace{[(q+k)^2-m^2]}_{\text{kills a propagator}} - k^2\} + 8q_\mu q_\nu + 4q_\mu k_\nu + 4k_\mu q_\nu$$

1-Loop Corrections: Photon Self Energy

$$\Rightarrow \Sigma^{AA}_{\mu\nu}(k) = \frac{\alpha}{2\pi} Q^2 \left\{ -g_{\mu\nu}\left[A_0(m) + A_0(m) - k^2 B_0\right] + 4 B_{\mu\nu} + 2 k_\mu B_\nu + 2 k_\nu B_\mu \right\} \qquad \left(\alpha = \frac{e^2}{4\pi}\right)$$

fine structure constant ~ 1/137

- tensor decomposition: $B_{\mu\nu} = g_{\mu\nu} B_{00} + k_\mu k_\nu B_{11}$, $B_\mu = k_\mu B_1$

$$\Rightarrow \Sigma^{AA}_{\mu\nu}(k) = \frac{\alpha}{2\pi} Q^2 \left\{ g_{\mu\nu}\left[k^2 B_0 - 2A_0 + 4 B_{00}\right] + 4 k_\mu k_\nu \left[B_{11} + B_1\right]\right\}$$

- reduction to scalar integrals:

$$B_1(k^2, m, m) = -\frac{1}{2} B_0$$

$$B_{00}(k^2, m, m) = \frac{1}{6}\left[A_0(m) + 2m^2 B_0 + k^2 B_1 + 2m^2 - \frac{k^2}{3}\right] + \mathcal{O}(\epsilon)$$

$$B_{11}(k^2, m, m) = \frac{1}{6k^2}\left[2A_0(m) - 2m^2 B_0 - 4k^2 B_1 - 2m^2 + \frac{k^2}{3}\right] + \mathcal{O}(\epsilon)$$

$$B_0(0, m, m) = \frac{\partial A_0(m)}{\partial m^2}$$

$$= (1-\epsilon)\frac{A_0(m)}{m^2}$$

$$\Rightarrow A_0(m) = m^2\left[B_0(0, m, m) + 1\right] + \mathcal{O}(\epsilon)$$

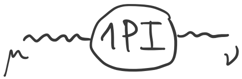
- decomposition into transversal & longitudinal parts:

EX 14

$$\Sigma^{AA}_{\mu\nu}(k) = \left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right)\Sigma^{AA}_T(k^2) + \frac{k_\mu k_\nu}{k^2}\Sigma^{AA}_L(k^2)$$

Scalars

1-Loop Corrections: Photon Self Energy

$$-i\,\Sigma^{AA}_{\mu\nu}(k) = $$  $$=$$  $$+\ \mathcal{O}(\alpha^2)$$

$$\Rightarrow \Sigma^{AA}_{\mu\nu}(k) = \left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right)\Sigma^{AA}_{T}(k^2) + \frac{k_\mu k_\nu}{k^2}\,\Sigma^{AA}_{L}(k^2)$$

$$\hookrightarrow \Sigma^{AA}_{T}(k^2) = \frac{\alpha}{3\pi}Q^2\left[k^2 B_0(k^2,m,m) + 2m^2\left(B_0(k^2,m,m) - B_0(0,m,m)\right) - \frac{k^2}{3}\right]$$

$$\hookrightarrow \Sigma^{AA}_{L}(k^2) = 0$$

- $\Sigma^{AA}_{L}(k^2) = 0$ to all orders due to Ward Identity ($\rightsquigarrow$ EX7: $k^\mu \Sigma^{AA}_{\mu\nu}(k) = 0$)
- UV divergence: $\Sigma^{AA}_{T}(k^2)\Big|_{div} = \frac{\alpha}{3\pi}k^2\frac{1}{\epsilon}$
- propagator ("vacuum polarization" $\Pi^{AA}(k^2) := \frac{\Sigma^{AA}_{T}(k^2)}{k^2}$)

$$G^{AA}_{\mu\nu}(k,-k) = \frac{-i}{\underbrace{k^2}\,[1+\Pi^{AA}(k^2)]}\left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right) - \xi\frac{i}{k^2}\frac{k_\mu k_\nu}{k^2}$$

no shift in the pole @ $k^2=0$ $\longleftrightarrow$ no photon mass!

EX 15

1-Loop Corrections : Elektron Self Energy

$$i\Sigma_1^{\bar{4}4}(k) = \longrightarrow \boxed{1PI} \longrightarrow =$$

$$+ \ \mathcal{O}(\alpha^2) \qquad (\text{Feynman gauge: } \xi = 1)$$

$$\Rightarrow \Sigma^{\bar{4}4}(k) = \not{k}\, \Sigma_V^{\bar{4}4}(k^2) + m\, \Sigma_S^{\bar{4}4}(k^2)$$

$$\longleftrightarrow \quad \Sigma_V^{\bar{4}4}(k^2) = -\frac{\alpha}{4\pi} Q^2 \frac{1}{k^2}\left[A_0(m) - (k^2+m^2)\, B_0(k^2; m, 0) + k^2\right]$$

$$\hookrightarrow \quad \Sigma_S^{\bar{4}4}(k^2) = -\frac{\alpha}{4\pi} Q^2 \left[4 B_0(k^2; m, 0) - 2\right]$$

- propagator

$$G^{4\bar{4}}(-p, p) = \frac{i}{\not{p} - m + \Sigma^{\bar{4}4}(p)} = i\, \frac{\not{p}\left[1 + \Sigma_V^{\bar{4}4}(p^2)\right] + m\left[1 - \Sigma_S^{\bar{4}4}(p^2)\right]}{\underbrace{p^2 - m^2 \left[\frac{1 - \Sigma_S^{\bar{4}4}(p^2)}{1 + \Sigma_V^{\bar{4}4}(p^2)}\right]}_{\text{radiative corrections shift the pole @ } p^2 = m^2}} \left[1 + \Sigma_V^{\bar{4}4}(p^2)\right]^{-2}$$

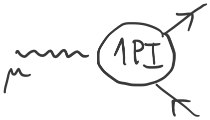
- UV divergence :

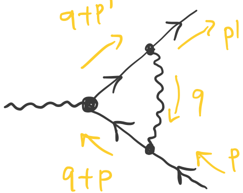
$$\Sigma_V^{\bar{4}4}(k^2)\Big|_{div} = -\frac{\alpha}{4\pi} Q^2 \frac{1}{k^2}\left[\frac{m^2}{\epsilon} - \frac{(k^2+m^2)}{\epsilon}\right] = \frac{\alpha}{4\pi} Q^2 \frac{1}{\epsilon}$$

$$\Sigma_S^{\bar{4}4}(k^2)\Big|_{div} = -\frac{\alpha}{4\pi} Q^2\, 4 \frac{1}{\epsilon} = -\frac{\alpha}{\pi} Q^2 \frac{1}{\epsilon}$$

1-Loop Corrections: Electron-Photon Vertex

(Feynman gauge: $\xi = 1$)

$$-ieQ\,\Lambda_\mu(p',p) =$$ 

$$=$$ 

$$+\ \mathcal{O}(\alpha^2)$$

$$\Rightarrow \Lambda_\mu(p',p) = \frac{\alpha}{4\pi} Q^2 \frac{(2\pi\mu)^{4-D}}{i\pi^2} \int d^Dq \, \frac{\gamma_\alpha (\not{q}+\not{p}'+m)\gamma_\mu(\not{q}+\not{p}-m)\gamma^\alpha}{(q^2-\lambda^2)\left((q+p')^2-m^2\right)\left((q+p)^2-m^2\right)}$$

- write in terms of $C_0, C_\mu, C_{\mu\nu}$ & reduce to A_0, B_0, C_0
  $\hookrightarrow$ lengthly & quite complicated result $\rightsquigarrow$ e.g. use computer algebra
- Ward identity: $k^\mu \Lambda_\mu(p',p) = \Sigma_1^{44}(p') - \Sigma_1^{44}(p) \Rightarrow \underbrace{\Lambda_\mu(p,p)}_{} = \frac{\partial}{\partial p^\mu}\Sigma_1^{44}(p)$
  photon momentum $k \to 0$
- UV divergence:

$$\Lambda_\mu(p',p)\Big|_{div} = \Lambda_\mu(p,p)\Big|_{div} = \frac{\partial \Sigma^{44}(p)}{\partial p^\mu}\Big|_{div} = \frac{\partial}{\partial p^\mu}\left\{ \not{p}\,\frac{\alpha}{4\pi}\frac{1}{\epsilon} - m\frac{\alpha}{\pi}\frac{1}{\epsilon}\right\}$$

$$= \gamma_\mu \frac{\alpha}{4\pi}\frac{1}{\epsilon} \longleftarrow$$ proportional to lowest order!

EX 16

Renormalization: Introduction

- from here on: add suffixes "$_0$" to indicate "bare" quantities

  ↪ $\mathcal{L}(\psi_0, A_0^\mu, m_0, e_0)$: Do m_0 & e_0 correspond to the <u>physical</u> mass & charge?

- the electron mass

  ↪ tree level: $G_0^{\psi\bar\psi}(-p,p) = \dfrac{i}{\not p - m_0} = \dfrac{i(\not p + m_0)}{p^2 - m_0^2}$ $\rightsquigarrow$ m_{pole} @ m_0

  ↪ rad. corr: $G^{\psi\bar\psi}(-p,p) = \dfrac{i}{\not p - m_0 + \Sigma^{\bar\psi\psi}(p)}$ $\rightsquigarrow$ m_{pole} @ $m_0\left[\dfrac{1-\Sigma_S^{\bar\psi\psi}}{1+\Sigma_V^{\bar\psi\psi}}\right]$

- similarly for the charge

Conclusion: radiative corrections have an impact on the physical interpretation of the bare quantities!

$\Rightarrow$ redefinition, "renormalization" necessary!

Renormalization: General Idea

- compute n physical observables: $Obs_i^{th.}(m_0, e_0)$ — divergent expressions in terms of m_0, e_0
- choose two (independent) observables to express m_0, e_0 in terms the measured quantities ($\longleftrightarrow$ input parameter scheme)

$$\left.\begin{aligned} \underbrace{Obs_1^{exp.}}_{\text{finite, obviously}} &\overset{!}{=} Obs_1^{th.}(m_0, e_0) \\ Obs_2^{exp.} &\overset{!}{=} Obs_2^{th.}(m_0, e_0) \end{aligned}\right\} \Rightarrow \begin{aligned} &m_0\left(Obs_1^{exp.}, Obs_2^{exp.}\right) \\ &e_0\left(Obs_1^{exp.}, Obs_2^{exp.}\right) \end{aligned}$$

divergent expressions in terms of $Obs_{1/2}^{exp.}$

- predictions for remaining $(n-2)$ observables

input parameters

$$Obs_i^{th.}\left(m_0\left(Obs_1^{exp.}, Obs_2^{exp.}\right), e_0\left(Obs_1^{exp.}, Obs_2^{exp.}\right)\right) = Obs_i^{th.}\left(Obs_1^{exp.}, Obs_2^{exp.}\right)$$

$\Rightarrow$ $Obs_i^{th.}$ in terms of $Obs_{1/2}^{exp.}$: FINITE!

Multiplicative Renormalization

- let's make the procedure from the previous slide more systematic
- write (divergent) bare quantities $\psi_0, A_0^\mu, m_0, e_0, \xi_0$ in terms of (finite) renormalized quantities times (divergent) renormalization constants

$$\psi_0 = \sqrt{Z_\psi}\,\psi, \qquad e_0 = Z_e\, e, \qquad \xi_0 = Z_\xi\, \xi$$

$$A_0^\mu = \sqrt{Z_A}\, A^\mu, \qquad m_0 = Z_m\, m,$$

- in perturbation theory:

$$Z_X = 1 + \alpha\Big(\underbrace{\frac{c_{-1}^X}{\epsilon}}_{\text{divergent part}} + \overbrace{c_0^X}^{\text{finite part}}\Big) + \mathcal{O}(\alpha^2)$$

finite part "free" choice: fixed by renormalization conditions

divergent part: uniquely determined

$$\Rightarrow \sqrt{Z_\psi}\,\psi = \left(1+\tfrac{1}{2}\delta Z_\psi\right)\psi, \qquad Z_e\, e = (1+\delta Z_e)\, e, \qquad Z_\xi\, \xi = (1+\delta Z_\xi)\,\xi$$

$$\sqrt{Z_A}\, A_\mu = \left(1+\tfrac{1}{2}\delta Z_A\right) A_\mu, \qquad Z_m\, m = (1+\delta Z_m)\, m \equiv m + \delta m$$

Counterterms

- We rewrite the Lagrangian in terms of renormalized quantities and "counterterms" (δZ_X):

$$\mathcal{L}(\psi_0, A_0^\mu, m_0, e_0) = -\frac{1}{4} F_{0,\mu\nu} F_0^{\mu\nu} + \bar{\psi}_0 (i\not{\partial} - Q e_0 \not{A}_0 - m_0)\psi_0 - \frac{1}{2\xi_0}(\partial A_0)^2$$

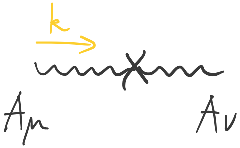
$$\begin{aligned}
= \underbrace{\begin{aligned} &-\frac{1}{2}(\partial_\mu A_\nu)(\partial^\mu A^\nu) \\ &+\frac{1}{2}\left(1-\frac{1}{\xi}\right)(\partial_\mu A^\mu)(\partial_\nu A^\nu) \\ &+\bar{\psi}(i\not{\partial}-m)\psi \\ &- eQ\bar{\psi}\not{A}\psi \end{aligned}}_{\mathcal{L}(\psi, A^\mu, m, e)}
\quad
\underbrace{\begin{aligned} &-\delta Z_A \frac{1}{2}(\partial_\mu A_\nu)(\partial^\mu A^\nu) \\ &+\left[\delta Z_A\left(1-\frac{1}{\xi}\right)+\delta Z_\xi \frac{1}{\xi}\right]\frac{1}{2}(\partial_\mu A_\nu)(\partial^\nu A^\mu) \\ &+\delta Z_\psi \bar{\psi}(i\not{\partial}-m)\psi - \delta m\, \bar{\psi}\psi \\ &-\left(\delta Z_e + \delta Z_\psi + \frac{1}{2}\delta Z_A\right)\cdot eQ\bar{\psi}\not{A}\psi \\ &+\mathcal{O}(\delta Z^2) \end{aligned}}_{\text{"counter terms"} \quad \mathcal{L}_{ct}}
\end{aligned}$$

- note: CT's are not "added": still the original theory (re-shuffling)
- Feynman rules: ($m = \mathcal{O}(1)$, $\delta m = \mathcal{O}(\alpha)$)

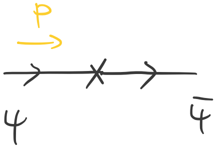
  1. "old" ones with renormalized quantities
  2. "new" CT vertices

EX 17

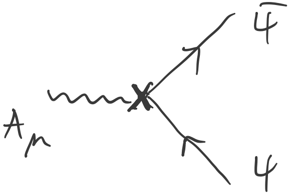
Counterterm Feynman Rules



$$= -i\,\delta Z_A \left[k^2 g_{\mu\nu} - \left(1-\frac{1}{\xi}\right) k_\mu k_\nu\right] + i\,\delta Z_\xi \,\frac{1}{\xi}\, k_\mu k_\nu$$



$$= i\,\delta Z_\psi (\not p - m) - i\,\delta m$$

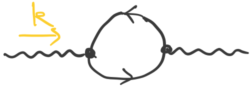


$$= -ieQ\,\gamma_\mu \left(\delta Z_e + \delta Z_\psi + \frac{1}{2}\delta Z_A\right)$$

EX 18

Renormalized Photon Self Energy

"hat" ≙ renormalized.

$$-i\hat{\Sigma}^{AA}_{\mu\nu}(k) =$$ 

$+$  $+ \mathcal{O}(\alpha^2)$

$$\Rightarrow \hat{\Sigma}^{AA}_{\mu\nu}(k) = \left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right)\hat{\Sigma}^{AA}_{T}(k^2) + \frac{k_\mu k_\nu}{k^2}\hat{\Sigma}^{AA}_{L}(k^2)$$

$$\hookrightarrow \hat{\Sigma}^{AA}_{T}(k^2) = \Sigma^{AA}_{T}(k^2) + k^2\,\delta Z_A$$

$$\hookrightarrow \hat{\Sigma}^{AA}_{L}(k^2) = \cancelto{0}{\Sigma^{AA}_{L}(k^2)} + \frac{1}{\xi}k^2\left[\delta Z_A - \delta Z_\xi\right]$$

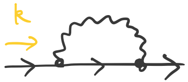
- UV divergences:

$$\delta Z_A\big|_{div} \overset{!}{=} -\frac{\Sigma^{AA}_{T}(k^2)}{k^2}\bigg|_{div} = -\Pi^{AA}(k^2)\big|_{div}$$

$$\delta Z_\xi\big|_{div} \overset{!}{=} \delta Z_A\big|_{div}$$ gauge-param. renormalization not necessary for phys. obs.

$\hookrightarrow$ we can choose $\delta Z_\xi = \delta Z_A \Leftrightarrow \hat{\Sigma}^{AA}_{L}(k^2) \equiv 0$

Renormalized Elektron Self Energy

$i\,\hat{\Sigma}_1^{\bar{4}4}(k) =$  $+$  $+ \mathcal{O}(\alpha^2)$

$$\Rightarrow \hat{\Sigma}^{\bar{4}4}(k) = \not{k}\,\hat{\Sigma}_V^{\bar{4}4}(k^2) + m\,\hat{\Sigma}_S^{\bar{4}4}(k^2)$$

$$\hookrightarrow \hat{\Sigma}_V^{\bar{4}4}(k^2) = \Sigma_V^{\bar{4}4}(k^2) + \delta Z_4$$

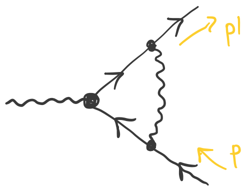
$$\hookrightarrow \hat{\Sigma}_S^{\bar{4}4}(k^2) = \Sigma_S^{\bar{4}4}(k^2) - \delta Z_4 - \frac{\delta m}{m}$$

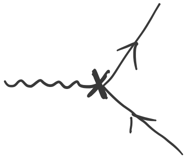
- UV divergences:

$$\delta Z_4\Big|_{div} \overset{!}{=} -\Sigma_V^{\bar{4}4}(k^2)\Big|_{div}$$

$$\delta m\Big|_{div} \overset{!}{=} m\left[\Sigma_S^{\bar{4}4}(k^2) - \delta Z_4\right]\Big|_{div} = m\left[\Sigma_S^{\bar{4}4}(k^2) + \Sigma_V^{\bar{4}4}\right]\Big|_{div}$$

Renormalized Electron-Photon Vertex

$-ieQ\,\hat{\Lambda}_\mu(p',p) =$ 

$+$  $+\ \mathcal{O}(\alpha^2)$

$$\Rightarrow \hat{\Lambda}_\mu(p',p) = \Lambda_\mu(p',p) + \gamma_\mu\left(\delta Z_e + \delta Z_\psi + \frac{1}{2}\delta Z_A\right)$$

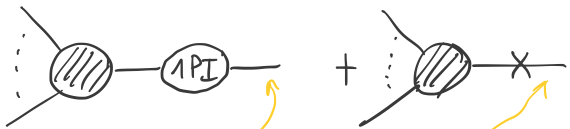
- UV divergences: (remember: $\Lambda_\mu\big|_{div} \sim \gamma_\mu$)

$$\delta Z_e = \left(-\Lambda_{div} - \delta Z_\psi - \frac{1}{2}\delta Z_A\right)\Big|_{div}$$

On-Shell Renormalization

- so far: determined the divergent part of δZ_X
  ↪ sufficient for MS, $\overline{\text{MS}}$ renormalization
- arbitrary finite contributions can be absorbed into the <u>definition</u> of δZ_X ⇝ renormalization scheme. (how we relate to phys. obs.)
  ↪ calculation at $\mathcal{O}(\alpha^n)$ ↔ scheme dependence $\mathcal{O}(\alpha^{n+1})$
- on-shell renormalization
  ↪ $m \overset{!}{=} m_{phys}$ ⇜ location of the propagator pole.
  ↪ normalize fields (Z_Φ) such that residue @ $p^2 = m^2$ is one
  $\Rightarrow$ no h.o. corrections to the residue

  equivalent:

 $\overset{!}{=} 0$

ext. leg with $p^2 = m^2$

Renormalization Conditions

(a) photon WF renormalization

$\Big($  $\overset{!}{=} 0 \Big)$

$$\Rightarrow \lim_{k^2\to 0} \left(\frac{-i\, g^{\mu\nu}}{k^2}\, \hat{\Gamma}^{AA}_{\nu\rho}(k,-k) \right) \varepsilon^{\rho}(k) \overset{!}{=} -\varepsilon^{\mu}(k) \qquad \left(k_\mu \varepsilon^\mu(k) = 0 \right)$$

$$\Rightarrow \lim_{k^2\to 0} \frac{1}{k^2}\, \hat{\Sigma}^{AA}_{T}(k^2) = \Pi^{AA}(0) + \delta Z_A \overset{!}{=} 0$$

$$\Rightarrow \delta Z_A = -\Pi^{AA}(0)$$

(b) gauge-parameter renormalization:

$$\hat{\Sigma}^{AA}_{L}(k^2) \overset{!}{=} 0 \Rightarrow \delta Z_\xi = \delta Z_A \qquad \left(Z_\xi = Z_A \right)$$

↪ as ξ, Z_ξ has no impact on phys. observables

↪ no renormalization for $\mathcal{L}_{fix} = -\frac{1}{2\xi_0}(\partial\cdot A_0)^2 = -\frac{1}{2\xi}(\partial\cdot A)^2$

Renormalization Conditions

(c) electron mass renormalization: ($p^2 = m^2 \hat{=}$ propagator pole)

$$\Rightarrow \quad 0 \overset{!}{=} \hat{\Gamma}^{\bar{4}4}(-p,p)\, u(p) \qquad (\text{Dirac Eq: } (\not{p}-m)\,u(p)=0)$$

$$= im \underbrace{\left(\hat{\Sigma}_V^{\bar{4}4}(m^2) + \hat{\Sigma}_S^{\bar{4}4}(m^2)\right)}_{\overset{!}{=}0}$$

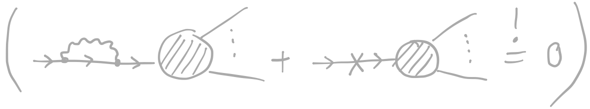
$$\Rightarrow \quad \Sigma_V^{\bar{4}4}(m^2) + \Sigma_S^{\bar{4}4}(m^2) - \frac{\delta m}{m} = 0$$

$$\Rightarrow \quad \frac{\delta m}{m} = \Sigma_V^{\bar{4}4}(m^2) + \Sigma_S^{\bar{4}4}(m^2)$$

EX 19

Renormalization Conditions

(d) electron WF renormalization



$$\Rightarrow \lim_{p^2 \to m^2} \left(\frac{i}{\not{p} - m} \hat{\Gamma}^{\bar{4}4}_{(-p,p)} \right) u(p) \overset{!}{=} - u(p)$$

$$\Rightarrow \lim_{p^2 \to m^2} \left(\frac{1}{\not{p} - m} \hat{\Sigma}^{\bar{4}4}(p) \right) u(p) \overset{!}{=} 0$$

$$\Rightarrow \delta Z_4 = - \Sigma_V^{\bar{4}4}(m^2) - 2m^2 \frac{\partial}{\partial p^2} \left[\Sigma_V^{\bar{4}4}(p^2) + \Sigma_S^{\bar{4}4}(p^2) \right] \Big|_{p^2 = m^2}$$

EX20

Renormalization Conditions

(e) charge renormalization:

$e \overset{!}{=}$ elementary charge from class. EDyn.

$\hat{=}$ coupling for photon momenta $k \to 0$

$$\Rightarrow \bar{u}(p)\, \hat{\Gamma}^{A\bar{4}4}_{\mu}(k=0,-p,p)\, u(p) \overset{!}{=} -ieQ\, \bar{u}(p)\gamma_\mu u(p)$$

$$\Rightarrow 0 \overset{!}{=} \bar{u}(p)\, \hat{\Lambda}_\mu(p,p)\, u(p) = \bar{u}(p)\left[\Lambda_\mu(p,p) + \gamma_\mu\left(\delta Z_e + \delta Z_4 + \tfrac{1}{2}\delta Z_A\right)\right]$$

$$* \quad \Lambda_\mu(p,p) = \frac{\partial}{\partial p^\mu}\Sigma^{\bar{4}4}(p) = \gamma_\mu \Sigma_V^{\bar{4}4}(p^2) + 2p_\mu\left[\not{p}\,\Sigma_V^{\bar{4}4\prime}(p^2) + m\Sigma_S^{\bar{4}4\prime}(p^2)\right]$$

$$* \quad \bar{u}(p)\gamma_\mu u(p) = \frac{p_\mu}{m}\bar{u}(p)u(p)$$

$$\Rightarrow 0 \overset{!}{=} \bar{u}(p)\gamma_\mu u(p)\left[\underbrace{\Sigma_V^{\bar{4}4}(m^2) + 2m^2\left(\Sigma_V^{\bar{4}4\prime}(m^2) + \Sigma_S^{\bar{4}4\prime}(m^2)\right) + \delta Z_4}_{=0} + \delta Z_e + \tfrac{1}{2}\delta Z_A\right]$$

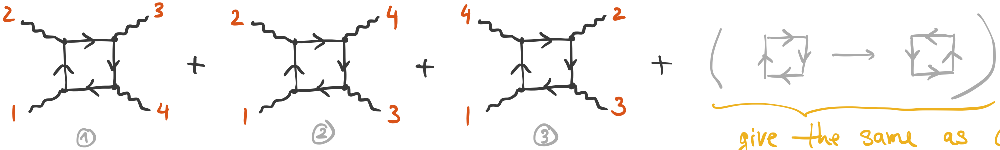
$$\Rightarrow \delta Z_e = -\frac{1}{2}\delta Z_A$$

Appendix: Four-Photon Vertex Function

- we saw that $\Gamma^{AAAA}_{\mu_1\mu_2\mu_3\mu_4}(k_1,k_2,k_3,k_4)$ has a superficial degree of divergence $=0$

  $\hookrightarrow$ if the leading UV behaviour is finite, we're ok!

$$\Gamma^{AAAA}_{\mu_1\mu_2\mu_3\mu_4} =$$



give the same as ①+②+③

- For the leading UV behaviour, we have

$$\Gamma^{①} \overset{\text{div}}{\sim} \int d^4q \, \frac{\mathrm{Tr}\left[\gamma^{\mu_1} \not{q}\, \gamma^{\mu_4} \not{q}\, \gamma^{\mu_3} \not{q}\, \gamma^{\mu_2} \not{q}\right]}{[q^2]^4}$$

we use isotropy: $\int d\Omega \; q^\mu q^\nu q^\rho q^\sigma \sim \left(g^{\mu\nu} g^{\rho\sigma} + g^{\mu\rho} g^{\nu\sigma} + g^{\mu\sigma} g^{\nu\rho}\right)$

and γ identities (can do this in $D=4$ since $(D-4)\sim\epsilon$ is sub-leading)

$$\Rightarrow \Gamma^{①} \overset{\text{div}}{\sim} g^{\mu_1\mu_4} g^{\mu_3\mu_2} + g^{\mu_1\mu_2} g^{\mu_3\mu_4} - 2\, g^{\mu_1\mu_3} g^{\mu_2\mu_4}$$

$$\Gamma^{②} = \Gamma^{①}\Big|_{\mu_3 \leftrightarrow \mu_4}$$

$$\Gamma^{③} = \Gamma^{①}\Big|_{\substack{\mu_2 \to \mu_4 \\ \mu_3 \to \mu_2 \\ \mu_4 \to \mu_3}}$$

$$\Rightarrow \Gamma^{①} + \Gamma^{②} + \Gamma^{③} \overset{\text{div}}{\sim} 0$$

$$\Rightarrow \Gamma^{AAAA}_{\mu_1\mu_2\mu_3\mu_4}(k_1,k_2,k_3,k_4)\Big|_{\text{div}} = 0$$

III. Yang-Mills Theory & QCD

QED from Gauge Symmetry

- the free Dirac theory $\mathcal{L}_\psi(\psi, \partial_\mu \psi)$ is invariant under global $U(1)$

$$\psi(x) \longrightarrow \psi'(x) = e^{-ieQ\theta}\,\psi(x)$$

- let's try to impose it as a local $(\theta = \theta(x))$ symmetry:

<u>problem</u>: $\partial_\mu \psi \longrightarrow \partial_\mu \psi' = \partial_\mu\left(e^{-ieQ\theta}\psi\right) = e^{-ieQ\theta}\left(\partial_\mu - ieQ\,(\partial_\mu\theta)\right)\psi \neq e^{-ieQ\theta}\partial_\mu\psi$

- solution: introduce the covariant derivative $D_\mu + ieQA_\mu(x)$ (minimal substitution)

$$\Rightarrow \quad D_\mu\psi \longrightarrow D'_\mu\psi' = e^{-ieQ\theta}D_\mu\psi\,, \qquad A_\mu \longrightarrow A'_\mu = A_\mu + (\partial_\mu\theta)$$

$\Rightarrow$ we generate interactions $\sim A_\mu\bar{\psi}\gamma^\mu\psi$ by imposing <u>local</u> $U(1)$!

- make the gauge field a dynamic quantity:

$$\mathcal{L}_{gauge} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} \qquad \left(F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu\right)$$

$\Rightarrow$ principle of minimal gauge-inv. coupling to a dynamic gauge field

$$\mathcal{L}_{QED} = \mathcal{L}_\psi(\psi, D_\mu\psi) + \mathcal{L}_{gauge}$$

Yang–Mills Theory

- apply the principle of gauge symmetry to (non-abelian) symmetries

① field multiplet $\vec{\Phi} = (\Phi_1, \dots, \Phi_n)^T$ & Lagrangian $\mathcal{L}_\Phi(\vec{\Phi}, \partial_\mu \vec{\Phi})$, invariant under global transformations of group G:

$$\vec{\Phi} \to \vec{\Phi}' = U(\underline{\theta})\,\vec{\Phi}\,, \qquad U(\underline{\theta}) = \exp\{-ig\,T^a\theta^a\}$$

generators of the group, $a = 1, \dots, \dim G$

arbitrary gauge coupling

structure constants: $[T^a, T^b] = i\,c^{abc}\,T^c$

normalization: $\mathrm{Tr}(T^a T^b) = \frac{1}{2}\delta^{ab}$

② make the symmetry local

$$D_\mu = \partial_\mu + ig\underbrace{A_\mu^a T^a}$$

$$=: \partial_\mu + ig\,\mathcal{A}_\mu$$

with property

$$D_\mu\vec{\Phi} \to D_\mu'\vec{\Phi}' = U(\underline{\theta})\,D_\mu\vec{\Phi}$$

EX21

Yang-Mills Theory

③ make the gauge field dynamic

↪ field-strength tensor: $\mathcal{F}_{\mu\nu} = -\frac{i}{g}[D_\mu, D_\nu] = \partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu + ig[\mathcal{A}_\mu, \mathcal{A}_\nu]$ ← transforms gauge co-variantly

$$\mathcal{F}_{\mu\nu} = F_{\mu\nu}^a T^a \rightsquigarrow F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a - g\, c^{abc} A_\mu^b A_\nu^c$$

$$\Rightarrow \mathcal{L}_A = -\frac{1}{2} \mathrm{Tr}[\mathcal{F}_{\mu\nu} \mathcal{F}^{\mu\nu}] = -\frac{1}{4} F_{\mu\nu}^a F^{a,\mu\nu}$$

↪ Lorentz- & gauge-invariant

↪ non-abelian: contains $(\partial A)A^2$, A^4 terms

⇒ gauge-boson self-interaction

↪ naive mass term $M^2(A_\mu^a A^{a,\mu})$ not allowed

⇒ from $\mathcal{L}_\Phi(\vec{\Phi}, \partial_\mu \vec{\Phi})$ (global symm.) $\rightsquigarrow$ gauge theory: $\mathcal{L}_\Phi(\vec{\Phi}, D_\mu \vec{\Phi}) + \mathcal{L}_A$

QCD

- gauge symmetry: $SU(3)_c$ — fundamental representation
- matter fields: quark fields

$$\psi_q(x) = \begin{pmatrix} q_r(x) \\ q_g(x) \\ q_b(x) \end{pmatrix} \longleftrightarrow \text{colour-triplett}$$

- generators: $T^a = \frac{\lambda^a}{2}$ (λ^a: Gell-Mann matrices, $a = 1, \dots, 8$)

$\leftrightarrow$ structure constants: $[T^a, T^b] = i f^{abc} T^c$

- eight gauge fields: G_μ^a ($a = 1 \dots 8$) $\rightsquigarrow$ gluons

$$\Rightarrow \mathcal{L}_{QCD} = \sum_q \bar{\psi}_q (i\not{D} - m_q)\psi_q - \frac{1}{4} F_{\mu\nu}^a F^{a,\mu\nu}$$

$\mathcal{L}_{free} \longrightarrow$ $$= -\frac{1}{4}(\partial_\mu G_\nu^a - \partial_\nu G_\mu^a)(\partial^\mu G^{a,\nu} - \partial^\nu G^{a,\mu}) + \sum_q \bar{\psi}_q (i\not{\partial} - m)\psi_q$$

$g\bar{q}q \longrightarrow$ $$- g_s G_\mu^a \bar{\psi}_q \gamma^\mu T^a \psi_q$$

g^3 & $g^4 \longrightarrow$ $$+ \frac{g_s}{2} f^{abc} (\partial_\mu G_\nu^a - \partial_\nu G_\mu^a) G^{b,\mu} G^{c,\nu} - \frac{g_s^2}{4} f^{abc} f^{ade} G_\mu^b G_\nu^c G^{d,\mu} G^{e,\nu}$$

Quantization of Gauge Theories

- the free part of the gluon

$$\mathcal{L}_{G,0} = -\frac{1}{4}\left(\partial_\mu G_\nu^a - \partial_\nu G_\mu^a\right)\left(\partial^\nu G^{a,\mu} - \partial^\mu G^{a,\nu}\right)$$

- from part I. $\rightsquigarrow$ problem with the generating functional:

$$Z_{G,0}[J_\mu^a] = \frac{1}{N}\int \mathcal{D}[G_\mu]\, \exp\left\{ i \int d^4x \left[\frac{1}{2} G_\mu^a \underbrace{\left(g^{\mu\nu}\Box - \partial^\mu\partial^\nu\right)}_{\text{has no inverse}} G_\nu^a + J_\mu^a G^{a,\mu} \right]\right\}$$

$\lightning$ deriving Green's function

- origin: gauge invariance!

$\mathcal{D}[G_\mu]$ includes physically equivalent field configurations

$$G_\mu \xrightarrow{U(\underline{\theta})} {}^{\underline{\theta}}G_\mu \qquad \left(U(\underline{\theta}) = \exp\left\{-i g_s T^a \theta^a(x)\right\} \right)$$

$\Rightarrow$ solution: restrict $\mathcal{D}[G_\mu]$ to only in-equivalent configurations

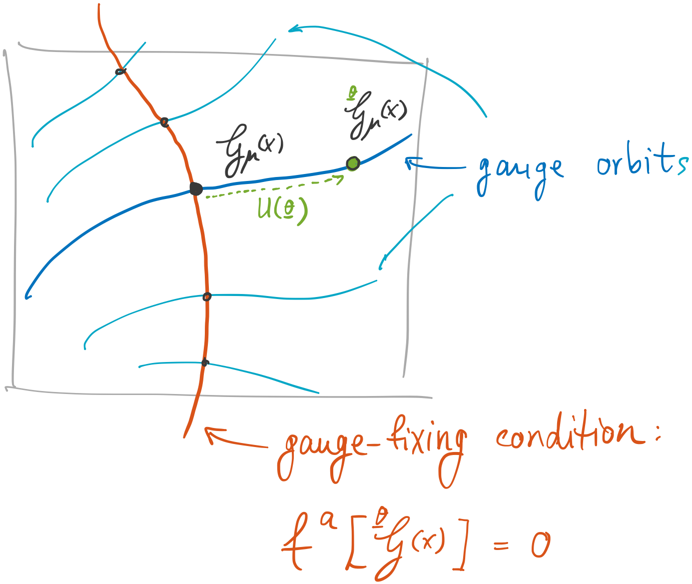
Faddeev-Popov Procedure

① define gauge-fixing conditions

$$f^a[\overset{\theta}{\mathcal{G}}_\mu(x)] \overset{!}{=} 0 \qquad a = 1, \dots 8$$

(QED in covariant gauge: $f[A_\mu(x)] = \partial^\mu A_\mu$)

$\Rightarrow$ selects one unique $\theta^a(x)$ from gauge orbit



Faddeev–Popov Procedure

② insert a clever one:

$$1 = \Delta[\mathcal{G}_\mu] \int \mathcal{D}[U(\theta)] \; \delta(f[\mathcal{G}^\theta])$$

Faddeev–Popov determinant:

Jacobi functional determinant

$$\underbrace{\Delta[\mathcal{G}_\mu]}_{} \propto \operatorname{Det}\left(\frac{\delta f^a[\mathcal{G}_\mu(x)]}{\delta\theta^b(y)}\right)\Bigg|_{f^a[\mathcal{G}_\mu]=0}$$

gauge invariant

"matrix" in a & x: $\quad M^{ab}(x,y) = \dfrac{\delta f^a[\mathcal{G}_\mu(x)]}{\delta\theta^b(y)}$

$$\Rightarrow (Mf)^a(x) = \int d^4y \; M^{ab}(x,y)\, f^b(y)$$

$\Rightarrow M(x,y) \sim \delta(x-y)$

(QED: $f[A_\mu(x)] = \partial^\mu A_\mu$, infinitesimal transf. $\delta A_\mu(x) = \partial_\mu \delta\theta(x)$

$$\Rightarrow \frac{\delta f[A_\mu(x)]}{\delta\theta(y)} = \delta(x-y)\,\Box_x \;)$$

Faddeev-Popov Procedure

③ insert into fuctional integral:

$$\int \mathcal{D}[G_\mu] \exp\{i S_0[G_\mu]\}$$

$$= \int \mathcal{D}[U(\underline{\theta})] \int \mathcal{D}[\underset{\downarrow\,\text{g.i.}}{G_\mu}] \; \Delta[\underset{\downarrow\,\text{g.i.}}{G_\mu}] \; \delta\left(f[{}^{\underline{\theta}}G_\mu]\right) \exp\{i S_0[\underset{\downarrow\,\text{g.i.}}{G_\mu}]\}$$

$$= \int \mathcal{D}[U(\underline{\theta})] \int \mathcal{D}[\underset{\downarrow\,\text{relabel}}{{}^{\underline{\theta}}G_\mu}] \; \Delta[{}^{\underline{\theta}}G_\mu] \; \delta\left(f[{}^{\underline{\theta}}G_\mu]\right) \exp\{i S_0[{}^{\underline{\theta}}G_\mu]\}$$

$$= \underbrace{\int \mathcal{D}[U(\underline{\theta})]}_{\substack{= \text{const} \\ (\infty \text{ volume}) \\ \rightsquigarrow \mathcal{N}}} \underbrace{\int \mathcal{D}[G_\mu] \; \Delta[G_\mu] \; \delta\left(f[G_\mu]\right) \exp\{i S_0[G_\mu]\}}_{\text{independent of } U(\underline{\theta})}$$

Faddeev-Popov Procedure

④ get rid of the $\delta(f[\mathcal{G}_\mu])$ constraint and write it as an extra term in the Lagrangian $\mathcal{L}_{fix}$ instead

↪ modify constraint

$$f^a[\mathcal{G}_\mu] \longrightarrow f^a[\mathcal{G}_\mu] - C^a(x)$$

(phys. obs. independent of $C^a(x)$)

↪ integrate with

$$\int \mathcal{D}[C^a] \;\dots\; \exp\left\{-i\int d^4y \, \frac{1}{2\xi} C_a(y)^2\right\}$$

arbitrary gauge param. could also choose ξ^a indiv.

$$\Rightarrow \int \mathcal{D}[C^a] \; \delta(f[\mathcal{G}_\mu]-C) \exp\left\{-i\int d^4y \, \frac{1}{2\xi} C_a(y)^2\right\} = \exp\left\{i\int d^4x \, \mathcal{L}_{fix}\right\}$$

with $\quad \mathcal{L}_{fix} = -\frac{1}{2\xi} f_a[\mathcal{G}_\mu]^2$

$\left(\text{QED: } \mathcal{L}_{fix} = -\frac{1}{2\xi}(\partial \cdot A)^2\right)$

Faddeev – Popov Procedure

⑤ rewrite the determinant as a Gaussian integral over Grassmann-valued fields ($\bar{u}^a, u^b$: ghosts)

$$\mathrm{Det}\left(M^{ab}(x,y)\right) \propto \int \mathcal{D}[u^b] \int \mathcal{D}[\bar{u}^a] \exp\left\{-i\int d^4x \int d^4y\ \bar{u}^a(x) M^{ab}(x,y)\, u^b(y)\right\}$$

$$M^{ab}(x,y) = \frac{\delta f^a[\mathcal{G}_\mu(x)]}{\delta\theta^b(y)} = \delta(x-y)\ \frac{\delta f^a[\mathcal{G}_\mu(x)]}{\delta\theta^b(x)}$$

EX 22

put it as an extra term in the Lagrangian:

$$\mathcal{L}_{FP} = -\bar{u}^a(x)\ M^{ab}(x)\ u^b(x)$$

$u^a, \bar{u}^a$: Grassmann-valued scalars
$\hookrightarrow$ unphysical!

(QED: only one ghost with $\mathcal{L}_{FP} = -\bar{u}(x)\ \Box\ u(x)$
$\hookrightarrow$ no interactions $\Rightarrow$ in abelian theories: ghosts decouple!)

Faddeev-Popov Procedure

- when the dust settles:

$$\mathcal{L}_{G,0} \longrightarrow \mathcal{L}_{G,0} + \mathcal{L}_{fix} + \mathcal{L}_{FP} = \mathcal{L}_{eff}$$

the effective Lagrangian

$$\Rightarrow Z[J_\mu^a, j^a, \bar{j}^a] = \frac{1}{N} \int \mathcal{D}[G_\mu^a] \int \mathcal{D}[u^a] \int \mathcal{D}[\bar{u}^a] \exp\left\{ i \int d^4x \left[\mathcal{L}_{G,0} + \mathcal{L}_{fix} + \mathcal{L}_{FP} + J_\mu^a G^{a,\mu} + \bar{j}^a u^a - \bar{u}^a j^a \right] \right\}$$

- QCD in covariant gauge: $f^a[G_\mu] = \partial^\mu G_\mu^a$

$$\Rightarrow \frac{\delta f^a[G_\mu(x)]}{\delta \theta^b(y)} = \delta(x-y) \left[\delta^{ab} \Box_x + g\, c^{abc} (\partial \cdot G^c) + g\, c^{abc} G_\mu^c \partial_x^\mu \right]$$

$$\hookrightarrow \quad \mathcal{L}_{fix} = -\frac{1}{2\xi} (\partial \cdot G^a)^2$$

ghosts only interact with the gluons.

$$\hookrightarrow \quad \mathcal{L}_{FP} = -\bar{u}^a \Box u^a - g\, c^{abc} \bar{u}^a \partial^\mu (G_\mu^c u^b)$$

BRS Symmetry

- Faddeev-Popov: $\mathcal{L}_{ferm} + \mathcal{L}_{gauge} \xrightarrow{FP} \underbrace{\mathcal{L}_{ferm} + \mathcal{L}_{gauge}}_{\text{gauge-inv.}} \underbrace{+ \mathcal{L}_{fix} + \mathcal{L}_{FP}}_{\text{not gauge inv.}}$ (the whole point)

  $\hookrightarrow$ phys. obs. gauge inv., manifest gauge inv. in $\mathcal{L}$ gone!

- what happened?

  $\hookrightarrow$ $\mathcal{L}_{fix}$: allows unphysical d.o.f. to propagate

  invertible diff. op $\Rightarrow$ compute Green's fn. ($\hat{=}$ propagator)

  $\hookrightarrow$ $\mathcal{L}_{FP}$: ghosts are there to compensate for them $\leftrightarrow$ cancel unphysical d.o.f introduced by $\mathcal{L}_{fix}$

- what is the symmetry that corresponds to gauge symmetry after FP?

  Becchi-Rouet-Stora (BRS) symmetry

  $\hookrightarrow$ gauge transformation with $\delta\theta^a(x) = \delta\lambda\, u^a(x)$

  $\left(U(\underline{\theta}) = \exp\{-ig_s T^a \theta^a\} \right)$

  Grassmann!

BRS Symmetry

- gauge transformation with $\delta\theta^a = \delta\bar{\lambda}\, u^a$ $\quad \left(U(\underline{\theta}) = \exp\{-ig_s T^a \theta^a\} \right)$

$\hookrightarrow \delta_{BRS} G_\mu^a = \delta\bar{\lambda}\left[g_s f^{abc} u^b G_\mu^c + \partial_\mu u^a \right]$

$\hookrightarrow \delta_{BRS} \psi = \delta\bar{\lambda}\left[-i g_s T^a u^a \psi \right]$

- clearly, both $\mathcal{L}_{ferm}$ & $\mathcal{L}_{gauge}$ are invariant under δ_{BRS}
- we need $\delta_{BRS}(\mathcal{L}_{fix} + \mathcal{L}_{FP}) = 0$ for $\mathcal{L}_{eff}$ to be invariant

$\hookrightarrow \delta_{BRS} \bar{u}^a \equiv \delta\bar{\lambda}\left[-\frac{1}{\xi} f_a[G_\mu] \right] = \delta\bar{\lambda}\left[-\frac{1}{\xi} \partial^\mu G_\mu^a \right]$

$\hookrightarrow \delta_{BRS} u^a = \delta\bar{\lambda}\left[-\frac{1}{2} g_s f^{abc} u^b u^c \right]$

$\Rightarrow$ Derive relations between n-pt functions from BRS symmetry

$\rightsquigarrow$ Slavnov-Taylor identities

Slavnov-Taylor Identities

- let us consider the transformation $\Phi_i \to \Phi_i + \delta_{BRS}\Phi_i$ in the P.I.

$$\hookrightarrow \int \mathcal{D}[\Phi_i + \delta_{BRS}\Phi_i] = \int \mathcal{D}[\Phi_i], \qquad \mathcal{L}_{eff}(\Phi_i + \delta_{BRS}\Phi_i) = \mathcal{L}_{eff}(\Phi_i)$$

$$\Rightarrow 0 = Z[J_i]\Big|_{\Phi_i + \delta_{BRS}\Phi_i} - Z[J_i]$$

only linear because $(\delta\bar{\lambda})^2 = 0$

$$= \frac{1}{N}\int \mathcal{D}[\Phi_i] \exp\left\{ i \int d^4x \left[\mathcal{L}_{eff} + J_i \Phi_i \right] \right\} \cdot \overbrace{i \int d^4y \, J_i(y) \left(\delta_{BRS} \Phi_i(y) \right)}$$

- functional differentiation $\Rightarrow$ Relations between n-pt functions

- Slavnov-Taylor identities:

$$0 \overset{!}{=} \delta_{BRS} \langle T\, \Phi_1(x_1) \dots \Phi_n(x_n) \rangle$$

$$= \left\langle T \left(\delta_{BRS}\Phi_1(x_1)\right) \Phi_2(x_2) \dots \Phi_n(x_n) \right\rangle + \dots + \left\langle T\, \Phi_1(x_1) \dots \Phi_{n-1}(x_{n-1}) \left(\delta_{BRS}\Phi_n(x_n)\right) \right\rangle$$

Slavnov–Taylor Identity: Gluon Propagator

$$0 \overset{!}{=} \delta_{BRS} \langle T\, \bar{u}^a(x)\, G^b_\mu(y) \rangle$$

$$= \langle T \underbrace{\left(\delta_{BRS}\, \bar{u}^a(x)\right)}_{\delta\bar{\lambda}\left[-\frac{1}{\xi}\partial^\nu G^a_\nu(x)\right]} G^b_\mu(y) \rangle + \langle T\, \bar{u}^a(x) \underbrace{\left(\delta_{BRS}\, G^b_\mu(y)\right)}_{\delta\bar{\lambda}\left[g_s f^{bcd} u^c(y) G^d_\mu(y) + \partial_\mu u^b(y)\right]} \rangle$$

Grassmann!

- bring $\delta\bar{\lambda}$ to the front & cancel
- differentiate w.r.t. y: $\partial^\nu_y(\ldots)$
- use the e.o.m. for the Green's function

$$\langle T\, \bar{u}^a(x)\; M^{bc}(y)\, u^c(y)\, |0\rangle = i\,\delta^{ab}\,\delta(x-y)$$

- $\rightsquigarrow$ momentum space: $\int d^4x \int d^4y\; e^{-ik(x-y)}$

$$\Rightarrow\; k^\mu k^\nu\, G^{g^a g^b}_{\mu\nu}(k,-k) = -i\,\xi\,\delta^{ab}$$

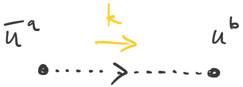
$\hookrightarrow$ no higher-order corrections to longitudinal component of the gluon!

EX23

QCD Feynman Rules (in the covariant gauge: $f^a[A_\mu(x)] = \partial^\mu A^a_\mu(x)$ $\rightsquigarrow$ Appendix: axial gauge)

- propagators

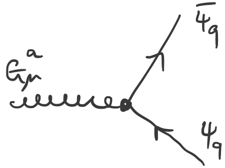
$$\frac{-i\,\delta^{ab}}{k^2+i0}\left[g_{\mu\nu} - (1-\xi)\frac{k_\mu k_\nu}{k^2}\right]$$



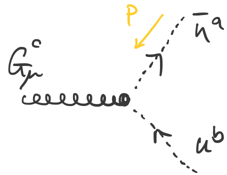
$$\frac{i\,\delta^{ab}}{k^2+i0}$$

$$\frac{i}{\not{k}-m_q+i0}$$

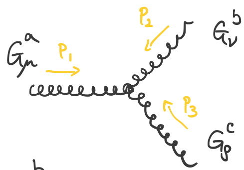
- vertices



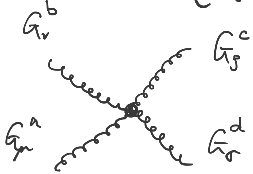
$$-ig_s T^a \gamma_\mu$$



$$p_\mu g_s f^{abc}$$



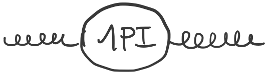
$$-g_s f^{abc}\left[g^{\mu\nu}(p_1-p_2)^\rho + g^{\nu\rho}(p_2-p_3)^\mu + g^{\rho\mu}(p_3-p_1)^\nu\right]$$



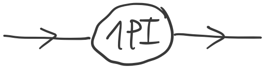
$$-ig_s^2\left[f^{abe}f^{cde}\left(g^{\mu\rho}g^{\nu\sigma} - g^{\mu\sigma}g^{\nu\rho}\right) + \text{cycl.}\right]$$

Superficially Divergent Vertex Functions

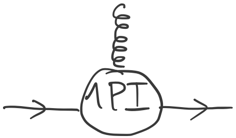
$\omega = 3 \longrightarrow 0$ (quant. number of vacuum)



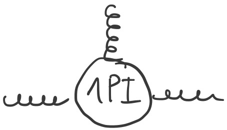
$\omega = 2$



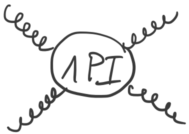
$\omega = 1$



$\omega = 0$



$\omega = 1$



$\omega = 0$

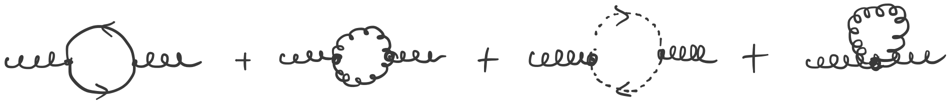
Ex 24

all divergent!

↪ more div. vertex functions than parameters!

↪ gauge symmetry essential

1-Loop Corrections: Gluon Self Energy

$$-i\,\Sigma^{G^aG^b}_{\mu\nu}(k) =$$ 

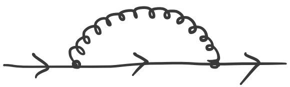
$$\hookrightarrow \Gamma^{G^aG^b}_{\mu\nu}(k) = -i\,\delta^{ab}\,k^2\left[g_{\mu\nu}-\frac{k_\mu k_\nu}{k^2}\right]\left[1+\Pi^{GG}(k^2)\right]-\frac{i}{\xi}\,\delta^{ab}\,k_\mu k_\nu$$

- gluon vacuum polarization

$$\Pi^{GG}(k^2) = \sum_q \frac{\alpha_s}{3\pi}\,T_F\,\frac{1}{k^2}\left\{(k^2+m_q^2)\,B_0(k^2,m_q,m_q)-\frac{k^2}{3}-2m_q^2\,B_0(0,m_q,m_q)\right\}$$
$$-\frac{\alpha_s}{12\pi}\,C_A\left\{\frac{13-3\xi}{2}\,B_0(k^2,0,0)+\frac{59}{12}+\frac{9}{2}\,\xi+\frac{3}{4}\,\xi^2\right\}$$

- fermion loop: analogue to QED $e^2Q^2 \longrightarrow g_s^2\,T_F\,\delta^{ab}$
- boson loop: $\Sigma_L^{GG}=0$ only after summation
- Π^{GG} gauge dependent already @ 1-loop (not in QED)

1-Loop Corrections: Quark Self Energy

$i\,\Sigma^{\bar{q}q}(k) =$ 

$\hookrightarrow \quad \Sigma^{\bar{q}q}(k) = \not{k}\,\Sigma_V^{\bar{q}q}(k^2) + m_q\,\Sigma_S^{\bar{q}q}(k^2)$

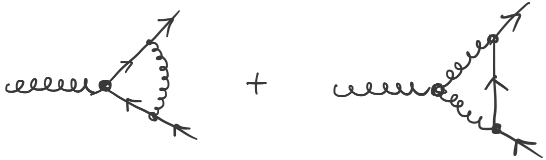
- result $(\xi = 1)$

$$\Sigma_V^{\bar{q}q}(k^2) = -\frac{\alpha_s}{4\pi} C_F \frac{1}{k^2}\left\{A_0(m_q) - (k^2+m_q^2)\,B_0(k^2, m_q, 0) + k^2\right\}$$

$$\Sigma_S^{\bar{q}q}(k^2) = -\frac{\alpha_s}{4\pi} C_F \left\{4\,B_0(k^2, m_q, 0) - 2\right\}$$

- from QED: $e^2 Q^2 \longrightarrow g_s^2\, C_F$

1-Loop Corrections: Quark-Gluon Vertex

$-ig_s \Lambda^a_\mu(p',p) =$ 

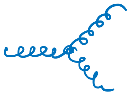
no analogue in QED

- 1st diagram from QED: $e^3 Q^3 \longrightarrow -g_s^3 \left(C_F - \frac{1}{2} C_A\right) T^a$
- UV divergence:

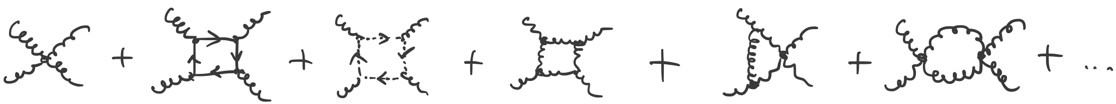
$$\Lambda^a_\mu(p',p)\Big|_{div} = \frac{\alpha_s}{4\pi}\left(C_F + C_A\right)\frac{1}{\epsilon}$$

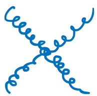
1-Loop Corrections: Gluon Vertices

$\Gamma^{G^aG^bG^c}_{\mu\nu\rho} =$ 

$\hookrightarrow \left.\Gamma^{G^aG^bG^c}_{\mu\nu\rho}\right|_{div} =$  $\cdot\, C^{GGG}_{div}$

(QED: zero)

$\Gamma^{G^aG^bG^cG^d}_{\mu\nu\rho\sigma} =$ 

$\hookrightarrow \left.\Gamma^{G^aG^bG^cG^d}_{\mu\nu\rho\sigma}\right|_{div} =$  C^{GGGG}_{div}

(QED: UV finite)

QCD Renormalization

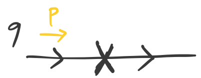
- multiplicative renormalization (1-loop: $Z_X = 1 + \delta Z_X$)

$$G^a_{\mu,0} = \sqrt{Z_G}\, G^a_\mu \,, \qquad \psi_{q,0} = \sqrt{Z_q}\, \psi_q$$

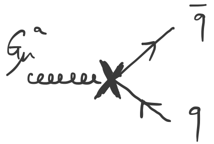
$$g_{s,0} = Z_g\, g_s \,, \qquad m_{q,0} = Z_{m_q} m_q = m_q + \delta m_q \,, \qquad \xi_0 = Z_\xi\, \xi$$

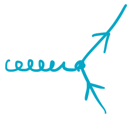
- Counterterm vertices:

$$-i\,\delta^{ab}\,\delta Z_G\, k^2\left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right) + i\,\delta^{ab}\,\frac{1}{\xi}\left(\delta Z_\xi - \delta Z_A\right) k_\mu k_\nu$$

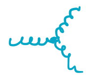


$$i\,\delta Z_q\,(\not{p} - m_q) - i\,\delta m_q$$

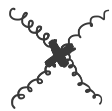


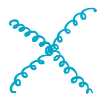


$$\times\left(\delta Z_g + \delta Z_q + \tfrac{1}{2}\delta Z_G\right)$$



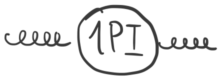
$$\times\left(\delta Z_g + \tfrac{3}{2}\delta Z_G\right)$$





$$\times\left(2\delta Z_g + 2\delta Z_G\right)$$

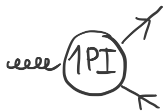
Renormalized Vertex Functions



$$= -i\delta^{ab}\left[g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right]\Big[k^2 + \overbrace{\Sigma_T^{GG}(k^2) + k^2\,\delta Z_G}^{=:\,\hat{\Sigma}_T^{GG}(k^2)}\Big]$$

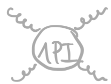
$$-i\delta^{ab}\,\frac{1}{\xi}\,\frac{k_\mu k_\nu}{k^2}\Big[k^2 + \underbrace{\cancelto{0}{\Sigma_L^{GG}(k^2)} + k^2\left(\delta Z_G - \delta Z_\xi\right)}_{=:\,\hat{\Sigma}_L^{GG}(k^2)}\Big]$$

$$= i\not{p}\Big[1 + \underbrace{\Sigma_V^{\bar{q}q}(p^2) + \delta Z_q}_{=:\,\hat{\Sigma}_V^{\bar{q}q}(p^2)}\Big] + i m_q\Big[-1 + \underbrace{\Sigma_S^{\bar{q}q}(p^2) - \delta Z_q - \frac{\delta m_q}{m_q}}_{=:\,\hat{\Sigma}_S^{\bar{q}q}(p^2)}\Big]$$



$$= -ig_s T^a \gamma_\mu - ig_s\Big[\underbrace{\Lambda_\mu^a(p',p) + T^a\gamma_\mu\left(\delta Z_g + \delta Z_q + \tfrac{1}{2}\delta Z_G\right)}_{=:\,\hat{\Lambda}_\mu^a(p',p)}\Big]$$

more:



Renormalization Conditions

- confinement in QCD

  $\hookrightarrow$ α_s definition through elementary qg scattering not accessible

  $\hookrightarrow$ no free propagation of quarks $\rightsquigarrow$ kin pole not accessible

- $\overline{\text{MS}}$ ("modified minimal subtraction") scheme

  δZ_X absorb only the standard divergence

  $$\Delta = \frac{1}{\epsilon} - \gamma_E + \ln(4\pi)$$ (1-loop)

$\Rightarrow$ predictions depend on the reference scale μ (DimReg)

  $\hookrightarrow$ running couplings and masses

$\overline{MS}$ Renormalization

- gluon self energy:

$$\delta Z_G = -\frac{\Sigma_T^{GG}(k^2)}{k^2}\Big|_\Delta = -\Pi^{GG}(k^2)\Big|_\Delta = -\frac{\alpha_s}{4\pi}\Delta\left[\frac{4}{3}N_f T_F - \frac{5}{3}C_A\right]$$

$$\delta Z_\xi = \delta Z_G \qquad \left(\Rightarrow \hat{\Sigma}_L^{GG}(k^2) = \Sigma_L^{GG}(k^2) = 0\right)$$

- quark self energy:

$$\delta Z_q = -\Sigma_V^{\bar{q}q}(k^2)\Big|_\Delta = -\frac{\alpha_s}{4\pi}\Delta\, C_F$$

$$\frac{\delta m_q}{m_q} = \left[\Sigma_V^{\bar{q}q}(k^2) + \Sigma_S^{\bar{q}q}(k^2)\right]\Big|_\Delta = -\frac{\alpha_s}{4\pi}\Delta\, 3C_F$$

↳ often $m_q = 0$ for light quarks $\Rightarrow \delta m_q = 0$ (chiral symmetry)

- quark-gluon vertex:

$$\Lambda_\mu^a(p',p)\Big|_\Delta + T^a\gamma_\mu\left(\delta Z_g + \delta Z_q + \frac{1}{2}\delta Z_G\right) \overset{!}{=} 0 \quad \Rightarrow \quad \delta Z_g = \frac{\alpha_s}{4\pi}\Delta\left[\frac{2}{3}N_f T_F - \frac{11}{6}C_A\right]$$

Renormalization Group

- equations for observables depend on μ
  $\Rightarrow$ extracted values for g, etc. are μ-dependent $g = g(\mu)$
- in the end: explicit & implicit μ-dependence compensate each other
  $\Rightarrow$ renormalization group equation

$$\mu \frac{d\alpha_s}{d\mu} = -\beta_0 \frac{\alpha_s^2}{\pi} \qquad (1\text{-loop})$$

EX25

$$\beta_0 = \frac{11}{6} C_A - \frac{2}{3} N_f T_F = \frac{11}{6} 3 - \frac{2}{3} N_f \frac{1}{2} > 0 \quad \text{for } N_f < 16.5 \quad (\checkmark)$$

beta function

- solution

$$\alpha_s(\mu_2) = \frac{\alpha_s(\mu_1)}{1 + \frac{\alpha_s(\mu_1)}{2\pi} \beta_0 \ln\left(\frac{\mu_2^2}{\mu_1^2}\right)} \Rightarrow \mu_2 > \mu_1 \Rightarrow \alpha_s(\mu_2) < \alpha_s(\mu_1)$$

asymptotic freedom

Appendix: QCD in the axial gauge

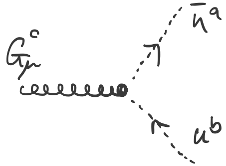
arb. ref. vector

- We choose a gauge-fixing condition $f^a[A_\mu(x)] = n^\mu A^a_\mu(x)$

$$\Rightarrow \mathcal{L}_{fix} = -\frac{1}{2\xi}\sum_a \left(n^\mu A^a_\mu\right)^2 \Rightarrow G^{A^a_\mu A^b_\nu}(p) = \frac{-i\,\delta^{ab}}{p^2+i0}\left\{ g_{\mu\nu} - \frac{p_\mu n_\nu + p_\nu n_\mu}{(p\cdot n)} + \frac{(n^2+\xi p^2)\,p_\mu p_\nu}{(p\cdot n)^2}\right\}$$

- $$n^\mu G^{A^a_\mu A^b_\nu}(p) = -i\,\xi\,\delta^{ab}\frac{p_\nu}{(p\cdot n)} \xrightarrow{\xi\to 0} 0 \qquad (1)$$

- ghost-gluon vertex:



$$-i\,g_s f^{abc}\,n_\mu \qquad (2)$$

- "axial" gauge through $\xi \to 0$

$\Rightarrow$ connected diag $\Rightarrow$ ghost couples to

(i) an internal gluon propagator $\overset{(1)}{\Rightarrow}$ 0

(ii) an external gluon $\overset{(2)}{\Rightarrow}$ 0 $\quad (n^\mu \varepsilon^a_\mu = 0)$

$\Rightarrow$ ghosts do not contribute to scattering amplitudes

IV. Infrared Singularities

Infrared Singularities

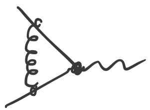
- UV ($q \to \infty$) divergences dealt with using renormalization
- IR singularities (mass singularities) associated with
  - ↪ vanishing (soft) momenta : $k^\mu \to 0$
  - ↪ collinear configurations : $p_i \parallel p_j$
- cancellation for "IR safe" observables between REAL & VIRTUAL :

1-loop   tree

$$\int d\Phi_m \; \sigma^{(1)}(p_1 \dots p_m) + \int d\Phi_{m+1} \; \sigma^{(0)}(p_1 \dots p_{m+1}) = \text{IR finite}$$

(Kinoshita-Lee-Nauenberg)

explicit poles in

 $\sim \frac{1}{\epsilon^2}, \frac{1}{\epsilon}$

implicit poles in phase-space integration

Singularities in Feynman Integrals

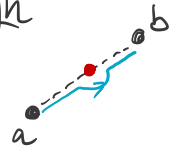
- arbitrary graph: $\int \prod_{l=1}^{L} d^4 q_l \, \frac{N(\{q_l\},\{p_i\})}{\prod_{n=1}^{N} [k_n^2 - m_n^2 + i\varnothing]}$

k_n lin. comb of
$\hookrightarrow q_l$: loop momenta
$\hookrightarrow p_i$: ext. momenta

$$\xrightarrow[\text{param.}]{\text{Feynman}} (n-1)! \int \prod_{l=1}^{L} d^4 q_l \int_0^1 \prod_{n=1}^{N} dx_n \, \delta\Big(1 - \sum_i x_i\Big) \frac{N(\{q_l\},\{p_i\})}{[\tilde{D} + i\varnothing]^N}$$

$$\tilde{D} = \sum_{n=1}^{N} x_n \left[k_n^2 - m_n^2\right]$$

- singularity $\longleftrightarrow$ $\tilde{D} = 0$ but can be avoided through contour deformation



Singularities in Feynman Integrals

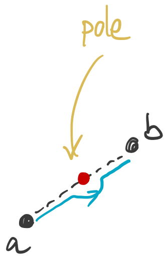
- arbitrary graph: $\displaystyle\int \prod_{l=1}^{L} d^4 q_l \, \frac{N(\{q_l\},\{p_i\})}{\prod_{n=1}^{N} \left[k_n^2 - m_n^2 + i0\right]}$

k_n lin. comb of
$\hookrightarrow q_l$: loop momenta
$\hookrightarrow p_i$: ext. momenta

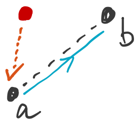
$$\xrightarrow[\text{param.}]{\text{Feynman}} \quad (n-1)! \int \prod_{l=1}^{L} d^4 q_l \int_0^1 \prod_{n=1}^{N} dx_n \; \delta\Big(1-\sum_i x_i\Big) \frac{N(\{q_l\},\{p_i\})}{[\tilde{D} + i0]^N}$$

$$\tilde{D} = \sum_{n=1}^{N} x_n \left[k_n^2 - m_n^2\right]$$

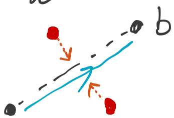
- singularity $\leftrightarrow$ $\tilde{D} = 0$ but can be avoided through contour deformation:



① end-point singularity:



② pinch singularity:



Landau Equations

- necessary condition for a singularity

$$x_n D_n = x_n \left[k_n^2 - m_n^2\right] = 0 \qquad \forall\ n = 1, \dots, N$$

$\Rightarrow$ either on-shell ($k_n^2 = m_n^2$) or reduced graph ($x_n = 0$)

$$\sum_n x_n \frac{\partial D_n}{\partial q_\ell} = \sum_n x_n k_n \frac{\partial k_n}{\partial q_\ell} = 0 \qquad \forall\ \ell = 1, \dots, L$$

$\Leftrightarrow$ pinch condition for $d^4 q_\ell$ (endpoint $\triangleq$ UV sing.)

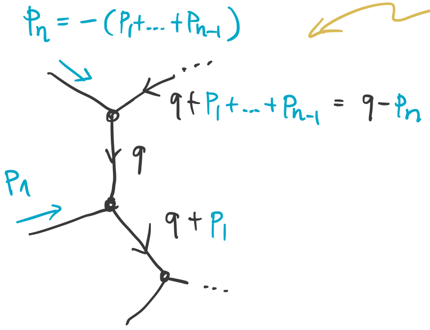
- IR (mass) singularities:

  independent of orientation of external momenta

  (no threshold, branch, ...)

IR Singularities in 1-loop integrals

n-point integral



$$D_0 = q^2 - m_0^2 \qquad (k_0 = q)$$

$$D_1 = (q+p_1)^2 - m_1^2 \qquad (k_1 = q + p_1)$$

$$\vdots$$

$$D_{n-1} = (q-p_n)^2 - m_{n-1}^2 \qquad (k_{n-1} = q - p_n)$$

- Landau Equations:

$$x_i D_i = x_i (k_i^2 - m_i^2) = x_i \left[\left(q + \sum_{j=1}^{i} p_j \right)^2 - m_i^2 \right] \quad \forall\, i = 1, \dots, n$$

$$\sum_{i=1}^{n} x_i k_i = 0$$

Soft Singularities

- consider only one propagator on-shell: $x_r \neq 0$ & $x_i = 0 \ \forall i \neq r$

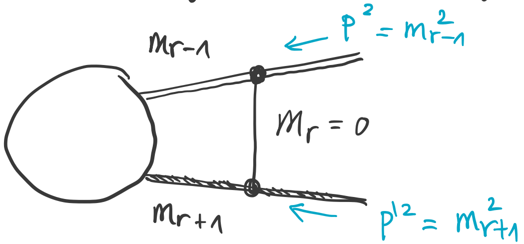
$$\Rightarrow \left.\begin{array}{l} k_r^2 = m_r^2 \\ k_r = 0 \end{array}\right\} \Rightarrow (\text{w.l.o.g. } k_r = q) \qquad q = 0, \quad m_r = 0$$

- not sufficient yet $\int \frac{d^4q}{q^2}$ finite $\rightarrow$ need enhancement:

in general, depends on orientation

$$\frac{1}{(q+P)^2 - M^2} \xrightarrow{P^2 = M^2} \frac{1}{q^2 + 2qP} \sim \frac{1}{q}$$

- for the two neighbouring propagators $(r-1)$ & $(r+1)$ pure on-shell const.



Collinear Singularities

- consider two propagators going on-shell: $x_r \neq 0 \neq x_s$ & $x_i = 0 \; \forall i \neq r,s$

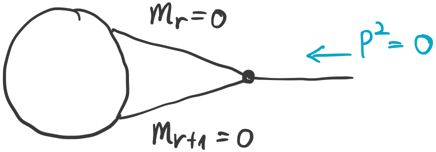
$$\left.\begin{array}{l} \Rightarrow \quad k_r^2 = m_r^2 \;, \quad k_s^2 = m_s^2 \\ x_r k_r + x_s k_s = 0 \end{array}\right\} \Rightarrow \quad \begin{array}{l} k_r = \pm \frac{m_r}{m_s} k_s \\ \underbrace{(k_r - k_s)^2}_{} = (m_r \pm m_s)^2 \end{array}$$

only indep. of orientation if neighbours

- $s = r+1$ $\quad (s = r-1$ via $s \leftrightarrow r)$

w.l.o.g.

$$\left.\begin{array}{l} D_r = q^2 - m_r^2 \\ D_{r+1} = (q + p_{r+1})^2 - m_{r+1}^2 \end{array}\right\} \text{not sufficient yet: } m_r = m_{r-1} = 0$$



$*$ soft & collinear can also overlap : $\frac{1}{\epsilon^2}$

Catani's I- Operator

* Singularities at 1-loop amplitudes universal

colour correlation

$$\mathcal{M}^{(1-loop)}\Big|_{IR} = \mathcal{M}^{(tree)} \otimes \hat{I}^{(1)}$$

[Catani, Seymour '97]

$$\hat{I}^{(1)} = -\frac{\alpha_s}{2\pi}\frac{1}{\Gamma(1-\epsilon)}\sum_i \frac{1}{\hat{T}_i^2}\mathcal{V}_i(\epsilon)\sum_{j\neq i}\hat{T}_i\cdot\hat{T}_j\left(\frac{4\pi\mu^2}{2(p_i\cdot p_j)}\right)^{\epsilon}$$

$$\mathcal{V}_i(\epsilon) = \hat{T}_i^2\left(\frac{1}{\epsilon^2}-\frac{\pi^2}{3}\right)+\gamma_i\frac{1}{\epsilon}+\gamma_i+K_i$$

constants dependent on parton type: q, g

* Similar expression for 2-loop amplitudes

$\hat{I}^{(2)}$ [Catani '98]

IR Singularities in Real Corrections

- soft $(k = \xi q,\ \xi \to 0)$

colour correlation

$$|\mathcal{M}_{m+1}(p_1,\dots,p_m,k)|^2 \xrightarrow{k \text{ soft}} -\frac{1}{\xi}\, 4\pi\mu^{2\epsilon}\alpha_s \left[\underline{\underline{J}}^\mu\right]^\dagger \left[\underline{\underline{J}}_\mu\right] \otimes |\mathcal{M}_m(p_1,\dots,p_m)|^2$$

$\hookrightarrow$ eikonal current:

$$\underline{\underline{J}}^\mu(q) = \sum_{i=1}^{m} \underline{\underline{T}}_i \frac{p_i^\mu}{(q\cdot p)}$$

$$\underline{\underline{T}}_i = \begin{cases} \text{gluon: } & T^b_{ac} = i f^{abc} \\ \text{quark: } & T^a_{\alpha\beta} = t^a_{\alpha\beta} \\ \text{anti-quark: } & T^a_{\alpha\beta} = -t^a_{\beta\alpha} \end{cases}$$

outgoing

- collinear $(p_i \parallel p_j)$

$$|\mathcal{M}_{m+1}(p_1,\dots,p_{m+1})|^2 \xrightarrow{p_i \parallel p_j} \frac{1}{(p_i p_j)}\, 4\pi\mu^{2\epsilon}\alpha_s\, \hat{P}_{ij} \otimes |\mathcal{M}_m(p_1,\dots(p_i+p_j),\dots p_{m+1})|^2$$

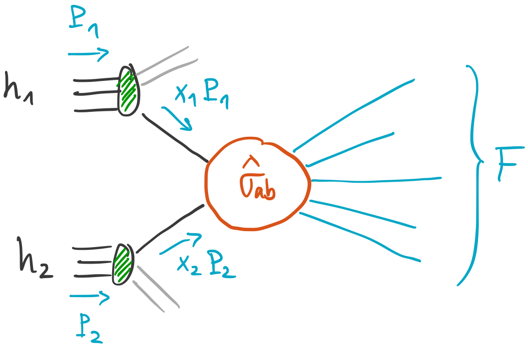
$\hookrightarrow$ Altarelli-Parisi splitting function

not regularized
$z \to 1$

$$\hat{P}_{qq}(z) = C_F\left[\frac{1+z^2}{1-z} - \epsilon(1-z)\right]$$

EX26

Hadron-Hadron Collisions



partonic
hard-scattering
cross section

$$\sigma_{h_1 h_2 \to F}(P_1, P_2) = \sum_{a,b} \int_0^1 dx_1 \int_0^1 dx_2 \, f_{a/h_1}(x_1) \, f_{b/h_2}(x_2) \, \hat{\sigma}_{ab \to F}(x_1 P_1, x_2 P_2)$$

parton distribution functions

$f_{a/h}(x)\,dx \mathrel{\hat{=}}$ # density
for parton a within h
to carry mom. frac. $[x, x+dx]$

EX 27

Drell-Yan: On-shell W^+ production

$$\frac{d\sigma^{LO}_{pp\to W^+}}{dy_W} = \int_0^1 dx_1 \int_0^1 dx_2 \, f_{u/p}(x_1)\, f_{\bar{d}/p}(x_2)$$

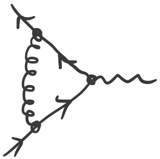
$(+\, u \leftrightarrow \bar{d})$

$$\times \frac{\pi}{12} \frac{e^2}{s_W^2}(1-\epsilon)\; \delta(M_W^2 - x_1 x_2 s)\; \delta\!\left(y_W - \frac{1}{2}\ln\left(\frac{x_1}{x_2}\right)\right)$$

$$= f_{u/p}(x_1)\, f_{\bar{d}/p}(x_2)\, \frac{\pi}{12}\frac{e^2}{s_W^2}(1-\epsilon) \Bigg|_{x_{1/2} = \sqrt{\frac{M_W^2}{s}}\, e^{\pm y_W}}$$

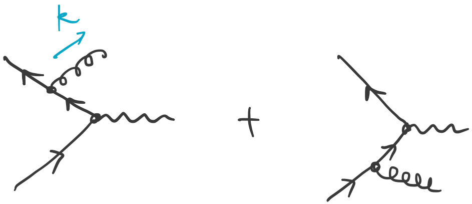
Drell-Yan @ NLO

- virtual corrections $\longleftrightarrow \int d\Phi_1(p_W)$



← the only diagram ( $= 0$)

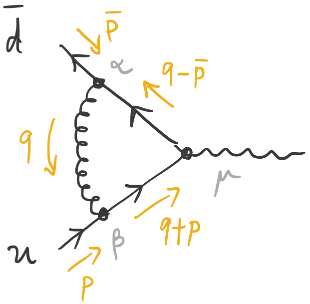
- real corrections $\longleftrightarrow \int d\Phi_2(p_W, k)$



qg & $\bar{q}g$ channels via crossing

new partonic channel that opens up @ NLO

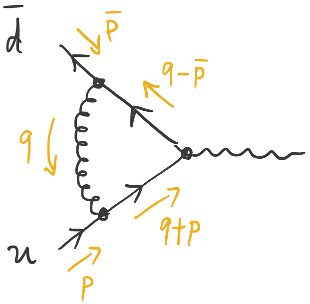
Drell-Yan: 1-loop corrections



$$i\mathcal{M}^{(1)}_{u\bar{d}\to W^+}(p,\bar{p}) = \frac{e}{\sqrt{2}\,s_W} g_s^2 C_F \delta_{\bar{c}c}\, \bar{v}_d(\bar{p}) \left\{ \mu^{2\epsilon} \int \frac{d^Dq}{(2\pi)^D} \frac{\gamma^\alpha (\not{q}-\not{\bar{p}}) \gamma^\mu (\not{q}+\not{p}) \gamma^\alpha}{q^2 (q+p)^2 (q-\bar{p})^2} \right\} \omega_- u_u(p)\, \varepsilon^*_\mu(p_W)$$

Ex 28

Drell-Yan: 1-loop corrections

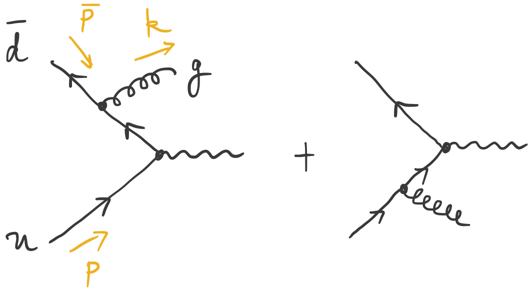


$$i\mathcal{M}^{(1)}_{u\bar{d}\to W^+}(p,\bar{p}) = \frac{e}{\sqrt{2}\,s_W} g_s^2 C_F \delta_{\bar{c}c}\, \bar{v}_d(\bar{p}) \left\{ \mu^{2\epsilon} \int \frac{d^D q}{(2\pi)^D} \frac{\gamma^\alpha(\cancel{q}-\cancel{\bar{p}})\gamma^\mu(\cancel{q}+\cancel{p})\gamma^\alpha}{q^2(q+p)^2(q-\bar{p})^2} \right\} \omega_- u_u(p)\, \varepsilon^*_\mu(p_W)$$

$$= i\mathcal{M}^{(0)}_{u\bar{d}\to W^+} \left\{ -\frac{\alpha_s}{2\pi} C_F \frac{(4\pi)^\epsilon}{\Gamma(1-\epsilon)} \left(\frac{\mu^2}{M_W^2}\right)^\epsilon \left[\frac{1}{\epsilon^2} + \frac{3}{2\epsilon} + 4 - \frac{\pi^2}{2} + i\pi\left(\frac{1}{\epsilon} + \frac{3}{2}\right) + \mathcal{O}(\epsilon) \right] \right\}$$

$$\Rightarrow \hat{\sigma}^{V}_{u\bar{d}\to W^+} = \hat{\sigma}^{LO}_{u\bar{d}\to W^+} \cdot 2\,\mathrm{Re}\{\dots\}$$

Drell-Yan: Real-emission Amplitude



$$i\mathcal{M}^{(0)}_{u\bar{d}\to W^+g}(p,\bar{p},k) = \frac{ie}{\sqrt{2}\,s_W}\, g_s T^a_{\bar{c}c}\, \varepsilon^{*a}_\nu(k)\, \varepsilon^*_\mu(p_W)\, \bar{v}_d(\bar{p}) \left\{ \frac{\gamma^\nu(\not{k}-\not{\bar{p}})\gamma^\mu}{-s_{\bar{d}g}} + \frac{\gamma^\mu(\not{p}-\not{k})\gamma^\nu}{-s_{ug}} \right\} \omega_- u_u(p)$$

Born: no kinematic dependence

$$\Rightarrow \langle |\mathcal{M}^{(0)}_{u\bar{d}\to W^+g}|^2 \rangle = \langle |\mathcal{M}_{u\bar{d}\to W^+}|^2 \rangle \frac{8\pi\alpha_s}{M_W^2} C_F$$

$$\times \left\{ (1-\epsilon)\left[\frac{s_{\bar{d}g}}{s_{ug}} + \frac{s_{ug}}{s_{\bar{d}g}} \right] + \frac{2 s_{u\bar{d}} M_W^2}{s_{ug}\, s_{\bar{d}g}} - 2\epsilon \right\}$$

divergence for $s_{ug}, s_{\bar{d}g} \to 0$!

Drell-Yan: Real Phase Space

- we want to integrate out the gluon emission analytically ($D=4-2\epsilon$)

$$d\Phi_2(p_W, k; p+\bar{p}) = [dp_W]\,[dk]\,(2\pi)^4\,\delta^{(4)}(p_W + k - (p+\bar{p}))$$

$$[dp] \equiv \frac{d^D p}{(2\pi)^D}(2\pi)\underbrace{\delta_+(p^2-m^2)}_{\Theta(p^0)\,\delta(p^2-m^2)}$$

- strategy: factor out the emission phase space

$$d\Phi_2(p_W, k; p+\bar{p}) \text{ ``}=\text{''} d\tilde{\Phi}_1(\tilde{p}_W) \otimes [dk]$$

in general, involves convolutions

EX 29

Drell-Yan: Real Phase Space

- we want to integrate out the gluon emission analytically ($D = 4-2\epsilon$)

$$d\Phi_2(p_W, k; p+\bar{p}) = [dp_W]\,[dk]\,(2\pi)^4\,\delta^{(4)}(p_W + k - (p+\bar{p}))$$

$$[dp] \equiv \frac{d^D p}{(2\pi)^D}\,(2\pi)\,\underbrace{\delta_+(p^2-m^2)}_{\Theta(p^0)\,\delta(p^2-m^2)}$$

- strategy: factor out the emission phase space

$$d\Phi_2(p_W, k; p+\bar{p}) = \frac{dz_1}{z_1}\frac{dz_2}{z_2}\, d\Phi_1(\tilde{p}_W; z_1 p + z_2 \bar{p})\,\frac{(4\pi)^\epsilon}{\Gamma(1-\epsilon)}\left(\frac{\mu^2}{M_W^2}\right)^\epsilon \frac{M_W^2}{8\pi^2}$$

$$\times\ \frac{z_1 z_2 (1+z_1 z_2)}{(z_1+z_2)^{2-2\epsilon}}\,(1-z_1^2)^{-\epsilon}\,(1-z_2^2)^{-\epsilon}$$

$$z_1 = \sqrt{\frac{M_W^2}{s_{u\bar{d}}}\;\frac{s_{u\bar{d}} - s_{\bar{d}g}}{s_{u\bar{d}} - s_{ug}}}\,, \quad z_2 = \sqrt{\frac{M_W^2}{s_{u\bar{d}}}\;\frac{s_{u\bar{d}} - s_{ug}}{s_{u\bar{d}} - s_{\bar{d}g}}}$$

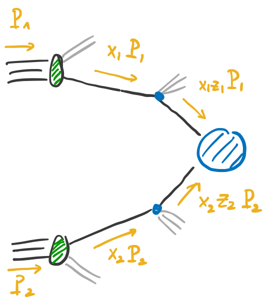
Drell-Yan: Real Corrections

- put the pieces together:

$$\frac{d\sigma^R_{pp\to W^+}}{dY_W} = \int_0^1 dx_1 \int_0^1 dx_2\, f_{u/p}(x_1)\, f_{\bar d/p}(x_2) \int_0^1 dz_1 \int_0^1 dz_2\, \delta(M_W^2 - x_1 x_2 z_1 z_2 S)\, \delta\left(Y_W - \frac{1}{2}\ln\left(\frac{x_1 z_1}{x_2 z_2}\right)\right)$$

$$\times \frac{(4\pi)^\epsilon}{\Gamma(1-\epsilon)}\left(\frac{\mu^2}{M_W^2}\right)^\epsilon \frac{z_1 z_2 (1-z_1 z_2)}{(z_1+z_2)^{2-2\epsilon}}(1-z_1^2)^{-\epsilon}(1-z_2^2)^{-\epsilon}$$

$$\times \frac{\alpha_s}{\pi} C_F \left\{(1-\epsilon)\left[\frac{S_{ug}}{S_{\bar d g}} + \frac{S_{\bar d g}}{S_{ug}}\right] + \frac{2 S_{u\bar d} M_W^2}{S_{ug} S_{\bar d g}} - 2\epsilon\right\} \underbrace{\frac{\pi}{z_1 z_2 S_{u\bar d}}}_{\text{flux}} \langle |\mathcal{M}^{(0)}_{u\bar d\to W^+}|^2\rangle$$



$$S_{ug} = S_{u\bar d} \cdot \frac{z_1(1-z_2^2)}{(z_1+z_2)}, \quad S_{\bar d g} = S_{u\bar d} \cdot \frac{z_2(1-z_1^2)}{(z_1+z_2)}$$

Drell-Yan: IR divergences

- virtual

$$\sim \delta(1-z_1)\,\delta(1-z_2)\left\{-\frac{\alpha_s}{\pi} C_F \frac{(4\pi)^\epsilon}{\Gamma(1-\epsilon)}\left(\frac{\mu^2}{M_W^2}\right)^\epsilon \cancel{\left[\frac{1}{\epsilon^2}+\frac{3}{2\epsilon}\right]}\right\}\hat{\sigma}^{LO}_{u\bar{d}\to W^+}(p,\bar{p})$$

- real

$\hookrightarrow$ use $$\frac{1}{(1-z)^{1+\epsilon}} = -\frac{1}{\epsilon}\delta(1-z)+\frac{1}{[1-z]_+}-\epsilon\left[\frac{\ln(1-z)}{1-z}\right]_+ + \dots$$

$$\int_0^1 dz\,[f(z)]_+\,g(z) = \int_0^1 dz\, f(z)\,[g(z)-g(1)]$$

$\hookrightarrow$ $$\frac{3}{2}\delta(1-z)+\frac{2}{[1-z]_+}-1-z = \left(\frac{1+z^2}{1-z}\right)_+$$

$$\sim \frac{\alpha_s}{\pi} C_F \frac{(4\pi)^\epsilon}{\Gamma(1-\epsilon)}\left(\frac{\mu^2}{M_W^2}\right)^\epsilon \left\{\delta(1-z_1)\,\delta(1-z_2)\cancel{\left[\frac{1}{\epsilon^2}+\frac{3}{2\epsilon}\right]} - \delta(1-z_1)\left(\frac{1+z_2^2}{1-z_2}\right)_+\frac{1}{2\epsilon} - \delta(1-z_2)\left(\frac{1+z_1^2}{1-z_1}\right)_+\frac{1}{2\epsilon}\right\}$$

left-over collinear initial-state singularities!

PDF renormalization

- universal collinear singularities are absorbed into the NLO PDFs $\rightsquigarrow$ additional "mass factorization terms"

$$d\sigma^{MF}_{ab}(p_a,p_b) = -\frac{\alpha_s}{2\pi}\frac{1}{\Gamma(1-\epsilon)}\left\{\sum_{a'}\int_0^1 dz\left[-\frac{1}{\epsilon}\left(\frac{4\pi\mu^2}{\mu_F^2}\right)^{\epsilon}P_{aa'}(z)+K^{F.S.}_{aa'}(z)\right]d\sigma^{LO}_{a'b}(zp_a,p_b)\right.$$
$$\left.+\sum_{b'}\int_0^1 dz\left[-\frac{1}{\epsilon}\left(\frac{4\pi\mu^2}{\mu_F^2}\right)^{\epsilon}P_{bb'}(z)+K^{F.S.}_{bb'}(z)\right]d\sigma^{LO}_{ab'}(p_a,zp_b)\right\}$$

$\Rightarrow$ Virtual + Real + MF : IR finite!