Exercise 1:

Remind yourself that

- $\not{p}\not{p} = p^2 \mathbb{1}$
- $\dfrac{1}{\not{p}-m} = \dfrac{\not{p}+m}{p^2-m^2}$
- $\gamma^\mu \gamma_\mu = 4 \cdot \mathbb{1}$
- $\gamma^\mu \gamma^\rho \gamma_\mu = -2\gamma^\rho$
- $\mathrm{Tr}(\gamma^\mu \gamma^\nu) = 4 g^{\mu\nu}$

$\not{p} = p^\mu \gamma_\mu$

$\{\gamma^\mu, \gamma^\nu\} = 2 g^{\mu\nu}$

$" \dfrac{1}{\not{p}-m} " = (\not{p}-m)^{-1}$

Exercise 2:

Show that translational invariance of $G^{\Phi_1 \dots \Phi_n}(x_1, \dots, x_n)$ implies momentum conservation:

$$\tilde{G}^{\Phi_1 \dots \Phi_n}(p_1, \dots, p_n) = (2\pi)^4 \delta^{(4)}(p_1 + \dots + p_n) \, f(p_1, \dots, p_{n-1})$$

$$\tilde{f}(p) = \int d^4x \, e^{-ipx} f(x) \, , \quad f(x) = \int \frac{d^4p}{(2\pi)^4} e^{+ipx} \tilde{f}(p)$$

Exercise 3

- consider the Lagrangian of the free photon field (+ gauge fixing term)

$$\mathcal{L}_{A,0} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2\xi} (\partial_\mu A^\mu)^2 \qquad (F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu)$$

$$\Rightarrow \text{e.o.m.} \qquad \left[g^{\mu\nu} \Box - \left(1 - \frac{1}{\xi}\right) \partial^\mu \partial^\nu \right] A_\nu(x) = 0$$

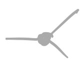
The Green's function is defined via

$$\left[g_{\mu\nu} \Box - \left(1 - \frac{1}{\xi}\right) \partial_\mu \partial_\nu \right] \Delta_F^{\nu\rho}(x) \stackrel{!}{=} g_\mu^{\ \rho} \, \delta(x)$$

Derive $\Delta_F^{\mu\nu}$ in momentum space using

$$\Delta_F^{\mu\nu}(x) = \int \frac{d^4p}{(2\pi)^4} \, e^{ipx} \, \Delta_F^{\mu\nu}(p)$$

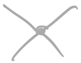
Exercise 4

(a) draw all graphs to the 2-pt function $G^{\phi\phi}$ up to $\mathcal{O}(g^2)$ in $\mathcal{L}_{int} = \frac{g}{3!}\phi^3$. ()

determine the symmetry factors.

- connected graphs $G_{con}^{\Phi_1 \dots \Phi_n}(x_1, \dots x_n) := G^{\Phi_1 \dots \Phi_n}(x_1, \dots x_n)\Big|_{\text{only connected graphs}}$

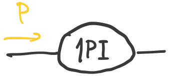
$$\hookrightarrow \quad G_{con}^{\Phi_1 \dots \Phi_n} = \frac{\delta}{i\delta J_1} \cdots \frac{\delta}{i\delta J_n} Z_{con}[\{J\}]\Big|_{J=0} \quad \left(Z_{con} = \ln(Z)\right)$$

(b) draw all connected graphs to the 2-pt function $G^{\phi \dots \phi}$ up to $\mathcal{O}(\lambda^2)$ in $\mathcal{L}_{int} = -\frac{\lambda}{4!}\phi^4$. ()

determine the symmetry factors.

Exercise 5

- consider a scalar theory with an arbitrary interaction ($\mathcal{L}_{int}$)

Let $i\Sigma_1^{\phi\phi}(p^2) \equiv$  be the sum of all 1PI graphs

"self energy"

↪ not allowed 

– Give a graphical representation for the 2-pt function $G^{\phi\phi}(p,-p)$ to all orders using $i\Sigma^{\phi\phi}(p^2)$

– resum the diagrammatic series ("Dyson series" $\sum_{n=0}^{\infty} a^n = \frac{1}{1-a}$)

– give an expression for the 2-pt vertex function $\Gamma^{\phi\phi}(p,-p)$

Exercise 6

- the generating functional $\Gamma[\{\Phi\}]$ for vertex functions is the Legendre-transformation of $Z_{con}[\{J\}]$

$$\Gamma^{\Phi_1 \dots \Phi_n}(x_1, \dots, x_n) = \frac{\delta}{\delta \Phi_1(x_1)} \cdots \frac{\delta}{\delta \Phi_n(x_n)} \Gamma[\{\Phi\}]$$

$\Leftrightarrow$ one can show that at tree-graph level $\rightsquigarrow$ Appendix

$$\Gamma^{(0)}[\{\Phi\}] = i \int d^4x \, \mathcal{L}(\{\Phi(x)\})$$

$\Rightarrow$ very easy derivation of Feynman rules $(\Gamma^{(0)\Phi_1 \dots \Phi_n})$

1. collect terms in $i\mathcal{L}$ that contain the fields $\Phi_1 \dots \Phi_n$
2. replace all derivatives by "$-i$" the incoming momentum $(\partial_\mu \to -i p_\mu)$
3. sum over all permutations of indices & momenta of identical external fields
4. drop all fields.

derive the Feynman rules for

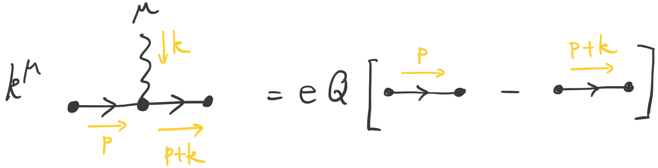
(a) $\mathcal{L}_g = g A^\mu (B \partial_\mu C - C \partial_\mu B)$

(b) $\mathcal{L}_g = g A^\mu A_\mu C D$

(c)  from $\bar{\psi}(i\not{\partial} - m)\psi$

Exercise 7

- using the tree-level identity



show the transversality of the photon self-energy

k^μ 

$= k^\mu \; \Sigma^{AA}_{1,\mu\nu}(k,-k) = 0$

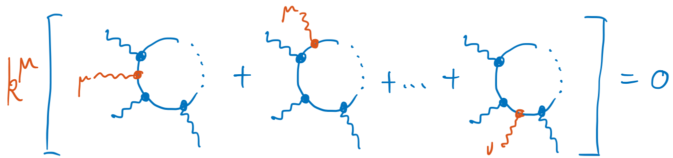
at 1-loop and 2-loop order.

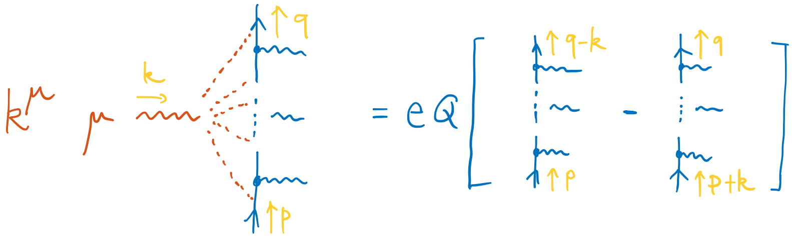
$\Rightarrow$ Ward Identity for the photon propagator:

$$-\frac{1}{\xi}\, k^2 k^\mu\, G^{AA}_{\mu\nu}(k,-k) = i k_\nu$$

(no h.o. corrections to the longitudinal part)

- similarly, one can show





Exercise 8

(a) determine a formula for the superficial degree of divergence

$$\omega(G) = \begin{cases} 0 \to \text{log. div} \\ 1 \to \text{lin. div.} \\ 2 \to \text{quad. div.} \\ \vdots \end{cases}$$

graph

(we're interested in 1PI graphs)

① write $\omega(G)$ in terms of L (# loops), I_A (# internal photon prop.), I_ψ (# internal fermion prop.).

② "Euler's loop equation": count the # of momentum-consv. constraints to write L in terms of I_A, I_ψ, and V (# vert.)

③ inspect the vertex ($A_\mu \bar{\psi}\psi$) and find relations

$$V = X(I_\psi, E_\psi) \quad \& \quad V = Y(I_A, E_A)$$

with $E_{\psi/A}$: # of external ψ/A fields

④ show that $\omega(G)$ only depends on $E_{\psi/A}$

cont.

Exercise 8

(b) find all superficially divergent vertex functions in QED

Exercise 9

- compute the D-dimensional solid angle $\Omega_D = \int d\Omega_D$

$\hookrightarrow$ the 1-dim Gaussian integral:

$$\sqrt{\pi} = \int_{-\infty}^{+\infty} dx \; e^{-x^2}$$

$\hookrightarrow$ Gamma function:

$$\Gamma(z) = \int_0^{\infty} t^{z-1} \; e^{-t} \; dt$$

Exercise 10

- expand $A_0(m) = -m^2 \left(\frac{m^2}{4\pi\mu^2}\right)^{\frac{D-4}{2}} \Gamma\left(\frac{2-D}{2}\right)$ $\quad (D = 4-2\epsilon)$

$$= -m^2 \left(\frac{m^2}{4\pi\mu^2}\right)^{-\epsilon} \Gamma(-1+\epsilon)$$

in ϵ $(D \to 4)$ up to finite terms

- Γ has poles for $z = 0, -1, -2, \ldots$
- $\Gamma(\epsilon) = \frac{1}{\epsilon} - \gamma_E + \mathcal{O}(\epsilon)$
- $\Gamma(z+1) = z\,\Gamma(z)$

Exercise 11

- derive the Feynman parametrisation: $\frac{1}{A_1 A_2} = \int_0^1 dx \frac{1}{[A_1(1-x) + A_2 x]^2}$

① convince yourself that $\frac{1}{A} = \int_0^\infty dt\, e^{-At}$ & write

$$\frac{1}{A_1 A_2} = \int_0^\infty dt_1\, dt_2\, e^{-A_1 t_1 - A_2 t_2}$$ (Schwinger parametrisation)

② hint: insert $1 = \int_0^\infty d\lambda\, \delta(\lambda - t_1 - t_2)$

- generalization to higher powers:

$$\frac{1}{A^\nu} = \frac{1}{\Gamma(\nu)} \int_0^\infty dt\, t^{\nu-1} e^{-At}$$ (by differentiation)

- Cheng-Wu Theorem

only a subset $S \subset \{1,\dots,N\}$ in the delta: $\delta(1 - \sum_{i=1}^N x_i) \rightsquigarrow \delta(1 - \sum_{i \in S} x_i)$

for $j \notin S$: $\int_0^\infty dx_j$

Exercise 12

imagine: 

- compute $B_0(p^2, 0, m)$ up to $\mathcal{O}(\epsilon)$

$\hookrightarrow$ result: $\Delta - \ln\left(\frac{m^2}{\mu^2}\right) + 2 + \frac{m^2-p^2}{p^2}\ln\left(\frac{m^2-p^2-i0}{m^2}\right) + \mathcal{O}(\epsilon)$

Exercise 13

- solve the system of linear equations for B_{00} & B_{11} in terms of A_0, B_0, B_1

$$D\, B_{00} + p^2 B_{11} = A_0(m_1) + m_0^2\, B_0$$

$$B_{00} + p^2 B_{11} = \frac{1}{2} A_0(m_1) - \frac{1}{2}(p^2 - m_1^2 + m_0^2)\, B_1$$

- give the divergent part of B_{00} & B_{11}

$$A_0(m)\Big|_{div} = \frac{m^2}{\epsilon}\,, \quad B_0(p^2, m_0, m_1)\Big|_{div} = \frac{1}{\epsilon}\,, \quad B_1(p^2, m_0, m_1) = -\frac{1}{2\epsilon}$$

Exercise 14

- the photon self energy is given by

$$\Sigma^{AA}_{\mu\nu}(k) = \frac{\alpha}{2\pi} Q^2 \left\{ g_{\mu\nu} \left[k^2 B_0 - 2A_0 + 4 B_{00} \right] + 4 k_\mu k_\nu \left[B_{11} + B_1 \right] \right\}$$

apply the reduction to scalar integrals:

$$B_1(k^2, m, m) = -\frac{1}{2} B_0$$

$$B_{00}(k^2, m, m) = \frac{1}{6} \left[A_0(m) + 2m^2 B_0 + k^2 B_1 + 2m^2 - \frac{k^2}{3} \right] + \mathcal{O}(\epsilon)$$

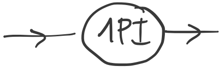
$$B_{11}(k^2, m, m) = \frac{1}{6k^2} \left[2A_0(m) - 2m^2 B_0 - 4k^2 B_1 - 2m^2 + \frac{k^2}{3} \right] + \mathcal{O}(\epsilon)$$

and identify the transversal & longitudinal parts in the decomposition

$$\Sigma^{AA}_{\mu\nu}(k) = \left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \right) \Sigma^{AA}_{T}(k^2) + \frac{k_\mu k_\nu}{k^2} \Sigma^{AA}_{L}(k^2)$$

Exercise 15

- the electron self energy is given by:

$i\,\Sigma_1^{\bar{4}4}(k) =$  $=$ 

$+ \dots$ (Feynman gauge: $\xi = 1$)

$$\Rightarrow \Sigma^{\bar{4}4}(k) = -\frac{\alpha}{4\pi} Q^2 \frac{(2\pi\mu)^{4-D}}{i\pi^2} \int d^Dq \; \frac{\gamma_\mu (\not{q}+m)\gamma^\mu}{(q^2-m^2)(q-k)^2}$$

and can be decomposed into its vector & scalar part as

$$\Sigma_1^{\bar{4}4}(k) = \not{k}\,\Sigma_V^{\bar{4}4}(k^2) + m\,\Sigma_S^{\bar{4}4}(k^2)$$

determine $\Sigma_V^{\bar{4}4}$ & $\Sigma_S^{\bar{4}4}$ in terms of scalar integrals.

careful with the sign!

① simplify the Dirac structure in the numerator

② perform a tensor reduction of $B_\mu(p, m_0, m_1) = p_\mu B_1(p, m_0, m_1)$

$$B_1(p^2, m_0, m_1) = \frac{1}{2p^2}\left[A_0(m_0) - A_0(m_1) - (p^2 - m_1^2 + m_0^2)\,B_0(p^2, m_0, m_1)\right]$$

③ get all finite terms using $A_0(m)\big|_{div} = \frac{m^2}{\epsilon}$, $B_0\big|_{div} = \frac{1}{\epsilon}$

Exercise 16

- explicitly calculate the divergent part of the vertex correction.

① for the UV behaviour ($q \to \infty$) external momenta are irrelevant

also neglect masses

$$\Lambda_\mu(p',p)\Big|_{\substack{uv\\div}} = \Lambda_\mu(0,0)\Big|_{\substack{uv\\div}} = \frac{\alpha}{4\pi}\, Q^2\, \frac{(2\pi\mu)^{4-D}}{i\pi^2}\int d^Dq\, \frac{\gamma_\alpha \not{q}\, \gamma_\mu \not{q}\, \gamma^\alpha}{[q^2]^3}\Bigg|_{\substack{uv\\div}}$$

② what is the possible Lorentz-structure of $\Lambda_\mu(0,0)$?

$\leadsto$ how can we extract the scalar coefficient of this decomposition?

③ determine the UV-divergent part

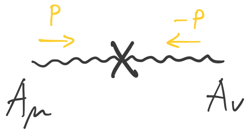
$$\frac{(2\pi\mu)^{4-D}}{i\pi^2}\int d^Dq\, \frac{1}{(q^2)^2}\Bigg|_{\substack{uv\\div}} = B_0(0,0,0)\Big|_{\substack{uv\\div}} = \frac{1}{\epsilon}$$

?

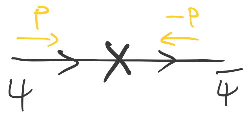
Exercise 17

- derive the counterterm Feynman rules for

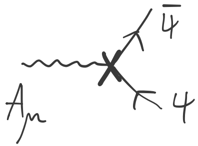
$$\mathcal{L}_{ct} = -\delta Z_A \frac{1}{2}(\partial_\mu A_\nu)(\partial^\mu A^\nu) + \left[\delta Z_A\left(1-\frac{1}{\xi}\right)+\delta Z_\xi \frac{1}{\xi}\right]\frac{1}{2}(\partial_\mu A_\nu)(\partial^\nu A^\mu) \Rightarrow$$



$$+\,\delta Z_\psi\, \bar{\psi}(i\not{\partial}-m)\psi - \delta m\, \bar{\psi}\psi \Rightarrow$$



$$-\left(\delta Z_e + \delta Z_\psi + \frac{1}{2}\delta Z_A\right) e Q\, \bar{\psi}\not{A}\psi \Rightarrow$$



Exercise 18

- field strength renormalization $\left(\Phi_i^0 = \sqrt{Z_{\Phi_i}}\,\Phi_i\right)$ is not needed in S-Matrix elements / physical observables.
  They are useful to make n-pt functions finite.
  how are the n-pt functions with & without Z_{Φ_i} related?

$$G^{\Phi_1^0 \dots \Phi_n^0} \quad \text{vs.} \quad G^{\Phi_1 \dots \Phi_n}$$

- what about truncated n-pt functions?

$$G_{trunc}^{\Phi_1^0 \dots \Phi_n^0} \quad \text{vs.} \quad G_{trunc}^{\Phi_1 \dots \Phi_n}$$

cont.

Exercise 18:

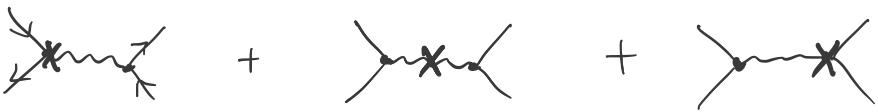
- The LSZ reduction formula is given in terms of bare fields:

$$i\mathcal{M}^{n\to m}(p_1,\dots,p_n,p_1',\dots,p_m') = \prod_{i=1}^{n} f_{in}^{\Phi_i}(p_i)\sqrt{R_{\Phi_i}}\ \prod_{j=1}^{m} f^{\Phi_j'}(p_j')\sqrt{R_{\Phi_j'}}$$

$$\times\ G_{trunc}^{\Phi_1^0\dots\Phi_n^0\,\Phi_1'^0\dots\Phi_m'^0}(p_1,\dots,p_n,-p_1',\dots,-p_m')\Big|_{on\text{-}shell}$$

what is the formula in terms of $G_{trunc}^{\Phi_1\dots\Phi_m'}$? (convenient choice?)

- Z_{Φ_i} are only relevant for <u>external</u> legs. Show that δZ_A cancels between these diagrams



Exercise 19

- determine the electron WF renormalization from

$$\lim_{p^2\to m^2}\left(\frac{i}{\not p - m}\hat{\Gamma}^{\bar{4}4}(-p,p)\right)u(p) \overset{!}{=} -u(p)$$

* $(\not p - m)\,u(p) = 0$
* $\hat{\Gamma}^{\bar{4}4}(-p,p) = i(\not p - m) + i\,\hat{\Sigma}^{\bar{4}4}(p)$
* $\hat{\Sigma}^{\bar{4}4}(p) = \not p\,\hat{\Sigma}_V^{\bar{4}4}(p^2) + m\,\hat{\Sigma}_S^{\bar{4}4}(p^2)$
* $\hat{\Sigma}_V^{\bar{4}4}(p^2) = \Sigma_V^{\bar{4}4}(p^2) + \delta Z_4$, $\quad\hat{\Sigma}_S^{\bar{4}4}(p^2) = \Sigma_S^{\bar{4}4}(p^2) - \delta Z_4 - \frac{\delta m}{m}$
* $\frac{\delta m}{m} = \Sigma_V^{\bar{4}4}(m^2) + \Sigma_S^{\bar{4}4}(m^2)$

Exercise 20

- determine the charge renormalization from the condition

$$0 \overset{!}{=} \bar{u}(p)\, \hat{\Lambda}_\mu(p,p)\, u(p)$$

① $\hat{\Lambda}_\mu(p',p) = \Lambda_\mu(p',p) + \gamma_\mu \left(\delta Z_e + \delta Z_4 + \frac{1}{2} \delta Z_A \right)$

use the Ward identity and $\Sigma^{\bar{4}4}(p) = \not{p}\, \Sigma_V^{\bar{4}4}(p^2) + m\, \Sigma_S^{\bar{4}4}(p^2)$

$$\Lambda_\mu(p,p) = \frac{\partial}{\partial p^\mu} \Sigma^{\bar{4}4}(p)$$

② Show the special Gordon identity $\bar{u}(p)\gamma_\mu u(p) = \frac{p_\mu}{m} \bar{u}(p) u(p)$ and use it to bracket out $\bar{u}(p)\gamma_\mu u(p)$

③ Use $\delta Z_4 = -\Sigma_V^{\bar{4}4}(m^2) - 2m^2 \left[\Sigma_V^{\bar{4}4\,\prime}(m^2) + \Sigma_S^{\bar{4}4\,\prime}(m^2) \right]$ to arrive at $\delta Z_e = -\frac{1}{2}\delta Z_A$

Exercise 21

- determine the transformation property of $\mathcal{A}_\mu := A_\mu^a T^a$ from

$$D_\mu \longrightarrow D_\mu' = U D_\mu U^\dagger \quad ; \quad D_\mu = \partial_\mu + ig A_\mu^a T^a = \partial_\mu + ig \mathcal{A}_\mu$$

$$U = U(\underline{\theta}) = \exp\{-ig\,\theta^a T^a\}$$

- show that for infinitesimal gauge transformations $\delta\underline{\theta}$ the gauge fields transform as

$$\delta A_\mu^a = g\, c^{abc}\, \delta\theta^b A_\mu^c + \partial_\mu(\delta\theta^a)$$

$[T^a, T^b] = i\, c^{abc}\, T^c$

Exercise 22

- Let $y_1, \dots y_N, y_1^*, \dots y_N^*$ be the 2N generators of a Grassmann alg.

  $\hookrightarrow \{y_i, y_j\} = \{y_i, y_j^*\} = \{y_i^*, y_j^*\} = 0$

  $\Rightarrow (y_i)^2 = (y_i^*)^2 = 0 \quad \Rightarrow$ only function $f(y) = a + b\,y$ $\quad \in \mathbb{C}$

  $\hookrightarrow$ derivatives:

  $$\frac{\partial}{\partial y_i} y_k = \delta_{ik} = \frac{\partial}{\partial y_i^*} y_k^* \;, \quad \frac{\partial}{\partial y_i} y_k^* = 0 = \frac{\partial}{\partial y_i^*} y_k$$

  $\hookrightarrow$ "integration" equiv. to derivative

  $$\int dy_i = 0 = \int dy_k^* \;, \quad \int dy_k\, y_i = \delta_{ik} = \int dy_k^*\, y_i^* \;, \quad \int dy_i\, y_k^* = \int dy_k^*\, y_i = 0$$

- show the substitution rule: (A = N x N regular matrix)

  $$z_i = A_{ij} y_j \quad \Rightarrow \quad dz_i = (A^{-1})_{ji}\, dz_j$$

- show $\displaystyle\int dz_1 \dots dz_N\, f(\underline{z}) = \int dy_1 \dots dy_N\, f(z(y))\, [\det A]^{-1}$

- Finally, show: $\displaystyle\int dy_1 \dots \int dy_N \int dy_N^* \dots \int dy_1^* \exp\{ y_i^* A_{ij} y_j \} = \det A$

Exercise 23

- derive the Ward identity for $k^\mu G_\mu^{A\psi\bar\psi}(k,p,p')$ using BRS symmetry

$$\delta_{BRS} A_\mu = \delta\bar\lambda\,[\partial_\mu u]$$

$$\delta_{BRS}\psi = \delta\bar\lambda\,[-ieQu\psi], \qquad \delta_{BRS}\bar\psi = \delta\bar\lambda\,[+ieQu\bar\psi]$$

$$\delta_{BRS}\bar u = \delta\bar\lambda\,[-\tfrac{1}{\xi}\partial^\mu A_\mu], \qquad \delta_{BRS} u = 0$$

① consider $0 \overset{!}{=} \delta_{BRS}\langle T\,\bar u(x)\,\psi(y)\,\bar\psi(z)\rangle$

② exploit the fact that the FP ghosts decouple:

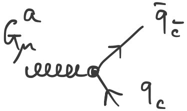
$$\langle T\,u(y_1)\bar u(y_2)\,\Phi_1(x_1)\dots\Phi_n(x_n)\rangle = \underbrace{\langle T\,u(y_1)\bar u(y_2)\rangle}_{=i\Delta_F(y_1-y_2)\,=\,\int\frac{d^4k}{(2\pi)^4}e^{-ik(y_1-y_2)}\frac{i}{k^2}}\langle T\,\Phi_1(x_1)\dots\Phi_n(x_n)\rangle$$

③ go into momentum space using

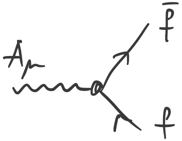
$$\int d^4x\,d^4y\,d^4z\;e^{-ikx-ipy-ip'z}\,(\dots)$$

Exercise 24

- compare the vertices in QCD vs QED



$-ig_s T^a_{\bar{c}c} \gamma_\mu$ vs.

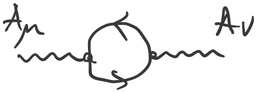


$-ie Q_f \gamma_\mu$

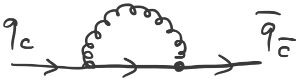
to derive conversion rules (looking at the colour structure) between:

(a)

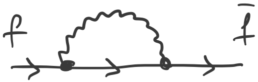
vs.



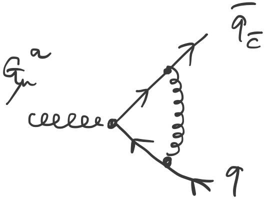
(b)



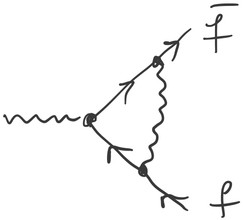
vs.



(c)



vs.



Exercise 25

- calculate the 1-loop beta function

① consider $g_{s,0} = \mu^{\epsilon} g_s Z_g$ with $Z_g = 1 + \delta Z_g$, $\delta Z_g = \frac{\alpha_s}{4\pi} \Delta \underbrace{\left(\frac{2}{3} N_f T_F - \frac{11}{6} C_A\right)}_{=: -\beta_0}$

and we obtain

$$\alpha_{s,0} = \mu^{2\epsilon} \alpha_s \left[1 - \frac{\alpha_s}{2\pi} \Delta \beta_0 + \dots\right]$$

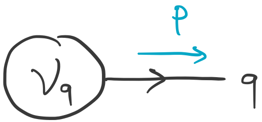
② use the fact that the bare coupling is scale-indep $\left(\mu \frac{d\alpha_{s,0}}{d\mu} = 0\right)$ to obtain an equation for

$$\mu \frac{d\alpha_s}{d\mu} = \dots$$

③ take the limit $D \to 4$ $(\epsilon \to 0)$

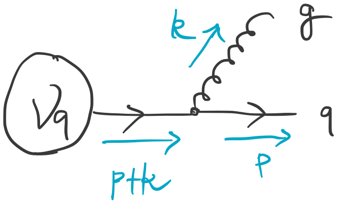
Exercise 26

- consider a diagram with an outgoing quark:



$$\equiv \mathcal{M}_q^c(p) = \bar{u}(p)\, V_q^c(p)$$

- and let's attach a gluon



$$\equiv \mathcal{M}_{qg}^{ca}(p,k)$$

- show:

$$\mathcal{M}_{qg}^{ca}(p,k) = \frac{g_s T^a_{cc'}}{2(p\cdot k)}\, \bar{u}(p)\, \gamma^\mu (\not{p}+\not{k})\, V_q^{c'}(p+k)\, \varepsilon^*_\mu(k)$$

cont.

Exercise 26 : soft limit

- use $k = \xi q$ with $\xi \longrightarrow 0$ and show

$$\mathcal{M}^{ca}_{qg}(p,k) \xrightarrow{\xi \to 0} \frac{1}{\xi}\, g_s\, J^{a,\mu}_{cc'}(q)\, \varepsilon^*_\mu(k) \times \mathcal{M}^{c'}_{q}(p)$$

- give $J^{a,\mu}_{cc'}(q)$

cont.

Exercise 26: collinear limit

- use the Sudakov decomposition

$$p^\mu = z\, q^\mu + k_\perp^\mu - \frac{k_\perp^2}{z} \frac{n^\mu}{2(q\cdot n)}$$

$$k^\mu = (1-z)\, q^\mu - k_\perp^\mu - \frac{k_\perp^2}{(1-z)} \frac{n^\mu}{2(q\cdot n)}$$

$n^2 = 0$

$k_\perp^2 < 0$

$k_\perp \cdot q = k_\perp \cdot n = 0$

defines how we approach $p \parallel k$

- and derive (use D-dimensional Dirac Algebra)

$$\sum_{\text{hel.}} \sum_{\text{col}} |\mathcal{M}_{qg}(p,k)|^2 \xrightarrow{p \parallel k} \frac{1}{(p\cdot k)} 4\pi\alpha_s\, C_F \left[\frac{1+z^2}{1-z} - \epsilon(1-z) \right]$$

$$\times\, |\mathcal{M}_q(p+k)|^2$$

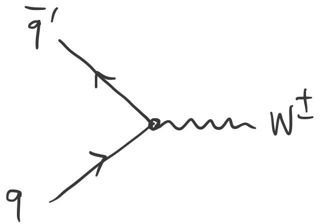
Exercise 27

- compute the sq. ME for on-shell W^+ production:

$$\langle |\mathcal{M}_{u\bar{d}\to W^+}|^2 \rangle$$

$\langle \dots \rangle \hat{=}$ average / sum over initial- / final-state d.o.f (col., hel.)

$$\omega_- = \frac{1}{2}(\mathbb{1} - \gamma^5)$$



$$= i \frac{e}{\sqrt{2}\, s_W} \gamma^\mu \omega_-$$

$$\sum_{\text{pol.}} \varepsilon^*_\mu(p_W)\, \varepsilon_\nu(p_W) = -g_{\mu\nu} + \frac{p_{W,\mu}\, p_{W,\nu}}{M_W^2}$$

- calculate the partonic cross section $\hat{\sigma}_{u\bar{d}\to W^+}(p, \bar{p})$
- calculate the differential hadronic cross section

$$\frac{d\sigma_{pp\to W^+}(P_1, P_2)}{d\, y_W}$$

rapidity: $y_W = \frac{1}{2} \ln\left(\frac{E_W + p_W^z}{E_W - p_W^z}\right)$

Exercise 28

- compute the 1-loop corrections: $\mu^{2\epsilon}\int\frac{d^Dq}{(2\pi)^D}\,\frac{\gamma^\alpha(\not{q}-\bar{\not{p}})\gamma^\mu(\not{q}+\not{p})\gamma_\alpha}{q^2\,(q+p)^2\,(q-\bar{p})^2}$

① combine the propagators & shift loop momentum $\quad \frac{1}{abc} = 2\int_0^1 dx\int_0^{1-x} dy\,\frac{1}{\left[a(1-x-y)+bx+cy\right]^3}$

② simplify the numerator:
- linear in $q \rightsquigarrow 0$
- $q^\mu q^\nu \sim g^{\mu\nu} \Rightarrow q^\mu q^\nu \rightsquigarrow g^{\mu\nu}\frac{q^2}{D}$
- Dirac Eqn. $\bar{V}_d(\bar{p})\,\bar{\not{p}} = 0 = \not{p}\,u_u(p)$

③ identify $I_{2,3}(-xy\hat{s})$ $\quad I_n(A) = \int d^Dq\,\frac{1}{[q^2-A]^n}$

④ expand in ϵ

Exercise 29

- determine ξ_i for the mapping $p+\bar{p} \longrightarrow p_W + k \rightsquigarrow \xi_1 p + \xi_2 \bar{p} \longrightarrow \tilde{p}_W$
  so that $p_W^2 = \tilde{p}_W^2$ & $y_W = \tilde{y}_W$

- insert into $d\Phi_2(p_W, k; p+\bar{p})$ some "1"'s

$$1 = \int_0^1 dz_1 \int_0^1 dz_2 \;\; \delta(z_1-\xi_1)\,\delta(z_2-\xi_2)$$

$$1 = \int \frac{d^D\tilde{p}_W}{(2\pi)^D}\,(2\pi)^D\;\delta^{(D)}\left(\tilde{p}_W - (z_1 p + z_2\bar{p})\right)$$

to get $d\Phi_2(p_W, k; p+\bar{p}) = \frac{dz_1}{z_1}\frac{dz_2}{z_2}\,d\Phi(\tilde{p}_W; z_1p+z_2\bar{p})\,\overbrace{[dk]\,z_1 z_2\,\delta(z_1-\xi_1)\,\delta(z_2-\xi_2)}^{\circledast}$

- integrate out $\circledast$

  $\hookrightarrow$ $d^Dk = dk^0\,dk^z\,d^{D-2}k_\perp \rightarrow dk_\perp^2\,\frac{1}{2}\left(k_\perp^2\right)^{\frac{D-4}{2}}\,d\Omega_{D-2}$

  $\hookrightarrow$ variable change: $k^{0,z} = \frac{1}{2\sqrt{s}}\left[\frac{s_{ug}}{x_1} \pm \frac{s_{\bar{d}g}}{x_2}\right] \rightsquigarrow \begin{cases} s_{ug} = s_{u\bar{d}}\,\dfrac{z_1(1-z_2^2)}{z_1+z_2} \\[2ex] s_{\bar{d}g} = s_{u\bar{d}}\,\dfrac{z_2(1-z_1^2)}{z_1+z_2} \end{cases}$