

Radiative Corrections

&

Renormalization

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The Plan

Ø Motivation & Conventions

I. QFT in a Nutshell

II. QED & Renormalization

III. Yang-Mills Theory & QCD

IV. Infrared Singularities

Ø. Motivation

&

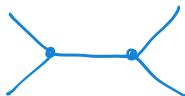
Conventions

Motivation

- radiative corrections \leftrightarrow perturbative evaluation of a QFT

\hookrightarrow diagrammatic representation: Feynman diagrams

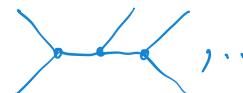
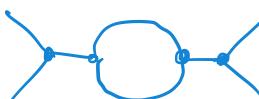
Born-level



, ...

“radiative corrections”

higher order



, ...

\hookrightarrow ultraviolet (UV) divergences

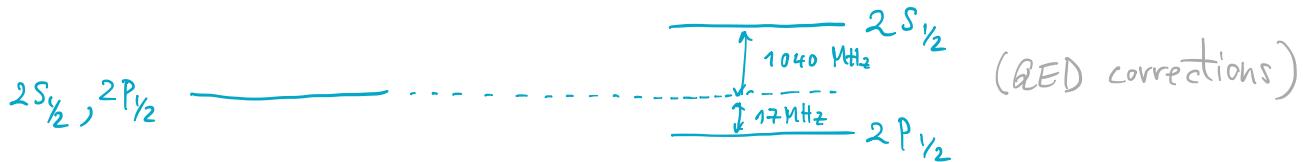
\Rightarrow renormalization

\hookrightarrow infrared (IR) divergences

\Rightarrow compensation between virtual & real-emission

Motivation: Measurable effects

- Lamb-Shift: (H -atom)



- anomalous magnetic moment (e, μ)

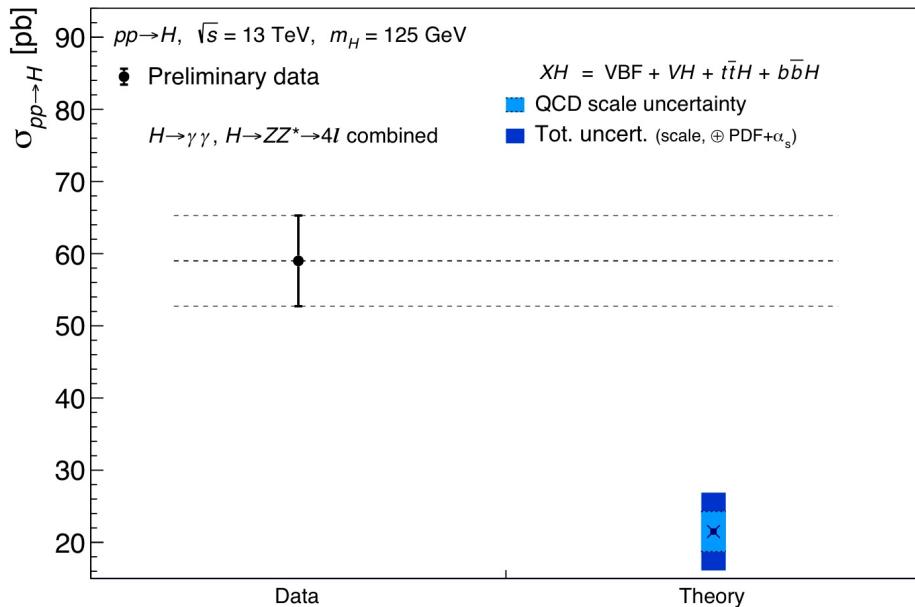
$$g_\mu = 2 \left[1 + \frac{\alpha}{2\pi} + \dots \right]$$

\uparrow \uparrow
Dirac Schrödinger

- electroweak precision tests Input parameters \rightarrow observables
 - ↪ fit Standard Model (SM) to Data: good agreement
 - ↪ predict m_t , constrain M_H

Predict & Compare

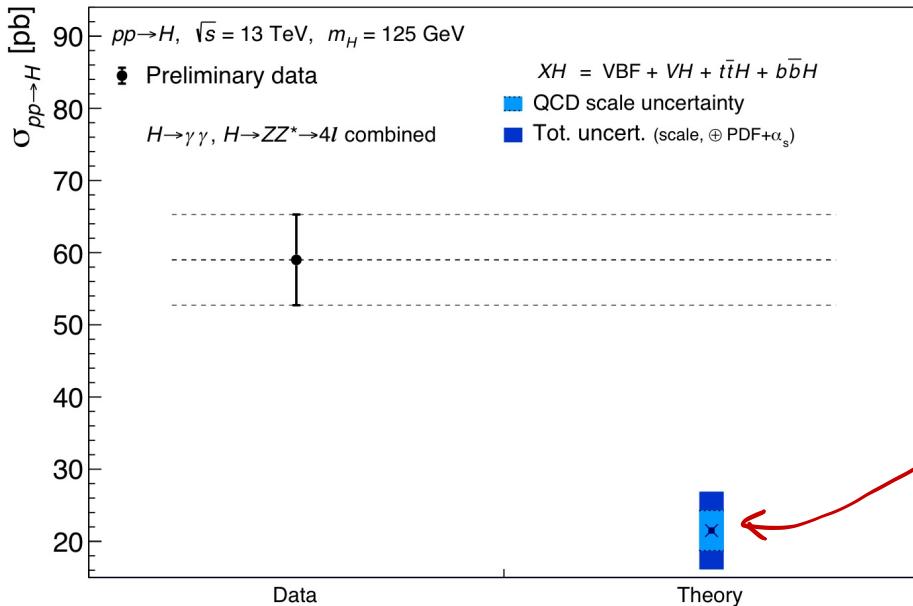
- Higgs production @ LHC:



- 3.8 σ deviation !!!
- Standard Model fails?!

Predict & Compare

- Higgs production @ LHC:



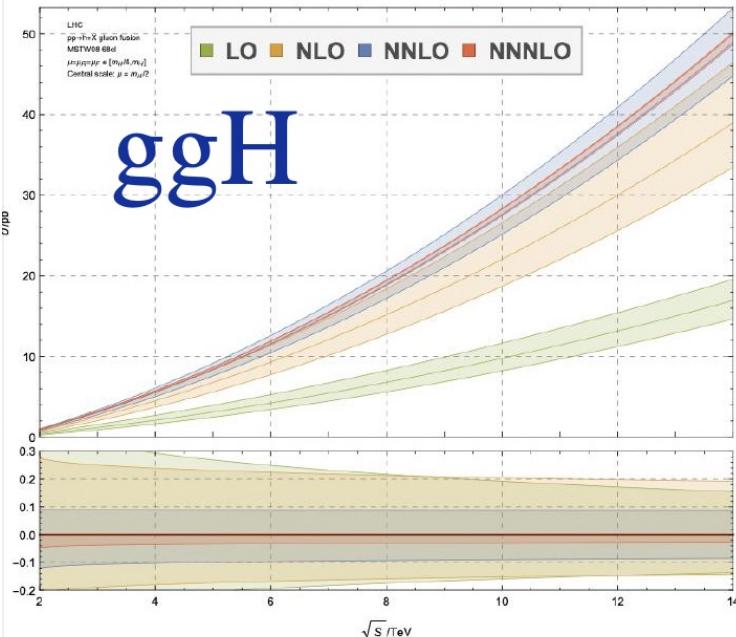
• 3.8σ
deviation !!!

• Standard Model fails?!

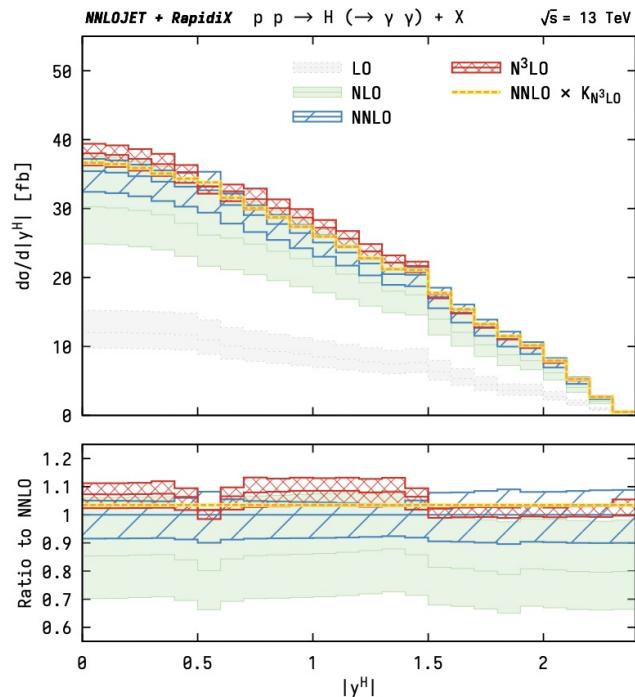
LEADING
ORDER

Higgs Production @ N³LO

[Anastasiou, Duhr, Dulat, Herzog, Mistlberger '15]



[Chen, Gehrmann, Glover, AH, Mistlberger, Pelloni '21]



Conventions & Notation

- natural units: $\hbar = c = 1$ one should avoid this...

- metric tensor $g_{\mu\nu} = g^{\mu\nu} = \text{diag } (1, -1, -1, -1)$

contravariant vector: $x^\mu = (x^0, \vec{x}) = (t, x, y, z)$

covariant vector: $x_\mu = g_{\mu\nu} x^\nu = (x^0, -\vec{x})$

derivative: $\partial_\mu \equiv \frac{\partial}{\partial x^\mu} = (\frac{\partial}{\partial t}, \vec{\nabla})$

scalar product: $x \cdot y = x^\mu y_\mu = x^\mu y^\nu g_{\mu\nu} = x^0 y^0 - \vec{x} \cdot \vec{y}$

$\square \equiv \partial^2 = \partial^\mu \partial_\mu = \partial_t^2 - \Delta$

Levi-Civita: $\epsilon^{\mu\nu\rho\sigma} = -\epsilon_{\mu\nu\rho\sigma} = \begin{cases} +1, & \{\mu, \nu, \rho, \sigma\} \text{ even perm } \{0, 1, 2, 3\} \\ -1, & \text{--- odd ---} \\ 0, & \text{else} \end{cases}$

Conventions & Notation

- Dirac algebra (4-dimensional)

$$\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu \gamma^\nu - \gamma^\nu \gamma^\mu = 2 g^{\mu\nu} \mathbb{1} \quad (\text{Tr}(\mathbb{1})=4)$$

$$\gamma^5 = -i \gamma_0 \gamma_1 \gamma_2 \gamma_3 = +i \gamma^0 \gamma^1 \gamma^2 \gamma^3 = -\frac{i}{4} \epsilon_{\mu\nu\rho\sigma} \gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma$$

$$\{\gamma^5, \gamma^\mu\} = 0, (\gamma^5)^2 = \mathbb{1}$$

$$\sigma^{\mu\nu} = \frac{i}{2} [\gamma^\mu, \gamma^\nu] = -\sigma^{\nu\mu}$$

Dirac-slash: $\not{a} = \gamma^\mu a_\mu$

- Dirac spinors: ($\bar{\psi} = \psi^\dagger \gamma^0$)

$$(\not{p}-m) u(p) = 0, \quad (\not{p}+m) v(p) = 0$$

$$\bar{u}(p) (\not{p}-m) = 0, \quad \bar{v}(p) (\not{p}+m) = 0$$

Ex 1

I. QFT in a
Nutshell

Overview

- starting point: (local) Lagrange density \mathcal{L} \leftrightarrow defines our theory

$$\hookrightarrow \phi^4\text{-theory: } \mathcal{L}_{\phi^4} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2}m^2 \phi^2 - \frac{\lambda}{4!} \phi^4$$

$$\hookrightarrow QED: \mathcal{L}_{QED} = \bar{\psi}(i\cancel{\partial} - m)\psi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - eQ \bar{\psi} \gamma_\mu \cancel{\partial} \gamma^\mu \psi$$

\hookrightarrow construction via symmetry considerations: $SU(3)_c \times SU(2)_L \times U(1)_Y$

\Rightarrow equation of motion (e.o.m)

$$\partial_\mu \frac{\delta \mathcal{L}}{\delta (\partial_\mu \Phi)} - \frac{\delta \mathcal{L}}{\delta \Phi} = 0$$

I write:

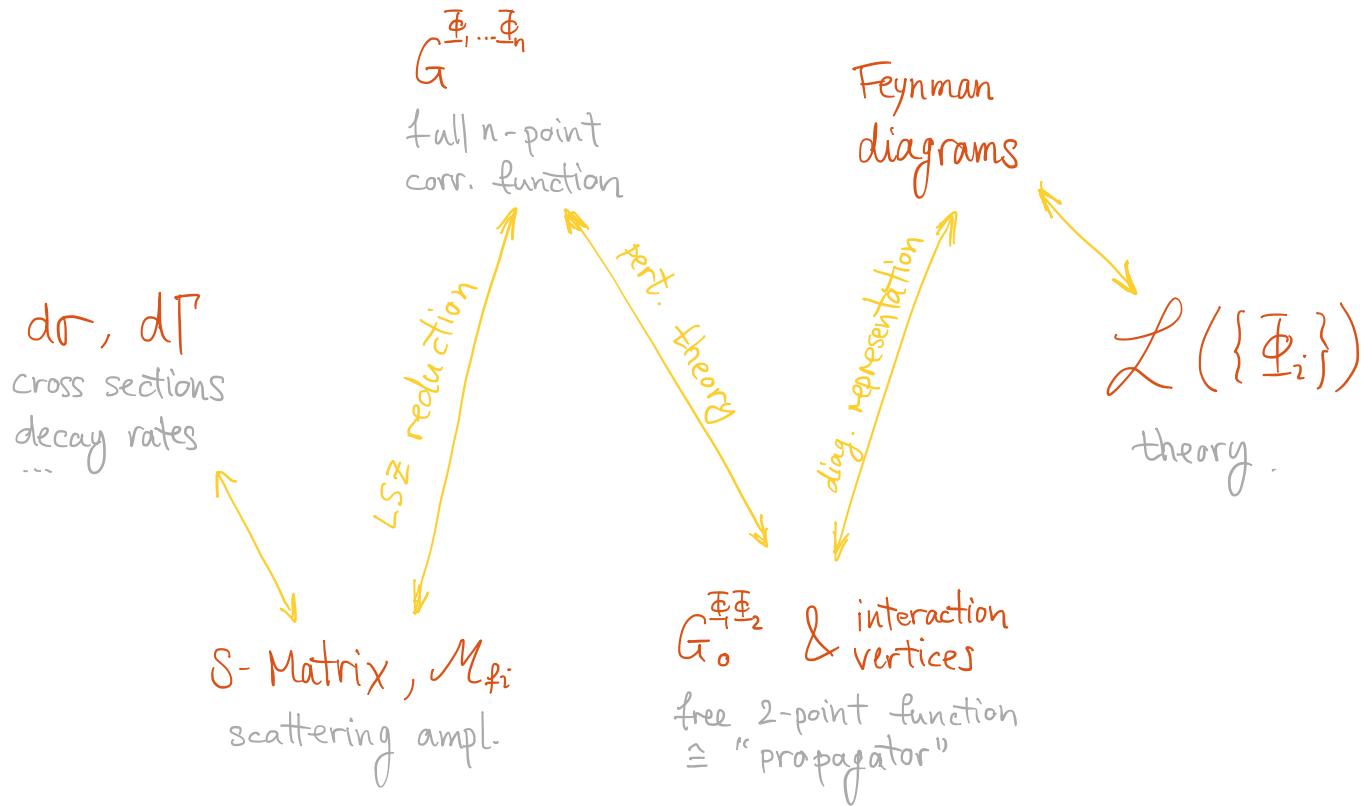
$$\Phi \in \{\phi, \psi, A^\mu, \dots\}$$

- fundamental objects: n -point correlation functions

$$G^{\Phi_1 \dots \Phi_n}(x_1, \dots, x_n)$$

Ex 2

The Road Map



Free Field theory

- "free" part of $\mathcal{L} \leftrightarrow$ bilinear in the fields:

$$\hookrightarrow \mathcal{L}_{\phi,0} = \frac{1}{2} (\partial_\mu \phi) (\partial^\mu \phi) - \frac{1}{2} m^2 \phi^2$$

$$\hookrightarrow \mathcal{L}_{\psi,0} = \bar{\psi} (i \not{D} - m) \psi$$

$$\hookrightarrow \mathcal{L}_{A,0} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \quad ; \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

- n-point correlation function

$$G_0^{\Phi_1 \dots \Phi_n}(x_1, \dots, x_n) = \langle 0 | T \Phi_1(x_1) \dots \Phi_n(x_n) | 0 \rangle \quad (\text{canonical quant.})$$

$$= \frac{1}{N_0} \int \mathcal{D}\{\Phi\} \Phi_1(x_1) \dots \Phi_n(x_n) \exp(i S_0[\{\Phi\}]) \quad (\text{path integral quant.})$$

time ordering: $\Phi_i(x_{i_n}) \dots \Phi_{i_1}(x_{i_1})$
for $x_{i_1} > \dots > x_{i_n}$

Free Scalar Theory

- $\mathcal{L}_{\phi,0} = \frac{1}{2} (\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2} m^2 \phi^2 \Rightarrow \text{e.o.m: } (\square + m^2) \phi = 0$
- Green's function: $(\square + m^2) \Delta_F(x) = -\delta(x)$

$$\Delta_F(x) = \int \frac{d^4 p}{(2\pi)^4} e^{-ipx} \frac{1}{p^2 - m^2 + i0} \quad (\text{Feynman prop.})$$

time ordering.

- n-point function:

$$G_o^{\phi \dots \phi}(x_1, \dots, x_n) = \langle 0 | T \phi(x_1) \dots \phi(x_n) | 0 \rangle = \frac{1}{N_o} \int \mathcal{D}[\phi] \phi(x_1) \dots \phi(x_n) e^{i S_0[\phi]}$$

- generating functional:

$$Z_{\phi,0}[J] = \frac{1}{N_o} \int \mathcal{D}[\phi] \exp \left\{ i \int d^4 x [\mathcal{L}_{\phi,0} + J(x) \phi(x)] \right\}$$

$$\Rightarrow G_o^{\phi \dots \phi}(x_1, \dots, x_n) = \frac{\delta}{i \delta J(x_1)} \dots \frac{\delta}{i \delta J(x_n)} Z_{\phi,0}[J] \Big|_{J=0}$$

Free Scalar Theory

- generating functional:

$$Z_{\phi,0} [J] = \frac{1}{N_0} \int \mathcal{D}[\phi] \exp \left\{ i \int d^4x \left[-\frac{1}{2} \phi (\square + m^2) \phi + J \phi \right] \right\}$$

... "complete the square" ($\phi = \phi' - \int d^4y \Delta_F(x-y) J(y)$)

... absorb $\int \mathcal{D}[\phi']$ into normalisation ($Z_{\phi,0}[0] \stackrel{!}{=} 1$)

$$= \exp \left\{ \int d^4x \int d^4y \frac{1}{2} i J(x) i \Delta_F(x-y) i J(y) \right\}$$

$$\Rightarrow G_0^{\phi\phi}(x_1, x_2) = \frac{\delta}{i \delta J(x_1)} \frac{\delta}{i \delta J(x_2)} Z_{\phi,0} [J] \Big|_{J=0} = i \Delta_F(x_1 - x_2) = \begin{array}{c} x_1 \\ \bullet \\ \text{---} \\ \bullet \\ x_2 \end{array}$$

$$G_0^{\phi\phi\phi\phi}(x_1, x_2, x_3, x_4) = \dots = \begin{array}{c} x_1 \bullet \text{---} \bullet x_2 \\ \bullet \text{---} \bullet x_3 \end{array} + \begin{array}{c} x_1 \bullet \\ | \\ x_3 \bullet \end{array} \begin{array}{c} x_2 \bullet \\ | \\ x_4 \bullet \end{array} + \begin{array}{c} x_1 \bullet \text{---} \bullet x_2 \\ \bullet \text{---} \bullet x_3 \end{array}$$

(no $G^{\overbrace{\phi \dots \phi}}$)

Free Dirac Theory

- $\mathcal{L}_{\psi,0} = \bar{\psi}(i\not{D} - m)\psi \Rightarrow \text{e.o.m. } (i\not{D} - m)\psi = 0$

- Green's function: $(i\not{D} - m)S_F(x) = \delta(x)$

$$S_F(x) = \int \frac{d^4 p}{(2\pi)^4} e^{-ipx} \frac{1}{p - m + i0}$$

- generating functional:

$$\begin{aligned} Z_{4,0}[\eta, \bar{\eta}] &= \frac{1}{N_0} \int \mathcal{D}[\psi] \mathcal{D}[\bar{\psi}] \exp \left\{ i \int d^4 x \left[\bar{\psi}(i\not{D} - m)\psi - \bar{\eta}\not{D}\psi + \bar{\eta}\psi \right] \right\} \\ &= \dots = \exp \left\{ - \int d^4 x \int d^4 y i\bar{\eta}(x) iS_F(x-y) i\eta(y) \right\} \end{aligned}$$

$$\begin{aligned} \Rightarrow G_0^{4\bar{4}}(x_1, x_2) &= \langle 0 | T \psi(x_1) \bar{\psi}(x_2) | 0 \rangle = \begin{cases} t_1 > t_2: \text{create } f @ x_2 \rightarrow \text{destroy } @ x_1 \\ t_2 > t_1: \text{create } \bar{f} @ x_1 \rightarrow \text{destroy } @ x_2 \end{cases} \\ &= \begin{array}{c} \bar{\psi}(x_2) \quad \psi(x_1) \\ \bullet \longrightarrow \bullet \end{array} \Leftrightarrow \text{definite fermion-number flow} \triangleq \underline{\text{arrow}} \\ &= \frac{\delta}{i\delta\bar{\eta}(x_1)} \frac{\delta}{i\delta\eta(x_2)} \Big|_{\eta=\bar{\eta}=0} Z_{4,0}[\eta, \bar{\eta}] \\ &= i S_F(x_1 - x_2) \end{aligned}$$

Grassmann-valued fields.

Free Photon Field

- $\mathcal{L}_{A,0} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$ ($F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$) \Rightarrow e.o.m. $[g^{\mu\nu} \square - \partial^\mu \partial^\nu] A_\nu = 0$

- Green's function: $(g^{\mu\nu} \square - \partial^\mu \partial^\nu) \Delta_{F, vp}^{(x)} = g^{\mu\nu} \delta(x)$ $\not\propto \partial_\mu [\dots]$

↪ origin gauge symmetry: $A_\mu(x) \rightarrow A_\mu(x) + \partial_\mu \Lambda(x)$
(unphysical d.o.f.)

$$(g^{\mu\nu} \square - \partial^\mu \partial^\nu) \partial_\mu \Lambda(x) = 0 \quad \hat{=} \text{ eigenvalue zero}$$

↪ Green's function $\Delta_{F, vp}$ is "inverse" of the differential operator $[g^{\mu\nu} \square - \partial^\mu \partial^\nu]$
eigenvalues zero \Rightarrow not invertible $\Rightarrow \nexists \Delta_{F, vp}$

EX 3

Free Photon Field (with covariant gauge fixing)

- $\mathcal{L}_{A,0} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2\xi} (\partial \cdot A)^2 \Rightarrow \text{e.o.m. } [g^{\mu\nu} \square - (1 - \frac{1}{\xi}) \partial^\mu \partial^\nu] A_\nu = 0$

- Green's function: $[g^{\mu\nu} \square - (1 - \frac{1}{\xi}) \partial^\mu \partial^\nu] \Delta_{F,0}(x) = g^\mu_\nu \delta(x)$

$$\Delta_F^{\mu\nu}(x) = \int \frac{d^4 p}{(2\pi)^4} e^{-ipx} \frac{-1}{p^2} \left[g^{\mu\nu} - (1 - \xi) \frac{p^\mu p^\nu}{p^2} \right]$$

- generating functional:

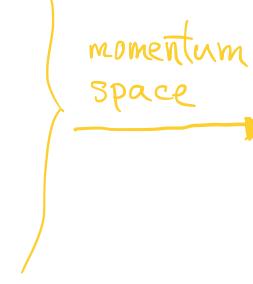
$$Z_{A,0}[J_\mu] = \exp \left\{ \frac{1}{2} \int d^4 x \int d^4 y i J_\mu(x) i \Delta_F^{\mu\nu}(x-y) i J_\nu(y) \right\}$$

$$\Rightarrow G_{\mu\nu,0}^{AA}(x_1, x_2) = \frac{\delta}{i \delta J^\mu(x_1)} \frac{\delta}{i \delta J^\nu(x_2)} Z_{A,0}[J^\mu] \Big|_{J^\mu=0}$$

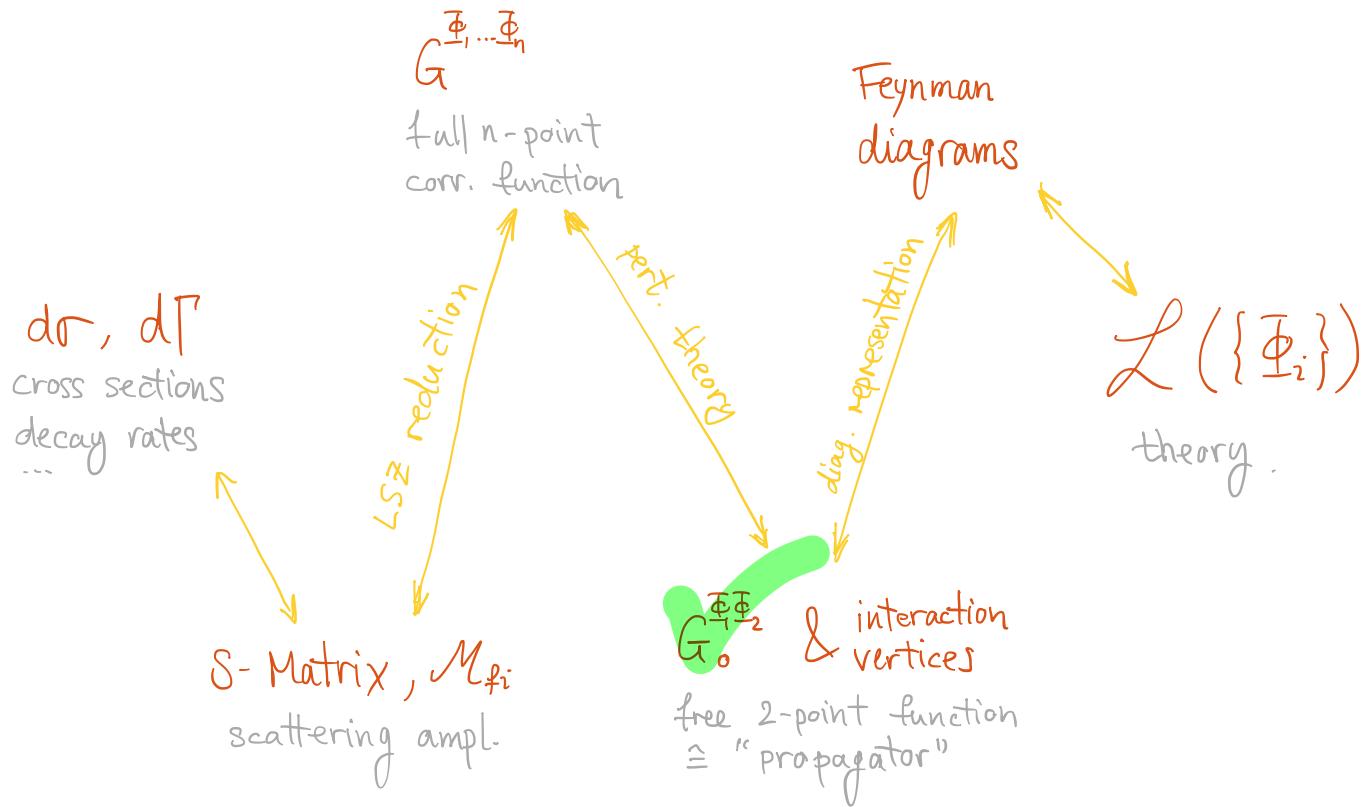
$$= \begin{matrix} \text{~~~~~} \\ A_\mu(x_1) \end{matrix} \quad \begin{matrix} \text{~~~~~} \\ A_\nu(x_2) \end{matrix} = i \Delta_{F,\mu\nu}(x_1 - x_2)$$

Summary : Free Field Theory

- exact solutions for $Z_0[J]$ \Rightarrow arbitrary n-point functions
- general procedure: $\mathcal{L}_0 = \bar{\Phi}^{(4)} D \bar{\Phi} \Rightarrow$ e.o.m. $D \bar{\Phi} = 0$
 \hookrightarrow solve using Green's functions: $D \Delta_F(k) \sim f(x)$
- \hookrightarrow perform path integral: $Z_0[J] = \exp \left\{ \pm \int d^4x \int d^4y [J(x) i\Delta_F(x-y) J(y)] \right\}$
- \Rightarrow basic building block: 2-point functions ("PROPAGATORS") $G_0^{\Phi\Phi}(x_1, x_2) = i\Delta_F(x-y)$

- scalar $\phi(x_2) \quad \phi(x_1) = i\Delta_F(x_1-x_2)$
 - fermion $\bar{\psi}(x_2) \quad \psi(x_1) = iS_F(x_1-x_2)$
 - photon $A_\nu(x_2) \quad A_\mu(x_1) = i\Delta_F^{\mu\nu}(x_1-x_2)$
 - so far: not very exciting: $G^{\phi\phi\phi\phi} = \text{---} + \text{---} + \text{X}$
- momentum space 
- | |
|--|
| $\frac{i}{p^2 - m^2}$ |
| $\frac{i}{p - m}$ |
| $\frac{-i}{p^2} \left[g^{\mu\nu} - (1-\xi) \frac{p^\mu p^\nu}{p^2} \right]$ |

The Road Map



Interacting Field Theory

- separate interactions from free part: $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{\text{int}}$

$$\hookrightarrow \mathcal{L}_{\phi^4} = -\frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2}m^2\phi^2 - \frac{\lambda^4}{4!}\phi^4$$

$$\hookrightarrow \mathcal{L}_{\text{QED}} = \bar{\psi}(i\gamma^\mu - m)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - eQ\bar{\psi}\gamma^\mu\psi$$

- n-point function

$$G^{\underline{\Phi}_1 \dots \underline{\Phi}_n}(x_1, \dots, x_n) = \langle \Omega | T \underline{\Phi}_1(x_1) \dots \underline{\Phi}_n(x_n) | \Omega \rangle \quad (\text{canonical quant.})$$

$$= \frac{1}{N} \int D[\{\underline{\Phi}\}] \underline{\Phi}_1(x_1) \dots \underline{\Phi}_n(x_n) \exp \left\{ i S[\{\underline{\Phi}\}] \right\} \quad (\text{path-integral quant.})$$

vacuum of the interacting theory ($\neq |\Omega\rangle$)

Perturbation Theory

- generating functional for n-point functions

$$\begin{aligned} Z[J] &= \frac{1}{N!} \int \mathcal{D}[\{\Phi\}] \exp \left\{ i \int d^4x \left[\mathcal{L}(\{\Phi(x)\}) + \sum_i J_i(x) \Phi_i(x) \right] \right\} \\ &= \frac{1}{N!} \int \mathcal{D}[\{\Phi\}] \exp \left\{ i \int d^4y \mathcal{L}_{\text{int}}(\{\Phi_i(y)\}) \right\} \exp \left\{ i \int d^4x \left[\mathcal{L}_0(\{\Phi_i(x)\}) + \sum_i J_i(x) \Phi_i(x) \right] \right\} \\ &= \frac{1}{N!} \exp \left\{ i \int d^4y \mathcal{L}_{\text{int}} \left(\left\{ \Phi_i(y) \rightarrow \frac{\delta}{i \delta J_i(y)} \right\} \right) \right\} Z_0[\{J_i\}] \end{aligned}$$

Series expansion $\hat{=}$ perturbative expansion in \mathcal{L}_{int}

- $Z[\{J=0\}] \stackrel{!}{=} 1$
 \leftrightarrow no "vacuum bubbles"

Example : ϕ^3 Theory

$$\mathcal{L}_{\text{int}} = \frac{g}{3!} \phi^3$$

permutation over vertices
($y_1 \leftrightarrow y_2$)

$$\mathcal{Z}[J] = \frac{1}{N} \left\{ 1 + \frac{i g}{3!} \int d^4y \left(\frac{\delta}{i \delta J(y)} \right)^3 + \underbrace{\frac{1}{2} \left(\frac{i g}{3!} \right)^2 \int d^4y_1 d^4y_2 \left(\frac{\delta}{i \delta J(y_1)} \right)^3 \left(\frac{\delta}{i \delta J(y_2)} \right)^3}_{\text{compensates:}} + \dots \right\}$$

$$x \left\{ 1 + \int d^4x \int d^4x' \frac{1}{2} i J(x) i \Delta_F(x-x') i J(x') + \dots \right\} \left(\frac{\delta}{i \delta J(y)} \right)^3 J(x_1) J(x_2) J(x_3)$$

- rules for $G^{\phi \dots \phi} = \frac{\delta}{i \delta J} \dots \frac{\delta}{i \delta J} \mathcal{Z}[J] \Big|_{J=0}$ at $\mathcal{O}(g^N)$: $= 3! \delta(x_1-y) \delta(x_2-y) \delta(x_3-y)$

① propagator  $= i \Delta_F(x-x')$, vertex:  $= i g \int d^4y$

- ② draw all graphs with N vertices and n external legs
(drop all "vacuum bubbles" $\leftrightarrow \mathcal{Z}[\emptyset] = 1$)

- ③ multiply by symmetry factor $\frac{1}{S_g}$; S_g : # of permutations leaving graph inv.

④ momentum space: (all momenta incoming)

①  $= \frac{i}{p^2 - m^2 + i0}$;  $= i g$

- ⑤ momentum cons. @ each vertex; undetermined $\rightarrow \int \frac{d^4p}{(2\pi)^4}$

EX 4

Vertex function

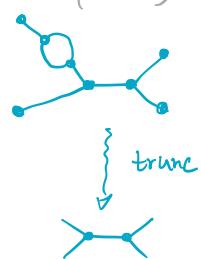
- we saw $G \rightarrow G_{\text{con}}$ (connected) $\rightarrow \text{EX 4}$

- truncated ("amputated") correlation function

$$G_{\text{trunc}} = G_{\text{con}} \quad \left| \begin{array}{l} \text{no external propagators /} \\ \text{propagator corrections} \end{array} \right. \quad (\text{do not draw ext. field points})$$

$$G_{\text{trunc}}^{\Phi_1 \dots \Phi_n}(p_1, \dots, p_n) = G_{\text{con}}^{\Phi_1 \dots \Phi_n}(p_1, \dots, p_n) \cdot \prod_{j=1}^n \left[G^{\Phi_j \Phi_j^{(*)}}(p_j, p_j) \right]^{-1}$$

$$\Rightarrow G_{\text{trunc}}^{\Phi \Phi^{(*)}} = \left[G^{\Phi \Phi} \right]^{-1}$$



- vertex functions elementary building blocks

$$\Gamma^{\Phi_1 \Phi_2}(p_1 - p) := - G_{\text{trunc}}^{\Phi_1 \Phi_2}(p_1 - p) = - \left[G^{\Phi_1 \Phi_2}(p_1 - p) \right]^{-1}$$

$$\Gamma^{\Phi_1 \dots \Phi_n}(p_1, \dots, p_n) := G_{\text{trunc}}^{\Phi_1 \dots \Phi_n}(p_1, \dots, p_n) \quad \left| \begin{array}{l} \text{only 1PI} \end{array} \right.$$

EX 5 & 6

"1PI": 1-particle irred.
no disconnected subgraphs
by cutting one internal line

Källén-Lehmann spectral representation

$$G^{\phi\phi}(x_1, x_2) = \langle \Omega | T\phi(x_1)\phi(x_2) | \Omega \rangle$$

interaction: ϕ also creates multi-particle states!

$$= \dots = \int_0^\infty \frac{d\mu^2}{2\pi} P(\mu^2) \int \frac{d^4 p}{(2\pi)^4} \frac{i e^{-ip(x_1-x_2)}}{p^2 - \mu^2 + i0}$$

• momentum space

$$G^{\phi\phi}(p, -p) = \frac{i R_\phi}{p^2 - m^2 + i0} + \int_{\sim 4m^2}^\infty \frac{d\mu^2}{2\pi} P(\mu^2) \frac{i}{p^2 - \mu^2 + i0}$$

$\hookrightarrow R_\phi$: residuum $\hat{=} |\langle p | \phi | \Omega \rangle|^2$
 (prob. for 1-particle state)

$\hookrightarrow m$: physical mass (in general \neq param. in \mathcal{L})

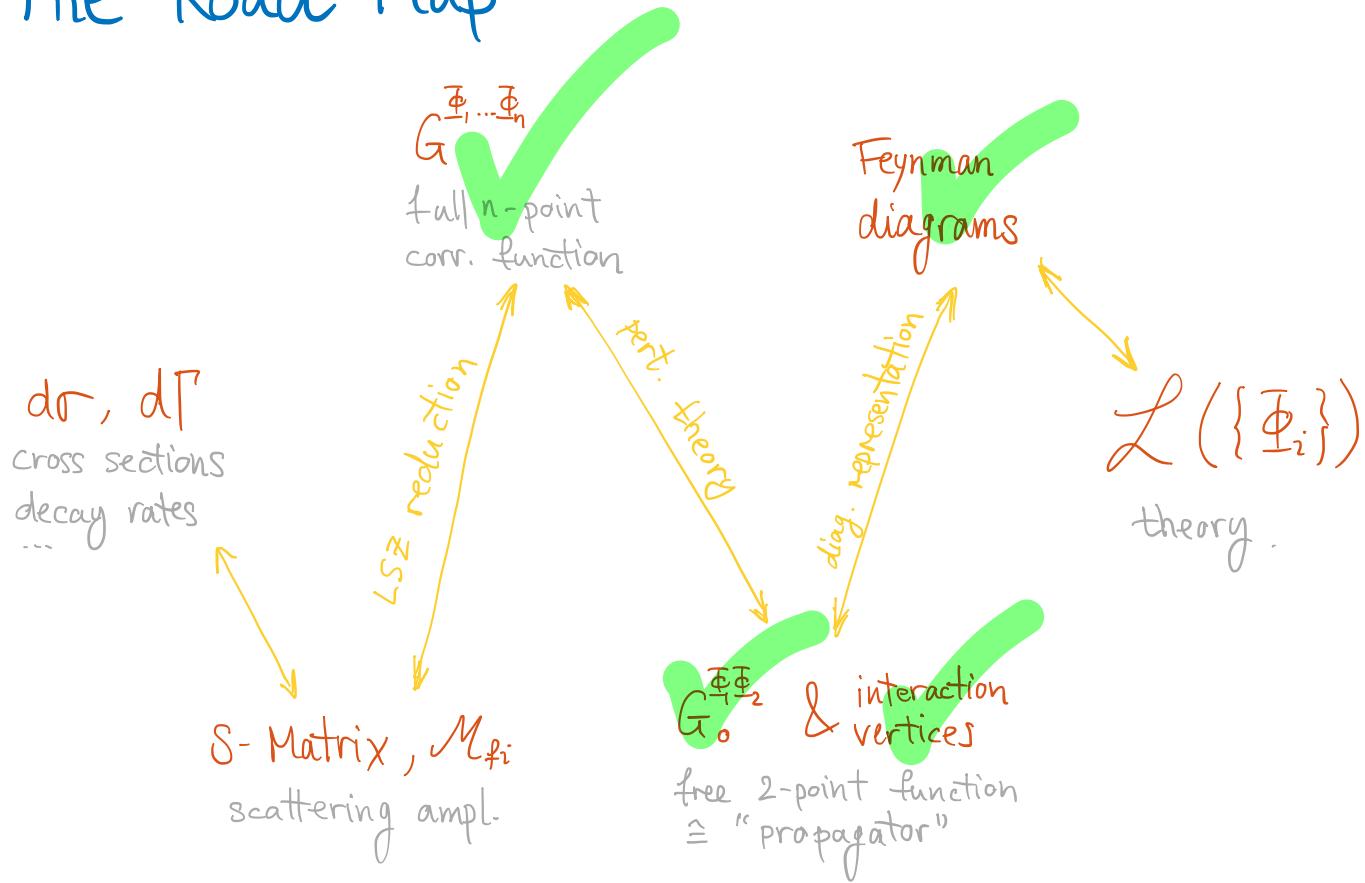
\rightsquigarrow free fields: $R_\phi = 1, m = m_0$

Done with n-pt functions

- G : all diags (no vacuum bubbles)
- G_{con} : only connected
- G_{trunc} : "amputate" all external legs
- Γ : only 1PI graphs of G_{trunc} ($\Gamma^{\phi\phi} = -[G^{\phi\phi}]^{-1}$)

Feynman diagrams & rules to compute them

The Road Map



The S Matrix & LSZ reduction

- central object in scattering & decays: **S matrix**
 ↵ time evolution operator from $t \rightarrow -\infty$ ("in") to $t \rightarrow +\infty$ ("out")

$$S_{fi} = \langle f | S | i \rangle = \langle f | 1 + i T | i \rangle$$

scattering amplitude

$$= \langle f | i \rangle + (2\pi)^4 \delta^4(p_i - p_f) i M_{fi}$$

- Lehmann Symanzik Zimmermann (LSZ) reduction formula:

$$i M^{n \rightarrow m}_{(p_1, \dots, p_n, p'_1, \dots, p'_m)} = \prod_{i=1}^n f_{in}^{\pm_i}(p_i) \sqrt{R_{\pm_i}} \prod_{j=1}^m f_{out}^{\pm'_j}(p'_j) \sqrt{R_{\pm'_j}}$$

$$\times G_{trunc}^{\pm_1 \dots \pm_n \pm'_1 \dots \pm'_m}(p_1, \dots, p_n, -p'_1, \dots, -p'_m)$$

on-shell

↪ truncate ext. legs & replace by 1-particle wave functions

$$f_{in/out}^\phi = 1, \quad f_{in/out}^4 = u(p), v(p), \quad f_{in/out}^{\bar{4}} = \bar{v}(p), \bar{u}(p), \quad f_{in/out}^{p_A} = \epsilon_\mu^{(A)}(p)$$

↪ go on-shell: $p_i^2 = m_i^2, \quad p_j'^2 = m_j'^2$

What we were after all this time

- differential cross section $p_a + p_b \rightarrow p_1 + \dots + p_n \quad (2 \rightarrow n)$

$$d\sigma = \frac{1}{4\sqrt{(p_a \cdot p_b)^2 - m_a^2 m_b^2}} |M_{fi}|^2 \frac{1}{S_{\{n\}}} d\Phi_n(p_1, \dots, p_n; p_a + p_b)$$

$\underbrace{|M_{fi}|^2}_{\text{flux}}$
 $\underbrace{\frac{1}{S_{\{n\}}}}_{\text{symmetry factor}}$
 $\underbrace{d\Phi_n(p_1, \dots, p_n; p_a + p_b)}_{\text{LIPS}}$

$S_{\{n\}} = n_1! \cdot \dots$

- decay width $p_a \rightarrow p_1 + \dots + p_n$

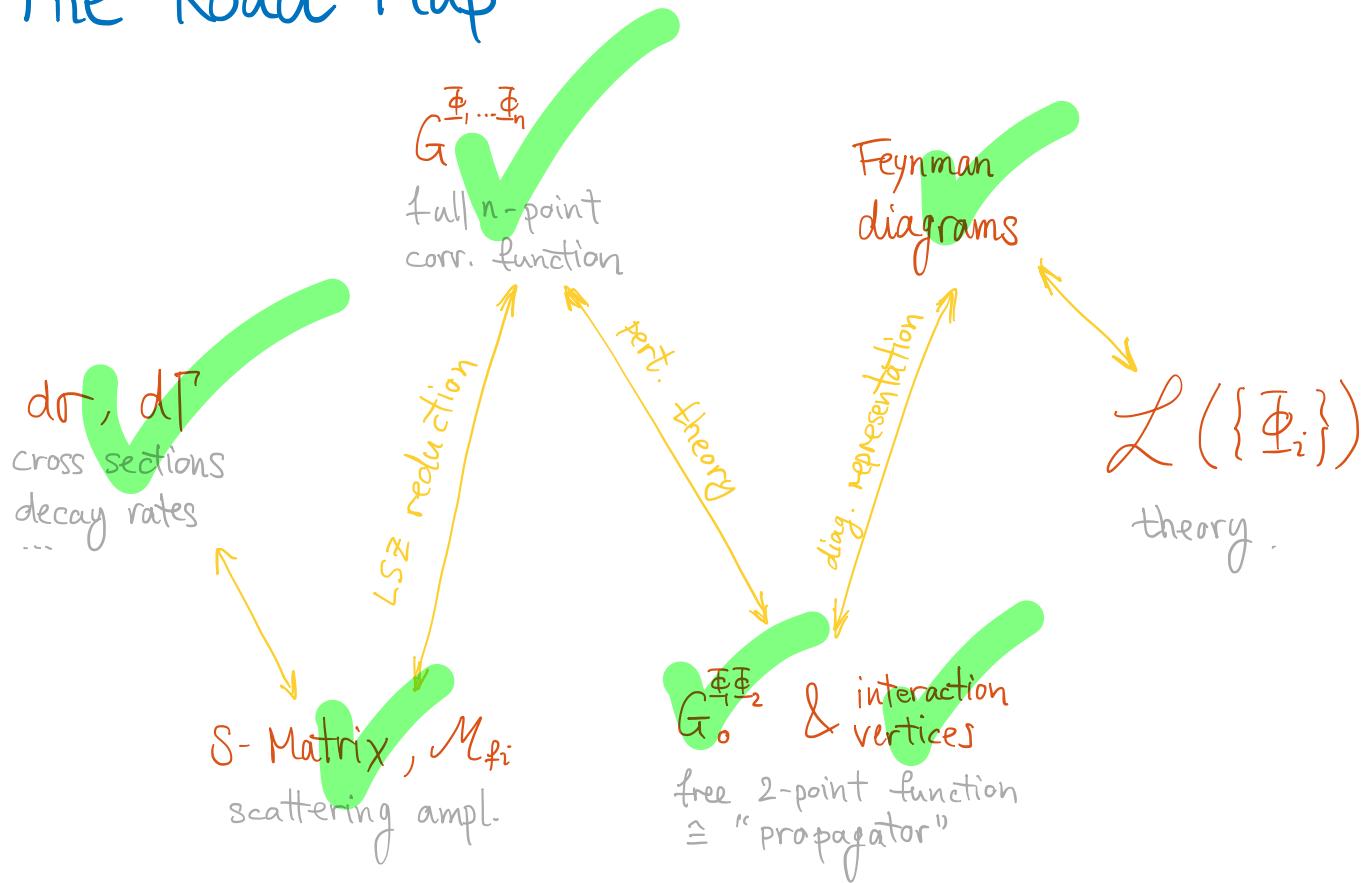
$$d\Gamma = \frac{1}{2E_a} |M_{fi}|^2 \frac{1}{S_{\{n\}}} d\Phi_n(p_1, \dots, p_n; p_a)$$

$|M_{fi}|^2$
 $\frac{1}{S_{\{n\}}}$
 $d\Phi_n(p_1, \dots, p_n; p_a)$

- LIPS

$$\begin{aligned}
 d\Phi_n(p_1, \dots, p_n; Q) &= \prod_{i=1}^n \left[\frac{d^4 p_i}{(2\pi)^3} \cdot \underbrace{\delta^+(p_i^2 - m_i^2)}_{\delta(p_i^2 - m_i^2) \Theta(p_i^0)} \right] \underbrace{(2\pi)^4}_{\text{over-all 4-momentum conservation}} \delta(Q - \sum_{i=1}^n p_i) \\
 &= \prod_{i=1}^n \left[\frac{d^3 p_i}{(2\pi)^3 2E_i} \right] (2\pi)^4 \delta(Q - \sum p_i)
 \end{aligned}$$

The Road Map



Appendix: Vertex Functions @ Tree-Graph Level

- reintroduce \hbar :

$$\begin{aligned} Z[J] &= \frac{1}{N} \int \mathcal{L}[\Phi] \exp \left\{ \frac{i}{\hbar} \int d^4x \left[\mathcal{L} + \hbar J(x) \bar{\Phi}(x) \right] \right\} \\ &= \frac{1}{N} \exp \left\{ \frac{i}{\hbar} \int d^4y \mathcal{Z}_{\text{int}} \left(\frac{\delta}{i \hbar J(y)} \right) \right\} \exp \left\{ \frac{\hbar}{2} \int d^4x d^4x' iJ(x) i\Delta_F(x-x') iJ(x') \right\} \end{aligned}$$

\Rightarrow each vertex (V): \hbar^{-1} , each propagator (I): \hbar^{+1}

\hookrightarrow for vertex graphs (1PI) $L = I - V + 1$

$\Rightarrow \hbar$ expansion $\hat{=}$ loop expansion: $\Gamma = \sum_L \hbar^L \Gamma^{(L)}$

- method of stationary phase:

\Rightarrow classical ($\hbar \rightarrow 0$) solution $\bar{\Phi}_{cl}$ from e.o.m.: $\frac{\delta \mathcal{L}}{\delta \bar{\Phi}} = \partial_\mu \frac{\delta \mathcal{L}}{\delta (\partial_\mu \bar{\Phi})}$

$\Rightarrow Z^{(0)}[J] = \frac{1}{N} \exp \left\{ \frac{i}{\hbar} \int d^4x \mathcal{L}(\bar{\Phi}_{cl}, \partial_\mu \bar{\Phi}_{cl}) \right\} \exp \left\{ i \int d^4y J(y) \bar{\Phi}_{cl}(y) \right\}$

$\Rightarrow Z_{\text{can}}^{(0)}[J] = \ln(Z[J]) = \text{const.} + \frac{i}{\hbar} \int d^4x \mathcal{L}(\bar{\Phi}_{cl}, \partial_\mu \bar{\Phi}_{cl}) + i \int d^4y J(y) \bar{\Phi}_{cl}(y)$

conjugate field: $\frac{\delta Z_{\text{can}}^{(0)}[J]}{i \delta J(x)} = \bar{\Phi}_{cl}(x)$

\Rightarrow Legendre-transf: $\Gamma^{(0)}[\bar{\Phi}_{cl}] = -i \int d^4x \bar{\Phi}_{cl}(x) J(x) + Z_{\text{can}}^{(0)}[J] = \frac{i}{\hbar} \int d^4x \mathcal{L}(\bar{\Phi}_{cl}, \partial_\mu \bar{\Phi}_{cl})$

II.

QED

&

Renormalization

The QED Feynman Rules

$$\mathcal{L}_{QED} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\psi} (i\gamma^\mu - Q e \not{A} - m) \psi - \frac{1}{2\pi} (\partial \cdot A)^2$$

let's take this
for granted ...
→ part III

- propagators:

$$A_\mu \quad A_\nu = \frac{-i}{k^2 + i0} \left[g^{\mu\nu} - (1-\xi) \frac{k^\mu k^\nu}{k^2} \right]$$

$$\bar{\psi} \quad \psi = \frac{i}{k - m + i0}$$

- vertex:

$$= -ie Q_f \gamma^\mu$$

- external legs:

incoming \rightarrow

$$\text{wavy line} = \epsilon_\mu(k, \lambda)$$

$$\rightarrow \text{line} = u(k, \sigma)$$

$$\leftarrow \text{line} = \bar{v}(k, \sigma)$$

outgoing \rightarrow

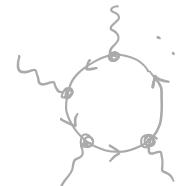
$$\text{wavy line} = \epsilon_\mu^*(k, \lambda)$$

$$\rightarrow \text{line} = \bar{u}(k, \sigma)$$

$$\leftarrow \text{line} = v(k, \sigma)$$

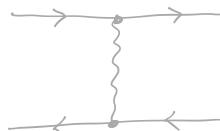
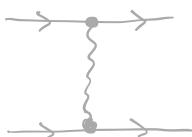
More Rules ...

- momentum conservation @ each vertex
- undetermined (loop) momenta $\rightarrow \frac{\int d^4 p}{(2\pi)^4}$
- traverse fermion lines in opposite direction to the arrow
- symmetry factors $1/S_A$
- (-1) for each closed fermion loop
- relative (-1) between diagrams related by exchange of external fermion legs

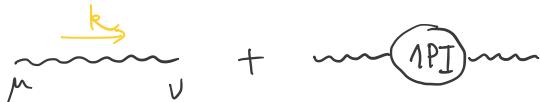


$e^- e^- \rightarrow e^- e^-$

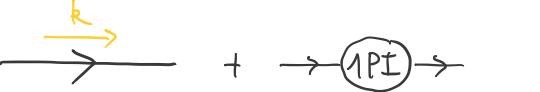
or Bhabha



QED Vertex Functions

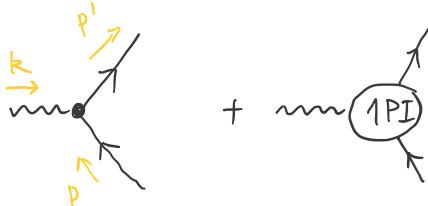
- photon 2-pt: 

$$\Gamma_{\mu\nu}^{AA}(k_1 - k) = -[G_{\mu\nu}^{AA}(k_1 - k)]^{-1} = -i \left[g_{\mu\nu} k^2 - (1 - \frac{1}{\xi}) k_\mu k_\nu \right] - i \sum_{\mu\nu}^{AA}(k)$$

- electron 2-pt: 

$$\Gamma^{\bar{q}q}(-k, k) = -[G^{\bar{q}q}(k_1 - k)]^{-1} = i(k - m) + i \sum^{\bar{q}q}(k)$$

- electron-photon vertex:



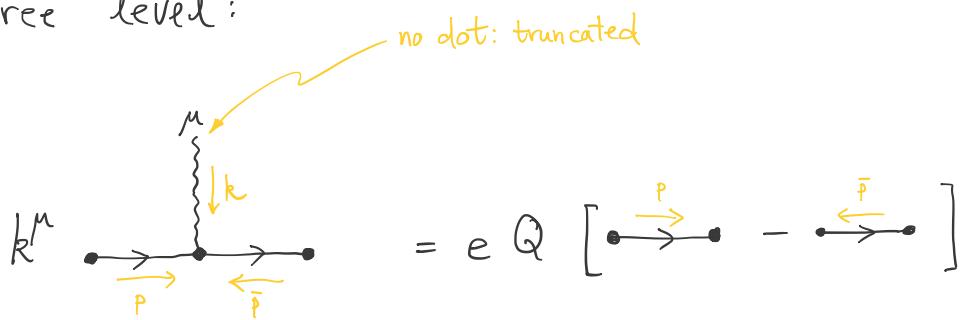
$$\Gamma_\mu^{A\bar{q}q}(k - p^1, p) = -ie Q \gamma_\mu - ie Q \Lambda_\mu(p^1, p)$$

The Ward-Takahashi Identity

- identities between correlation functions due to electromagnetic U(1) symmetry
- Ward identity for the electron-photon vertex:

$$k^\mu G^{4\bar{4}}(\bar{p}, -\bar{p}) \Gamma_\mu^{\bar{4}4}(k, \bar{p}, p) G^{4\bar{4}}(-p, p) = e Q \left[G^{4\bar{4}}(-p, p) - G^{4\bar{4}}(\bar{p}, -\bar{p}) \right]$$

↔ tree level:



EX 7

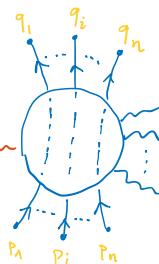
(not using the vertex $\Gamma_\mu^{\bar{4}4}$ but the Green's function)

$$\left(\frac{i}{\hbar} k^2 k^\mu G_\mu^{4\bar{4}}(k, \bar{p}, p) = e Q \left[G^{4\bar{4}}(-p, p) - G^{4\bar{4}}(\bar{p}, -\bar{p}) \right] \right)$$

The Ward-Takahashi Identity

- general amplitude with an external photon:

$$\mathcal{A} = \mathcal{E}_\mu^{(*)}(k) A^\mu$$



$$\Rightarrow k_p A^{\mu} = e Q \sum_{i=1}^n [q_i (P_i - k) q_n] -$$

- for scattering amplitudes (LSZ \leftrightarrow truncation / on-shell residue)

$$\mathcal{M} = \varepsilon^{(*)}_\mu(k) \mathcal{M}^\mu \quad \Rightarrow \quad k_\mu \mathcal{M}^\mu = 0$$

1-Loop Corrections: Photon Self Energy

$$-i \sum_{\mu\nu}^{AA}(k) = \text{[1PI]} + \dots$$

$$\begin{aligned} \Rightarrow \sum_{\mu\nu}^{AA}(k) &= i(-1) \int \frac{d^4 q}{(2\pi)^4} \text{Tr} \left\{ (-ieQ)\gamma_\mu \frac{i}{q-m} (-ieQ)\gamma_\nu \frac{i}{q+k-m} \right\} \\ &= -ie^2 Q^2 \int \frac{d^4 q}{(2\pi)^4} \frac{\text{Tr} [\gamma_\mu (q+m) \gamma_\nu (q+k+m)]}{(q^2 - m^2) ((q+k)^2 - m^2)} \end{aligned}$$

↪ UV behaviour ($q \rightarrow \infty$)

$$\int d^4 q \sim \int dq q^3 ; \quad (q^2 - m^2), ((q+k)^2 - m^2) \sim q^2$$

$$\Rightarrow \sum_{\mu\nu}^{AA} \sim \int dq \left\{ \frac{1}{q}, 1, q \right\} \Rightarrow \text{quadratically divergent!}$$

1-Loop Corrections : Elektron Self Energy

$$i \bar{\Sigma}_l^{\bar{4}4}(k) = \rightarrow \textcircled{1PI} \rightarrow = \text{Feynman diagram} + \dots \quad (\text{Feynman gauge: } \xi = 1)$$

The Feynman diagram shows a loop with an incoming electron line labeled k and an outgoing electron line labeled p . A wavy line labeled q connects the two lines at the loop vertex.

$$\begin{aligned} \Rightarrow \bar{\Sigma}_l^{\bar{4}4}(k) &= (-i) \int \frac{d^4 q}{(2\pi)^4} (-ieQ\gamma_\mu) \frac{i}{q-m} (-ieQ\gamma_\nu) \cdot \frac{-i g^{\mu\nu}}{(q-k)^2 - \lambda^2} \\ &= ie^2 Q^2 \int \frac{d^4 q}{(2\pi)^4} \frac{\gamma_\mu (q+m) \gamma^\nu}{(q^2-m^2)((q-k)^2-\lambda^2)} \end{aligned}$$

\hookleftarrow UV behaviour ($q \rightarrow \infty$)

$$\Rightarrow \bar{\Sigma}_l^{\bar{4}4} \sim \int d^4 q \left\{ \frac{1}{q}, 1 \right\} \Rightarrow \text{linearly divergent!}$$

1-Loop Corrections: Electron-Photon Vertex

$$-ieQ\Lambda_\mu(p', p) = \text{---} \circlearrowleft \text{---} \quad \text{---} \circlearrowright \text{---} \quad = \text{---} \circlearrowleft \text{---} \quad \text{---} \circlearrowright \text{---} + \dots$$

(Feynman gauge: $\xi = 1$)

$$\begin{aligned} \Rightarrow \Lambda_\mu(p', p) &= \frac{i}{eQ} \int \frac{d^4 q}{(2\pi)^4} \frac{-ig^{\alpha\beta}}{q^2 - \lambda^2} (-ieQ\gamma_\alpha) \frac{i}{q+p^1-m} (-ieQ\gamma_\mu) \frac{i}{q+p^2-m} (-ieQ\gamma_\beta) \\ &= -ie^2 Q^2 \int \frac{d^4 q}{(2\pi)^4} \frac{\gamma_\alpha (q+p^1+m) \gamma_\mu (q+p^2-m) \gamma^\alpha}{(q^2 - \lambda^2)((q+p^1)^2 - m^2)((q+p^2)^2 - m^2)} \end{aligned}$$

↔ UV behaviour ($q \rightarrow \infty$)

EX 8

$$\Rightarrow \Lambda_\mu(p', p) \sim \int dq \left\{ \frac{1}{q^3}, \frac{1}{q^2}, \frac{1}{q} \right\} \Rightarrow \text{logarithmically divergent!}$$

Superficially Divergent Vertex Functions

• EX 8 $\Rightarrow \omega(\text{f}) = 4 - \frac{3}{2} E_4 - E_A$



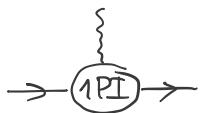
$\omega = 3 \rightarrow = 0$ (quant. number of vacuum)



$\omega = 2$



$\omega = 1$

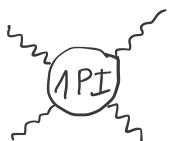


$\omega = 0$

} actually divergent ones



$\omega = 1 \rightarrow = 0$ (Furry's theorem)



$\omega = 0 \rightarrow = \text{finite}$ (Ward Id.) \rightsquigarrow Appendix

(gauge symmetry essential in renormalizability!)

The General Idea

Problem: radiative corrections give infinities (UV)

① regularization: tame the infinities by modifying the theory such that calculations become formally well-defined.

↔ regularization parameter δ for $f \rightarrow f_0$: recover div. expressions

② renormalization: how do we know what λ_0 (bare param) is?

↪ we have to measure it

↪ prescription/observable how to measure it: scheme

↪ once measurable quantities are re-expressed in terms of other measurable quantities $\Rightarrow \delta$ -dependence cancels!

(independent of how it was regularized)

Regularization Schemes

- momentum cut-off: $\int d^4q \rightarrow \int_{|q^0|, |q^i| < \Lambda} d^4q \quad (\delta = \Lambda, \delta_0 = \infty)$
- "Pauli-Villars": $\frac{1}{q^2 - m^2} \rightarrow \frac{1}{q^2 - m^2} - \frac{1}{q^2 - M^2} \quad (\delta = M, \delta_0 = \infty)$
- "Momentum subtraction": ex. $\Gamma(p_h^2, \dots) \rightarrow \Gamma(p_h^2, \dots) - \Gamma(p_h^2 = q_1^2, \dots)$
- put it on the lattice: $\int d^4x \rightarrow \sum_{x_i} \Delta_i \quad (\delta = a, \delta_0 = 0)$
 $x_i \leftarrow$ lattice spacing
- dimensional regularization: $\delta = D, \delta_0 = 4$
⊕ Lorentz- & gauge-invariance, IR regularization, simple

Dimensional Regularization (Dim Reg)

$$\int \frac{d^4 q}{(2\pi)^4} \rightarrow \int \frac{d^D q}{(2\pi)^D} \quad (D = 4 - 2\epsilon)$$

- defined as a set of axioms that integrals should satisfy:
 - shifts of loop momenta always allowed:

$$\int d^D q f(k+p) \equiv \int d^D q f(q)$$

- analog of D-dim. rotational symmetry / Lorentz transformations

$$\int d^D q f(\lambda q) \equiv \int d^D q f(q) \quad \text{scale-less integrals vanish: } \int d^D q (q^2)^\alpha \equiv 0$$

$$\text{scaling property} \quad \int d^D q f(a \cdot q) \equiv a^{-D} \int d^D q f(q)$$

$$\text{linearity: } \int d^D q [f(q) + g(q)] \equiv \int d^D q f(q) + \int d^D q g(q)$$

- differentiation & integration commute:

$$\frac{\partial}{\partial p_\mu} \int d^D q f(p, q) \equiv \int d^D q \frac{\partial}{\partial p_\mu} f(p, q)$$

More on Dim Reg

- metric $\delta_\mu^\nu = g_\mu^\nu = D$
- Dirac matrices : $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} \Rightarrow \gamma^\mu \gamma_\nu = g^\mu_\nu = D$
 $\gamma^\mu \gamma_\nu \gamma_\mu = (2-D) \gamma_\nu, \dots$
 can define $\text{Tr}(\mathbb{1}) = 4$
- no natural extension of γ^5 in D dimensions

$$\text{Tr} [\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma \gamma^5] \sim \epsilon^{\mu\nu\rho\sigma} \quad \text{no need to adapt } \gamma^5 \text{ algebra}$$

- field / coupling dimensions ($[S]=0 \Rightarrow [L]=D$)
 $[A] = \frac{D-1}{2}, [A_\mu] = \frac{D-2}{2}$
 $\Rightarrow [e] = \frac{4-D}{2}; \text{ keep } [e] = 0 \Rightarrow e \rightarrow \mu^{\frac{4-D}{2}} e$ scale of Dim Reg!
 $\omega(\mathfrak{f})$ in D -dim \rightsquigarrow EX 8
 $\omega(\mathfrak{f}) = D - \frac{D-1}{2} E_4 - \frac{D-2}{2} E_A + V\left(\frac{D-4}{2}\right)$

(often absorbed into loop integration)

1-Loop Integrals — The Roadmap

generally confronted with
tensor integrals $T_{\mu_1 \dots \mu_N}^N$

$$\int d^D q \frac{q_{\mu_1} \dots q_{\mu_N}}{\prod_{i=1}^N D_i}$$
$$(D_i = (q + p_i)^2 - m_i^2)$$

merge propagators
to isolate loop integral

$$\int \left(\prod_{i=1}^N dx_i \right) \frac{\delta(1 - \sum x_i)}{[\sum D_i x_i]^N}$$

scalar integrals S^N

$$\int d^D q \frac{1}{\prod_{i=1}^N D_i}$$

Rosenfeld-Veltman reduction

Feynman parameters

tadpole

generic integral
perform $\int d^D q$

$$I_N(A) := \int d^D q [q^2 - A + i0]^{-N}$$

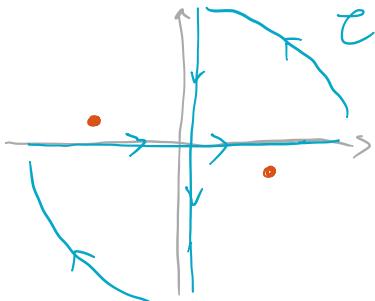
Our First Integral

$$I_n(A) := \int d^D q \frac{1}{(q^2 - A + i0)^n} \quad (\text{D} < 2n, A > 0)$$

convergence

① Wick-rotation in the complex q° plane

↪ poles @ $q^\circ = \pm \sqrt{\vec{q}^2 + A^2 - i0} = \pm \sqrt{\vec{q}^2 + A^2} \mp i0$



$$\oint_C dq^\circ (q^2 - A^2 + i0)^{-n} = 0$$

$$\Rightarrow \int_{-\infty}^{+\infty} dq^\circ (...)^{-n} = \int_{-i\infty}^{i\infty} dq^\circ (...)^{-n} = i \int_{-\infty}^{\infty} dq_E^\circ (...)^{-n}$$

Eucleidian coordinates:
 $q_E^\circ = i q^\circ$

$$(q^2 = (q^\circ)^2 - \vec{q}^2 = -(q_E^\circ)^2 - \vec{q}^2 = -q_E^2)$$

② Integration in polar coordinates:

$$\int d^D q_E = \int d\Omega_D \int_0^\infty dq_E (q_E)^{D-1} = \int d\Omega_D \int_0^\infty dq_E^2 \frac{1}{2} (q_E^2)^{\frac{D}{2}-1}$$

EX 9

Result for $I_n(A)$

$$I_n(A) \stackrel{(y = q^{\frac{z}{2}})}{=} i(-1)^n \frac{\pi^{\frac{D}{2}}}{\Gamma(\frac{D}{2})} (A - i0)^{\frac{D}{2}-n} \int_0^\infty dy y^{\frac{D}{2}-1} (1+y)^{-n}$$

$$B\left(\frac{D}{2}, n - \frac{D}{2}\right)$$

$$B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$$

$$= i(-1)^n \pi^{\frac{D}{2}} \frac{\Gamma(n - \frac{D}{2})}{\Gamma(n)} (A - i0)^{\frac{D}{2}-n}$$

↪ analytic continuation in D & A possible!

- properties of the Gamma function

↪ $\Gamma(z)$ is meromorphic : Poles @ $z = -n$, $n \in \mathbb{N}_0$

↪ $\Gamma(z+1) = z \Gamma(z)$

↪ $\Gamma(n+1) = n!$ $\forall n \in \mathbb{N}$

↪ $\Gamma(\epsilon) = \left(\frac{1}{\epsilon}\right) - \gamma_\epsilon + \mathcal{O}(\epsilon)$ with $\gamma_\epsilon = 0.57\dots$ (Euler const)

UV divergences as $\frac{1}{\epsilon}$ poles

1-Loop Integrals — The Roadmap

generally confronted with
tensor integrals $T_{\mu_1 \dots \mu_N}^N$

$$\int d^D q \frac{q_{\mu_1} \dots q_{\mu_N}}{\prod_{i=1}^N D_i}$$
$$(D_i = (q + p_i)^2 - m_i^2)$$

scalar integrals S^N

$$\int d^D q \frac{1}{\prod_{i=1}^N D_i}$$

merge propagators
to isolate loop integral

$$\int \left(\prod_{i=1}^N dx_i \right) \frac{\delta(1 - \sum x_i)}{[\sum D_i x_i]^N}$$

generic integral
perform $\int d^D q$

$$I_N(A) := \int d^D q [q^2 - A + i\epsilon]^{-N}$$

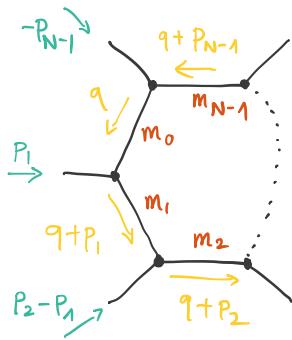
Rosenfeld-Veltman reduction

Feynman parameters

tadpole

Scalar Integrals

$$S^N(p_1, \dots, p_{N-1}, m_0, \dots, m_{N-1}) := \frac{(2\pi\mu)^{4-D}}{i\pi^2} \int d^D q \prod_{n=0}^{N-1} \frac{1}{(q+p_n)^2 - m_n^2 + i0} \quad (p_0 = 0)$$



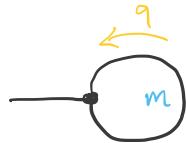
Notation: $S^1 = A_0$ (tadpole)

$S^2 = B_0$ (bubble)

$S^3 = C_0$ (triangle)
⋮

- all scalar integrals can be brought into the form of $I_n(A)$
↪ D-dimensional momentum integration ✓

1-pt Function

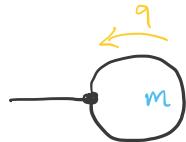


$$A_0(m) = \frac{(2\pi\mu)^{4-D}}{i\pi^2} \underbrace{\int d^D q}_{\text{I}_1(m^2)} (q^2 - m^2 + i0)^{-1}$$

$$= -m^2 \left(\frac{m^2}{4\pi\mu^2} \right)^{\frac{D-4}{2}} \frac{\Gamma(\frac{2-D}{2})}{\Gamma(\frac{2-D}{2})}$$

EX10

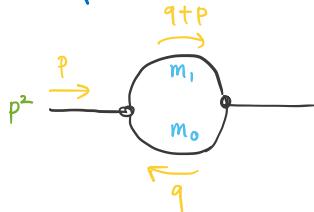
1-pt Function



$$\begin{aligned}
 A_0(m) &= \frac{(2\pi\mu)^{4-D}}{i\pi^2} \underbrace{\int d^D q (q^2 - m^2 + i0)^{-1}}_{I_1(m^2)} \\
 &= -m^2 \left(\frac{m^2}{4\pi\mu^2} \right)^{\frac{D-4}{2}} \Gamma\left(\frac{2-D}{2}\right) \\
 &= m^2 \left[\frac{1}{\epsilon} - \gamma_E + \ln(4\pi) - \ln\left(\frac{m^2}{\mu^2}\right) + 1 \right] + \mathcal{O}(\epsilon) \\
 &\quad \underbrace{=} \Delta \quad (\text{divergent constant}) \quad \text{and} \quad \overline{\text{MS}} \text{ scheme}
 \end{aligned}$$

- note: $A_0(m)|_{\text{div}} = \frac{m^2}{\epsilon}$

2-pt Function



$$B_0(p, m_0, m_1) = \frac{(2\pi i)^{4-D}}{i\pi^2} \int d^D q \left(q^2 - m_0^2 + i0 \right)^{-1} \left((q+p)^2 - m_1^2 + i0 \right)^{-1}$$

$\underbrace{\hspace{10em}}$

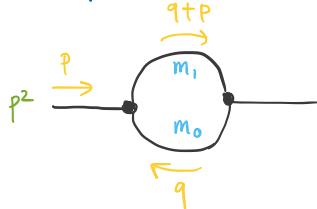
$$\frac{1}{a} \cdot \frac{1}{b}$$

- to reduce S^N to $I_n(A)$ we use Feynman parameters

$$\frac{1}{A_1 \cdot \dots \cdot A_N} = \Gamma(N) \int_0^1 \prod_{i=1}^N dx_i \frac{\delta(1 - \sum_{i=1}^N x_i)}{\left[\sum_{i=1}^N x_i A_i \right]^N}$$

EX 11

2-pt Function



$$\begin{aligned}
 B_o(p, m_0, m_1) &= \frac{(2\pi\mu)^{4-D}}{i\pi^2} \int d^D q \underbrace{\left(q^2 - m_0^2 + i0\right)^{-1}}_{(Feynman param)} \underbrace{\left((q+p)^2 - m_1^2 + i0\right)^{-1}}_{(Feynman param)} \\
 &= \int_0^1 dx \left\{ (q^2 - m_0^2 + i0)(1-x) + [(q+p)^2 - m_1^2 + i0]x \right\}^{-2} \\
 &= \int_0^1 dx \left\{ (q+xp)^2 - x^2 p^2 + x(p^2 - m_1^2 + m_0^2) - m_0^2 + i0 \right\}^{-2} \\
 &\quad =: -A
 \end{aligned}$$

$$\begin{aligned}
 \Rightarrow B_o(p, m_0, m_1) &= \frac{(2\pi\mu)^{4-D}}{i\pi} \int_0^1 dx I_2(A) = (4\pi/\mu^2)^{\frac{4-D}{2}} \Gamma\left(\frac{4-D}{2}\right) \int_0^1 dx \left[x^2 p^2 + x(p^2 - m_1^2 + m_0^2) + m_0^2 - i0 \right]^{\frac{D-4}{2}} \\
 &= \Delta - \int_0^1 dx \ln \left[\frac{x^2 p^2 - x(p^2 - m_1^2 + m_0^2) + m_0^2 - i0}{\mu^2} \right] + \theta(\epsilon)
 \end{aligned}$$

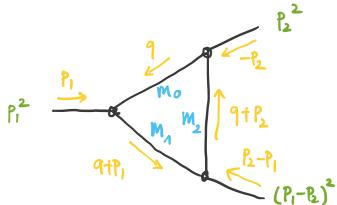
- note: $B_o(p, m_0, m_1) \Big|_{\text{div}} = \frac{1}{\epsilon}$

- alternative notation: $B_o(p^2, m_0, m_1)$

- symmetry: $B_o(p^2, m_0, m_1) = B_o(p^2, m_1, m_0)$

EX 12

3-pt Function



finite in $D=4$ dimensions!

$$C_0(p_1, p_2, m_0, m_1, m_2) = \frac{1}{i\pi^2} \int d^4q \underbrace{\left[q^2 - m_0^2 + i0 \right]^{-1}}_{(Feynman param)} \underbrace{\left[(q+p_1)^2 - m_1^2 + i0 \right]^{-1}}_{(Feynman param)} \underbrace{\left[(q+p_2)^2 - m_2^2 + i0 \right]^{-1}}_{(Feynman param)}$$

... complete the square in q , shift momentum $q \rightarrow q + \dots$, identify $I_3(A)$...

$$\Rightarrow C_0(p_1, p_2, m_0, m_1, m_2) = - \int_0^1 dx \int_0^{1-x} dy \left[x^2 p_1^2 + y^2 p_2^2 + 2xy p_1 p_2 - x(p_1 - m_1^2 + m_0^2) - y(p_2^2 - m_2^2 + m_0^2) + m_0^2 + i0 \right]^{-1}$$

- steps:
 - linearize in x or y : $x \rightarrow x + \alpha y$ to kill all y^2 terms $\rightsquigarrow y$ -integration
 - decompose quad. forms into lin. factors $\rightsquigarrow \int dx \frac{\ln(ax+b)}{cx+d} \rightsquigarrow \ln, \ln^2, \text{Li}_2$

• note : $C_0(p_1, p_2, m_0, m_1, m_2) \Big|_{\text{div}} = 0$

• alternative notation : $C_0(p_1^2, (p_1-p_2)^2, p_2^2, m_0, m_1, m_2)$

1-Loop Integrals — The Roadmap

generally confronted with
tensor integrals $T_{\mu_1 \dots \mu_N}^N$

$$\int d^D q \frac{q_{\mu_1} \dots q_{\mu_N}}{\prod_{i=1}^N D_i}$$
$$(D_i = (q + p_i)^2 - m_i^2)$$

merge propagators
to isolate loop integral

$$\int \left(\prod_{i=1}^N dx_i \right) \frac{\delta(1 - \sum x_i)}{[\sum D_i x_i]^N}$$

scalar integrals S^N

$$\int d^D q \frac{1}{\prod_{i=1}^N D_i}$$

Rosenfeld-Veltman reduction

Feynman parameters

tadpole

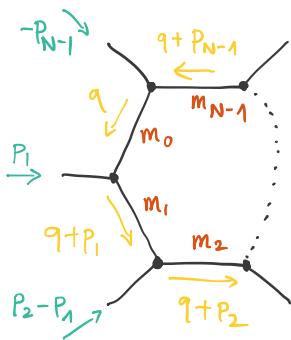
generic integral

perform $\int d^D q$

$$I_N(A) := \int d^D q [q^2 - A + i\epsilon]^{-N}$$

Tensor Integrals

$$T_{\mu_1 \dots \mu_M}^N(p_1, \dots, p_{N-1}, m_0, \dots, m_{N-1}) := \frac{(2\pi\mu)^{4-0}}{i\pi^2} \int d^D q \, q_{\mu_1} \dots q_{\mu_M} \prod_{n=0}^{N-1} \frac{1}{(q+p_n)^2 - m_n^2 + i0} \quad (p_0 = 0)$$



Notation: $T^1 = S^1 = A_0$

$$T^2 = S^2 = B_0, \quad T_\mu^2 = B_\mu, \quad T_{\mu\nu}^2 = B_{\mu\nu}$$

$$T^3 = S^3 = C_0, \quad T_\mu^3 = C_\mu, \quad T_{\mu\nu}^3 = C_{\mu\nu}$$

- symmetric in $\mu_1 \dots \mu_M$, Lorentz-covariance

\Rightarrow decomposition into symmetric tensors of degree M
example:

$$C_{\mu\nu}(p_1, p_2, m_0, m_1, m_2) = g_{\mu\nu} C_{00} + p_{1\mu} p_{1\nu} C_{11} + p_{2\mu} p_{2\nu} C_{22} + (p_{1\mu} p_{2\nu} + p_{2\mu} p_{1\nu}) C_{12}$$

scalar!

Passatino - Veltman Reduction

- algebraic reduction of tensor coefficients $B_1, B_{11}, B_{00}, C_1, \dots$
to scalar integrals A_0, B_0, C_0, \dots
- recursive algorithm: $(T_{\mu_1 \dots \mu_N}^N)$

① contraction of integral representation & tensor decomposition
with external momenta p_i^μ and the metric $g^{\mu\nu}$

↪ in the integrand:

$$\begin{aligned} p_i^\mu q_\mu &= \frac{1}{2} [(q+p_i)^2 - q^2 - p_i^2] = \underbrace{\frac{1}{2} [(q+p_i) - m_i^2]}_{\text{kill prop. } i} - \underbrace{\frac{1}{2} [q^2 - m_0^2]}_{(N-1)\text{-pt integrals}} - \underbrace{\frac{1}{2} (p_i^2 - m_i^2 + m_0^2)}_{\text{rank } (M-1)} \\ g^{\mu\nu} q_\mu q_\nu &= \underbrace{[q^2 - m_0^2]}_{(N-1)\text{-pt}} + \underbrace{m_0^2}_{\text{rank } (M-2)} \end{aligned}$$

↪ in the tensor decomposition \rightsquigarrow linear combination of tensor coefficients

② solve the system of linear equations for $T_{ij\dots}^N$

2-pt Function

- short-hand notation $\langle \dots \rangle_q \equiv \frac{(2\pi\mu)^{D-4}}{i\pi^2} \int d^D q (\dots)$
- rank 1: $B_\mu(p, m_0, m_1) = \left\langle \frac{q_\mu}{(q^2 - m_0^2)[(q+p)^2 - m_1^2]} \right\rangle_q = p_\mu B_1(p^2, m_0, m_1)$
 \hookrightarrow contraction with p^μ

$$\Rightarrow p^2 B_1 = \left\langle \frac{p \cdot q}{(q^2 - m_0^2)[(q+p)^2 - m_1^2]} \right\rangle_q$$

$$= \left\langle \frac{\frac{1}{2}[(q+p)^2 - m_1^2] - \frac{1}{2}[q^2 - m_0^2] - \frac{1}{2}(p^2 - m_1^2 + m_0^2)}{(q^2 - m_0^2)[(q+p)^2 - m_1^2]} \right\rangle_q$$

$$= \frac{1}{2} \left\langle \frac{1}{q^2 - m_0^2} \right\rangle_q - \frac{1}{2} \left\langle \frac{1}{(q+p)^2 - m_1^2} \right\rangle_q - \frac{1}{2} \left\langle \frac{p^2 - m_1^2 + m_0^2}{(q^2 - m_0^2)[(q+p)^2 - m_1^2]} \right\rangle_q$$

$$= \frac{1}{2} A_0(m_0) - \frac{1}{2} A_0(m_1) - \frac{1}{2} (p^2 - m_1^2 + m_0^2) B_0$$
- $\Rightarrow B_1 = \frac{1}{2p^2} [A_0(m_0) - A_0(m_1) - (p^2 - m_1^2 + m_0^2) B_0]$
- \hookrightarrow note: $B_1(p^2, m_0, m_1) \Big|_{\text{div}} = \frac{1}{2p^2} \left[\frac{m_0^2}{\epsilon} - \frac{m_1^2}{\epsilon} - (p^2 - m_1^2 + m_0^2) \frac{1}{\epsilon} \right] = -\frac{1}{2\epsilon}$

2-pt Function

• rank 2 : $B_{\mu\nu}(p, m_0, m_1) = \left\langle \frac{q_\mu q_\nu}{(q^2 - m_0^2)[(q+p)^2 - m_1^2]} \right\rangle_q = g_{\mu\nu} B_{00} + p_\mu p_\nu B_{11}$

↪ contraction with $g^{\mu\nu}$

$$\Rightarrow D B_{00} + P^2 B_{11} = \left\langle \frac{q^2 - m_0^2 + m_0^2}{(q^2 - m_0^2)[(q+p)^2 - m_1^2]} \right\rangle_q = A_0(m_1) + m_0^2 B_0$$

↪ contraction with p^μ

$$\Rightarrow P_\nu (B_{00} + P^2 B_{11}) = \left\langle \frac{q_\nu \left\{ \frac{1}{2} [(q+p)^2 - m_1^2] - \frac{1}{2} [q^2 - m_0^2] - \frac{1}{2} (P^2 - m_1^2 + m_0^2) \right\}}{(q^2 - m_0^2)[(q+p)^2 - m_1^2]} \right\rangle_q$$

$$= \underbrace{\frac{1}{2} \left\langle \frac{q_\nu}{q^2 - m_0^2} \right\rangle_q}_0 - \underbrace{\frac{1}{2} \left\langle \frac{q_\nu}{(q+p)^2 - m_1^2} \right\rangle_q}_{\left\langle \frac{q'_\nu - P_\nu}{(q')^2 - m_1^2} \right\rangle_{q'} = -P_\nu A_0(m_1)} - \underbrace{\frac{1}{2} (P^2 - m_1^2 + m_0^2) B_\nu}_{P_\nu B_1}$$

$$= P_\nu \left[\frac{1}{2} A_0(m_1) - \frac{1}{2} (P^2 - m_1^2 + m_0^2) B_1 \right]$$

EX13

3-pt Function

- rank 1: $C_\mu(p_1, p_2, m_0, m_1, m_2) = p_{1\mu} C_1 + p_{2\mu} C_2$

↪ contraction with p_1^μ

$$\Rightarrow p_1^2 C_1 + p_1 \cdot p_2 C_2 = \frac{1}{2} B_0(p_2^2, m_0, m_2) - \frac{1}{2} B_0((p_1-p_2)^2, m_1, m_2) - \frac{1}{2} (p_1^2 - m_1^2 + m_0^2) C_0$$

↪ contraction with p_2^μ : as above, $1 \leftrightarrow 2$

$$f_i = p_i^2 - m_i^2 + m_0$$

$$\Rightarrow \begin{pmatrix} p_1^2 & p_1 \cdot p_2 \\ p_2 \cdot p_1 & p_2^2 \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} B_0(p_2^2, m_0, m_2) - \frac{1}{2} B_0((p_1-p_2)^2, m_1, m_2) - \frac{1}{2} f_1 C_0 \\ \frac{1}{2} B_0(p_1^2, m_0, m_1) - \frac{1}{2} B_0((p_1-p_2)^2, m_1, m_2) - \frac{1}{2} f_2 C_0 \end{pmatrix}$$

$$\Rightarrow C_1, C_2 = \dots$$

↪ note: $C_1 \Big|_{\text{div}} = C_2 \Big|_{\text{div}} = 0$

1-Loop Integrals — The Roadmap

generally confronted with
tensor integrals $T_{\mu_1 \dots \mu_N}^N$

$$\int d^D q \frac{q_{\mu_1} \dots q_{\mu_N}}{\prod_{i=1}^N D_i}$$
$$(D_i = (q + p_i)^2 - m_i^2)$$

merge propagators
to isolate loop integral

$$\int \left(\prod_{i=1}^N dx_i \right) \frac{\delta(1 - \sum x_i)}{[\sum D_i x_i]^N}$$

scalar integrals S^N

$$\int d^D q \frac{1}{\prod_{i=1}^N D_i}$$

Rosenfeld-Veltman reduction

Feynman parameters

tadpole

generic integral
perform $\int d^D q$

$$I_N(A) := \int d^D q [q^2 - A + i\epsilon]^{-N}$$

A Bird's Eye View

- ① QED Ward identities

$$k^\mu \sum_{\mu\nu}^{AA}(k) = 0$$

$$k^\mu \Lambda_\mu(p', p) = \sum^{\bar{4}\bar{4}}(p') - \sum^{\bar{4}\bar{4}}(p)$$

- ② Study UV divergent behaviour (superficial degree of divergence)

our problem childs: $\textcircled{1PI}$, $\rightarrow \textcircled{1PI}$, $\textcircled{1PI}$ ↗ $\textcircled{1PI}$ ↗
($\textcircled{1PI}$) ↗
⇒ Appendix

- ③ Let's get a handle on infinities

dimensional regularization & 1-loop integrals

- ④ compute $\textcircled{1PI}$, $\rightarrow \textcircled{1PI}$, $\textcircled{1PI}$ ↗ in DimReg

- ⑤ renormalize the theory

- ⑥ the on-shell scheme

REGULARIZATION

RENORMALIZATION

1-Loop Corrections: Photon Self Energy

$$-i \sum_{\mu\nu}^{AA}(k) = \text{[1PI]} = \text{[loop diagram with } k \text{ and } q \text{]} + \mathcal{O}(\alpha^2)$$

$$\Rightarrow \sum_{\mu\nu}^{AA}(k) = -ie^2 Q^2 \int \frac{d^D q}{(2\pi)^D} \frac{\text{Tr} [\gamma_\mu (q+m) \gamma_\nu (q+k+m)]}{(q^2-m^2) ((q+k)^2-m^2)} \quad (\text{Dim Reg})$$

- trace in the numerator:

$$\begin{aligned}
 & \text{Tr} [\gamma_\mu (q+m) \gamma_\nu (q+k+m)] \quad (\text{only even # of } \gamma \text{'s}) \\
 &= \underbrace{\text{Tr} [\gamma_\mu q \gamma_\nu (q+k)]}_{4[-g_{\mu\nu} q \cdot (q+k) + 2q_\mu q_\nu + q_\mu k_\nu + k_\mu q_\nu]} + m^2 \underbrace{\text{Tr} [\gamma_\mu \gamma_\nu]}_{4g_{\mu\nu}} \quad (\text{Tr} [\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma] = 4(g^{\mu\nu} g^{\rho\sigma} - g^{\mu\rho} g^{\nu\sigma} + g^{\mu\sigma} g^{\nu\rho})) \\
 &= -2g_{\mu\nu} \left\{ [q^2 - m^2] + [(q+k)^2 - m^2] - k^2 \right\} + 8q_\mu q_\nu + 4q_\mu k_\nu + 4k_\mu q_\nu \\
 &\quad \text{tells a propagator} \quad \text{tells a propagator}
 \end{aligned}$$

1-Loop Corrections: Photon Self Energy

$$\Rightarrow \sum_{\mu\nu}^{AA}(k) = \frac{\alpha}{2\pi} Q^2 \left\{ -g_{\mu\nu} [A_0(m) + A_0(m) - k^2 B_0] + 4 B_{\mu\nu} + 2 k_\mu B_\nu + 2 k_\nu B_\mu \right\} \quad (\alpha = \frac{e^2}{4\pi})$$

- tensor decomposition: $B_{\mu\nu} = g_{\mu\nu} B_{00} + k_\mu k_\nu B_{11}$, $B_m = k_\mu B_\mu$

fine structure
constant $\sim 1/137$

$$\Rightarrow \sum_{\mu\nu}^{AA}(k) = \frac{\alpha}{2\pi} Q^2 \left\{ g_{\mu\nu} [k^2 B_0 - 2A_0 + 4B_{00}] + 4 k_\mu k_\nu [B_1 + B_1] \right\}$$

- reduction to scalar integrals:

$$B_1(k^2, m, m) = -\frac{1}{2} B_0$$

$$B_{00}(k^2, m, m) = \frac{1}{6} [A_0(m) + 2m^2 B_0 + k^2 B_1 + 2m^2 - \frac{k^2}{3}] + \theta(\epsilon)$$

$$B_{11}(k^2, m, m) = \frac{1}{6k^2} [2A_0(m) - 2m^2 B_0 - 4k^2 B_1 - 2m^2 + \frac{k^2}{3}] + \theta(\epsilon)$$

$$\begin{aligned} B_0(0, m, m) &= \frac{\partial A_0(m)}{\partial m^2} \\ &= (1-\epsilon) \frac{A_0(m)}{m^2} \\ \Rightarrow A_0(m) &= m^2 [B_0(0, m, m) + 1] + \theta(\epsilon) \end{aligned}$$

- decomposition into transversal & longitudinal parts:

$$\sum_{\mu\nu}^{AA}(k) = \left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \right) \sum_T^{AA}(k^2) + \frac{k_\mu k_\nu}{k^2} \sum_L^{AA}(k^2)$$

Scalars

EX 14

1-Loop Corrections: Photon Self Energy

$$-i \sum_{\mu\nu}^{AA}(k) = \text{propagator} \circ \text{(1PI)} \circ = \text{propagator} \circ \text{loop diagram} + \mathcal{O}(\alpha^2)$$

$$\Rightarrow \sum_{\mu\nu}^{AA}(k) = \left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \right) \sum_T^{AA}(k^2) + \frac{k_\mu k_\nu}{k^2} \sum_L^{AA}(k^2)$$

$$\hookrightarrow \sum_T^{AA}(k^2) = \frac{\alpha}{3\pi} \left[k^2 B_0(k^2, m_1 m) + 2m \left(B_0(k^2, m_1 m) - B_0(0, m_1 m) \right) - \frac{k^2}{3} \right]$$

$$\hookrightarrow \sum_L^{AA}(k^2) = 0$$

- $\sum_L^{AA}(k^2) = 0$ to all orders due to Ward Identity ($\leadsto \text{EX7: } k^\mu \sum_{\mu\nu}^{AA}(k) = 0$)
- UV divergence: $\sum_T^{AA}(k^2) \Big|_{\text{div}} = \frac{\alpha}{3\pi} k^2 \frac{1}{\epsilon}$
- propagator ("vacuum polarization" $\Pi^{AA}(k^2) := \frac{\sum_T^{AA}(k^2)}{k^2}$)

$$G_{\mu\nu}^{AA}(k, -k) = \frac{-i}{k^2 [1 + \Pi^{AA}(k^2)]} \left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \right) - \frac{i}{3} \frac{k_\mu k_\nu}{k^2}$$

no shift in the pole @ $k^2 = 0 \leftrightarrow$ no photon mass!

EX15

1-Loop Corrections : Elektron Self Energy

$$i \sum^{\bar{4}4}(k) = \rightarrow (1PI) \rightarrow = \xrightarrow{k} \text{---} \xrightarrow{q} + \mathcal{O}(\alpha^2) \quad (\text{Feynman gauge: } \xi = 1)$$

$$\Rightarrow \sum^{\bar{4}4}(k) = k \sum_V^{\bar{4}4}(k^2) + m \sum_S^{\bar{4}4}(k^2)$$

$$\hookleftarrow \sum_V^{\bar{4}4}(k^2) = -\frac{\alpha}{4\pi} Q^2 \frac{1}{k^2} [A_0(m) - (k^2 + m^2) B_0(k^2, m, 0) + k^2]$$

$$\hookleftarrow \sum_S^{\bar{4}4}(k^2) = -\frac{\alpha}{4\pi} Q^2 [4B_0(k^2, m, 0) - 2]$$

- propagator

$$G^{\bar{4}\bar{4}}(-p, p) = \frac{i}{p - m + \sum^{\bar{4}4}(p)} = i \frac{p [1 + \sum_V^{\bar{4}4}(p^2)] + m [1 - \sum_S^{\bar{4}4}(p^2)]}{p^2 - m^2 \left[\frac{1 - \sum_S^{\bar{4}4}(p^2)}{1 + \sum_V^{\bar{4}4}(p^2)} \right]} [1 + \sum_V^{\bar{4}4}(p^2)]^2$$

- UV divergence :

$$\sum_V^{\bar{4}4}(k^2) \Big|_{\text{div}} = -\frac{\alpha}{4\pi} Q^2 \frac{1}{k^2} \left[\frac{m^2}{\epsilon} - \frac{(k^2 + m^2)}{\epsilon} \right] = \frac{\alpha}{4\pi} Q^2 \frac{1}{\epsilon}$$

$$\sum_S^{\bar{4}4}(k^2) \Big|_{\text{div}} = -\frac{\alpha}{4\pi} Q^2 4 \frac{1}{\epsilon} = -\frac{\alpha}{\pi} Q^2 \frac{1}{\epsilon}$$

radiative corrections shift
the pole @ $p^2 = m^2$

1-Loop Corrections: Electron-Photon Vertex

$$-ieQ \Lambda_\mu(p', p) = \text{---} \circled{1\text{PI}} \text{---} = \text{---} \xrightarrow{\text{---}} q \xleftarrow{\text{---}} q+p' \quad + \mathcal{O}(\alpha^2)$$

(Feynman gauge: $\xi = 1$)

$$\Rightarrow \Lambda_\mu(p', p) = \frac{\alpha}{4\pi} Q^2 \frac{(2\pi/\lambda)^{4-D}}{i\pi^2} \int d^D q \frac{\gamma_\mu (q+p')^\lambda + m \gamma_\mu (q+p'-m) \gamma^\lambda}{(q^2 - \lambda^2) ((q+p')^2 - m^2) ((q+p)^2 - m^2)}$$

- write in terms of $C_0, C_\mu, C_{\mu\nu}$ & reduce to A_0, B_0, C_0
 \hookrightarrow lengthly & quite complicated result \leadsto e.g. use computer algebra
- Ward identity: $k^\mu \Lambda_\mu(p', p) = \sum \bar{\psi}^\mu(p') - \sum \bar{\psi}^\mu(p) \Rightarrow \Lambda_\mu(p', p) = \underbrace{\frac{\partial}{\partial p^\mu} \sum \bar{\psi}^\mu(p)}_{\text{photon momentum } k \rightarrow 0}$
- UV divergence:

$$\Lambda_\mu(p', p) \Big|_{\text{div}} = \Lambda_\mu(p, p) \Big|_{\text{div}} = \frac{\partial \sum \bar{\psi}^\mu(p)}{\partial p^\mu} \Big|_{\text{div}} = \frac{\partial}{\partial p^\mu} \left\{ p \frac{\alpha}{4\pi} \frac{1}{e} - m \frac{\alpha}{\pi} \frac{1}{e} \right\}$$

 $= \gamma_\mu \frac{\alpha}{4\pi} \frac{1}{e}$ \leftarrow proportional to lowest order!

EX 16

Renormalization: Introduction

- from here on: add suffixes " \circ " to indicate "bare" quantities

$\hookrightarrow \mathcal{L}(\psi_0, A_0^M, m_0, e_0)$: Do m_0 & e_0 correspond to the physical mass & charge?

- the electron mass

\hookrightarrow tree level: $G_0^{\psi\bar{\psi}}(-p, p) = \frac{i}{p - m_0} = \frac{i(p + m_0)}{p^2 - m_0^2}$ $\Rightarrow m_{\text{pole}} @ m_0$

\hookrightarrow rad. corr.: $G^{\psi\bar{\psi}}(-p, p) = \frac{i}{p - m_0 + \Sigma^{\bar{\psi}\psi}(p)}$ $\Rightarrow m_{\text{pole}} @ m_0 \left[\frac{1 - Z_s^{\bar{\psi}\psi}}{1 + \Sigma_v^{\bar{\psi}\psi}} \right]$

- similarly for the charge

Conclusion: radiative corrections have an impact on the physical interpretation of the bare quantities!

\Rightarrow redefinition, "renormalization" necessary!

Renormalization: General Idea

- compute n physical observables: $\text{Obs}_i^{\text{th.}}(m_0, e_0)$ divergent expressions in terms of m_0, e_0
- choose two (independent) observables to express m_0, e_0 in terms of the measured quantities (\leftrightarrow input parameter scheme)

$$\left. \begin{array}{l} \text{Obs}_1^{\text{exp.}} \stackrel{!}{=} \text{Obs}_1^{\text{th.}}(m_0, e_0) \\ \text{Obs}_2^{\text{exp.}} \stackrel{!}{=} \text{Obs}_2^{\text{th.}}(m_0, e_0) \end{array} \right\} \Rightarrow \begin{array}{l} m_0 \left(\text{Obs}_1^{\text{exp.}}, \text{Obs}_2^{\text{exp.}} \right) \\ e_0 \left(\text{Obs}_1^{\text{exp.}}, \text{Obs}_2^{\text{exp.}} \right) \end{array}$$

divergent expressions in terms of $\text{Obs}_{1/2}^{\text{exp.}}$

finite, obviously

- predictions for remaining $(n-2)$ observables

$$\text{Obs}_i^{\text{th.}} \left(m_0 \left(\text{Obs}_1^{\text{exp.}}, \text{Obs}_2^{\text{exp.}} \right), e_0 \left(\text{Obs}_1^{\text{exp.}}, \text{Obs}_2^{\text{exp.}} \right) \right) = \text{Obs}_i^{\text{th.}} \left(\text{Obs}_1^{\text{exp.}}, \text{Obs}_2^{\text{exp.}} \right)$$

input parameters

$\Rightarrow \text{Obs}_i^{\text{th.}}$ in terms of $\text{Obs}_{1/2}^{\text{exp.}}$: FINITE!

Multiplicative Renormalization

- let's make the procedure from the previous slide more systematic
- write (divergent) bare quantities $\psi_0, A_0^{\mu}, m_0, e_0, \xi_0$ in terms of (finite) renormalized quantities times (divergent) renormalization constants

$$\psi_0 = \sqrt{z_4} \psi, \quad e_0 = z_e e, \quad \xi_0 = z_{\xi} \xi$$

$$A_0^{\mu} = \sqrt{z_A} A^{\mu}, \quad m_0 = z_m m,$$

- in perturbation theory:

$$z_x = 1 + \alpha \left(\underbrace{\frac{c_{-1}^x}{e}}_{\text{divergent part}} + \underbrace{c_0^x}_{\text{finite part}} \right) + \mathcal{O}(\alpha^2)$$

"free" choice:
fixed by renormalization
conditions

divergent part: $\underbrace{c_{-1}^x}_{\text{uniquely determined}}$

$$\Rightarrow \sqrt{z_4} \psi = (1 + \frac{1}{2} \delta z_4) \psi, \quad z_e e = (1 + \delta z_e) e, \quad z_{\xi} \xi = (1 + \delta z_{\xi}) \xi$$

$$\sqrt{z_A} A^{\mu} = (1 + \frac{1}{2} \delta z_A) A^{\mu}, \quad z_m m = (1 + \delta z_m) m = m + \delta m$$

Counterterms

- We rewrite the Lagrangian in terms of renormalized quantities and "counterterms" (δZ_X):

$$\begin{aligned} \mathcal{L}(q_0, A_0^\mu, m_0, e_0) &= -\frac{1}{4} F_{\mu\nu} F_0^{\mu\nu} + \bar{q}_0 (i\cancel{D} - Q e_0 \cancel{A}_0 - m_0) q_0 - \frac{1}{2\epsilon_0} (\partial A_0)^2 \\ &= -\frac{1}{2} (\partial_\mu A_0) (\partial^\mu A_0) - \delta Z_A \frac{1}{2} (\partial_\mu A_0) (\partial^\mu A') \\ &\quad + \frac{1}{2} \left(1 - \frac{1}{3}\right) (\partial_\mu A_0^\mu) (\partial_\nu A_0^\nu) + [\delta Z_A (1 - \frac{1}{3}) + \delta Z_F \frac{1}{3}] \frac{1}{2} (\partial_\mu A_0^\mu) (\partial^\mu A') \\ &\quad + \bar{q}_0 (i\cancel{D} - m) q_0 + \delta Z_q \bar{q}_0 (i\cancel{D} - m) q_0 - \delta m \bar{q}_0 q_0 \\ &\quad - e Q \bar{q}_0 \cancel{A} q_0 + (\delta Z_e + \delta Z_F + \frac{1}{2} \delta Z_A) \cdot e Q \bar{q}_0 \cancel{A} q_0 \\ &\quad + \mathcal{O}(\delta Z^2) \end{aligned}$$

$\mathcal{L}(q, A^\mu, m, e)$ "counter terms" \mathcal{L}_{ct}

- note: CT's are not "added": still the original theory (re-shuffling)
- Feynman rules: ($m = \mathcal{O}(1)$, $\delta m = \mathcal{O}(\alpha)$)
 - "old" ones with renormalized quantities
 - "new" CT vertices

EX 17

Counterterm Feynman Rules

$$= -i \delta Z_A \left[k^2 g_{\mu\nu} - \left(1 - \frac{1}{\xi}\right) k_\mu k_\nu \right] + i \delta Z_\xi \frac{1}{\xi} k_\mu k_\nu$$

$$= i \delta Z_4 (p-m) - i \delta m$$

$$= -ieQ Y_\mu \left(\delta Z_e + \delta Z_4 + \frac{1}{2} \delta Z_A \right)$$

EX 18

Renormalized Photon Self Energy

"hat" $\hat{\Sigma}$ $\hat{\equiv}$ renormalized.

$$-i \hat{\Sigma}_{\mu\nu}^{AA}(k) = \text{Diagram with loop} + \text{Diagram with crossed lines} + \mathcal{O}(\alpha^2)$$

$$\Rightarrow \hat{\Sigma}_{\mu\nu}^{AA}(k) = \left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \right) \hat{\Sigma}_T^{AA}(k^2) + \frac{k_\mu k_\nu}{k^2} \hat{\Sigma}_L^{AA}(k^2)$$

$$\hookrightarrow \hat{\Sigma}_T^{AA}(k^2) = \hat{\Sigma}_T^{AA}(k^2) + k^2 \delta Z_A$$

$$\hookrightarrow \hat{\Sigma}_L^{AA}(k^2) = \hat{\Sigma}_{IL}^{AA}(k^2) + \frac{1}{3} k^2 [\delta Z_A - \delta Z_{\bar{A}}]$$

- UV divergences:

$$\delta Z_A|_{\text{div}} \stackrel{!}{=} - \frac{\hat{\Sigma}_T^{AA}(k^2)}{k^2}|_{\text{div}} = - \Pi^{AA}(k^2)|_{\text{div}}$$

$$\delta Z_{\bar{A}}|_{\text{div}} \stackrel{!}{=} \delta Z_A|_{\text{div}} \quad \text{gauge-param. renormalization not necessary for phys. obs.}$$

$$\hookrightarrow \text{we can choose } \delta Z_{\bar{A}} = \delta Z_A \Leftrightarrow \hat{\Sigma}_L^{AA}(k^2) = 0$$

Renormalized Elektron Self Energy

$$i \hat{\Sigma}^{\bar{4}4}(k) = \text{Diagram with } k \text{ arrow} + \text{Diagram with } \times \text{ arrow} + O(\alpha^2)$$

$$\Rightarrow \hat{\Sigma}^{\bar{4}4}(k) = k \hat{\Sigma}_V(k^2) + m \hat{\Sigma}_S(k^2)$$

$$\hookrightarrow \hat{\Sigma}_V(k^2) = \hat{\Sigma}_V(k^2) + \delta Z_4$$

$$\hookrightarrow \hat{\Sigma}_S(k^2) = \hat{\Sigma}_S(k^2) - \delta Z_4 - \frac{\delta m}{m}$$

• UV divergences:

$$\delta Z_4 \Big|_{\text{div}} \stackrel{!}{=} - \hat{\Sigma}_V(k^2) \Big|_{\text{div}}$$

$$\delta m \Big|_{\text{div}} \stackrel{!}{=} m \left[\hat{\Sigma}_S(k^2) - \delta Z_4 \right] \Big|_{\text{div}} = m \left[\hat{\Sigma}_S(k^2) + \hat{\Sigma}_V(k^2) \right] \Big|_{\text{div}}$$

Renormalized Electron-Photon Vertex

$$-ieQ \hat{\Lambda}_\mu(p', p) = \text{Diagram with loop} + \text{Diagram with crossed lines} + \mathcal{O}(\alpha^2)$$

$$\Rightarrow \hat{\Lambda}_\mu(p', p) = \Lambda_\mu(p', p) + \gamma_\mu (\delta z_e + \delta z_4 + \frac{1}{2} \delta z_A)$$

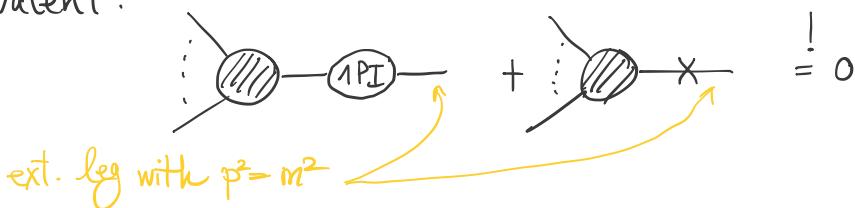
• UV divergences: (remember: $\Lambda_\mu|_{\text{div}} \sim \gamma_\mu$)

$$\delta z_e = \left(\Lambda_{\text{div}} - \delta z_4 - \frac{1}{2} \delta z_A \right) \Big|_{\text{div}}$$

On-Shell Renormalization

- so far: determined the divergent part of δZ_X
↳ sufficient for MS, $\overline{\text{MS}}$ renormalization
- arbitrary finite contributions can be absorbed into the definition of δZ_X ↳ renormalization scheme. (how we relate to phys. obs.)
↳ calculation at $\mathcal{O}(\alpha^n)$ ↳ scheme dependence $\mathcal{O}(\alpha^{n+1})$
- on-shell renormalization
 - ↳ $m \stackrel{!}{=} m_{\text{phys}}$ ↳ location of the propagator pole.
 - ↳ normalize fields (\mathcal{Z}_Φ) such that residue @ $p^2 = m^2$ is one
⇒ no h.o. corrections to the residue

equivalent:



Renormalization Conditions

(a) photon WF renormalization

$$\left(\text{unshaded loop} + \text{shaded loop} \stackrel{!}{=} 0 \right)$$

$$\Rightarrow \lim_{k^2 \rightarrow 0} \left(\frac{-i g^{\mu\nu}}{k^2} \hat{\Gamma}_{\nu p}^{AA}(k, -k) \right) \varepsilon^p(k) \stackrel{!}{=} -\varepsilon^\mu(k) \quad (k_\mu \varepsilon^\mu(k) = 0)$$

$$\Rightarrow \lim_{k^2 \rightarrow 0} \frac{1}{k^2} \hat{\Sigma}_T^{AA}(k^2) = \Pi^{AA}(0) + \delta Z_A \stackrel{!}{=} 0$$

$$\Rightarrow \delta Z_A = -\Pi^{AA}(0)$$

(b) gauge-parameter renormalization:

$$\hat{\Sigma}_L^{AA}(k^2) \stackrel{!}{=} 0 \Rightarrow \delta Z_{\bar{\xi}} = \delta Z_A \quad (\bar{Z}_{\bar{\xi}} = Z_A)$$

↪ as $\bar{\xi}$, $Z_{\bar{\xi}}$ has no impact on phys. observables

$$\hookrightarrow \text{no renormalization for } \mathcal{L}_{\text{fix}} = -\frac{1}{2\bar{\xi}_0} (\partial \cdot A_0)^2 = -\frac{1}{2\bar{\xi}} (\partial \cdot A)^2$$

Renormalization Conditions

(c) electron mass renormalization: ($p^2 = m^2 \hat{=} \text{propagator pole}$)

$$\Rightarrow 0 \stackrel{!}{=} \Gamma^{\bar{\psi}\psi}(-p, p) u(p) \quad (\text{Dirac Eq: } (\not{p} - m) u(p) = 0)$$
$$= i m \left(\underbrace{\sum_{IV}^{\bar{\psi}\psi}(m^2) + \sum_S^{\bar{\psi}\psi}(m^2)}_{=0} \right)$$

$$\Rightarrow \sum_{IV}^{\bar{\psi}\psi}(m^2) + \sum_S^{\bar{\psi}\psi}(m^2) - \frac{8m}{m} = 0$$

$$\rightarrow \frac{\delta m}{m} = \sum_{IV}^{\bar{\psi}\psi}(m^2) + \sum_S^{\bar{\psi}\psi}(m^2)$$

EX 19

Renormalization Conditions

(d) electron WF renormalization

$$\left(\text{Diagram with wavy line} + \text{Diagram with cross} \right) \stackrel{!}{=} 0$$

$$\Rightarrow \lim_{p^2 \rightarrow m^2} \left(\frac{i}{p-m} \hat{\Gamma}_{(-p, p)}^{\bar{q}4} \right) u(p) \stackrel{!}{=} -u(p)$$

$$\Rightarrow \lim_{p^2 \rightarrow m^2} \left(\frac{1}{p-m} \hat{\Sigma}^{\bar{q}4}(p) \right) u(p) \stackrel{!}{=} 0$$

$$\Rightarrow \delta Z_4 = - \sum_V \bar{q}4(m^2) - 2m^2 \frac{\partial}{\partial p^2} \left[\sum_V \bar{q}4(p^2) + \sum_S \bar{q}4(p^2) \right] \Big|_{p^2=m^2}$$

EX 20

Renormalization Conditions

(e) charge renormalization:

$e \stackrel{!}{=} \text{elementary charge from class. E}\text{Dyn.}$

$\stackrel{!}{=} \text{coupling for photon momenta } k \rightarrow 0$

$$\Rightarrow \bar{u}(p) \hat{\Gamma}_\mu^{A\bar{q}q} (k=0, -p, p) u(p) \stackrel{!}{=} -ieQ \bar{u}(p) \gamma_\mu u(p)$$

$$\Rightarrow 0 \stackrel{!}{=} \bar{u}(p) \hat{\Delta}_\mu(p, p) u(p) = \bar{u}(p) [\Delta_\mu(p, p) + \gamma_\mu (\delta z_e + \delta z_A + \frac{1}{2} \delta z_A)]$$

$$* \Delta_\mu(p, p) = \frac{\partial}{\partial p^\mu} \sum \bar{q}^4(p) = \gamma_\mu \sum_v \bar{q}^4(p^2) + 2p_\mu \left[\cancel{\not{p}} \sum_v \bar{q}^4(p^2) + m \sum_s \bar{q}^4(p^2) \right]$$

$$* \bar{u}(p) \gamma_\mu u(p) = \frac{p_\mu}{m} \bar{u}(p) u(p)$$

$$\Rightarrow 0 \stackrel{!}{=} \bar{u}(p) \gamma_\mu u(p) \left[\underbrace{\sum_v \bar{q}^4(m^2) + 2m^2 (\sum_v \bar{q}^4(m^2) + \sum_s \bar{q}^4(m^2))}_{=0} + \delta z_A + \delta z_e + \frac{1}{2} \delta z_A \right]$$

$$\Rightarrow \delta z_e = -\frac{1}{2} \delta z_A$$

Appendix : Four - Photon Vertex Function

- we saw that $\Gamma_{\mu_1 \mu_2 \mu_3 \mu_4}^{AAAA}(k_1, k_2, k_3, k_4)$ has a superficial degree of divergence = 0

↪ if the leading UV behaviour is finite, we're ok!

$$\Gamma_{\mu_1 \mu_2 \mu_3 \mu_4}^{AAAA} = \text{Diagram } ① + \text{Diagram } ② + \text{Diagram } ③ + (\text{Diagram } ④ \xrightarrow{\text{give the same as}} \text{Diagram } ① + \text{Diagram } ② + \text{Diagram } ③)$$

- For the leading UV behaviour, we have

$$\Gamma^{\text{①}} \stackrel{\text{div}}{\sim} \int d^4 q \frac{\text{Tr} [\gamma^{\mu_1} \gamma^{\mu_4} \gamma^{\mu_3} \gamma^{\mu_2}]}{[q^2]^4}$$

we use isotropy : $\int d\Omega q^\mu q^\nu q^\rho q^\sigma \sim (g^{\mu\nu} g^{\rho\sigma} + g^{\mu\rho} g^{\nu\sigma} + g^{\mu\sigma} g^{\nu\rho})$

and Γ identities (can do this in $D=4$ since $(D-4) \sim \epsilon$ is sub-leading)

$$\Rightarrow \Gamma^{\text{①}} \stackrel{\text{div}}{\sim} g^{\mu_1 \mu_4} g^{\mu_3 \mu_2} + g^{\mu_1 \mu_2} g^{\mu_3 \mu_4} - 2 g^{\mu_1 \mu_3} g^{\mu_2 \mu_4}$$

$$\Gamma^{\text{②}} = \Gamma^{\text{①}} \Big| \mu_3 \leftrightarrow \mu_4$$

$$\Gamma^{\text{③}} = \Gamma^{\text{①}} \Big| \begin{array}{l} \mu_2 \rightarrow \mu_4 \\ \mu_3 \rightarrow \mu_2 \\ \mu_4 \rightarrow \mu_3 \end{array}$$

$$\Rightarrow \Gamma^{\text{①}} + \Gamma^{\text{②}} + \Gamma^{\text{③}} \stackrel{\text{div}}{\sim} 0$$

$$\Rightarrow \Gamma_{\mu_1 \mu_2 \mu_3 \mu_4}^{AAAA}(k_1, k_2, k_3, k_4) \Big|_{\text{div}} = 0$$

III. Yang-Mills Theory

& QCD

QED from Gauge Symmetry

- the free Dirac theory $\mathcal{L}_4(4, \partial_\mu 4)$ is invariant under global U(1)

$$4(x) \rightarrow 4'(x) = e^{-ieQ\theta} 4(x)$$

- let's try to impose it as a local ($\theta = \theta(x)$) symmetry:

problem: $\partial_\mu 4 \rightarrow \partial_\mu 4' = \partial_\mu (e^{-ieQ\theta} 4) = e^{-ieQ\theta} (\partial_\mu - ieQ \underline{\partial_\mu \theta}) 4 \neq e^{-ieQ\theta} \partial_\mu 4$

- solution: introduce the covariant derivative $D_\mu + ieQA_\mu(x)$ (minimal substitution)

$$\Rightarrow D_\mu 4 \rightarrow D'_\mu 4' = e^{-ieQ\theta} D_\mu 4 , \quad A_\mu \rightarrow A'_\mu = A_\mu + (\partial_\mu \theta)$$

\Rightarrow we generate interactions $\sim A_\mu \bar{4} \gamma^\mu 4$ by imposing local U(1)!

- make the gauge field a dynamic quantity:

$$\mathcal{L}_{\text{gauge}} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \quad (F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu)$$

\Rightarrow principle of minimal gauge-inv. coupling to a dynamic gauge field

$$\mathcal{L}_{\text{QED}} = \mathcal{L}_4(4, D_\mu 4) + \mathcal{L}_{\text{gauge}}$$

Yang - Mills Theory

- apply the principle of gauge symmetry to (non-abelian) symmetries

① field multiplet $\vec{\Phi} = (\Phi_1, \dots, \Phi_n)^T$ & Lagrangian $\mathcal{L}_{\vec{\Phi}}(\vec{\Phi}, \partial_\mu \vec{\Phi})$, invariant under global transformations of group G :

$$\vec{\Phi} \rightarrow \vec{\Phi}' = U(\theta) \vec{\Phi},$$

$$U(\theta) = \exp \left\{ -ig T^a \theta^a \right\}$$

generators of
the group,
 $a = 1, \dots, \dim G$

arbitrary gauge coupling

② make the symmetry local

$$\begin{aligned} D_\mu &= \partial_\mu + ig \underbrace{A_\mu^a T^a}_{} \\ &=: \partial_\mu + ig A_\mu \end{aligned}$$

with property

$$D_\mu \vec{\Phi} \rightarrow D'_\mu \vec{\Phi}' = U(\theta) D_\mu \vec{\Phi}$$

EX21

Yang - Mills Theory

③ make the gauge field dynamic

transforms gauge co-variantly

↪ field-strength tensor: $\tilde{F}_{\mu\nu} = -\frac{i}{g} [D_\mu, D_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu + ig [A_\mu, A_\nu]$

$$\tilde{F}_{\mu\nu} = F_{\mu\nu}^a T^a \Rightarrow F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a - g C^{abc} A_\mu^b A_\nu^c$$

$$\Rightarrow \mathcal{L}_A = -\frac{1}{2} \text{Tr} [\tilde{F}_{\mu\nu} \tilde{F}^{\mu\nu}] = -\frac{1}{4} F_{\mu\nu}^a F^{a,\mu\nu}$$

↪ Lorentz- & gauge-invariant

↪ non-abelian: contains $(\partial A) A^2, A^4$ terms

⇒ gauge-boson self-interaction

↪ naive mass term $M^2 (A_\mu^a A^{a\mu})$ not allowed

⇒ from $\mathcal{L}_{\vec{\Phi}} (\vec{\Phi}, \partial_\mu \vec{\Phi})$ (global symm.) \Rightarrow gauge theory: $\mathcal{L}_{\vec{\Phi}} (\vec{\Phi}, D_\mu \vec{\Phi}) + \mathcal{L}_A$

QCD

- gauge symmetry: $SU(3)_c$
- matter fields: quark fields

fundamental representation

$$\psi_q(x) = \begin{pmatrix} q_r(x) \\ q_g(x) \\ q_b(x) \end{pmatrix} \quad \longleftrightarrow \text{colour-triplett}$$

- generators: $T^a = \frac{\lambda^a}{2}$ (λ^a : Gell-Mann matrices, $a=1, \dots, 8$)
 ↳ structure constants: $[T^a, T^b] = i f^{abc} T^c$
- eight gauge fields: g_μ^a ($a=1 \dots 8$) \rightarrow gluons

$$\Rightarrow \mathcal{L}_{QCD} = \sum_q \bar{\psi}_q (i\cancel{D} - m_q) \psi_q - \frac{1}{4} F_{\mu\nu}^a F^{a,\mu\nu}$$

$$\mathcal{L}_{free} \rightarrow = -\frac{1}{4} (\partial_\mu g_\nu^a - \partial_\nu g_\mu^a) (\partial^\mu g^{a,\nu} - \partial^\nu g^{a,\mu}) + \sum_q \bar{\psi}_q (i\cancel{D} - m) \psi_q$$

$$g\bar{q}q \rightarrow -g_S g_\mu^a \bar{\psi}_q \gamma^\mu T^a \psi_q$$

$$g^3 \& g^4 \rightarrow +\frac{g_S}{2} f^{abc} (\partial_\mu g_\nu^a - \partial_\nu g_\mu^a) G^{b,\mu} G^{c,\nu} - \frac{g_S^2}{4} f^{abc} f^{ade} G_\mu^b G_\nu^c G^{d,\mu} G^{e,\nu}$$

Quantization of Gauge Theories

- the free part of the gluon

$$\mathcal{L}_{G,0} = -\frac{1}{4} (\partial_\mu G_\nu^a - \partial_\nu G_\mu^a) (\partial^\nu G_{\mu\rho}^a - \partial^\mu G_{\nu\rho}^a)$$

- from part I. no problem with the generating functional:

$$\mathcal{Z}_{G,0}[J_\mu^a] = \frac{1}{N} \int \mathcal{D}[G_\mu] \exp \left\{ i \int d^4x \left[\frac{1}{2} G_\mu^a \underbrace{(g^{\mu\nu} \square - \partial^\mu \partial^\nu)}_{\text{has no inverse}} G_\nu^a + J_\mu^a G_{\mu\nu}^a \right] \right\}$$

↳ deriving Green's function

- origin: gauge invariance!

$\mathcal{D}[G_\mu]$ includes physically equivalent field configurations

$$G_\mu \xrightarrow{U(\theta)} \overset{\circ}{G}_\mu \quad (U(\theta) = \exp \{ -i g_s T^a \theta^a(x) \})$$

⇒ solution: restrict $\mathcal{D}[G_\mu]$ to only in-equivalent configurations

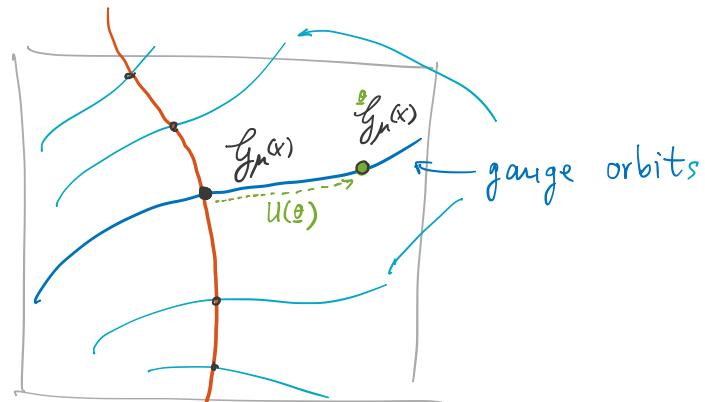
Faddeev-Popov Procedure

① define gauge-fixing conditions

$$f^a \left[\frac{\partial}{\partial} g_\mu^{(x)} \right] = 0 \quad a = 1, \dots, 8$$

(QED in covariant gauge:
 $f[A_\mu(x)] = \partial^\mu A_\mu$)

⇒ selects one unique $\theta^a(x)$ from gauge orbit



gauge-fixing condition:

$$f^a \left[\frac{\partial}{\partial} g_\mu^{(x)} \right] = 0$$

Faddeev-Popov Procedure

② insert a clever one:

$$1 = \Delta[g_\mu] \int D[U(\theta)] \delta(f[\frac{\partial}{\partial} g])$$

Faddeev-Popov determinant:

$$\underbrace{\Delta[g_\mu]}_{\text{gauge invariant}} \propto \det \left(\frac{\delta f^a[g_\mu(x)]}{\delta \theta^b(y)} \right) \Big|_{f^a[g_\mu] = 0}$$

Jacobi functional determinant

"matrix" in $a \& x$:

$$M^{ab}(x, y) = \frac{\delta f^a[g_\mu(x)]}{\delta \theta^b(y)}$$

$$\Rightarrow (Mf)^a(x) = \int d^4y M^{ab}(x, y) f^b(y)$$

$$\Rightarrow M(x, y) \sim \delta(x-y)$$

(QED: $f[A_\mu(x)] = \partial^\mu A_\mu$, infinitesimal transf. $\delta A_\mu(x) = \partial_\mu \delta \theta(x)$)

$$\Rightarrow \frac{\delta f[A_\mu(x)]}{\delta \theta(y)} = \delta(x-y) \square_x$$

Faddeev - Popov Procedure

③ insert into functional integral:

$$\int \mathcal{D}[g_\mu] \exp\{i S_0[g_\mu]\}$$

$$= \int \mathcal{D}[u(\underline{\theta})] \int \mathcal{D}[g_\mu] \Delta[g_\mu] \delta(f[\overset{\theta}{g}_\mu]) \exp\left\{i S_0[\overset{\theta}{g}_\mu]\right\}$$

$\downarrow \text{g.i.}$ $\downarrow \text{g.i.}$ $\downarrow \text{g.i.}$

$$= \int \mathcal{D}[u(\underline{\theta})] \int \mathcal{D}[\overset{\theta}{g}_\mu] \Delta[\overset{\theta}{g}_\mu] \delta(f[\overset{\theta}{g}_\mu]) \exp\left\{i S_0[\overset{\theta}{g}_\mu]\right\}$$

$\downarrow \text{relabel}$

$$= \int \mathcal{D}[u(\underline{\theta})] \int \mathcal{D}[g_\mu] \Delta[g_\mu] \delta(f[g_\mu]) \exp\{i S_0[g_\mu]\}$$

$\underbrace{\quad}_{= \text{const}}$

(\propto volume)

$m > N$

independent of $u(\underline{\theta})$

Faddeev-Popov Procedure

- ④ get rid of the $\delta(f[g_\mu])$ constraint and write it as an extra term in the Lagrangian \mathcal{L}_{fix} instead

↪ modify constraint

$$f^a[g_\mu] \rightarrow f^a[g_\mu] - C^a(x) \quad (\text{phys. obs. independent of } C^a(x))$$

↪ integrate with

$$\int \mathcal{D}[C^a] \dots \exp \left\{ -i \int d^4y \frac{1}{2\xi} C_a(y)^2 \right\}$$

arbitrary gauge param.
could also choose ξ^a indir.

$$\Rightarrow \int \mathcal{D}[C^a] \delta(f^a[g_\mu] - C^a) \exp \left\{ -i \int d^4y \frac{1}{2\xi} C_a(y)^2 \right\} = \exp \left\{ i \int d^4x \mathcal{L}_{\text{fix}} \right\}$$

with $\mathcal{L}_{\text{fix}} = -\frac{1}{2\xi} f_a[g_\mu]^2$

(QED: $\mathcal{L}_{\text{fix}} = -\frac{1}{2\xi} (\partial A)^2$)

Faddeev-Popov Procedure

⑤ rewrite the determinant as a Gaussian integral over Grassmann-valued fields (\bar{u}^a, u^b : ghosts)

$$\text{Det}(M^{ab}(x,y)) \propto \int \mathcal{D}[u^b] \int \mathcal{D}[\bar{u}^a] \exp \left\{ -i \int d^4x \int d^4y \bar{u}^a(x) M^{ab}(x,y) u^b(y) \right\}$$

$$M^{ab}(x,y) = \frac{\delta f^a[g_m(x)]}{\delta \theta^b(y)} = \delta(x-y) \frac{\delta f^a[g_m(x)]}{\delta \theta^b(x)}$$

EX 22

put it as an extra term in the Lagrangian:

$$\mathcal{L}_{FP} = -\bar{u}^a(x) M^{ab}(x) u^b(x)$$

u^a, \bar{u}^a : Grassmann-valued scalars
↪ unphysical!

(QED: only one ghost with $\mathcal{L}_{FP} = -\bar{u}(x) \square u(x)$)
↪ no interactions \Rightarrow in abelian theories: ghosts decouple!

Faddeev-Popov Procedure

- when the dust settles:

the effective Lagrangian

$$\mathcal{L}_{G,0} \longrightarrow \mathcal{L}_{G,0} + \mathcal{L}_{\text{fix}} + \mathcal{L}_{\text{FP}} = \mathcal{L}_{\text{eff}}$$

$$\Rightarrow Z[J_\mu^a, j^a, \bar{j}^a] = \frac{1}{N} \int \mathcal{D}[G_\mu^a] \int \mathcal{D}[u^a] \int \mathcal{D}[\bar{u}^a] \exp \left\{ i \int d^4x [\mathcal{L}_{G,0} + \mathcal{L}_{\text{fix}} + \mathcal{L}_{\text{FP}} + J_\mu^a G_\mu^{aj} + j^a u^a - \bar{u}^a j^a] \right\}$$

- QCD in covariant gauge: $f^a[G_\mu] = \partial^\mu G_\mu^a$

$$\Rightarrow \frac{\delta f^a[G_\mu(x)]}{\delta \theta^b(y)} = \delta(x-y) \left[\delta^{ab} \square_x + g C^{abc} (\partial \cdot G^c) + g C^{abc} G_\mu^c \partial_x^\mu \right]$$

$$\hookrightarrow \mathcal{L}_{\text{fix}} = - \frac{1}{2\xi} (\partial \cdot G^a)^2$$

ghosts only interact with the gluons.

$$\hookrightarrow \mathcal{L}_{\text{FP}} = - \bar{u}^a \square u^a - g C^{abc} \bar{u}^a \partial^\mu (G_\mu^c u^b)$$

BRS Symmetry

$$\textcircled{1} \text{ Faddeev-Popov: } \mathcal{L}_{\text{ferm}} + \mathcal{L}_{\text{gauge}} \xrightarrow{\text{FP}} \underbrace{\mathcal{L}_{\text{ferm}} + \mathcal{L}_{\text{gauge}}}_{\text{gauge-inv.}} + \underbrace{\mathcal{L}_{\text{fix}} + \mathcal{L}_{\text{FP}}}_{\text{not gauge inv. (the whole point)}}$$

\hookrightarrow phys. abs. gauge inv., manifest gauge inv. in \mathcal{L} gone!

$\textcircled{2}$ what happened?

$\hookrightarrow \mathcal{L}_{\text{fix}}$: allows unphysical d.o.f. to propagate

$\hookrightarrow \mathcal{L}_{\text{FP}}$: ghosts are there to compensate for them

invertible diff. op
 \Rightarrow compute Green's fn.
 $(\Leftarrow$ propagator)

\hookleftarrow cancel unphysical d.o.f.
introduced by \mathcal{L}_{fix}

$\textcircled{3}$ what is the symmetry that corresponds to gauge symmetry after FP?

Becchi-Rouet-Stora (BRS) symmetry

\hookrightarrow gauge transformation with $\delta\theta^a(x) = \delta\bar{\lambda} u^a(x)$

$$(U(\theta) = \exp \{-i g_s T^a \theta^a\})$$

Grassmann!

BRS Symmetry

- gauge transformation with $\delta\theta^a = \delta\bar{\lambda} u^a$ ($U(\theta) = \exp\{-i\bar{\lambda} T^a \theta^a\}$)

$$\hookrightarrow \delta_{\text{BRS}} G_\mu^a = \delta\bar{\lambda} [g_s f^{abc} u^b G_\mu^c + \partial_\mu u^a]$$

$$\hookrightarrow \delta_{\text{BRS}} \Psi = \delta\bar{\lambda} [-i g_s T^a u^a \Psi]$$

- clearly, both $\mathcal{L}_{\text{ferm}}$ & $\mathcal{L}_{\text{gauge}}$ are invariant under δ_{BRS}

- we need $\delta_{\text{BRS}} (\mathcal{L}_{\text{fix}} + \mathcal{L}_{\text{FP}}) = 0$ for \mathcal{L}_{eff} to be invariant

$$\hookrightarrow \delta_{\text{BRS}} \bar{u}^a \equiv \delta\bar{\lambda} \left[-\frac{1}{5} f_a[G_\mu] \right] = \delta\bar{\lambda} \left[-\frac{1}{5} \partial^\mu G_\mu^a \right]$$

$$\hookrightarrow \delta_{\text{BRS}} u^a = \delta\bar{\lambda} \left[-\frac{1}{2} g_s f^{abc} u^b u^c \right]$$

\Rightarrow Derive relations between n-pt functions from BRS symmetry

and Slavnov-Taylor identities

Slavnov-Taylor Identities

- let us consider the transformation $\Phi_i \rightarrow \bar{\Phi}_i + \delta_{\text{BRS}} \bar{\Phi}_i$ in the P.I.

$$\hookrightarrow \int D[\Phi_i + \delta_{\text{BRS}} \bar{\Phi}_i] = \int D[\bar{\Phi}_i], \quad L_{\text{eff}}(\bar{\Phi}_i + \delta_{\text{BRS}} \bar{\Phi}_i) = L_{\text{eff}}(\bar{\Phi}_i)$$

$$\begin{aligned} \Rightarrow 0 &= Z[J_i] \Big|_{\bar{\Phi}_i + \delta_{\text{BRS}} \bar{\Phi}_i} - Z[J_i] && \text{only linear because } (\delta \bar{x})^2 = 0 \\ &= \frac{1}{N} \int D[\bar{\Phi}_i] \exp \left\{ i \int d^4x [L_{\text{eff}} + J_i \bar{\Phi}_i] \right\} \cdot \underbrace{i \int d^4y J_i(y)}_{\delta_{\text{BRS}} \bar{\Phi}_i(y)} \end{aligned}$$

- functional differentiation \Rightarrow Relations between n-pt functions

- Slavnov-Taylor identities:

$$0 \stackrel{!}{=} \delta_{\text{BRS}} \langle T \bar{\Phi}_1(x_1) \dots \bar{\Phi}_n(x_n) \rangle$$

$$= \langle T (\delta_{\text{BRS}} \bar{\Phi}_1(x_1)) \bar{\Phi}_2(x_2) \dots \bar{\Phi}_n(x_n) \rangle + \dots + \langle T \bar{\Phi}_1(x_1) \dots \bar{\Phi}_{n-1}(x_{n-1}) (\delta_{\text{BRS}} \bar{\Phi}_n(x_n)) \rangle$$

Slavnov-Taylor Identity: Gluon Propagator

$$\begin{aligned}
 0 &\stackrel{!}{=} \delta_{\text{BRS}} \left\langle T \bar{u}^a(x) G^b_{\mu}(y) \right\rangle \\
 &= \underbrace{\left\langle T (\delta_{\text{BRS}} \bar{u}^a(x)) G^b_{\mu}(y) \right\rangle}_{\delta \bar{\lambda} \left[-\frac{1}{3} \partial^\nu \bar{G}^a_\nu(x) \right]} + \underbrace{\left\langle T \bar{u}^a(x) (\delta_{\text{BRS}} G^b_{\mu}(y)) \right\rangle}_{\text{Grassmann!} \rightarrow \delta \bar{\lambda} \left[\delta_S f^{bcd} \bar{u}^c(y) G^d_\mu(y) + \partial_\mu \bar{u}^b(y) \right]}
 \end{aligned}$$

- bring $\delta \bar{\lambda}$ to the front & cancel
- differentiate w.r.t. y : $\partial_y^\nu (\dots)$
- use the e.o.m. for the Green's function

EX 23

$$\left\langle T \bar{u}^a(x) M^{bc}(y) u^c(y) | 0 \right\rangle = i \delta^{ab} \delta(x-y)$$

- \mapsto momentum space: $\int d^4x \int d^4y e^{-ik(x-y)}$

$$\Rightarrow k^\mu k^\nu G_{\mu\nu}^{g^a g^b}(k, -k) = -i \not{\delta}^{ab}$$

\hookrightarrow no higher-order corrections to longitudinal component of the gluon!

QCD Feynman Rules

- propagators

$$G_\mu^a \xrightarrow{k} G_\nu^b$$

$$\frac{-i \delta^{ab}}{k^2 + i0} \left[g_{\mu\nu} - (1-\xi) \frac{k_\mu k_\nu}{k^2} \right]$$

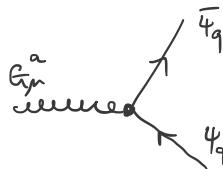
$$\bar{u}^a \xrightarrow{k} u^b$$

$$\frac{i \delta^{ab}}{k^2 + i0}$$

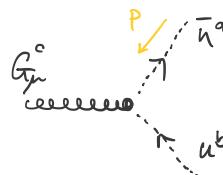
$$\bar{q}_i \xrightarrow{k} q_j$$

$$\frac{i}{k - m_q + i0}$$

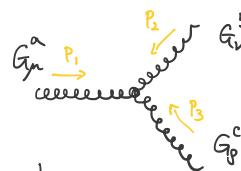
- vertices



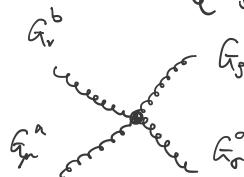
$$-ig_s T^a \gamma_\mu$$



$$P_\mu g_s f^{abc}$$



$$-g_s f^{abc} \left[g^{\mu\nu} (p_1 - p_2)^{\rho} + g^{\nu\rho} (p_2 - p_3)^{\mu} + g^{\rho\mu} (p_3 - p_1)^{\nu} \right]$$



$$-ig_s^2 \left[f^{abe} f^{cde} (g^{\mu\nu} g^{\rho\sigma} - g^{\mu\rho} g^{\nu\sigma}) + \text{cycl.} \right]$$

Superficially Divergent Vertex Functions



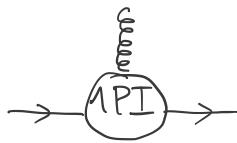
$\omega = 3 \rightarrow 0$ (quant. number of vacuum)



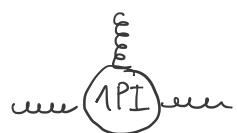
$\omega = 2$



$\omega = 1$



$\omega = 0$



$\omega = 1$



$\omega = 0$

}

all divergent!

↪ more div. vertex functions than parameters!

↪ gauge symmetry essential

EX 24

1-Loop Corrections : Gluon Self Energy

$$-i \sum_{\mu\nu}^{G^a G^b}(k) = \text{one loop} + \text{two loops} + \text{three loops} + \text{four loops}$$

$$\hookrightarrow \Gamma_{\mu\nu}^{G^a G^b}(k) = -i \delta^{ab} k^2 \left[g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \right] \left[1 + \Pi^{GG}(k^2) \right] - \frac{i}{\xi} \delta^{ab} k_\mu k_\nu$$

- gluon vacuum polarization

$$\Pi^{GG}(k^2) = \sum_q \frac{\alpha_s}{3\pi} T_F \frac{1}{k^2} \left\{ (k^2 + m_q^2) B_0(k^2, m_q, m_q) - \frac{k^2}{3} - 2m_q^2 B_0(0, m_q, m_q) \right\} \\ - \frac{\alpha_s}{12\pi} C_A \left\{ \frac{13 - 3\xi}{2} B_0(k^2, 0, 0) + \frac{59}{12} + \frac{9}{2}\xi + \frac{3}{4}\xi^2 \right\}$$

- fermion loop: analogue to QED $e^2 Q^2 \rightarrow g_s^2 T_F \delta^{ab}$
- boson loop: $\sum_L^{GG} = 0$ only after summation
- Π^{GG} gauge dependent already @ 1-loop (not in QED)

1-Loop Corrections: Quark Self Energy

$$i \sum^{\bar{q}q}(k) = \text{Diagram: A horizontal line with arrows pointing right, with a semi-circular loop attached to it.}$$

$$\hookrightarrow \sum^{\bar{q}q}(k) = \not{k} \sum_V^{\bar{q}q}(k^2) + m_q \sum_S^{\bar{q}q}(k^2)$$

- result ($\gamma_5 = 1$)

$$\sum_V^{\bar{q}q}(k^2) = -\frac{\alpha_s}{4\pi} C_F \frac{1}{k^2} \left\{ A_0(m_q) - (k^2 + m_q^2) B_0(k^2, m_q, 0) + k^2 \right\}$$

$$\sum_S^{\bar{q}q}(k^2) = -\frac{\alpha_s}{4\pi} C_F \left\{ 4 B_0(k^2, m_q, 0) - 2 \right\}$$

- from QED: $e^2 Q^2 \rightarrow g_s^2 C_F$

1-Loop Corrections: Quark-Gluon Vertex

$$-i g_s \Lambda_\mu^a(p', p) = \text{Diagram 1} + \text{Diagram 2}$$

no analogue in QED

- 1st diagram from QED: $e^3 Q^3 \rightarrow -g_s^3 (C_F - \frac{1}{2} C_A) T^a$
- UV divergence:

$$\Lambda_\mu^a(p', p) \Big|_{\text{div}} = \frac{\alpha_s}{4\pi} (C_F + C_A) \frac{1}{e}$$

1-Loop Corrections: Gluon Vertices

$$\Gamma_{\mu\nu\rho}^{G^a G^b G^c} = \text{tree} + \text{one loop} + \text{Furry} + \text{two loops} + \dots$$

$$\hookrightarrow \Gamma_{\mu\nu\rho}^{G^a G^b G^c} \Big|_{\text{div}} = \text{tree} \cdot C_{\text{div}}^{GGG}$$

(QED: zero)

$$\Gamma_{\mu\nu\rho\sigma}^{G^a G^b G^c G^d} = \text{tree} + \text{one loop} + \text{two loops} + \text{three loops} + \text{four loops} + \text{five loops} + \dots$$

$$\hookrightarrow \Gamma_{\mu\nu\rho\sigma}^{G^a G^b G^c G^d} \Big|_{\text{div}} = \text{tree} \cdot C_{\text{div}}^{GGGG}$$

(QED: UV finite)

QCD Renormalization

- multiplicative renormalization (1-loop: $Z_X = 1 + \delta Z_X$)

$$G_{\mu,0}^a = \sqrt{Z_G} G_\mu^a, \quad 4_{q,0} = \sqrt{Z_q} 4_q$$

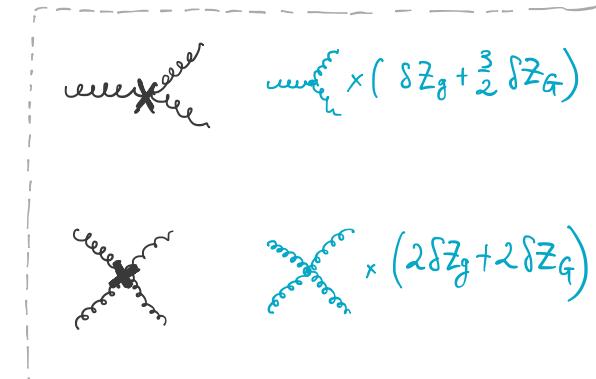
$$g_{s,0} = Z_g g_s, \quad m_{q,0} = Z_{m_q} m_q = m_q + \delta m_q, \quad \xi_0 = Z_\xi \xi$$

- Counterterm vertices:

~~G_μ^a~~ $\times \cancel{\text{---}} \times \cancel{\text{---}}$ G_ν^b $-i \int^{ab} \delta Z_G k^2 \left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \right) + i \int^{ab} \frac{1}{\xi} (\delta Z_g - \delta Z_A) k_\mu k_\nu$

$q \xrightarrow{P} \cancel{x} \rightarrow \bar{q}$ $i \delta Z_q (\not{p} - m_q) - i \delta m_q$

G_μ^a $\times \cancel{\text{---}} \times \bar{q}$ $\cancel{x} \times \left(\delta Z_g + \delta Z_q + \frac{1}{2} \delta Z_A \right)$



Renormalized Vertex Functions



$$= -i \delta^{ab} \left[g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \right] \left[k^2 + \overbrace{\sum_T^{\text{GF}}(k^2) + k^2 \delta Z_G}^{\stackrel{\wedge}{\sum}_T^{\text{GF}}(k^2)} \right]$$

$$- i \delta^{ab} \frac{1}{\pi} \frac{k_\mu k_\nu}{k^2} \left[k^2 + \overbrace{\sum_L^{\text{GF}}(k^2) + k^2 (\delta Z_G - \delta Z_F)}^{\stackrel{\wedge}{\sum}_L^{\text{GF}}(k^2)} \right]$$

$$=: \stackrel{\wedge}{\sum}_L^{\text{GF}}(k^2)$$



$$= i \not{p} \left[1 + \overbrace{\sum_V^{\bar{q}q}(p^2) + \delta Z_q}^{\stackrel{\wedge}{\sum}_V^{\bar{q}q}(p^2)} \right] + i m_q \left[-1 + \overbrace{\sum_S^{\bar{q}q}(p^2) - \delta Z_q - \frac{\delta m_q}{m_q}}^{\stackrel{\wedge}{\sum}_S^{\bar{q}q}(p^2)} \right]$$



$$= -i g_s T^a \gamma_\mu - i g_s \left[\overbrace{\Lambda_\mu^a(p'_1 p) + T^a \gamma_\mu (\delta Z_g + \delta Z_q + \frac{1}{2} \delta Z_F)}^{\stackrel{\wedge}{\Lambda}_\mu^a(p'_1 p)} \right]$$

$$=: \stackrel{\wedge}{\Lambda}_\mu^a(p'_1 p)$$

More:





Renormalization Conditions

- confinement in QCD
 - ↪ definition through elementary gg scattering not accessible
 - ↪ no free propagation of quarks \Rightarrow kin pole not accessible
- $\overline{\text{MS}}$ ("modified minimal subtraction") scheme

δZ_X absorb only the standard divergence

$$\Delta = \frac{1}{\epsilon} - \gamma_E + \ln(4\pi) \quad (1\text{-loop})$$

\Rightarrow predictions depend on the reference scale μ (DimReg)
↪ running couplings and masses

MS Renormalization

- gluon self energy:

$$\delta Z_G = - \frac{\sum_I^{GG}(k^2)}{k^2} \Big|_\Delta = - \Pi^{GG}(k^2) \Big|_\Delta = - \frac{\alpha_s}{4\pi} \Delta \left[\frac{4}{3} N_f T_F - \frac{5}{3} C_A \right]$$

$$\delta Z_{\xi} = \delta Z_G \quad (\Rightarrow \hat{\sum}_L^{GG}(k^2) = \sum_L^{GG}(k^2) = 0)$$

- quark self energy:

$$\delta Z_q = - \sum_V^{\bar{q}q}(k^2) \Big|_\Delta = - \frac{\alpha_s}{4\pi} \Delta C_F$$

$$\frac{\delta m_q}{m_q} = \left[\sum_V^{\bar{q}q}(k^2) + \sum_S^{\bar{q}q}(k^2) \right] \Big|_\Delta = - \frac{\alpha_s}{4\pi} \Delta 3 C_F$$

↑ often $m_q=0$ for light quarks $\Rightarrow \delta m_q=0$ (chiral symmetry)

- quark-gluon vertex:

$$\Gamma_\mu^a(p'_1 p) \Big|_\Delta + T^a \gamma_\mu \left(\delta Z_g + \delta Z_q + \frac{1}{2} \delta Z_G \right) \Big|_0 \Rightarrow \delta Z_g = \frac{\alpha_s}{4\pi} \Delta \left[\frac{2}{3} N_f T_F - \frac{11}{6} C_A \right]$$

Renormalization Group

- equations for observables depend on μ
⇒ extracted values for g , etc. are μ -dependent $g = g(\mu)$
- in the end: explicit & implicit μ -dependence compensate each other
⇒ renormalization group equation

$$\mu \frac{d\alpha_s}{d\mu} = -\beta_0 \frac{\alpha_s^2}{\pi} \quad (1\text{-loop})$$

EX 25

$$\beta_0 = \frac{11}{6} C_A - \frac{2}{3} N_f T_F = \frac{11}{6} 3 - \frac{2}{3} N_f \frac{1}{2} > 0 \quad \text{for } N_f < 16.5 \quad (\checkmark)$$

↗ beta function

- solution

$$\alpha_s(\mu_2) = \frac{\alpha_s(\mu_1)}{1 - \frac{\alpha_s(\mu_1)}{2\pi} \beta_0 \ln\left(\frac{\mu_2}{\mu_1}\right)} \Rightarrow \mu_2 > \mu_1 \Rightarrow \alpha_s(\mu_2) < \alpha_s(\mu_1)$$

asymptotic freedom

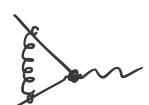
IV. Infrared Singularities

Infrared Singularities

- UV ($q \rightarrow \infty$) divergences dealt with using renormalization
- IR singularities (mass singularities) associated with
 - ↪ vanishing (soft) momenta: $k^{\mu} \rightarrow 0$
 - ↪ collinear configurations: $p_i \parallel p_j$
- cancellation for "IR safe" observables between REAL & VIRTUAL:

$$\int d\Phi_m \sigma^{(1)}(p_1, \dots, p_m) + \int d\Phi_{m+1} \sigma^{(0)}(p_1, \dots, p_{m+1}) = \text{IR finite}$$

(Kinoshita-Lee-Nauenberg)

explicit poles in implicit poles in
 $\sim \frac{1}{\epsilon^2} / \frac{1}{\epsilon}$ phase-space integration

Singularities in Feynman Integrals

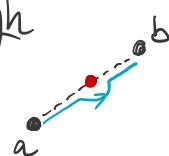
- arbitrary graph: $\int \prod_{\ell=1}^L d^4 q_\ell \frac{N(\{q_\ell\}, \{p_i\})}{\prod_{n=1}^N [k_n^2 - m_n^2 + i\epsilon]}$
- k_n lin. comb of
 $\hookrightarrow q_\ell$: loop momenta
 $\hookrightarrow p_i$: ext. momenta

Feynman param.

$$(n-1)! \int \prod_{\ell=1}^L d^4 q_\ell \int_0^1 \prod_{n=1}^N dx_n f(1 - \sum_i x_i) \frac{N(\{q_\ell\}, \{p_i\})}{[\tilde{D} + i\epsilon]^N}$$

$$\tilde{D} = \sum_{n=1}^N x_n [k_n^2 - m_n^2]$$

- singularity $\leftrightarrow \tilde{D} = 0$ but can be avoided through contour deformation



Singularities in Feynman Integrals

- arbitrary graph: $\int \prod_{\ell=1}^L d^4 q_\ell \frac{N(\{q_\ell\}, \{p_i\})}{\prod_{n=1}^N [k_n^2 - m_n^2 + i\epsilon]}$
- k_n lin. comb of
 $\hookrightarrow q_\ell$: loop momenta
 $\hookrightarrow p_i$: ext. momenta

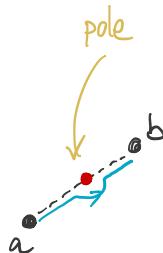
Feynman param.

$$(n-1)! \int \prod_{\ell=1}^L d^4 q_\ell \int_0^1 \prod_{n=1}^N dx_n f(1 - \sum_i x_i) \frac{N(\{q_\ell\}, \{p_i\})}{[\tilde{D} + i\epsilon]^N}$$

$$\tilde{D} = \sum_{n=1}^N x_n [k_n^2 - m_n^2]$$

- singularity $\leftrightarrow \tilde{D} = 0$

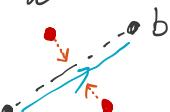
but can be avoided through contour deformation:



- end-point singularity:



- pinch singularity:



Landau Equations

- necessary condition for a singularity

$$x_n D_n = x_n [k_n^2 - m_n^2] = 0 \quad \nexists n=1,\dots,N$$

\Rightarrow either on-shell ($k_n^2 = m_n^2$) or reduced graph ($x_n = 0$)

$$\sum_n x_n \frac{\partial D_n}{\partial q_e} = \sum_n x_n k_n \frac{\partial k_n}{\partial q_e} = 0 \quad \nexists l=1,\dots,L$$

\Leftrightarrow pinch condition for $d^4 q_e$ (endpoint \triangleq UV sing.)

- IR (mass) singularities:

independent of orientation of external momenta
(no threshold, branch, ...)

IR Singularities in 1-loop integrals

$\cancel{p}_n = -(p_1 + \dots + p_{n-1})$ *n-point integral*

$$q + p_1 + \dots + p_{n-1} = q - \cancel{p}_n$$
$$D_0 = q^2 - m_0^2 \quad (k_0 = q)$$
$$D_1 = (q + p_1)^2 - m_1^2 \quad (k_1 = q + p_1)$$
$$\vdots$$
$$D_{n-1} = (q - \cancel{p}_n)^2 - m_{n-1}^2 \quad (k_{n-1} = q - p_n)$$

- Landau Equations:

$$x_i D_i = x_i (k_i^2 - m_i^2) = x_i \left[\left(q + \sum_{j=1}^i p_j \right)^2 - m_i^2 \right] \quad \forall i = 1, \dots, n$$

$$\sum_{i=1}^n x_i k_i = 0$$

Soft Singularities

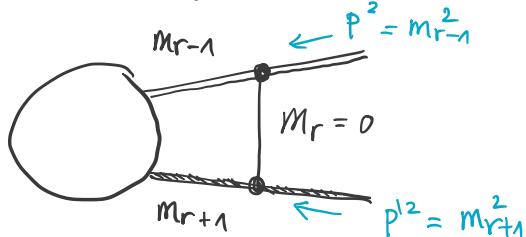
- consider only one propagator on-shell: $x_r \neq 0$ & $x_i = 0 \neq i \neq r$

$$\Rightarrow \begin{array}{l} k^2 = m_r^2 \\ k_r = 0 \end{array} \quad \left. \right\} \Rightarrow (\text{w.l.o.g. } k_r = q) \quad q = 0, \quad m_r = 0$$

- not sufficient yet $\int \frac{d^4 q}{q^2}$ finite \rightarrow need enhancement:
in general, depends on orientation

$$\frac{1}{(q+p)^2 - M^2} \xrightarrow{P^2 = M^2} \frac{1}{q^2 + 2qP} \sim \frac{1}{q}$$

- for the two neighbouring propagators $(r-1)$ & $(r+1)$ pure on-shell const.



Collinear Singularities

- consider two propagators going on-shell: $x_r \neq 0 \neq x_s$ & $x_i = 0 \forall i \neq r, s$

$$\Rightarrow k_r^2 = m_r^2, k_s^2 = m_s^2 \quad \left. \begin{array}{l} \\ \end{array} \right\} \Rightarrow k_r = \pm \frac{m_r}{m_s} k_s$$

$$x_r k_r + x_s k_s = 0 \quad \left. \begin{array}{l} \\ \end{array} \right\} \Rightarrow (k_r - k_s)^2 = (m_r \pm m_s)^2$$

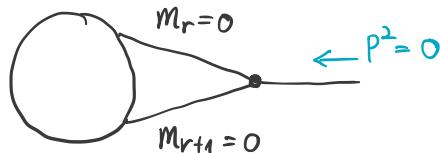
 only indep. of orientation if neighbours

- $s = r+1$ ($s = r-1$ via $s \leftrightarrow r$)

w.l.o.g $D_r = q^2 - m_r^2$

$$D_{r+1} = (q + p_{r+1})^2 - m_{r+1}^2 \quad \left. \begin{array}{l} \\ \end{array} \right\}$$

not sufficient yet: $m_r = m_{r+1} = 0$



* soft & collinear can also overlap: $\frac{1}{\epsilon^2}$

IR Singularities in Real Corrections

- soft ($k = \xi q, \xi \rightarrow 0$)

$$|\mathcal{M}_{m+n}(p_1, \dots, p_m, k)|^2 \xrightarrow{k \text{ soft}} -\frac{1}{\xi} 4\pi \mu^{2\varepsilon} \propto [\underline{\mathbb{J}}^\mu]^+ [\underline{\mathbb{J}}_\mu] \otimes |\mathcal{M}_m(p_1, \dots, p_m)|^2$$

colour correlation

↪ eikonal current:

$$\underline{\mathbb{J}}^\mu(q) = \sum_{i=1}^m \underline{\mathbb{T}}_i \frac{p_i^\mu}{(q \cdot p)}$$

$$\underline{\mathbb{T}}_i = \begin{cases} \text{gluon: } T_{abc}^b = i f^{abc} \\ \text{quark: } T_{\alpha\beta}^a = t_{\alpha\beta}^a \\ \text{anti-quark: } T_{\bar{\alpha}\beta}^a = -t_{\bar{\alpha}\beta}^a \end{cases}$$

outgoing

- collinear ($p_i \parallel p_j$)

$$|\mathcal{M}_{m+n}(p_1, \dots, p_m, p_j)|^2 \xrightarrow{p_i \parallel p_j} \frac{1}{(p_i \cdot p_j)} 4\pi \mu^{2\varepsilon} \hat{P}_{ij} \otimes |\mathcal{M}_m(p_1, \dots, (p_i + p_j), \dots, p_{m+n})|^2$$

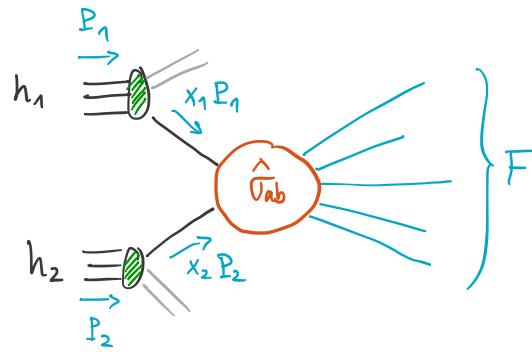
↪ Altarelli-Parisi splitting function

not regularized $\xrightarrow{z \rightarrow 1}$

$$\hat{P}_{qq}(z) = C_F \left[\frac{1+z^2}{1-z} - \epsilon(1-z) \right]$$

EX26

Hadron-Hadron Collisions



partonic
hard-scattering
cross section

$$\sigma_{h_1 h_2 \rightarrow F}(P_1, P_2) = \sum_{a/b} \int_0^1 dx_1 \int_0^1 dx_2 f_{a/h_1}(x_1) f_{b/h_2}(x_2) \hat{\sigma}_{ab \rightarrow F}(x_1 P_1, x_2 P_2)$$

parton distribution functions

$f_{a/h}(x)dx \hat{=} \# \text{ density}$

for parton a within h
to carry mom. frac. $[x, x+dx]$

EX 27)

Drell-Yan : On-shell W^+ production

$$\begin{aligned}
 \frac{d\sigma_{pp \rightarrow W^+}^{LO}}{dy_W} &= \int_0^1 dx_1 \int_0^1 dx_2 f_{u/p}(x_1) f_{\bar{d}/p}(x_2) \\
 &\quad \times \frac{\pi}{12} \frac{e^2}{S_W^2} (1-\epsilon) \delta(M_W^2 - x_1 x_2 S) \delta(y_W - \frac{1}{2} \ln(\frac{x_1}{x_2})) \\
 &= f_{u/p}(x_1) f_{\bar{d}/p}(x_2) \frac{\pi}{12} \frac{e^2}{S_W^2} (1-\epsilon) \left| x_{1,2} = \sqrt{\frac{M_W^2}{S}} e^{\pm y_W} \right.
 \end{aligned}$$

$(+ u \leftrightarrow \bar{d})$
↓

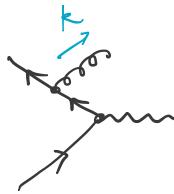
Drell-Yan @ NLO

- virtual corrections $\leftrightarrow \int d\bar{\Phi}_1(P_w)$

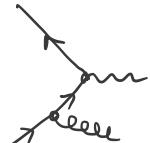


\leftarrow the only diagram $(\rightarrow \text{loop} \rightarrow = 0)$

- real corrections $\leftrightarrow \int d\bar{\Phi}_2(P_w, k)$

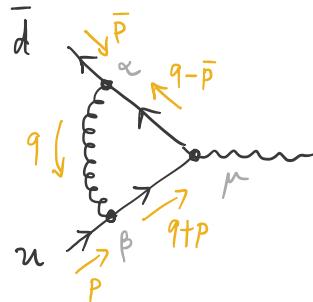


+



qg & $\bar{q}g$ channels via crossing
new partonic channel that
opens up @ NLO

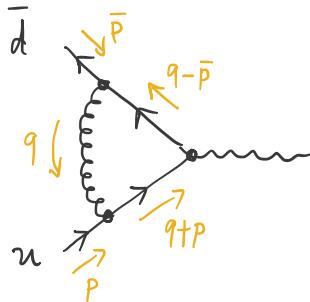
Drell-Yan : 1-loop corrections



$$iM_{u\bar{d} \rightarrow W^+}^{(1)}(p, \bar{p}) = \frac{e}{\sqrt{2} s_W} g_S^2 C_F \delta_{cc} \bar{V}_d(\bar{p}) \left\{ \frac{\int_0^{2e} \frac{d^D q}{(2\pi)^D}}{q^2 (q+p)^2 (q-\bar{p})^2} \frac{\gamma^\alpha (q-\bar{p}) \gamma^\mu (q+p) \gamma^\alpha}{\omega_u(p)} \right\} \omega_u(p) \epsilon_\mu^*(p_W)$$

EX 28

Drell-Yan : 1-loop corrections

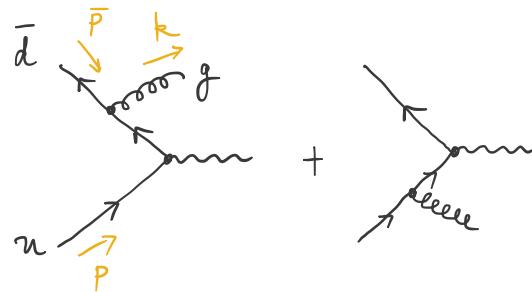


$$i\mathcal{M}_{u\bar{d} \rightarrow W^+}^{(1)}(p, \bar{p}) = \frac{e}{\sqrt{2} s_W} g_S^2 C_F \delta_{cc} \bar{V}_d(\bar{p}) \left\{ i \int \frac{d^D q}{(2\pi)^D} \frac{\gamma^\alpha(q-\bar{p}) \gamma^\mu(q+p) \gamma^\nu}{q^2 (q+p)^2 (q-\bar{p})^2} \right\} \omega_u(p) \epsilon_\mu^*(p_W)$$

$$= i\mathcal{M}_{u\bar{d} \rightarrow W^+}^{(0)} \left\{ -\frac{\alpha_S}{2\pi} C_F \frac{(4\pi)^\epsilon}{\Gamma(1-\epsilon)} \left(\frac{\mu^2}{M_W^2}\right)^\epsilon \left[\frac{1}{\epsilon^2} + \frac{3}{2\epsilon} + 4 - \frac{\pi^2}{2} + i\pi \left(\frac{1}{\epsilon} + \frac{3}{2} \right) + \mathcal{O}(\epsilon) \right] \right\}$$

$$\Rightarrow \hat{\Gamma}_{u\bar{d} \rightarrow W^+}^V = \hat{\Gamma}_{u\bar{d} \rightarrow W^+}^{LO} \cdot 2 \operatorname{Re} \{ \dots \}$$

Drell-Yan: Real-emission Amplitude



$$i\mathcal{M}_{u\bar{d} \rightarrow W^+ g}^{(0)}(p, \bar{p}, k) = \frac{ie}{\sqrt{2} S_W} g_S T_{cc}^a \epsilon_v^{*a}(k) \epsilon_\mu^*(p_W) \bar{V}_d(\bar{p}) \left\{ \frac{\gamma^\nu(k - \bar{p}) \gamma^\mu}{-S_{\bar{d}g}} + \frac{\gamma^\mu(p - k) \gamma^\nu}{-S_{ug}} \right\} \omega_u U_u(p)$$

Born: no kinematic dependence

$$\Rightarrow \langle |\mathcal{M}_{u\bar{d} \rightarrow W^+ g}^{(0)}|^2 \rangle = \langle |\mathcal{M}_{u\bar{d} \rightarrow W^+}|^2 \rangle \frac{8\pi\alpha_s}{M_W^2} C_F$$

$$\times \left\{ (1-\epsilon) \left[\frac{S_{\bar{d}g}}{S_{ug}} + \frac{S_{ug}}{S_{\bar{d}g}} \right] + \frac{2 S_{u\bar{d}} M_W^2}{S_{ug} S_{\bar{d}g}} - 2\epsilon \right\}$$

divergence for
 $S_{ug}, S_{\bar{d}g} \rightarrow 0$!

Drell-Yan: Real Phase Space

- we want to integrate out the gluon emission analytically ($D=4-2\epsilon$)

$$d\Phi_2(p_w, k; p + \bar{p}) = [dp_w] [dk] (2\pi)^4 \delta^{(4)}(p_w + k - (p + \bar{p}))$$

$$[dp] \equiv \frac{d^D p}{(2\pi)^D} (2\pi) \underbrace{\delta_{+}(p^2 - m^2)}_{\Theta(p^0) \delta(p^2 - m^2)}$$

- strategy: factor out the emission phase space

$$d\tilde{\Phi}_2(p_w, k; p + \bar{p}) \stackrel{!}{=} \tilde{d}\Phi_1(\tilde{p}_w) \otimes [dk]$$

in general, involves convolutions

EX 29

Drell-Yan: Real Phase Space

- we want to integrate out the gluon emission analytically ($D=4-2\epsilon$)

$$d\Phi_2(p_W, k; p + \bar{p}) = [dp_W] [dk] (2\pi)^4 \delta^{(4)}(p_W + k - (p + \bar{p}))$$

$$[dp] \equiv \frac{d^D p}{(2\pi)^D} (2\pi) \underbrace{\delta_{+}(p^2 - m^2)}_{\Theta(p^0) \delta(p^2 - m^2)}$$

- strategy: factor out the emission phase space

$$d\Phi_2(p_W, k; p + \bar{p}) = \frac{dz_1}{z_1} \frac{dz_2}{z_2} d\Phi_1(\tilde{p}_W; z_1 p + z_2 \bar{p}) \frac{(4\pi)^{\epsilon}}{\Gamma(1-\epsilon)} \left(\frac{\mu^2}{M_W^2}\right)^{\epsilon} \frac{M_W^2}{8\pi^2}$$

$$\times \frac{z_1 z_2 (1 + z_1 z_2)}{(z_1 + z_2)^{2-2\epsilon}} (1-z_1^{-\epsilon})(1-z_2^{-\epsilon})$$

$$z_1 = \sqrt{\frac{M_W^2}{S_{\text{ud}}} \frac{S_{\text{ud}} - S_{\text{sig}}}{S_{\text{ud}} - S_{\text{bg}}}}, \quad z_2 = \sqrt{\frac{M_W^2}{S_{\text{ud}}} \frac{S_{\text{ud}} - S_{\text{sig}}}{S_{\text{ud}} - S_{\text{bg}}}}$$

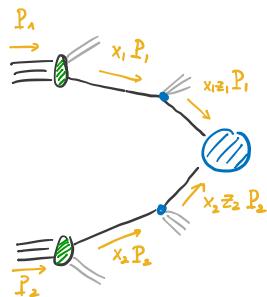
Drell-Yan: Real Corrections

- put the pieces together:

$$\frac{d\sigma_{pp \rightarrow W^+}^R}{dy_W} = \int_0^1 dx_1 \int_0^1 dx_2 f_{u/p}(x_1) f_{\bar{d}/p}(x_2) \int_0^1 dz_1 \int_0^1 dz_2 \delta(M_W^2 - x_1 x_2 z_1 z_2 s) \delta(y_W - \frac{1}{2} \ln \left(\frac{x_1 z_1}{x_2 z_2} \right))$$

$$\times \frac{(4\pi)^{\epsilon}}{\Gamma(1-\epsilon)} \left(\frac{\mu^2}{M_W^2} \right)^{\epsilon} \frac{z_1 z_2 (1-z_1 z_2)}{(z_1 + z_2)^{2-2\epsilon}} (1-z_1^2)^{-\epsilon} (1-z_2^2)^{-\epsilon}$$

$$\times \frac{L_S}{\pi} C_F \left\{ (1-\epsilon) \left[\frac{S_{u\bar{g}}}{S_{d\bar{g}}} + \frac{S_{d\bar{g}}}{S_{u\bar{g}}} \right] + \frac{2 S_{u\bar{d}} M_W^2}{S_{u\bar{g}} S_{d\bar{g}}} - 2\epsilon \right\} \underbrace{\frac{\pi}{z_1 z_2 S_{u\bar{d}}}}_{\text{flux}} \langle |M_{u\bar{d} \rightarrow W^+}^{(0)}|^2 \rangle$$



$$S_{u\bar{g}} = S_{u\bar{d}} \cdot \frac{z_1 (1-z_2^2)}{(z_1 + z_2)}, \quad S_{d\bar{g}} = S_{u\bar{d}} \cdot \frac{z_2 (1-z_1^2)}{(z_1 + z_2)}$$

Drell-Yan: IR divergences

- virtual

$$\sim \delta(1-z_1) \delta(1-z_2) \left\{ -\frac{\alpha_s}{\pi} C_F \frac{(4\pi)^{\epsilon}}{\Gamma(1-\epsilon)} \left(\frac{m^2}{M_W^2}\right)^{\epsilon} \left[\frac{1}{\epsilon^2} + \frac{3}{2\epsilon} \right] \right\} \hat{f}_{n\bar{n} \rightarrow W^+}(p, \bar{p})$$

~~$\frac{1}{\epsilon^2}$~~ ~~$\frac{3}{2\epsilon}$~~

- real

$$\hookrightarrow \text{use } \frac{1}{(1-z)^{1+\epsilon}} = -\frac{1}{\epsilon} \delta(1-z) + \frac{1}{[1-z]_+} - \epsilon \left[\frac{\ln(1-z)}{1-z} \right]_+ + \dots$$

$$\hookrightarrow \frac{3}{2} \delta(1-z) + \frac{2}{[1-z]_+} - 1-z = \left(\frac{1+z^2}{1-z} \right)_+$$

$$\begin{aligned} & \int_a^b dz [f(z)]_+ g(z) \\ &= \int_a^b dz f(z) [g(z) - g(a)] \end{aligned}$$

$$\sim \frac{\alpha_s}{\pi} C_F \frac{(4\pi)^{\epsilon}}{\Gamma(1-\epsilon)} \left(\frac{m^2}{M_W^2}\right)^{\epsilon} \left\{ \delta(1-z_1) \delta(1-z_2) \left[\frac{1}{\epsilon^2} + \frac{3}{2\epsilon} \right] \right. \\ & \quad \left. - \delta(1-z_1) \left(\frac{1+z_2^2}{1-z_2} \right)_+ \frac{1}{2\epsilon} - \delta(1-z_2) \left(\frac{1+z_1^2}{1-z_1} \right)_+ \frac{1}{2\epsilon} \right\}$$

left-over collinear initial-state singularities!

PDF renormalization

- universal collinear singularities are absorbed into the NLO PDFs \rightsquigarrow additional "mass factorization terms"

$$d\Gamma_{ab}^{MF}(p_a, p_b) = -\frac{ds}{2\pi} \frac{1}{\Gamma(1-\epsilon)} \left\{ \sum_{a'} \int_0^1 dz \left[-\frac{1}{\epsilon} \left(\frac{4\pi\mu^2}{\mu_F^2} \right)^\epsilon P_{aa'}(z) + K_{aa'}^{\text{F.S.}}(z) \right] d\Gamma_{a'b}^{\text{LO}}(zp_a, p_b) \right. \\ \left. + \sum_{b'} \int_0^1 dz \left[-\frac{1}{\epsilon} \left(\frac{4\pi\mu^2}{\mu_F^2} \right)^\epsilon P_{bb'}(z) + K_{bb'}^{\text{F.S.}}(z) \right] d\Gamma_{ab'}^{\text{LO}}(p_a, zp_b) \right\}$$

\Rightarrow Virtual + Real + MF : IR finite!