On the Complexity and Solution of Inverse Mixed Integer Linear Programs

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Outline

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2. Complexity
3. Algorithm
4. Computational Results

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Definitions

For a given $d \in \mathbb{R}^n$ and $\mathcal{P}$, we consider a MILP

$$z_{IP} = \min_{x \in \mathcal{P}} d^T x, \tag{1}$$

where,

$$\mathcal{P} = \{x \in \mathbb{R}^n | Ax = b, x \geq 0\} \cap (\mathbb{Z}^r \times \mathbb{R}^{n-r}).$$

For a given $c \in \mathbb{R}^n$, $x^0 \in \mathcal{P}$, the inverse problem is defined as follows.

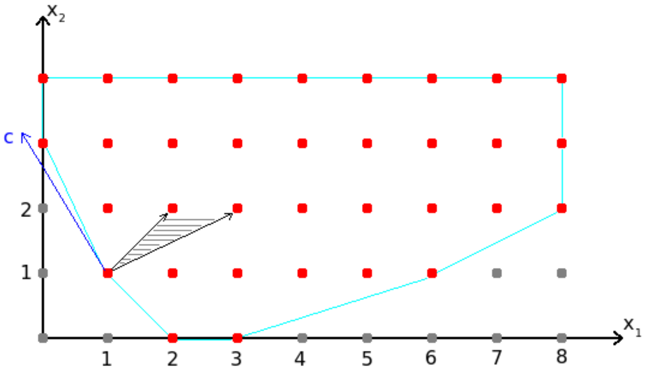
$$\begin{aligned} & \min \|c - d\| \\ & s.t. \\ & d^T x^0 \leq d^T x \qquad \forall x \in \mathcal{P}. \end{aligned} \tag{2}$$

Assumption: $\mathcal{P}$ is bounded.

A Small Example

Define forward problem feasible set $\mathcal{P}$ as follows and let $c = (-3, 5)$, $x^0 = (1, 1)$.

Figure: $\mathcal{P}$, its convex hull and feasible d cone



Definitions

$$
\begin{aligned}
& \min \|c - d\| \\
& s.t. \\
& d^T x^0 \le d^T x \qquad \forall x \in \mathcal{P}.
\end{aligned} \tag{4}
$$

- Model can be linearized for l_1 and l_∞ norms.
- Convex hull of $\mathcal{P}$ is a polytope.
- Last constraint set can be represented with the set of extreme points of convex hull of $\mathcal{P}$.
- Let $\mathcal{E}$ be the set of extreme points of convex hull of $\mathcal{P}$, $\mathcal{E}$ is finite.

Inverse MILP with l_1 norm

$$
\begin{aligned}
z_{IP}^1 &= \min \sum_{i=1}^{n} \theta_i \\
s.t. \\
c_i - d_i &\leq \theta_i && \forall i \in \{1, 2, \ldots, n\} \\
d_i - c_i &\leq \theta_i && \forall i \in \{1, 2, \ldots, n\} \\
d^T x^0 &\leq d^T x && \forall x \in \mathcal{E}.
\end{aligned}
\tag{3}
$$

Inverse MILP with l_∞ norm

$$
\begin{aligned}
z_{IP}^{\infty} &= \min y \\
& s.t. \\
& c_i - d_i \le y && \forall i \in \{1, 2, \ldots, n\} \\
& d_i - c_i \le y && \forall i \in \{1, 2, \ldots, n\} \\
& d^T x^0 \le d^T x && \forall x \in \mathcal{E}.
\end{aligned}
\tag{4}
$$

For the remainder of the presentation, we deal with the case of l_∞ norm. Let $\mathcal{S}$ represent feasible set of the inverse IP, defined as

$$\mathcal{S} = \left\{ (y, d) \in \mathbb{R} \times \mathbb{R}^n | y \ge \|c - d\|_\infty, d^T (x^0 - x) \le 0 \; \forall x \in \mathcal{E} \right\}.$$

Note that $\mathcal{S}$ is a polyhedron.

Polynomially Solvable Forward Problems

Ahuja and Orlin [1] determines the complexity of inverse problem when the forward problem is polynomially solvable.

Theorem

(Ahuja and Orlin [1]) If a forward problem is polynomially solvable for each linear cost function, then corresponding inverse problems under l_1 and l_∞ are polynomially solvable.

Forward Problems

Define the following problems related to MILP and inverse MILP.

Definition

MILP decision problem: Given $\gamma \in \mathbb{Q}$ decide whether $d^T x \leq \gamma$ holds for some $x \in \mathcal{P}$.

Definition

MILP optimization problem: Find solution vector x^* such that $x^* \in \text{argmin}_{x \in \mathcal{P}} d^T x$ or decide the problem is unbounded or decide the problem is infeasible.

Inverse Problems

Definition

Inverse MILP decision problem: Given $\gamma \in \mathbb{Q}$ decide whether $y \leq \gamma$ holds for some $(y, d) \in \mathcal{S}$.

Definition

Inverse MILP optimization problem: Find solution vector (y^*, d^*), such that $(y^*, d^*) \in \text{argmin}_{(y,d)\in\mathcal{S}} y$.

Definition

Inverse MILP separation problem: Given a vector $(\bar{y}, \bar{d}) \in \mathbb{Q} \times \mathbb{Q}^n$, decide whether $(\bar{y}, \bar{d})$ is in $\mathcal{S}$, and if not, find a hyperplane that separates $(\bar{y}, \bar{d})$ from $\mathcal{S}$, i.e., find $\pi \in \mathbb{Q}^{n+1}$ such that $\pi^T \begin{bmatrix} \bar{y} \\ \bar{d} \end{bmatrix} < \min \left\{ \pi^T \begin{bmatrix} y \\ d \end{bmatrix} | (y, d) \in \mathcal{S} \right\}$.

Component 1–An Observation

Recall inverse MILP optimization problem.

$$
\begin{aligned}
z_{IP}^{\infty} &= \min y \\
s.t. & \\
c_i - d_i &\leq y && \forall i \in \{1, 2, \ldots, n\} \\
d_i - c_i &\leq y && \forall i \in \{1, 2, \ldots, n\} \\
d^T x^0 &\leq d^T x && \forall x \in \mathcal{E}.
\end{aligned}
$$

Definition

Inverse MILP separation problem: Given a vector $(\bar{y}, \bar{d}) \in \mathbb{Q} \times \mathbb{Q}^n$, decide whether $(\bar{y}, \bar{d})$ is in $\mathcal{S}$, and if not, find a hyperplane that separates $(\bar{y}, \bar{d})$ from $\mathcal{S}$, i.e., find $\pi \in \mathbb{Q}^{n+1}$ such that $\pi^T \begin{bmatrix} \bar{y} \\ \bar{d} \end{bmatrix} < \min \left\{ \pi^T \begin{bmatrix} y \\ d \end{bmatrix} \middle| (y, d) \in \mathcal{S} \right\}$.

Component 2–GLS Theorem

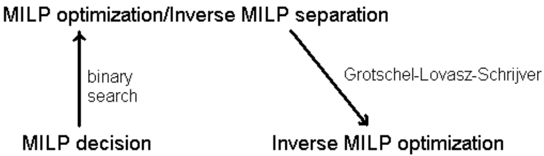
The following theorem by Grötschel et al. indicates the relationship between separation and optimization problems.

Theorem

(Grötschel et al. [2]) Given an oracle for the separation problem, the optimization problem over a given polyhedron with linear objective can be solved in time, polynomial in φ, n and the encoding length of objective coefficient vector, where facet complexity of polyhedron is at most φ.

Complexity of MILP optimization/decision problems

Figure: Problem relations



Theorem

Inverse MILP optimization problem under l_∞/l_1 norm is solvable in time polynomial of φ, $n + 1/2n$, and encoding length of $(1, 0, \ldots, 0)/(1, \ldots, 1, 0, \ldots, 0)$, given an oracle for the MILP decision problem.

Corollary

Inverse MILP decision problem is in Δ_2^{P}.

Inverse MILP decision is Δ_2^{P}–complete

Inverse Decision problem is Δ_2^{P}–complete by a reduction from DSAT problem.

Definition

Deterministic formula: Let $Y_1, Y_2, \ldots, Y_k, \{x_1, x_2, \ldots, x_{k-1}\}$ be sets of Boolean variables. A formula F involving these variables is said to be *deterministic* if it consists of the conjuction of the following clauses.

(a) For each $y \in Y_1 \cup Y_k$ either (y) or $(\bar{y})$ is a clase of F.

(b) For each $j = 1, 2, \ldots, k-1$ and each $y \in Y_{j+1}$, there are two sets of conjuctions of literals from $Y_j \cup \{x_j\}$, say $\{C_i\}$ and $\{\mathsf{D}_i\}$, such that $(C_i \rightarrow y)$ and $(\mathsf{D}_i \rightarrow \bar{y})$ are all clauses of F, and, furthermore, for any truth assignment for $Y_j \cup \{x_j\}$ exactly one of the conjuctions in $\{C_i\} \cup \{D_i\}$ is true.

Definition

Deterministic Satisfiability (DSAT) is the following computational problem. Given k formulas $F_0(x_1, \ldots, x_{k-1}, Y_1, \ldots, Y_k), F_1(Y_1, Z_1), \ldots, F_{k-1}(Y_{k-1}, Z_{k-1})$, where $\{x_1, \ldots, x_{k-1}\}, Y_1, \ldots, Y_k, Z_1, \ldots, Z_{k-1}$ are disjoint sets of variables and F_0 is a deterministic formula, is there a truth assignment $\hat{x}_1, \ldots, \hat{x}_{k-1}, \hat{Y}_1, \ldots, \hat{Y}_k$ for x and y variables such that

(a) F_0 is satisfied, and

(b) $F_j(\hat{Y}_j, Z_j))$ is satisfiable iff $\hat{x}_j = true$.

Theorem

(Papadimitriou [3]) DSAT is Δ_2^{P}–complete.

Proof follows by reducing a Deterministic Truing Machine with an NP oracle to DSAT.

Inverse MILP decision is Δ_2^{P}–complete

- DSAT is canonical Δ_2^{P}–complete problem, like SAT for NP–complete.
- Papadimitriou defines DSAT to reduce it to unique optimum problem and shows that it is Δ_2^{P}–complete [3].
- Using similar techniques DSAT can be reduced to inverse TSP decision version.
- Inverse TSP decision version is Δ_2^{P}–complete. Since it is a special case of inverse MILP decision, inverse MILP decision is also Δ_2^{P}–complete.

Theorem

Inverse MILP decision problem is Δ_2^{P}–complete.

Proposed Algorithm

First, we define two parametric problems named P_k and $InvP_k$ as follows

$$\min_{x \in \mathcal{P}} d^{kT} x \tag{P_k}$$

$$\begin{aligned}
& \min y \\
& s.t. \\
& c_i - d_i \leq y && \forall i \in \{1, 2, \dots, n\} \\
& d_i - c_i \leq y && \forall i \in \{1, 2, \dots, n\} \\
& d^T x^0 \leq d^T x && \forall \mathcal{E}^{k-1}
\end{aligned} \tag{$InvP_k$}$$

where $\mathcal{E}^{k-1}$ is the set of extreme points found so far.

Proposed Algorithm

Algorithm 1

$k \leftarrow 1$
$d^k \leftarrow c$
Solve P_k, $x^k \leftarrow x^*$
while $d^{kT}(x^0 - x^k) > 0$ **do**
  $k \leftarrow k + 1$
  Solve $InvP_k$, $d^k \leftarrow d^*$
  Solve P_k, $x^k \leftarrow x^*$
end while

- x^k is an optimal solution to P_k, whereas d^k is an optimal solution to $invP_k$.
- The algorithm stops when a generated cut is not violated by the current solution.

Table: MIPLIB Iteration number–l_1

p. name	pert. 0.1	pert. 0.2	pert. 0.3	pert. 0.4	pert. 0.5	pert. 0.6
bell3a	1	1	2	2	11	7
blend2	2	2	2	2	2	2
dcmulti	1	5	1	27	21	47
egout	20	20	20	20	20	20
enigma	1	1	1	1	1	1
flugpl	1	1	1	1	1	1
gen	1	142	188	852	844	1578
gt2	1	1	1	1	1	1
l152lav	16	23	46	46	46	46
lseu	1	1	1	1	1	1
mas74	1	1	1	1	1	1
misc03	1	1	1	1	1	1
misc06	1	1	1	1	1	1
mod008	3	3	6	6	6	6
mod010	51	67	311	390	390	390
modglob	1	1	1	1	1	1
p0201	1	1	4	4	4	4
p0282	1	1	1	1	1	1
pk1	1	1	1	1	1	1
pp08aCUTS	11	12	4	4	14	3
qiu	1	16	10	10	10	10
rgn	1	1	1	1	1	1
rout	1	1	1	1	1	1
stein27	1	1	1	1	1	1
vpm1	1	1	1	1	1	1

Table: MIPLIB Iteration number–l_{inf}

p. name	pert. 0.1	pert. 0.2	pert. 0.3	pert. 0.4	pert. 0.5	pert. 0.6
air03	2	3	2	2	2	2
blend2	3	3	3	2	3	2
cap6000	6	27	30	30	30	32
dcmulti	1	3	1	10	17	21
egout	8	8	8	9	9	9
enigma	1	1	1	1	1	1
flugpl	1	1	1	1	1	1
gen	1	21	35	37	220	179
gt2	1	1	1	1	1	1
harp2	6	6	6	6	7	8
l152lav	4	3	4	4	4	4
lseu	1	1	1	1	1	1
mas74	1	1	1	1	1	1
misc03	1	1	1	1	1	1
misc06	1	1	1	1	1	1
mod008	2	2	3	3	3	2
mod010	4	3	3	3	3	3
modglob	1	1	1	1	1	1
p0201	1	1	2	3	3	3
p0282	1	1	1	1	1	1
pk1	1	1	1	1	1	1
pp08aCUTS	11	11	5	5	8	5
qnet1	1	1	1	1	1	1
rgn	1	1	1	1	1	1
rout	1	1	1	1	1	1
stein27	1	1	1	1	1	1
vpm1	1	1	1	1	1	1

Table: TSPLIB Iteration number–l_1

p. name	pert. 0.1	pert. 0.2	pert. 0.3	pert. 0.4	pert. 0.5	pert. 0.6
att48	1	1	3	4	9	24
berlin52	1	1	1	1	1	1
bier127	73	103	186	307	298	358
burma14	1	1	1	1	1	1
ch130	1	14	31	52	88	81
ch150	1	7	15	20	49	49
eil101	1	4	7	4	8	6
eil51	1	2	4	4	7	16
eil76	1	3	2	4	4	6
kroA100	3	3	6	10	17	38
kroC100	1	3	4	4	8	14
kroD100	1	4	13	24	43	138
kroE100	1	13	13	17	27	26
lin105	1	2	6	18	20	20
pr107	2	13	23	30	38	39
pr124	2	3	6	8	15	13
pr152	3	41	18	18	24	45
rat99	1	10	4	13	21	22
rd100	1	3	8	8	11	11
st70	1	4	13	14	11	14
u159	1	1	1	8	9	13
ulysses16	1	1	1	1	1	1
ulysses22	1	1	3	3	3	5

Table: TSPLIB Iteration number–l_{inf}

p. name	pert. 0.1	pert. 0.2	pert. 0.3	pert. 0.4	pert. 0.5	pert. 0.6
att48	1	1	2	2	3	5
berlin52	1	1	1	1	1	1
bier127	6	5	6	5	6	6
burma14	1	1	1	1	1	1
ch130	1	5	4	6	6	6
ch150	1	4	4	5	5	5
eil101	1	2	2	2	2	2
eil51	1	2	2	2	2	3
eil76	1	3	4	3	3	2
kroA100	2	2	2	3	3	3
kroA150	4	3	5	5	5	5
kroB200	3	4	3	5	6	6
kroC100	1	3	3	3	2	3
kroD100	1	2	4	4	4	5
kroE100	1	5	5	5	5	4
lin105	1	1	3	2	4	4
pr107	1	4	4	6	4	7
pr124	2	3	2	3	3	3
pr136	2	3	4	6	6	6
pr144	3	4	5	5	6	4
pr152	2	3	5	5	3	3
pr76	1	2	3	3	3	3
rat99	1	3	2	4	4	4
rd100	1	2	3	3	3	3
st70	1	2	4	5	4	5
u159	1	1	2	3	3	4
ulysses16	1	1	1	1	1	1
ulysses22	1	1	3	3	2	5

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Operations Research, 49(5):771–783, September/October 2001.

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End of presentation

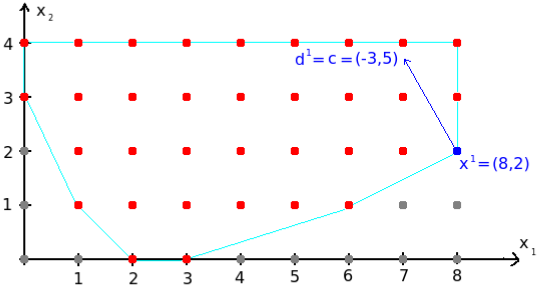
This is end of presentation!

Thank you for listening!

A Small Example: Iteration 1

Let $\mathcal{P}$ and convex hull of $\mathcal{P}$ given as in Figure. Let $c = (-3, 5)$ and $x^0 = (1, 1)$. We know $d^1 = c$. When objective coefficient of forward problem is d^1, optimal solution is $x^1 = (8, 2)$.

Figure: Iteration 1, d^1 and x^1

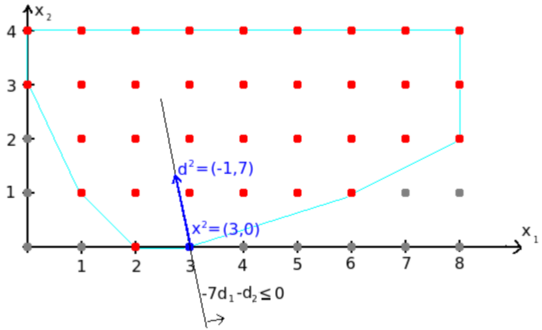


Using x^1, we generate cut $d^\top(x^0 - x^1) \leq 0$, i.e. $-7d_1 - d_2 \leq 0$.

A Small Example: Iteration 2

Following figure shows feasible cone for d. d^2 is the inverse optimal with the current cut. x^2 is the forward optimal solution when d^2 is objective coefficient vector.

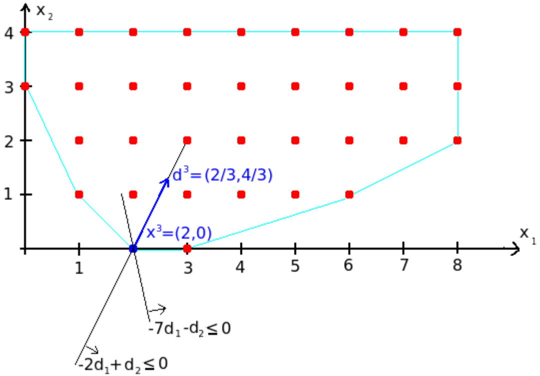
Figure: Iteration 2, feasible d cone ($(3,0)$ is apex), d^2 and x^2



Using x^2, we generate cut $d^\top(x^0 - x^2) \leq 0$, i.e. $-2d_1 + d_2 \leq 0$.

A Small Example: Iteration 3

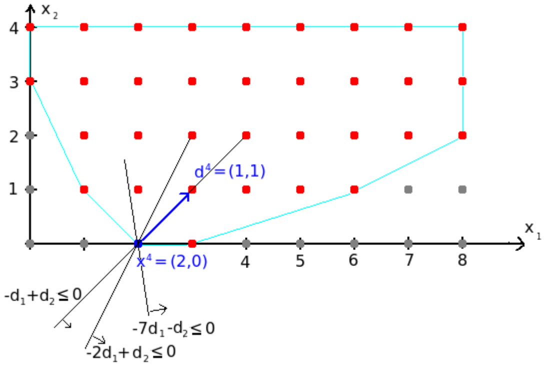
Figure: Iteration 3, feasible d cone, d^3 and x^3



Using x^3, we generate cut $d^\top(x^0 - x^3) \leq 0$, i.e. $-d_1 + d_2 \leq 0$.

A Small Example: Iteration 4

Figure: Iteration 4, feasible d cone, d^4 and x^4



Using x^4, we generate cut $d^\top(x^0 - x^4) \leq 0$, i.e. $-d_1 + d_2 \leq 0$. Note that current d^4 does not violate this cut, then d^4 is the optimal solution of inverse problem.