On the Complexity of Inverse Mixed–Integer Linear Programming

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Abstract

The *inverse optimization problem* with respect to a given linear optimization problem with objective function $c \in \Re^n$ and feasible region $\mathcal{P} \subseteq \Re^n$ is defined with respect to a given $x^0 \in \mathbb{R}^n$. Informally, the problem is to find the objective function vector $d \in \Re^n$ closest to c according to a given norm for which x^0 is an optimal solution to $\max_{x\in\mathcal{P}} d^\top x$. Inverse optimization arises in systems where some parameters of an underlying optimization problem, which we refer to as the *forward problem*, are unknown (cannot be observed directly). By observing an optimal solution to the underlying problem, we wish to determine values for these parameters.

The contribution of this study is to establish the complexity of inverse MILP. Ahuja and Orlin established the complexity of the inverse linear programming problem using the Grötschel, Lovász and Schrijver (GLS) separation–optimization procedure. The same technique can be applied straightforwardly to show that inverse MILP is in Δ_2^{P}, but we show that it is in fact coNP–complete, a stronger (and somewhat surprising) result. This result can be generalized to other inverse problems and yields insight into the nature of the GLS framework.

Introduction

We consider the inverse problem with respect to a given MILP for which the feasible set is

$$\mathcal{P} = \{x \in \mathbb{R}^n | Ax = b, x \geq 0\} \cap (\mathbb{Z}^r \times \mathbb{R}^{n-r}). \qquad (1)$$

Formally, the problem can be formulated as follows,

$$\min_{d\in\mathcal{K}(y)\cap\mathcal{D}} y \qquad (2)$$

where $\mathcal{K}$(y) is cone $\{\alpha d \in \mathbb{R}^n : \|c - d\| \leq y, \alpha \in \mathbb{R}_+\}$, $\mathcal{D}$ is cone $\{d \in \mathbb{R}^n : d^\top(x^0 - x) \geq 0 \forall x \in \mathcal{P}\}$ and $\|\cdot\|$ is the chosen norm.

Inverse MILP under l_∞ norm

$$\begin{aligned}
& z_{IP}^\infty = \min y \\
& s.t. \\
& c_i - d_i \leq y && \forall i \in \{1, 2, \ldots, n\} \\
& d_i - c_i \leq y && \forall i \in \{1, 2, \ldots, n\} \\
& d^\top(x^0 - x) \geq 0 && \forall x \in \mathcal{P}.
\end{aligned} \qquad (3)$$

An Algorithmic Framework for Inverse MILP under l_∞ norm

We propose a cutting plane algorithm for solving inverse MILPs. We can cut a given $(\overline{y}, \overline{d})$ by optimizing $\overline{d}^\top x$ over $\mathcal{P}$. We define two parametric problems named P_k and $InvP_k$ as follows,

$$\begin{aligned}
& \min y \\
& s.t. \\
& c_i - d_i \leq y && \forall i \in \{1, 2, \ldots, n\} \\
& d_i - c_i \leq y && \forall i \in \{1, 2, \ldots, n\} \\
& d^\top(x^0 - x) \geq 0 && \forall x \in \mathcal{E}^k
\end{aligned} \qquad (InvP_k)$$

$$\max_{x\in\mathcal{P}} d^{kT} x \qquad (P_k)$$

where $\mathcal{E}^k$ is the set of solutions found by solving $P_1, \ldots, P_{k-1}$.

Algorithm 1. Cutting plane for inverse MILP under l_∞ norm

$k \leftarrow 0$, $\mathcal{E}^1 \leftarrow \emptyset$

do

  $k \leftarrow k + 1$

  Solve $InvP_k$, $d^k \leftarrow d^*$

  Solve P_k, if unbounded $y^* \leftarrow \|c\|_\infty$, STOP; else $x^k \leftarrow x^*$

  $\mathcal{E}^{k+1} \leftarrow \mathcal{E}^k \cup \{x^k\}$

while $d^{k\top}(x^0 - x^k) < 0$

Complexity of Inverse MILP

Theorem 1. *Inverse MILP optimization problem under l_∞/l_1 norm is solvable in time polynomial in the size of the problem input, given an oracle for the MILP decision problem. Existence of the described procedure is the proof for the theorem.*

Inverse MILP decision problem (INV): Given $\gamma \in \mathbb{Q}$ decide whether the system $\mathcal{K}(\gamma) \cap \mathcal{D}$ is feasible.
Inputs: γ, c, x^0 and set $\mathcal{P}$.

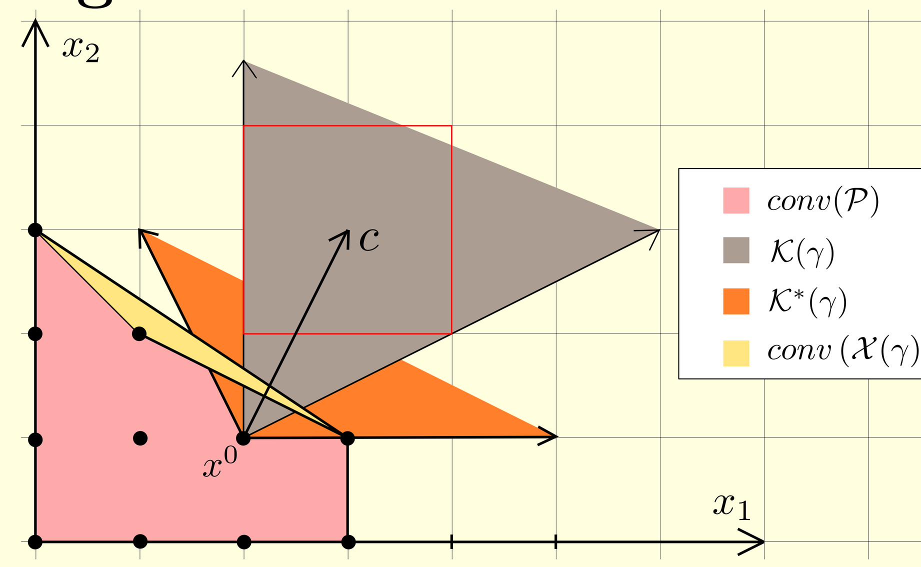
Theorem 2. *INV is* coNP*–complete.*

We prove this result by showing the existence of a short certificate for the negative answer. We define the following sets,

$$\begin{aligned}
\mathcal{X}(\gamma) &= \{x \in \mathcal{P} | \exists d \in \mathcal{K}(\gamma) \ s.t. \ d^\top(x - x^0) > 0\}, \\
\mathcal{K}^*(\gamma) &= \{x \in \mathbb{R}^n | d^\top(x - x^0) \geq 0 \ \forall d \in \mathcal{K}(\gamma)\}.
\end{aligned}$$

$conv(\mathcal{X}(\gamma)) \cap int(\mathcal{K}^*(\gamma)) \neq \emptyset$ when answer is negative. For any $\overline{x} \in conv(\mathcal{X}(\gamma)) \cap int(\mathcal{K}^*(\gamma))$, set of points in $\mathcal{X}(\gamma)$ that gives $\overline{x}$ as a convex combination is a short certificate. Figure 1 shows defined sets under l_∞ norm where $x^0 = (2, 1)$, $c = (1, 2)$, $\gamma = 1$ and $\mathcal{P}$ is pure integer set. Answer is negative.

Figure 1 Two dimensional example



Conclusions and Future Research

Testing performance of algorithm computationally. Investigating the reasons of GSL leading loose complexity results.