

COMP547

DEEP UNSUPERVISED LEARNING

Lecture #6 – Normalizing Flow Models



KOÇ
UNIVERSITY

Aykut Erdem // Koç University // Spring 2021

Good news, everyone!

- Assignment 1 is out, due March 17, 23:59!
- PyTorch tutorial at 20:00 today!
- Paper presentations will be prerecorded!
- Don't forget to check the rubrics before preparing your slide decks!



Previously on COMP547

- Motivation
- Simple generative models:
histograms
- Parameterized distributions and
maximum likelihood
- Autoregressive Models
 - Recurrent Neural Nets
 - Masking-based Models



Our Goal Today

- How to fit a density model $p_{\theta}(x)$ with continuous $x \in \mathbb{R}^n$
- What do we want from this model?
 - Good fit to the training data (really, the underlying distribution!)
 - For new x , ability to evaluate $p_{\theta}(x)$
 - Ability to sample from $p_{\theta}(x)$
 - And, ideally, a latent representation that's meaningful

Our Goal Today

- How to fit a density model $p_{\theta}(x)$ with **continuous** $x \in \mathbb{R}^n$
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 - For new x , ability to evaluate $p_{\theta}(x)$
 - Ability to sample from $p_{\theta}(x)$
 - And, ideally, **a latent representation** that's meaningful
- **Differences from Autoregressive Models from last lecture**

Lecture overview

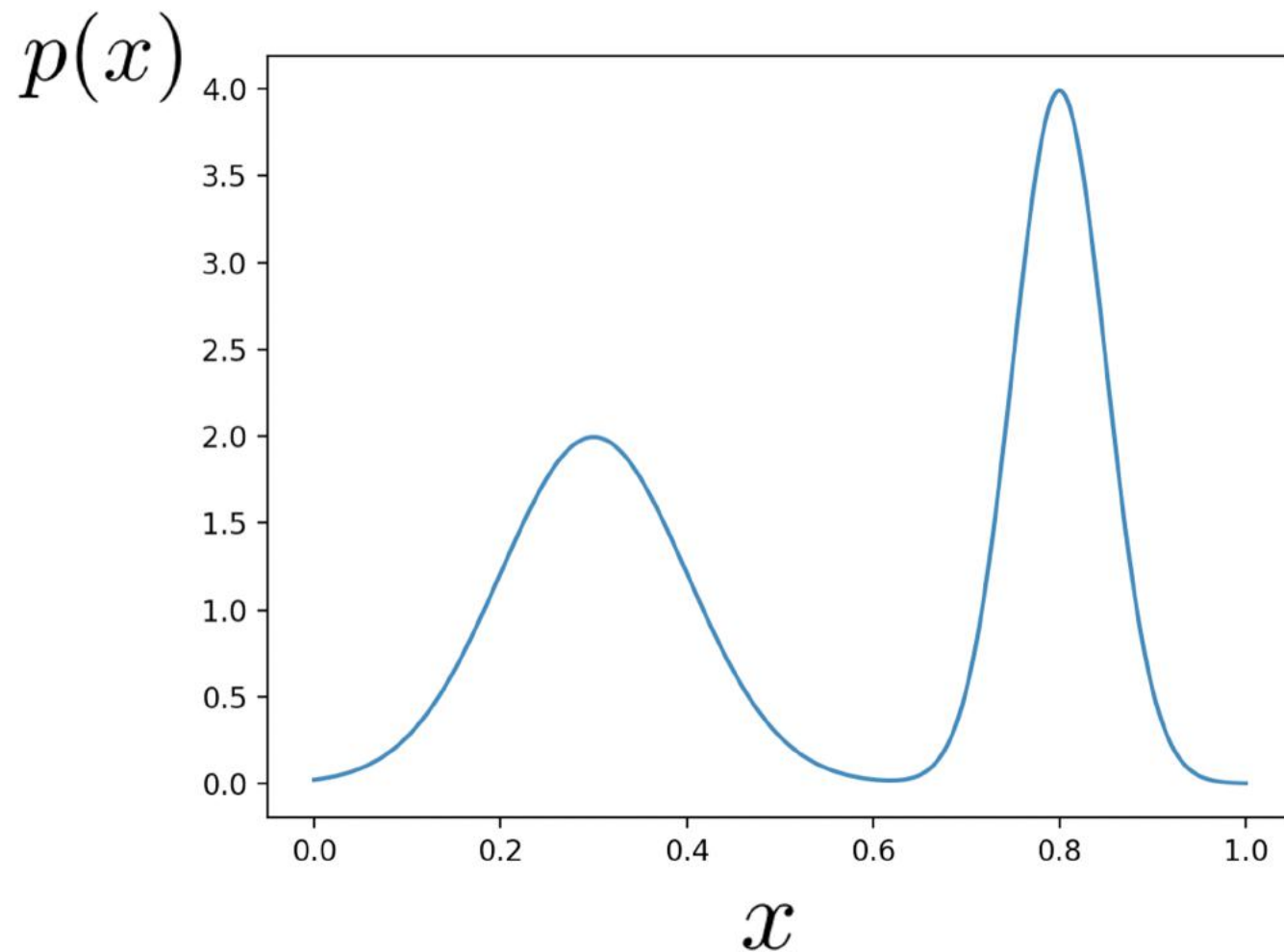
- Foundations of Flows (1-D)
- 2-D Flows
- N-D Flows
- Dequantization

Disclaimer: Much of the material and slides for this lecture were borrowed from
—Pieter Abbeel, Peter Chen, Jonathan Ho, Aravind Srinivas' Berkeley CS294-158 class
—Chin-Wei Huang slides on Normalizing Flows

Lecture overview

- Foundations of Flows (1-D)
- 2-D Flows
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Quick Refresher: Probability Density Models



$$P(x \in [a, b]) = \int_a^b p(x) dx$$

How to fit a density model?

Continuous data

0.22159854, 0.84525919, 0.09121633, 0.364252 , 0.30738086,
0.32240615, 0.24371194, 0.22400792, 0.39181847, 0.16407012,
0.84685229, 0.15944969, 0.79142357, 0.6505366 , 0.33123603,
0.81409325, 0.74042126, 0.67950372, 0.74073271, 0.37091554,
0.83476616, 0.38346571, 0.33561352, 0.74100048, 0.32061713,
0.09172335, 0.39037131, 0.80496586, 0.80301971, 0.32048452,
0.79428266, 0.6961708 , 0.20183965, 0.82621227, 0.367292 ,
0.76095756, 0.10125199, 0.41495427, 0.85999877, 0.23004346,
0.28881973, 0.41211802, 0.24764836, 0.72743029, 0.20749136,
0.29877091, 0.75781455, 0.29219608, 0.79681589, 0.86823823,
0.29936483, 0.02948181, 0.78528968, 0.84015573, 0.40391632,
0.77816356, 0.75039186, 0.84709016, 0.76950307, 0.29772759,
0.41163966, 0.24862007, 0.34249207, 0.74363912, 0.38303383, ...

Maximum Likelihood:

$$\max_{\theta} \sum_i \log p_{\theta}(x^{(i)})$$

Equivalently:

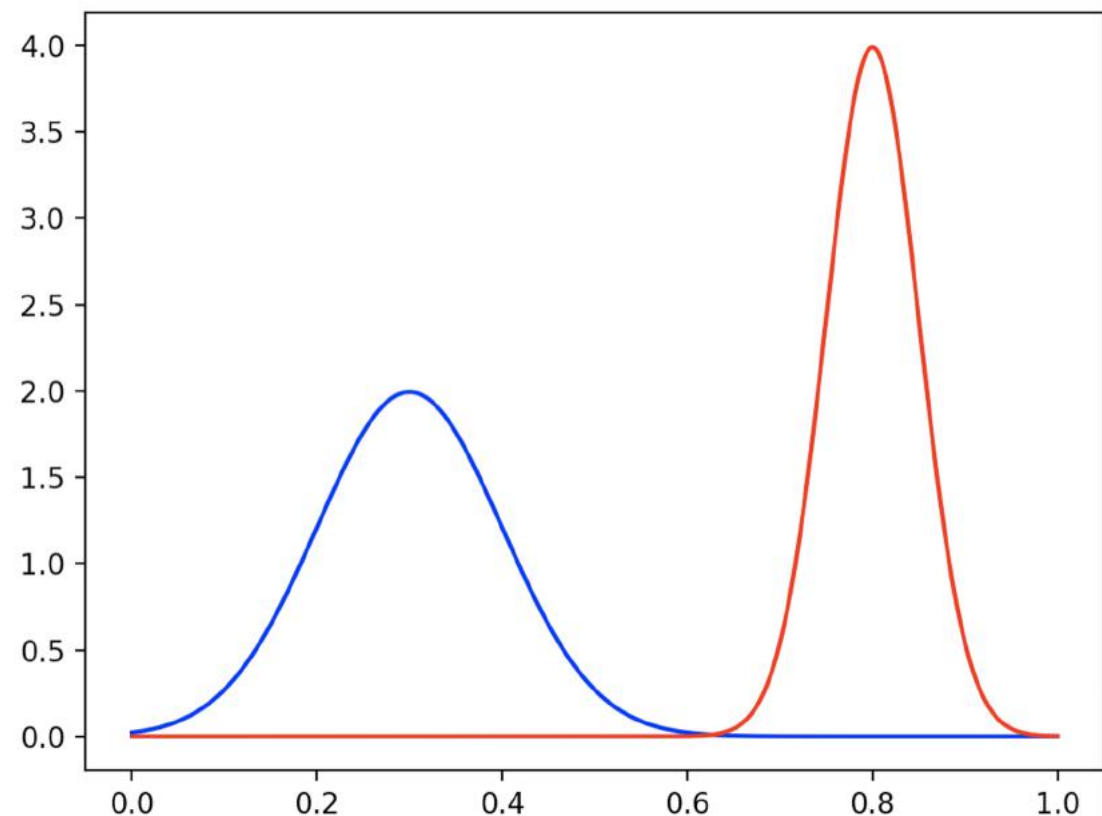
$$\min_{\theta} \mathbb{E}_x [-\log p_{\theta}(x)]$$

Example Density Model: Mixtures of Gaussians

$$p_{\theta}(x) = \sum_{i=1}^k \pi_i \mathcal{N}(x; \mu_i, \sigma_i^2)$$

Parameters: means and variances of components, mixture weights

$$\theta = (\pi_1, \dots, \pi_k, \mu_1, \dots, \mu_k, \sigma_1, \dots, \sigma_k)$$



Aside on Mixtures of Gaussians

Do mixtures of Gaussians work for high-dimensional data?

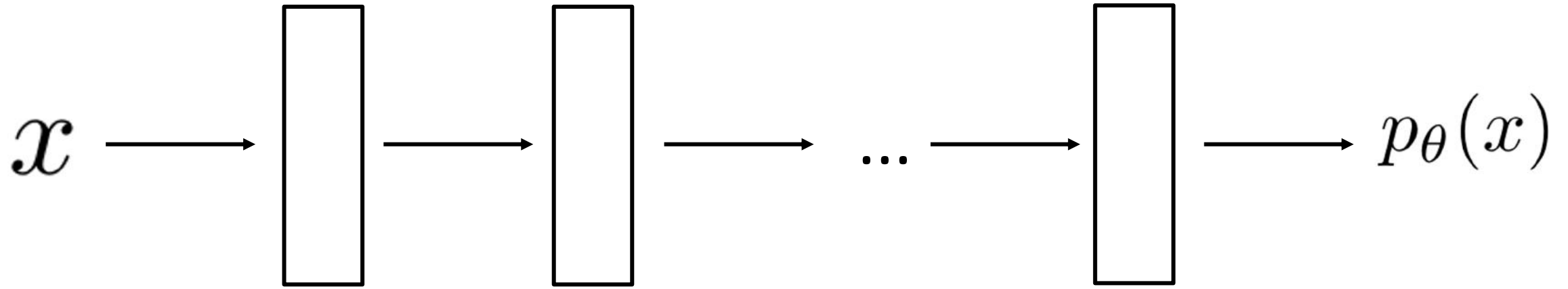
Not really. The sampling process is:

1. Pick a cluster center
2. Add Gaussian noise

Imagine this for modeling natural images! The only way a realistic image can be generated is if it is a cluster center, i.e. if it is already stored directly in the parameters.



How to fit a general density model?



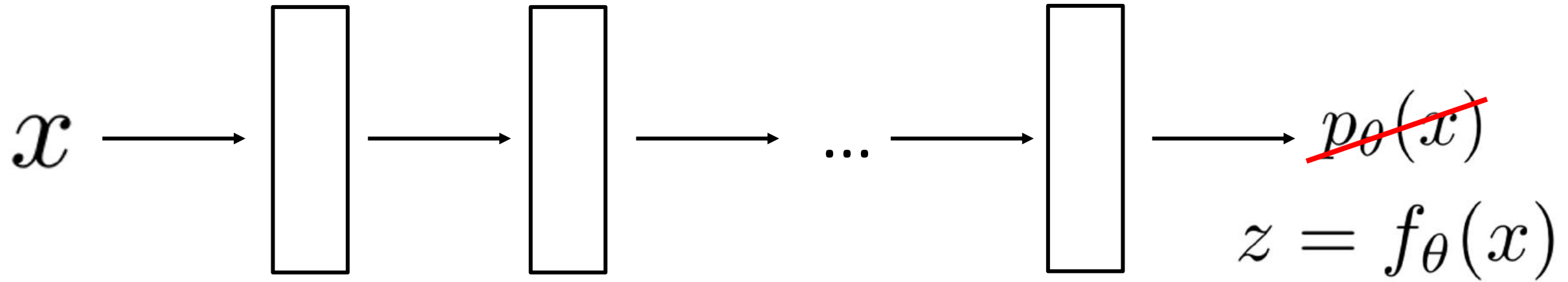
- How to ensure proper distribution?

$$\int_{-\infty}^{+\infty} p_\theta(x) dx = 1 \quad p_\theta(x) \geq 0 \quad \forall x$$

- How to sample?
- Latent representation?

Easily achieved for discrete data, using softmax
What about continuous data?

Flows: Main Idea

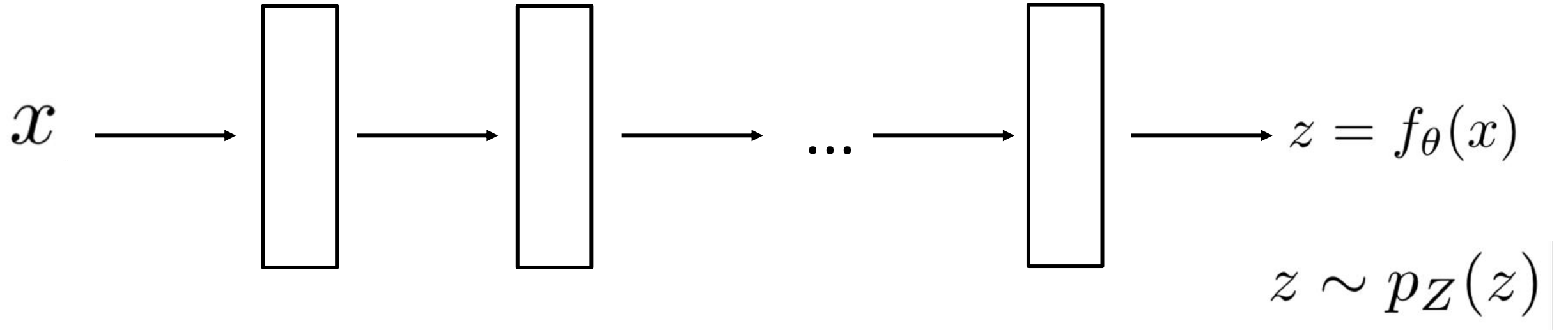


Generally: $z \sim p_Z(z)$

Normalizing Flow: $z \sim \mathcal{N}(0, 1)$

How to train? How to evaluate $p_\theta(x)$? How to sample?

Flows: Training



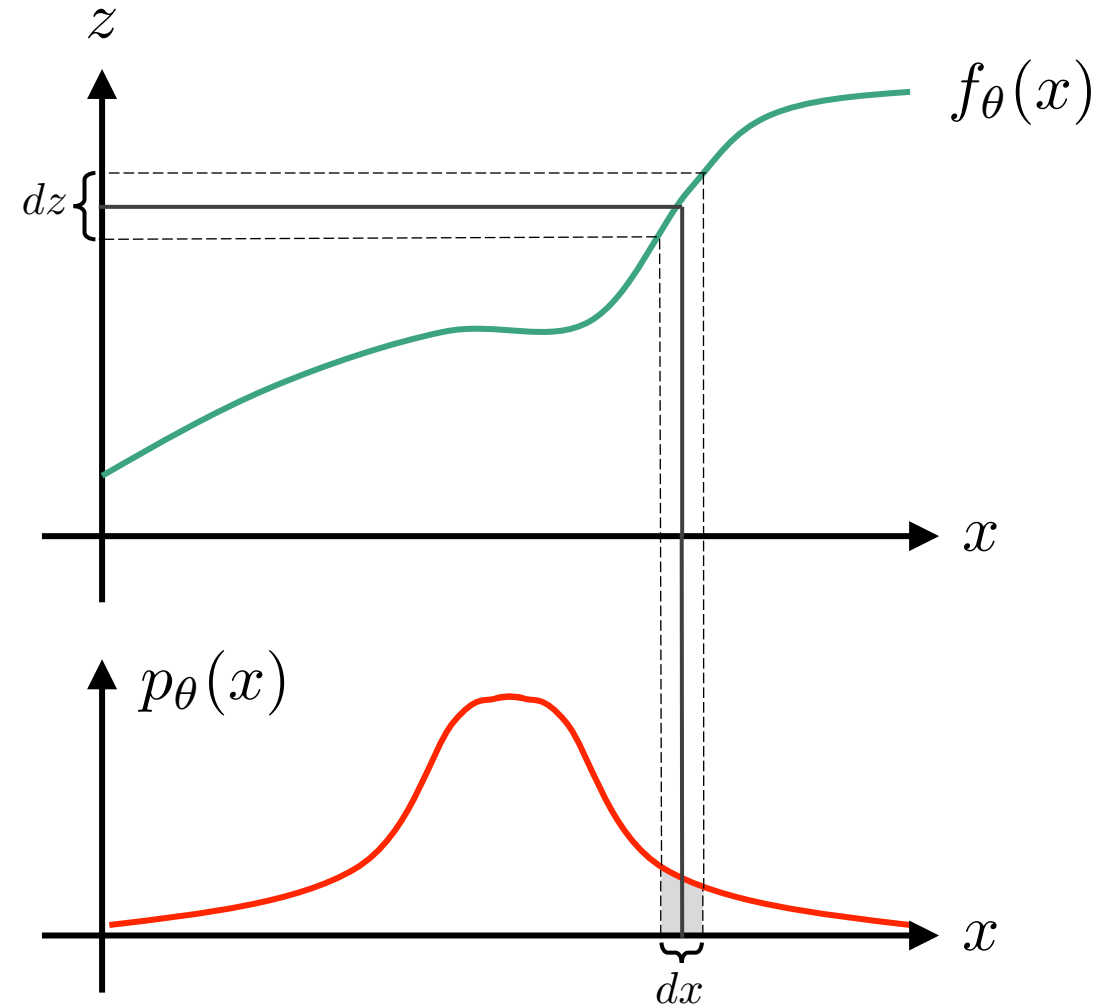
$$\max_{\theta} \sum_i \log p_{\theta}(x^{(i)})$$

Change of Variables

$$z = f_{\theta}(x)$$

$$p_{\theta}(x) dx = p(z) dz$$

$$p_{\theta}(x) = p(f_{\theta}(x)) \left| \frac{\partial f_{\theta}(x)}{\partial x} \right|$$

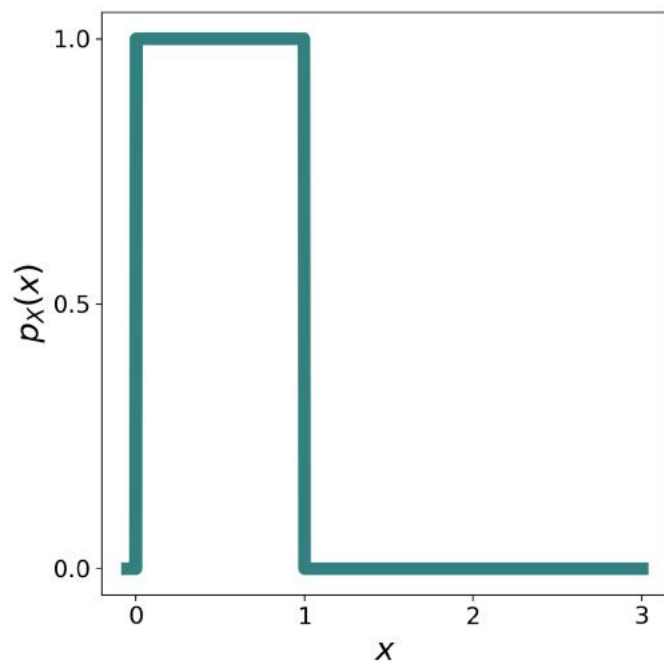


Note: requires f_{θ} invertible & differentiable

Change of Variable Density Needs to Be Normalized

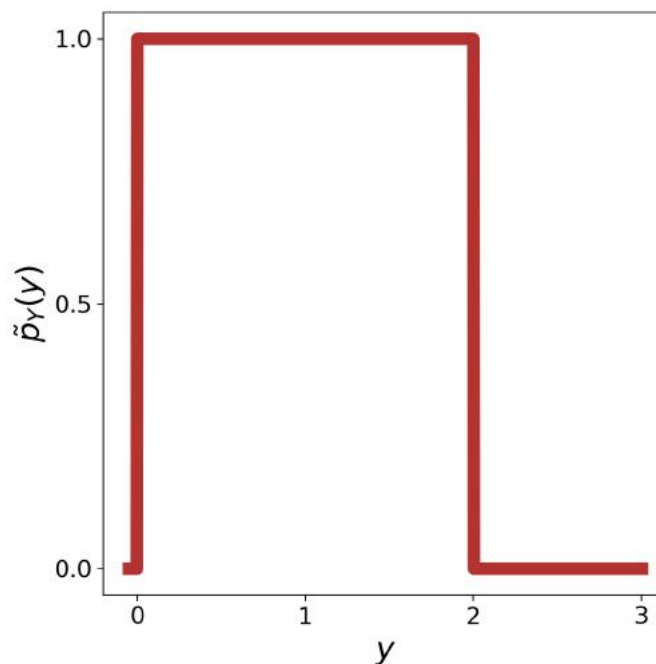
$$X \sim p_X$$

$$p_X(x) = \begin{cases} 1 & \text{for } 0 \leq x \leq 1 \\ 0 & \text{else} \end{cases}$$

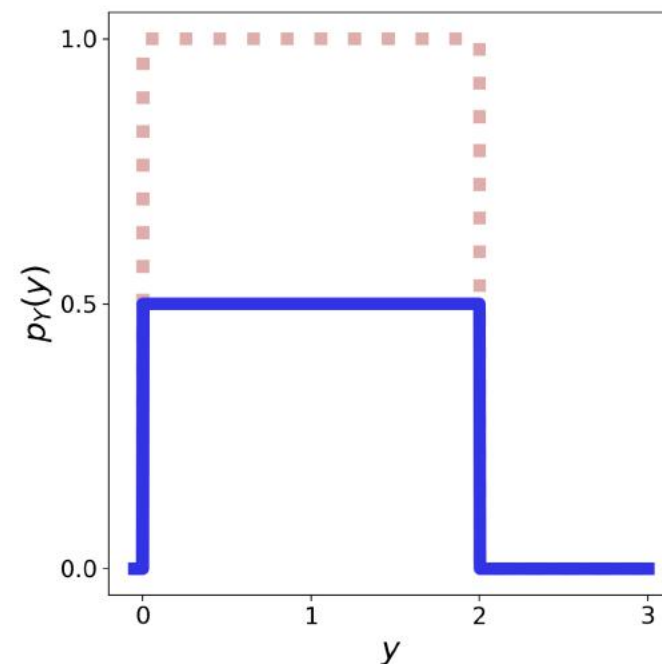


$$Y := 2X$$

$$\tilde{p}_Y(y) = p_X(y/2)$$



$$p_Y(y) = p_X(y/2)/2$$



Flows: Training

$$\max_{\theta} \sum_i \log p_{\theta}(x^{(i)})$$

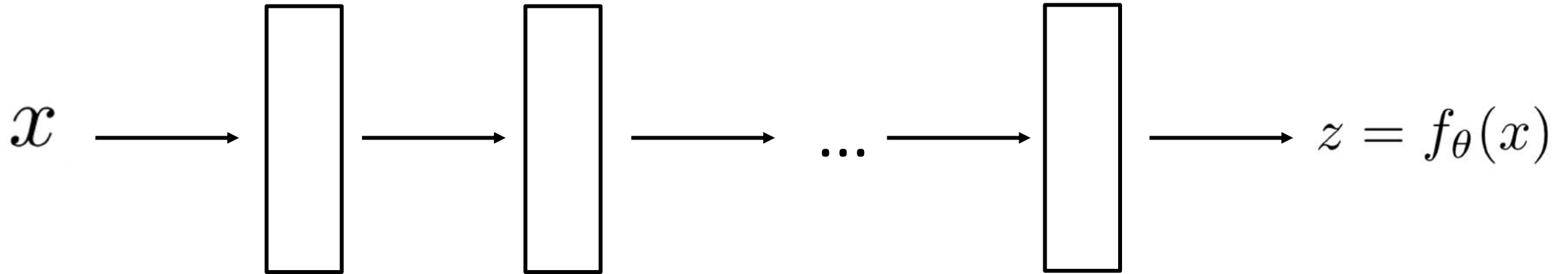
$$z^{(i)} = f_{\theta}(x^{(i)})$$

$$\begin{aligned} p_{\theta}(x^{(i)}) &= p_Z(z^{(i)}) \left| \frac{\partial z}{\partial x}(x^{(i)}) \right| \\ &= p_Z(f_{\theta}(x^{(i)})) \left| \frac{\partial f_{\theta}}{\partial x}(x^{(i)}) \right| \end{aligned}$$

$$\max_{\theta} \sum_i \log p_{\theta}(x^{(i)}) = \max_{\theta} \sum_i \log p_Z(f_{\theta}(x^{(i)})) + \log \left| \frac{\partial f_{\theta}}{\partial x}(x^{(i)}) \right|$$

→ assuming we have an expression for p_Z ,
this can be optimized with Stochastic Gradient Descent

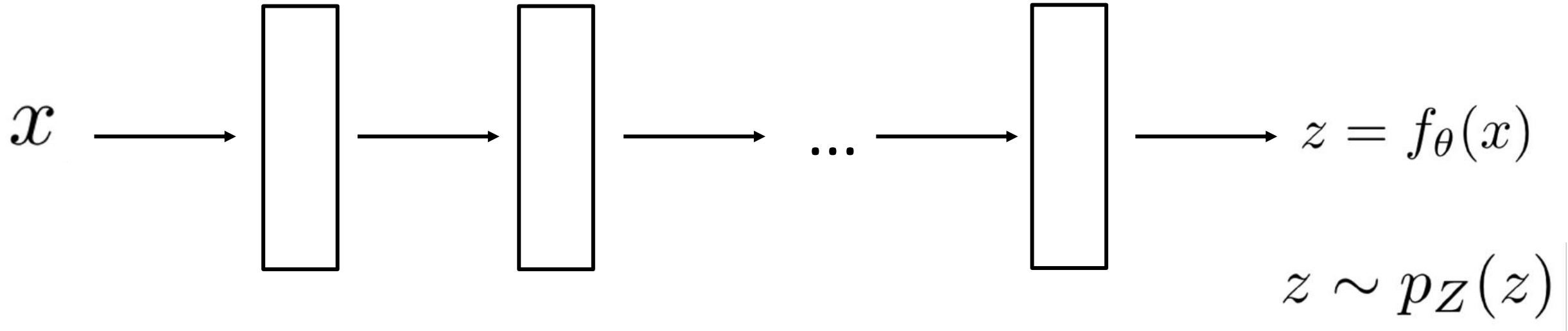
Flows: Sampling



Step 1: sample $z \sim p_Z(z)$

Step 2: $x = f_{\theta}^{-1}(z)$

What do we need to keep in mind for f ?

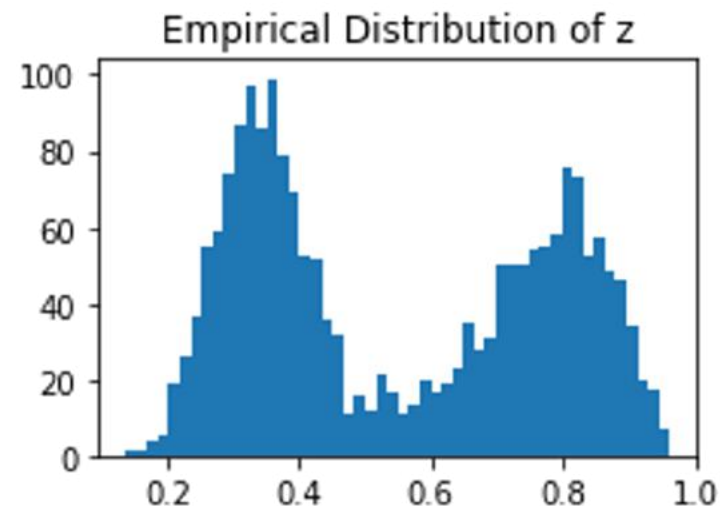
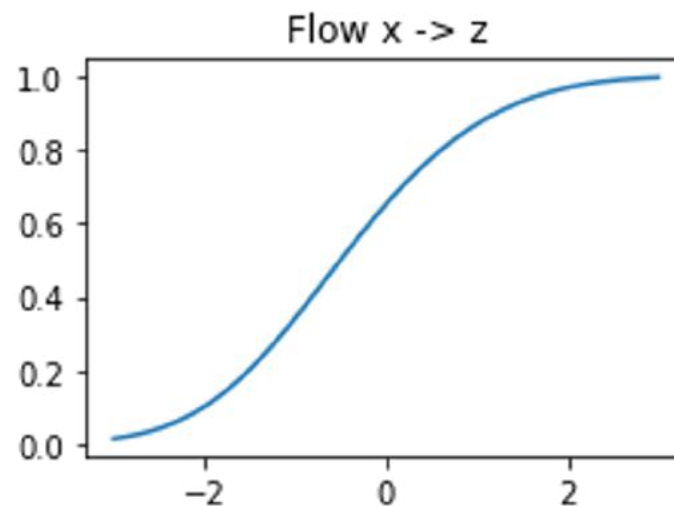
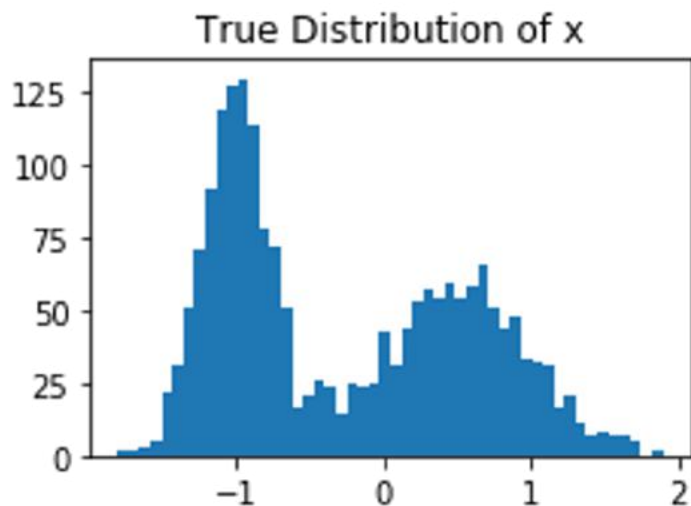


Recall, change of variable formula requires

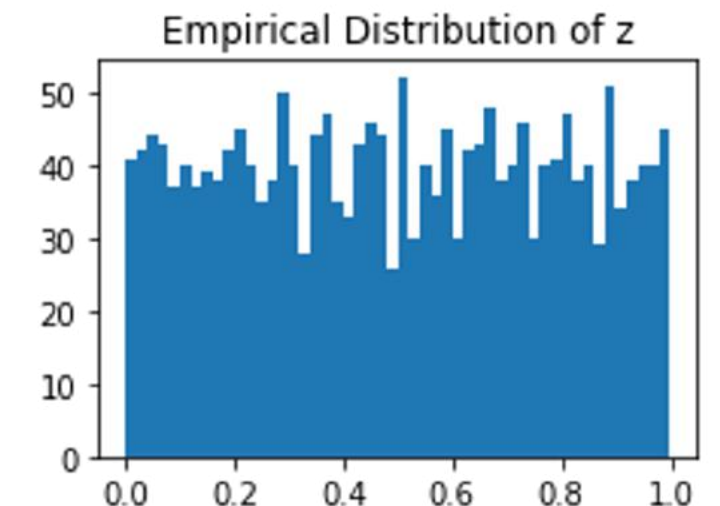
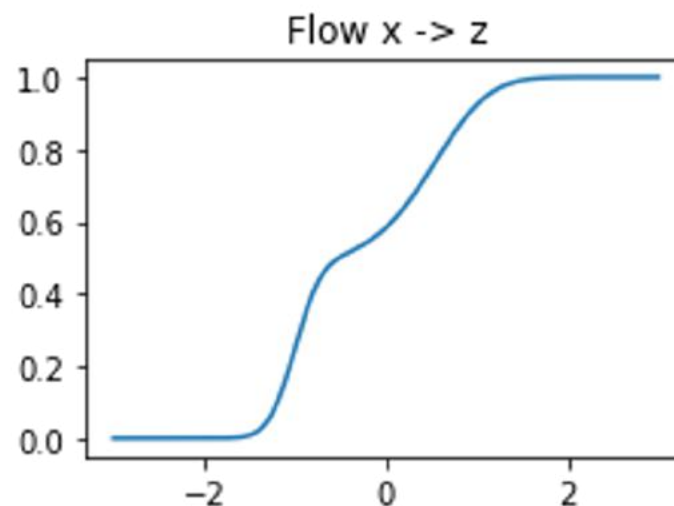
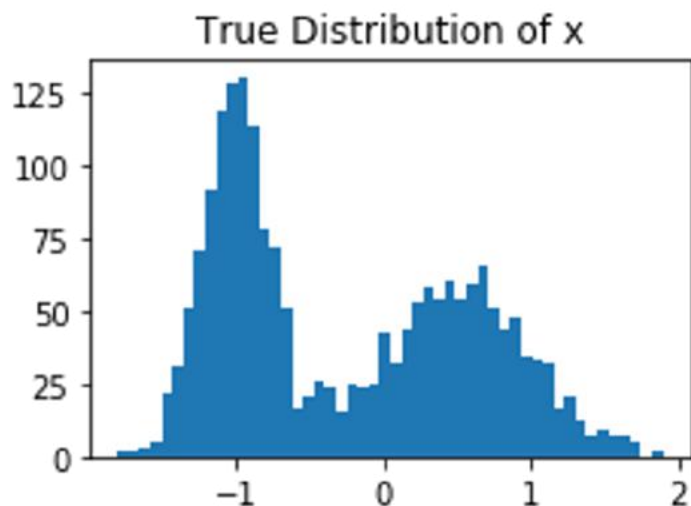
- f_{θ} Invertible & differentiable

Example: Flow to Uniform z

Before training



After training



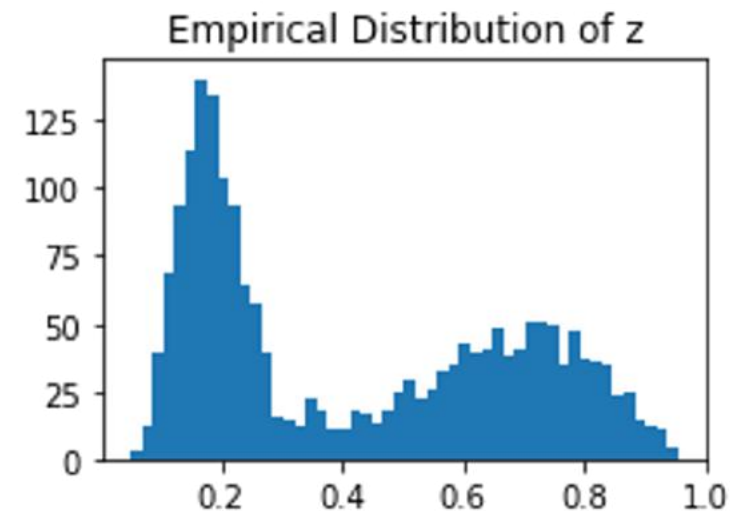
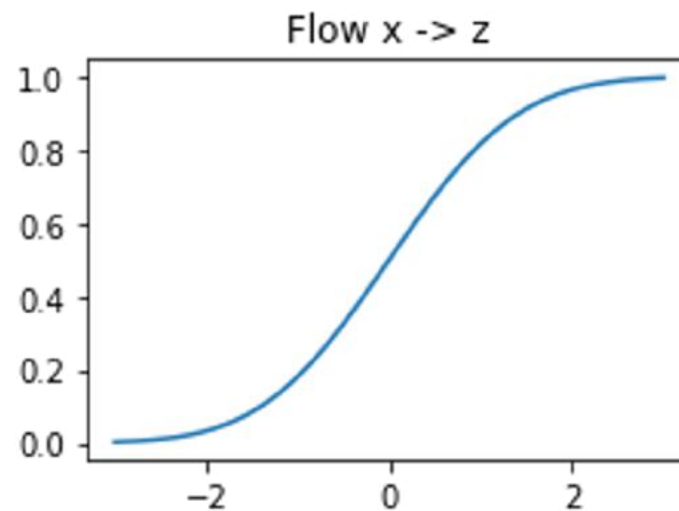
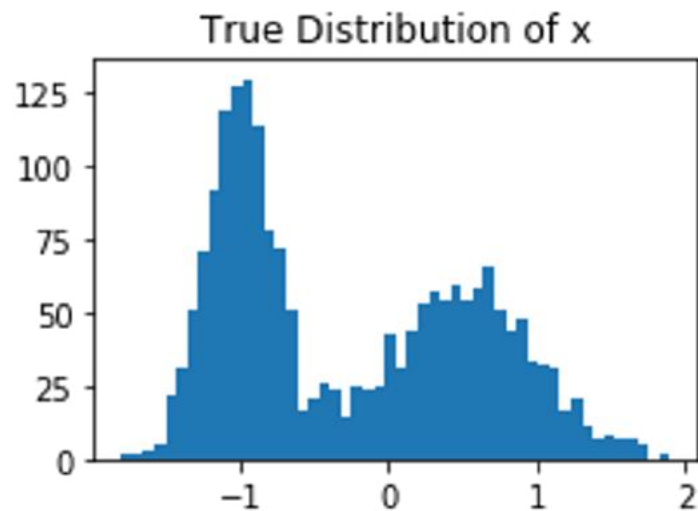
True distribution of x

Flow x \rightarrow z

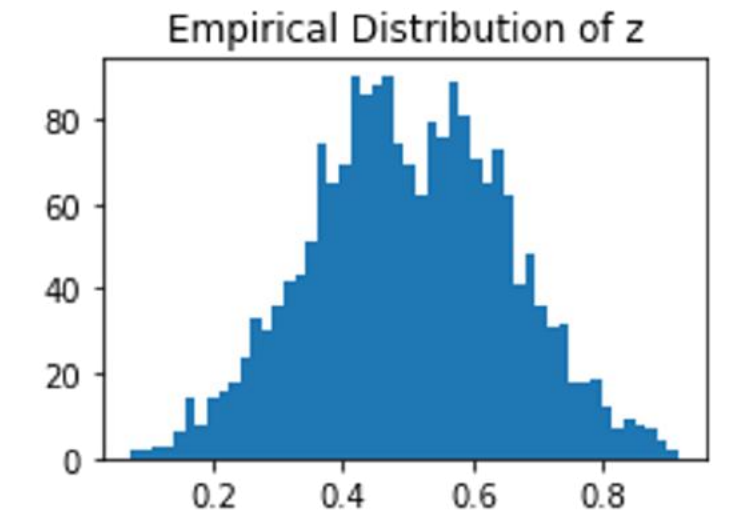
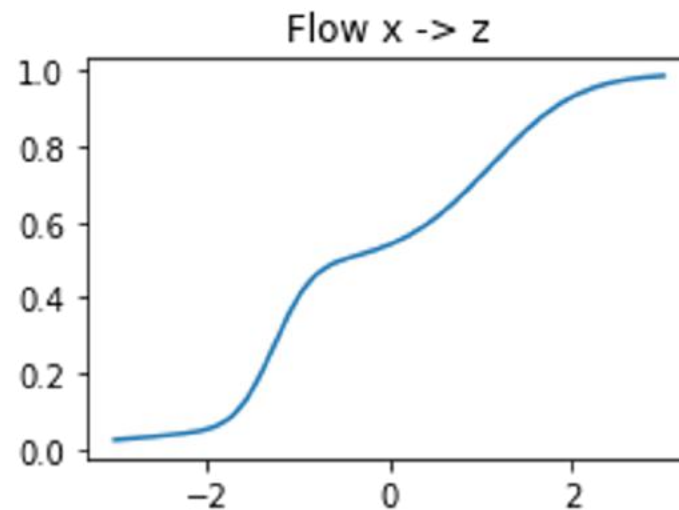
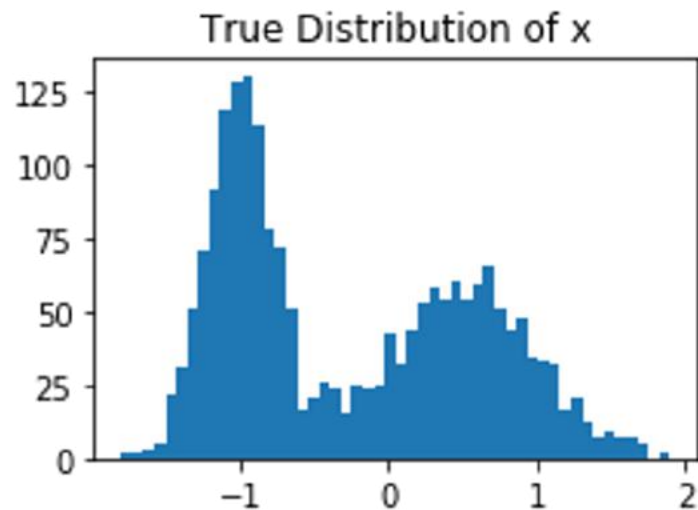
Empirical distribution of z

Example: Flow to Beta(5,5) z

Before training



After training



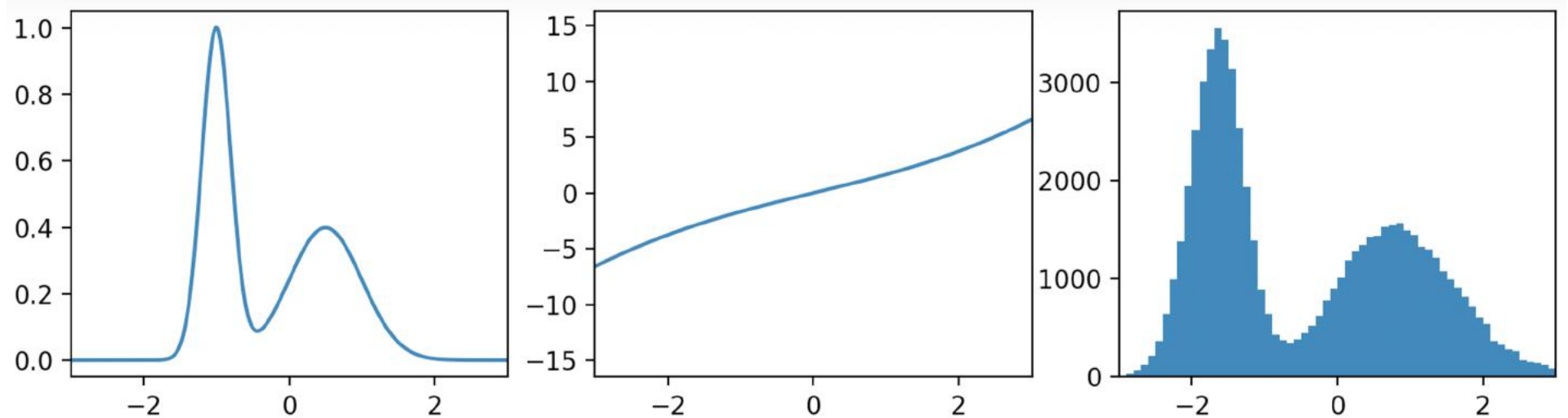
True distribution of x

Flow x \rightarrow z

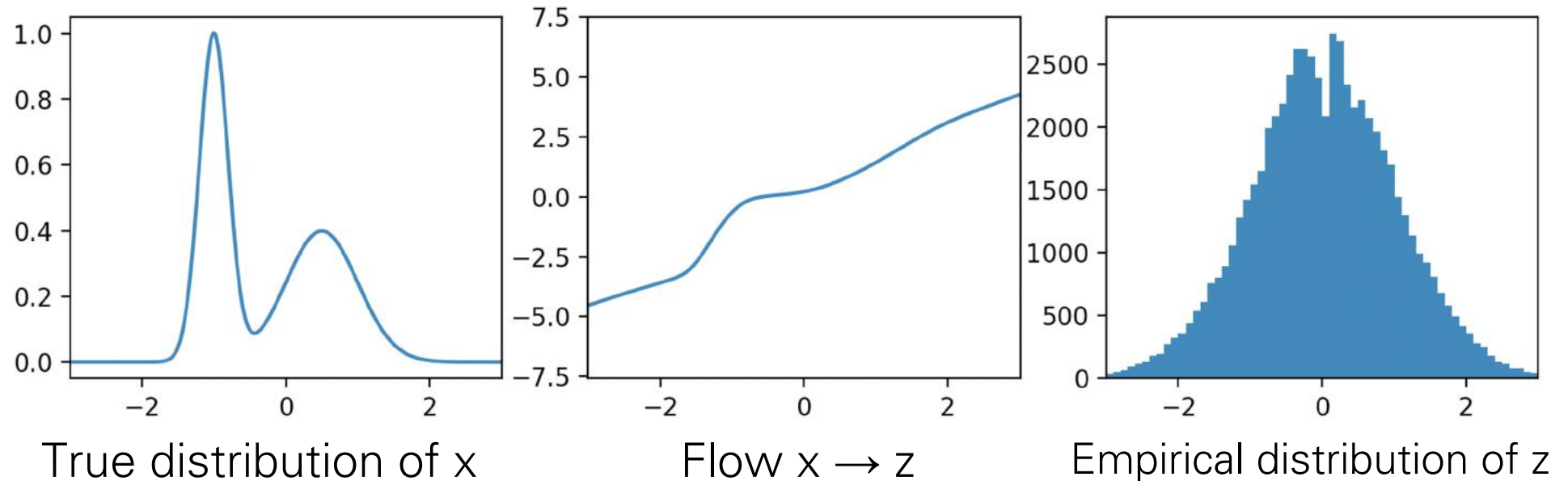
Empirical distribution of z

Example: Flow to Gaussian z

Before training



After training



Practical Parameterizations of Flows

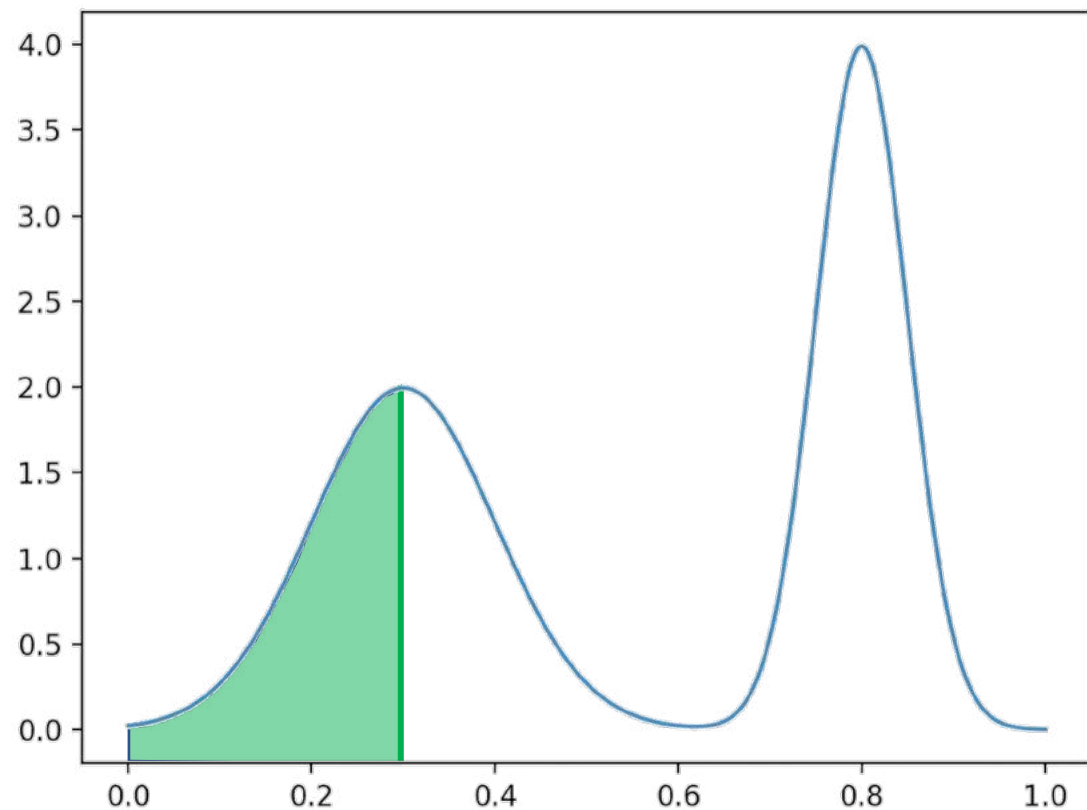
Requirement: Invertible and Differentiable

- Cumulative Density Functions
 - E.g. Gaussian mixture density, mixture of logistics
- Neural Net
 - If each layer flow, then sequencing of layers = flow
 - Each layer:
 - ReLU?
 - Sigmoid?
 - Tanh?

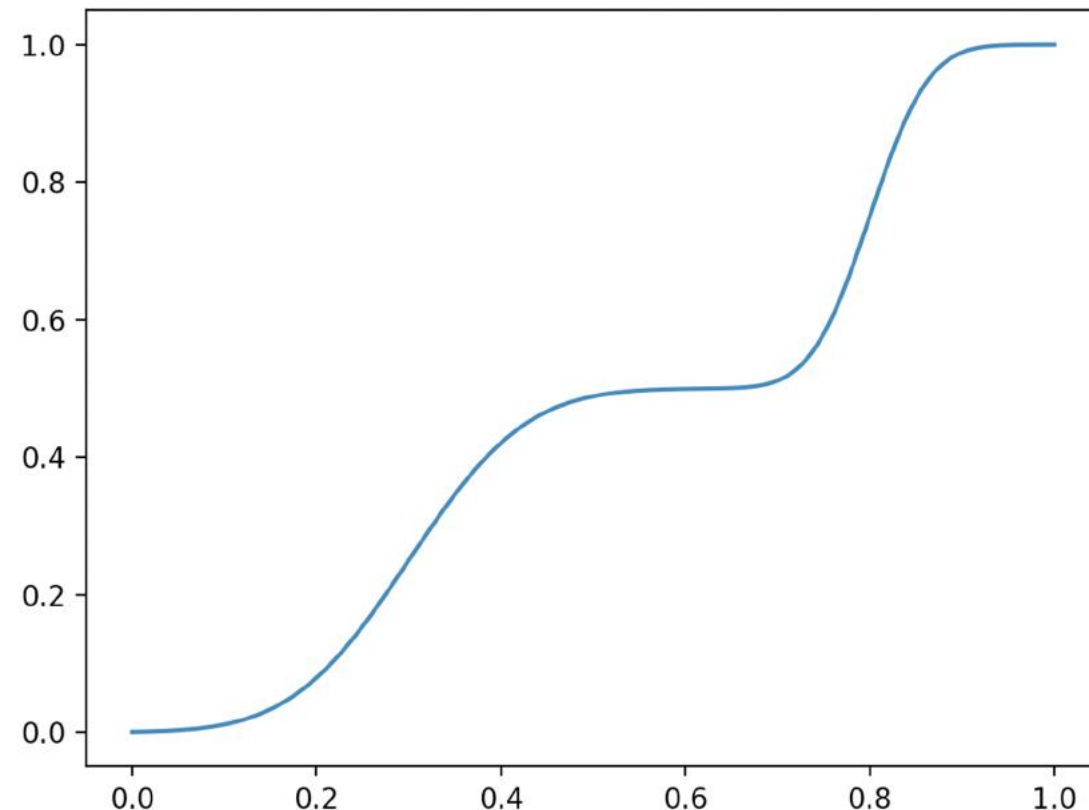
How general are flows?

- Can every (smooth) distribution be represented by a (normalizing) flow? [considering 1-D for now]

Refresher: Cumulative Density Function (CDF)



$p_{\theta}(x)$



$$f_{\theta}(x) = \int_{-\infty}^x p_{\theta}(t) dt$$

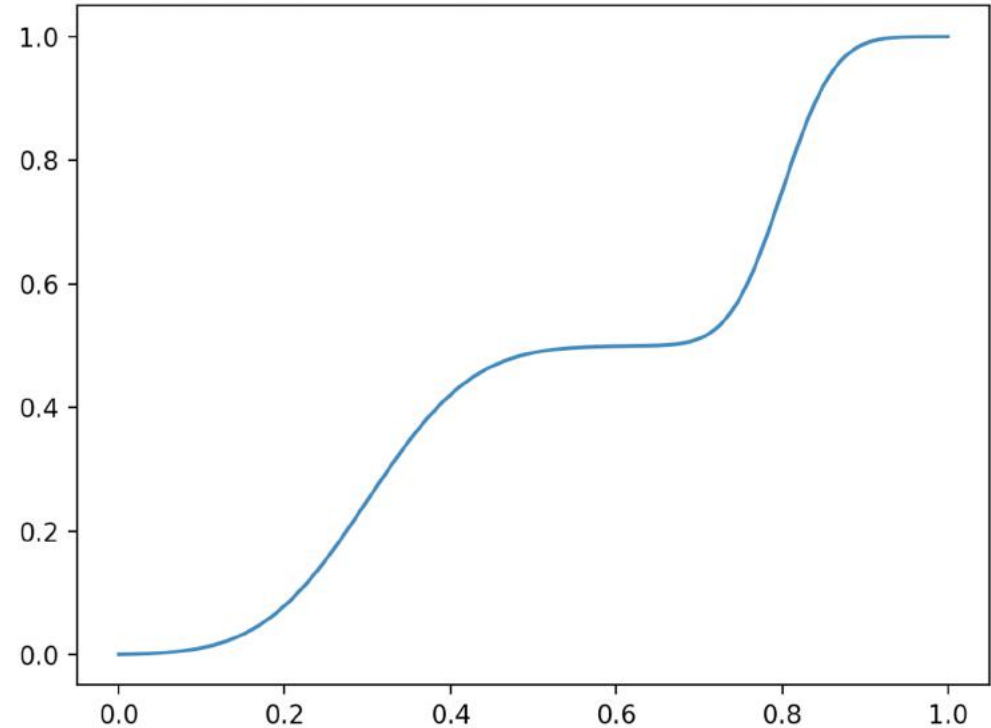
Sampling via inverse CDF

Sampling from the model:

$$z \sim \text{Uniform}([0, 1])$$

$$x = f_{\theta}^{-1}(z)$$

The CDF is an invertible, differentiable map from data to $[0, 1]$



$$f_{\theta}(x) = \int_{-\infty}^x p_{\theta}(t) dt$$

How general are flows?

- CDF turns any density into uniform
- Inverse flow is flow

$$x \xrightarrow{\text{CDF}} u$$

$$z \xrightarrow{\text{CDF}} u$$

$$x \xrightarrow{\text{CDF}} u \xrightarrow{\text{CDF}} z$$

→ can turn any (smooth) $p(x)$ into any (smooth) $p(z)$

Lecture overview

- Foundations of Flows (1-D)
- **2-D Flows**
- N-D Flows
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2-D Autoregressive Flow

$$x_1 \rightarrow z_1 = f_\theta(x_1)$$

$$x_2 \rightarrow z_2 = f_\phi(x_1, x_2)$$

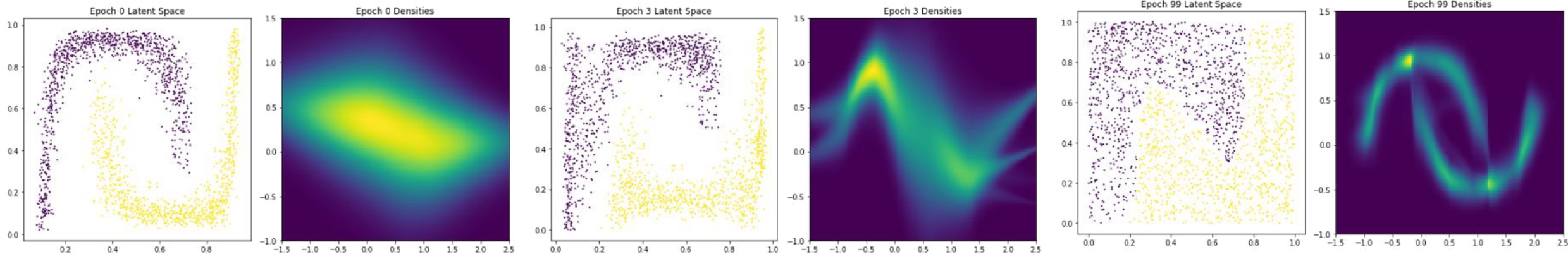
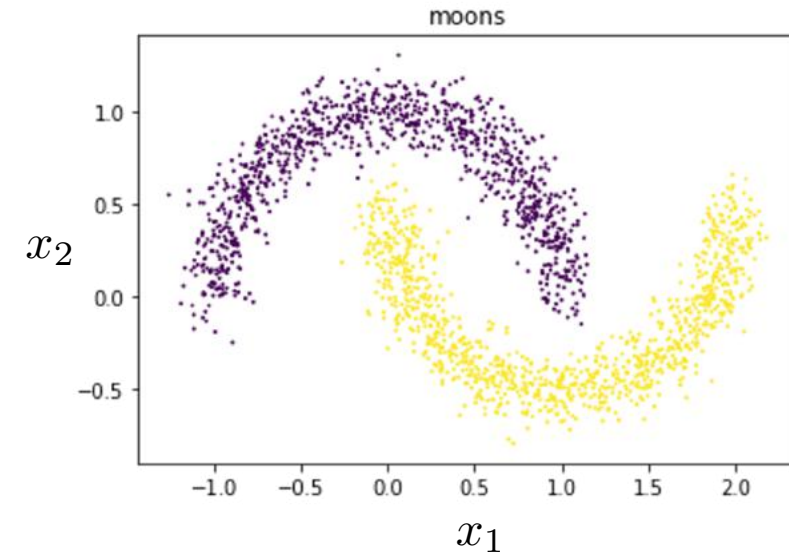
$$\max_{\theta, \phi} \sum_i \log p_{z_1}(f_\theta(x_1)) + \log \left| \frac{dz_1}{dx_1} \right| + \log p_{z_2}(f_\phi(x_1, x_2)) + \log \left| \frac{dz_2}{dx_2} \right|$$

$$\frac{dz_1}{dx_1} = \frac{df_\theta(x_1)}{dx_1}, \quad \frac{dz_2}{dx_2} = \frac{df_\phi(x_1, x_2)}{dx_2}$$

2-D Autoregressive Flow: Two Moons

Architecture:

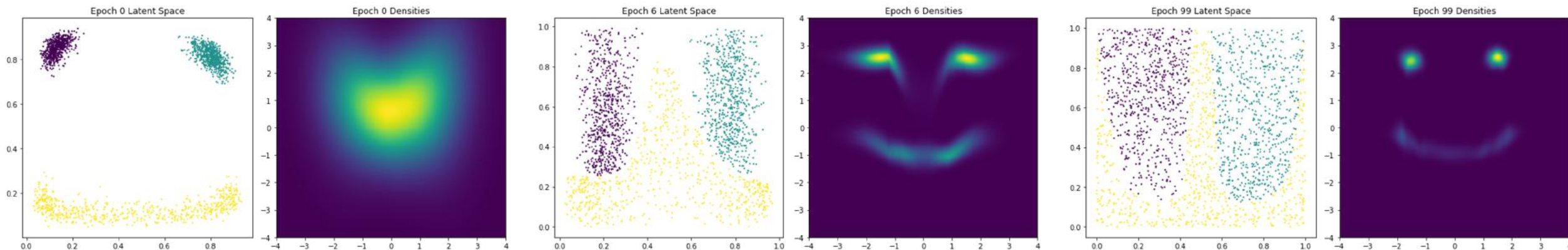
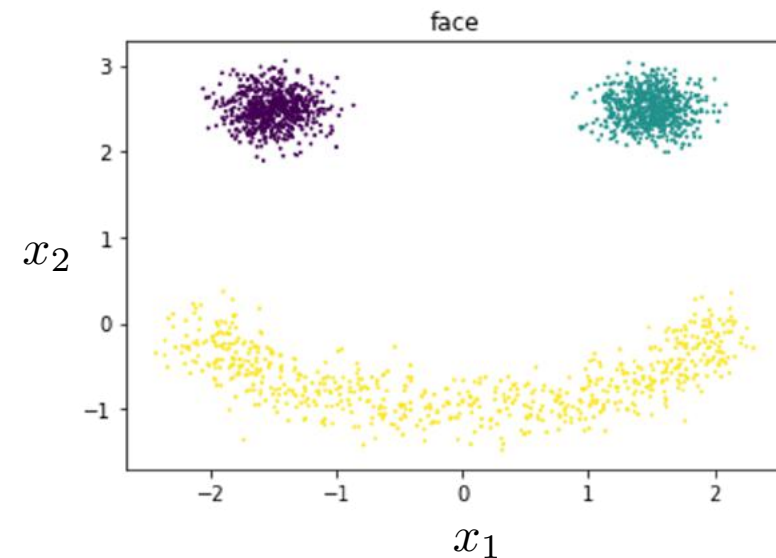
- Base distribution: $\text{Uniform}[0,1]^2$
- x_1 : mixture of 5 Gaussians
- x_2 : mixture of 5 Gaussians, conditioned on x_1



2-D Autoregressive Flow: Face

Architecture:

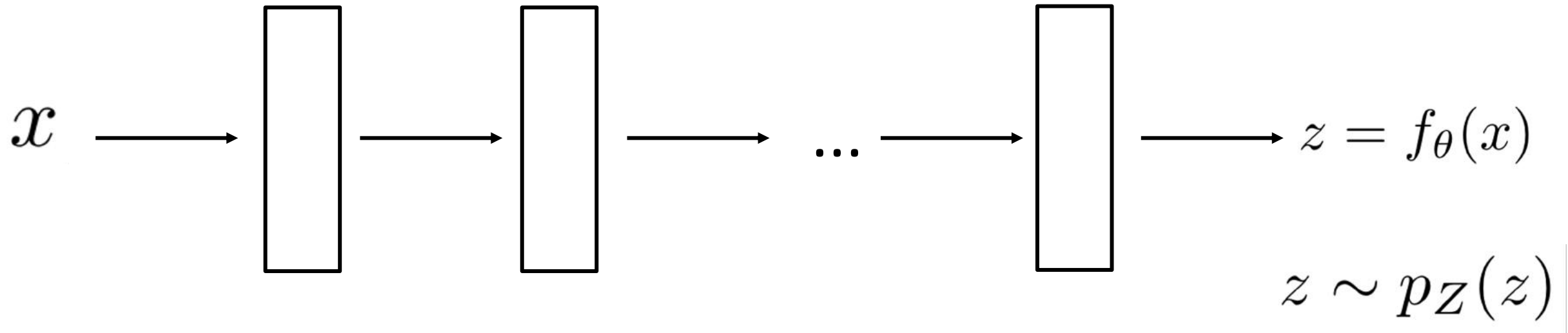
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Recap: Normalizing Flows

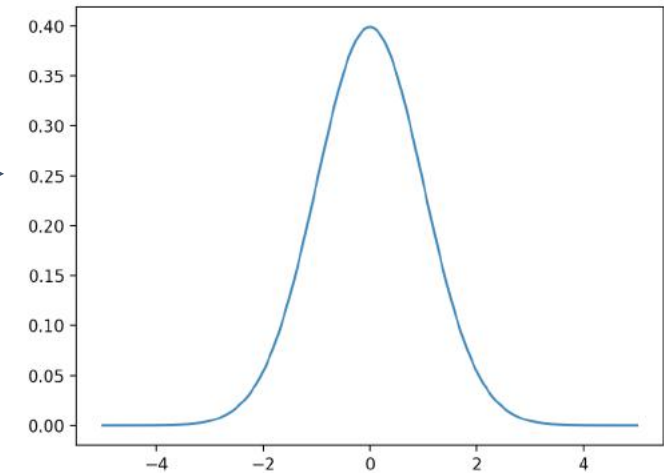
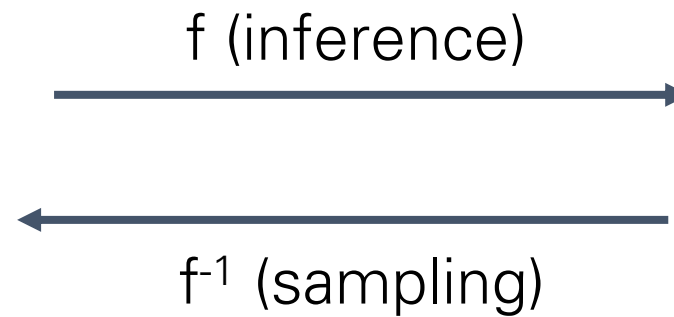
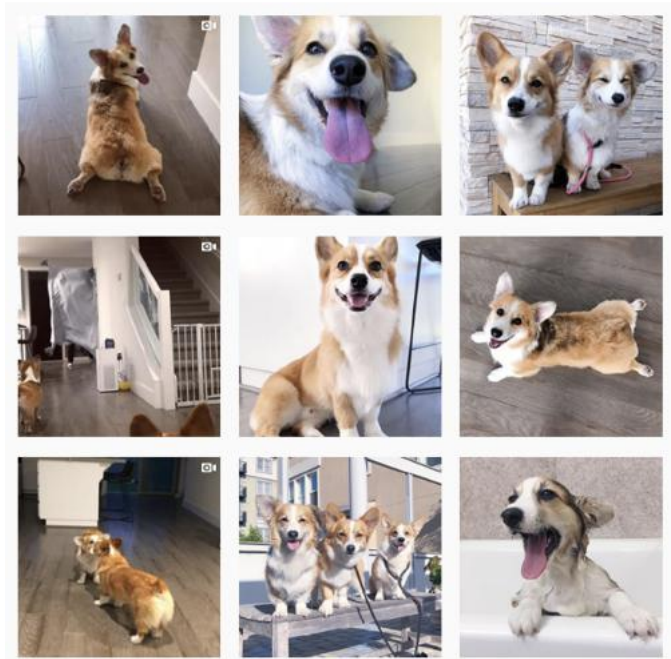


- f_{θ} invertible & differentiable

Training objective:

$$\max_{\theta} \sum_i \log p_{\theta}(x^{(i)}) = \max_{\theta} \sum_i \log p_Z(f_{\theta}(x^{(i)})) + \log \left| \frac{\partial f_{\theta}}{\partial x}(x^{(i)}) \right|$$

High-dimensional data



x and z must have the same dimension

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 - RealNVP (like) architectures
 - Glow, Flow++, FFJORD
- Dequantization

Autoregressive flows

- The sampling process of a Bayes net is a flow
 - If autoregressive, this flow is called an **autoregressive flow**

$$\begin{array}{lll} x_1 \sim p_\theta(x_1) & x_1 = f_\theta^{-1}(z_1) & z_1 = f_\theta(x_1) \\ x_2 \sim p_\theta(x_2|x_1) & x_2 = f_\theta^{-1}(z_2; x_1) & z_2 = f_\theta(x_1, x_2) \\ x_3 \sim p_\theta(x_3|x_1, x_2) & x_3 = f_\theta^{-1}(z_3; x_1, x_2) & z_3 = f_\theta(x_1, x_2, x_3) \end{array}$$

- Sampling is an **invertible** mapping from z to x

Autoregressive flows

- How to fit autoregressive flows?

- Map \mathbf{x} to \mathbf{z}
- Fully parallelizable

$$p_{\theta}(\mathbf{x}) = p(f_{\theta}(\mathbf{x})) \left| \det \frac{\partial f_{\theta}(\mathbf{x})}{\partial \mathbf{x}} \right|$$

- Notice

- $\mathbf{x} \rightarrow \mathbf{z}$ has the same structure as the **log likelihood** computation of an autoregressive model
- $\mathbf{z} \rightarrow \mathbf{x}$ has the same structure as the **sampling** procedure of an autoregressive model

$$z_1 = f_{\theta}(x_1)$$

$$z_2 = f_{\theta}(x_2; x_1)$$

$$z_3 = f_{\theta}(x_3; x_1, x_2)$$

$$x_1 = f_{\theta}^{-1}(z_1)$$

$$x_2 = f_{\theta}^{-1}(z_2; x_1)$$

$$x_3 = f_{\theta}^{-1}(z_3; x_1, x_2)$$

Inverse autoregressive flows

- The inverse of an autoregressive flow is also a flow, called the **inverse autoregressive flow (IAF)**
 - $\mathbf{x} \rightarrow \mathbf{z}$ has the same structure as the **sampling** in an autoregressive model
 - $\mathbf{z} \rightarrow \mathbf{x}$ has the same structure as **log likelihood** computation of an autoregressive model. So, **IAF sampling is fast**

$$z_1 = f_{\theta}^{-1}(x_1)$$

$$x_1 = f_{\theta}(z_1)$$

$$z_2 = f_{\theta}^{-1}(x_2; z_1)$$

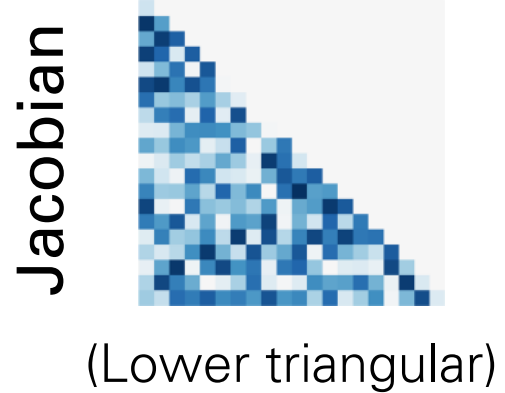
$$x_2 = f_{\theta}(z_2; z_1)$$

$$z_3 = f_{\theta}^{-1}(x_3; z_1, z_2)$$

$$x_3 = f_{\theta}(z_3; z_1, z_2)$$

AF vs IAF

- Autoregressive flow
 - **Fast** evaluation of $p(x)$ for arbitrary x
 - **Slow** sampling
- Inverse autoregressive flow
 - **Slow** evaluation of $p(x)$ for arbitrary x , so training directly by maximum likelihood is slow.
 - **Fast** sampling
 - **Fast** evaluation of $p(x)$ if x is a sample
- There are models (Parallel WaveNet, IAF-VAE) that exploit IAF's fast sampling



AF and IAF

Naively, both end up being as deep as the number of variables!

- E.g. 1MP image \rightarrow 1M layers/sampling steps...

Can do parameter sharing as in Autoregressive Models from previous lecture [e.g. RNN, masking]

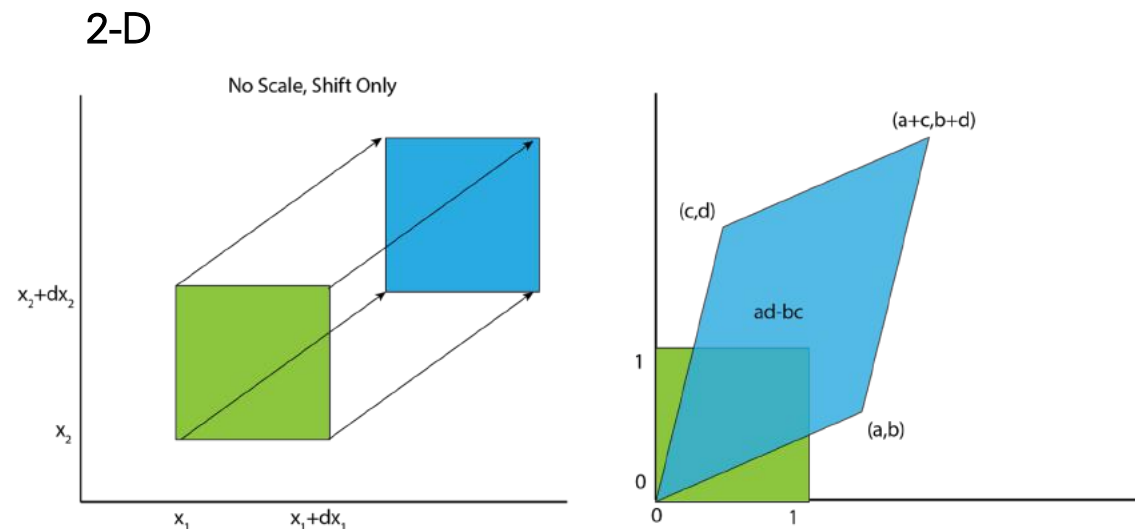
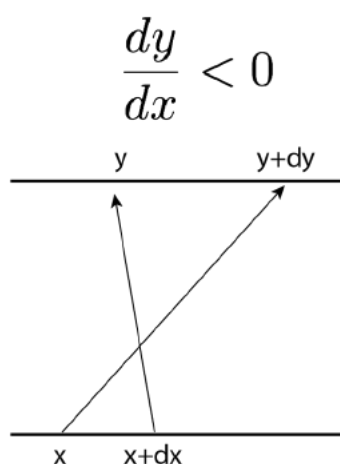
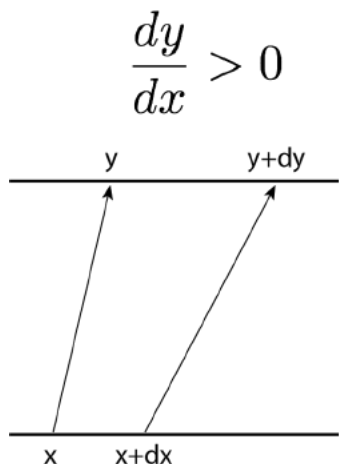
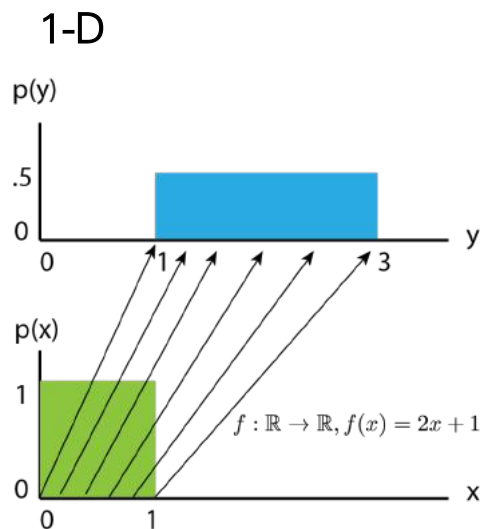
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Change of MANY variables

For a multivariable invertible mapping $f : \mathbb{R}^m \rightarrow \mathbb{R}^m$ $X \sim p_X$ $Y := f(X)$

$$p_Y(y) = p_X(f^{-1}(y)) \left| \det \frac{\partial f^{-1}(y)}{\partial y} \right|$$



Change of MANY variables

For $z \sim p(z)$, sampling process f^{-1} linearly transforms a small cube dz to a small parallelepiped dx . Probability is conserved:

$$p(x) = p(z) \frac{\text{vol}(dz)}{\text{vol}(dx)} = p(z) \left| \det \frac{dz}{dx} \right|$$

Intuition: x is likely if it maps to a “large” region in z space

Flow models: training

Change-of-variables formula lets us compute the density over \mathbf{x} :

$$p_{\theta}(\mathbf{x}) = p(f_{\theta}(\mathbf{x})) \left| \det \frac{\partial f_{\theta}(\mathbf{x})}{\partial \mathbf{x}} \right|$$

Train with maximum likelihood:

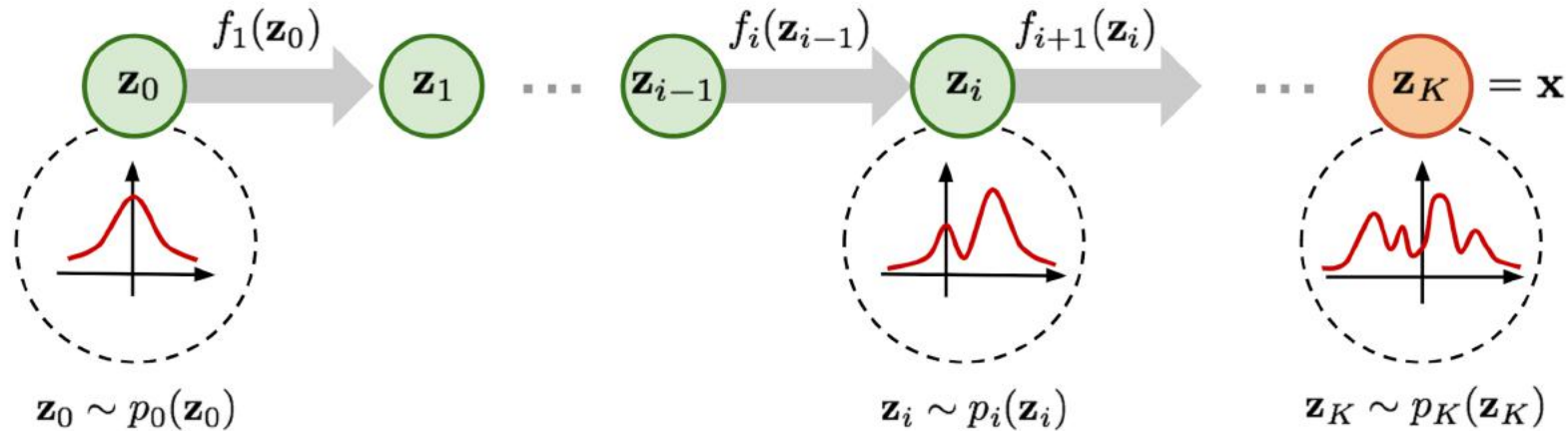
$$\arg \min_{\theta} \mathbb{E}_{\mathbf{x}} [-\log p_{\theta}(\mathbf{x})] = \mathbb{E}_{\mathbf{x}} \left[-\log p(f_{\theta}(\mathbf{x})) - \log \det \left| \frac{\partial f_{\theta}(\mathbf{x})}{\partial \mathbf{x}} \right| \right]$$

New key requirement: the Jacobian determinant must be easy to calculate and differentiate!

Chaining Invertible Mappings

$$f = f_S \circ \cdots \circ f_2 \circ f_1$$

$$f(x) = f_S(\cdots f_2(f_1(x)))$$



$$\frac{\partial f(x)}{\partial x} = \frac{f_S(x_{S-1})}{\partial x_{S-1}} \cdots \frac{f_2(x_1)}{\partial x_1} \frac{f_1(x_0)}{\partial x_0} \quad \begin{array}{l} x_s = f_s(x_{s-1}) \\ x_0 = x \end{array}$$

Chain rule

$$\det \left(\frac{\partial f(x)}{\partial x} \right) = \det \left(\frac{f_S(x_{S-1})}{\partial x_{S-1}} \right) \cdots \det \left(\frac{f_2(x_1)}{\partial x_1} \right) \det \left(\frac{f_1(x_0)}{\partial x_0} \right)$$

Determinant of
matrix product

Constructing flows: composition

- Flows can be composed

$$x \rightarrow f_1 \rightarrow f_2 \rightarrow \dots f_k \rightarrow z$$

$$z = f_k \circ \dots \circ f_1(x)$$

$$x = f_1^{-1} \circ \dots \circ f_k^{-1}(z)$$

$$\log p_\theta(x) = \log p_\theta(z) + \sum_{i=1}^k \log \left| \det \frac{\partial f_i}{\partial f_{i-1}} \right|$$

- Easy way to increase expressiveness

Affine flows

- Another name for affine flow: multivariate Gaussian.
 - Parameters: an invertible matrix A and a vector b
 - $f(x) = A^{-1}(x - b)$
- Sampling: $x = Az + b$, where $z \sim \mathcal{N}(0, I)$ $x \sim \mathcal{N}(b, AA^T)$
- Log likelihood is expensive when dimension is large.
 - The Jacobian of f is A^{-1}
 - Log likelihood involves calculating $\det(A)$

Elementwise flows

$$f_{\theta}((x_1, \dots, x_d)) = (f_{\theta}(x_1), \dots, f_{\theta}(x_d))$$

- Lots of freedom in elementwise flow
 - Can use elementwise affine functions or CDF flows.
- The Jacobian is diagonal, so the determinant is easy to evaluate.

$$\frac{\partial \mathbf{z}}{\partial \mathbf{x}} = \text{diag}(f'_{\theta}(x_1), \dots, f'_{\theta}(x_d))$$

$$\det \frac{\partial \mathbf{z}}{\partial \mathbf{x}} = \prod_{i=1}^d f'_{\theta}(x_i)$$

NICE/RealNVP

Affine coupling layer

- Split variables in half: $\mathbf{x}_{1:d/2}$, $\mathbf{x}_{d/2+1:d}$

$$\mathbf{z}_{1:d/2} = \mathbf{x}_{1:d/2}$$

$$\mathbf{z}_{d/2+1:d} = \mathbf{x}_{d/2+1:d} \cdot s_{\theta}(\mathbf{x}_{1:d/2}) + t_{\theta}(\mathbf{x}_{1:d/2})$$

- Invertible! Note that s_{θ} and t_{θ} can be arbitrary neural nets with **no restrictions**.
 - Think of them as **data-parameterized elementwise flows**.

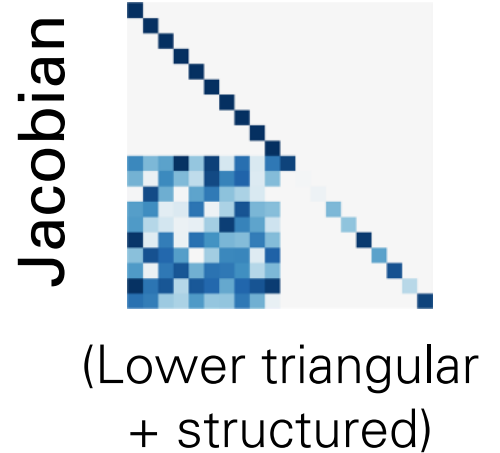
NICE/RealNVP

- It also has a tractable Jacobian determinant

$$\mathbf{z}_{1:d/2} = \mathbf{x}_{1:d/2}$$

$$\mathbf{z}_{d/2:d} = \mathbf{x}_{d/2:d} \cdot s_{\theta}(\mathbf{x}_{1:d/2}) + t_{\theta}(\mathbf{x}_{1:d/2})$$

$$\frac{\partial \mathbf{z}}{\partial \mathbf{x}} = \begin{bmatrix} I & 0 \\ \frac{\partial \mathbf{z}_{d/2:d}}{\partial \mathbf{x}_{1:d/2}} & \text{diag}(s_{\theta}(\mathbf{x}_{1:d/2})) \end{bmatrix}$$



- The Jacobian is triangular, so its determinant is the product of diagonal entries.

$$\det \frac{\partial \mathbf{z}}{\partial \mathbf{x}} = \prod_{k=1}^d s_{\theta}(\mathbf{x}_{1:d/2})_k$$

RealNVP

- Takeaway: coupling layers allow unrestricted neural nets to be used in flows, while preserving invertibility and tractability



[Dinh et al. Density estimation using Real NVP. ICLR 2017]

RealNVP Architecture

Input x : $32 \times 32 \times c$ image

- Layer 1: (Checkerboard $\times 3$, channel squeeze, channel $\times 3$)
 - Split result to get x_1 : $16 \times 16 \times 2c$ and z_1 : $16 \times 16 \times 2c$ (fine-grained latents)
- Layer 2: (Checkerboard $\times 3$, channel squeeze, channel $\times 3$)
 - Split result to get x_2 : $8 \times 8 \times 4c$ and z_2 : $8 \times 8 \times 4c$ (coarser latents)
- Layer 3: (Checkerboard $\times 3$, channel squeeze, channel $\times 3$)
 - Get z_3 : $4 \times 4 \times 16c$ (latents for highest-level details)

Can be better??

RealNVP: How to partition variables?

Partitioning can be implemented using a binary mask b , and using the functional form for y

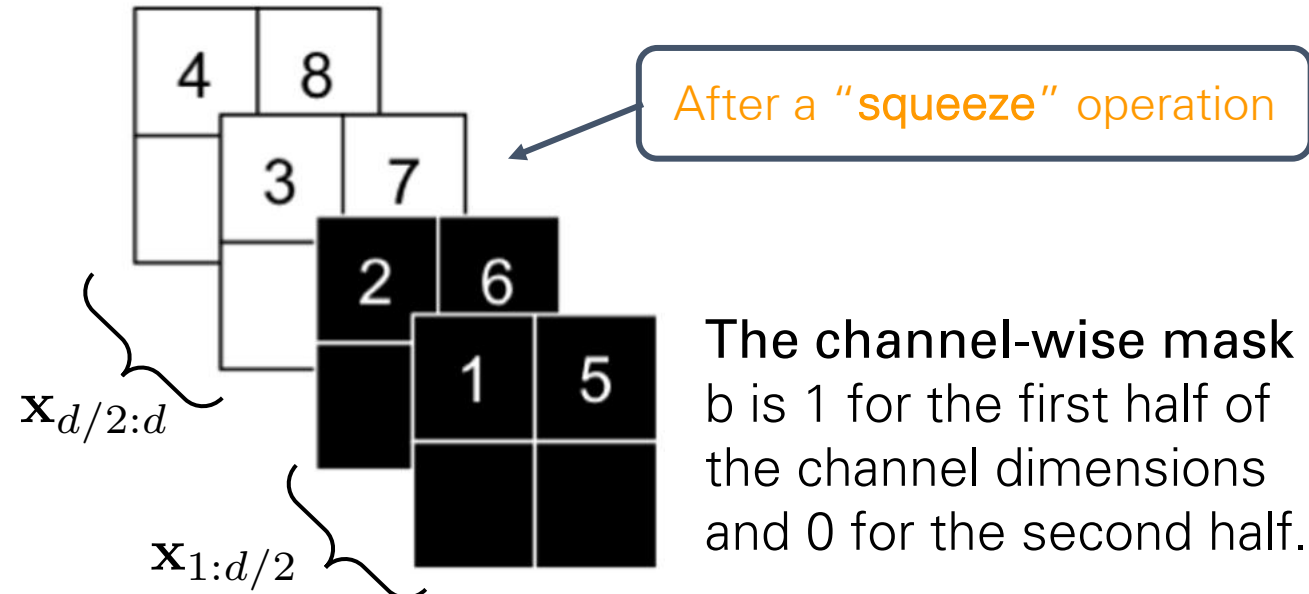
$$f(x) = b \odot x + (1 - b) \odot (x \odot \exp(s_-(b \odot x)) + m(b \odot x))$$

RealNVP: How to partition variables?

Partitioning can be implemented using a binary mask b , and using the functional form for y

$$f(x) = b \odot x + (1 - b) \odot (x \odot \exp(s_-(b \odot x)) + m(b \odot x))$$

The **spatial checkerboard pattern mask** has value 1 where the sum of spatial coordinates is odd, and 0 otherwise.

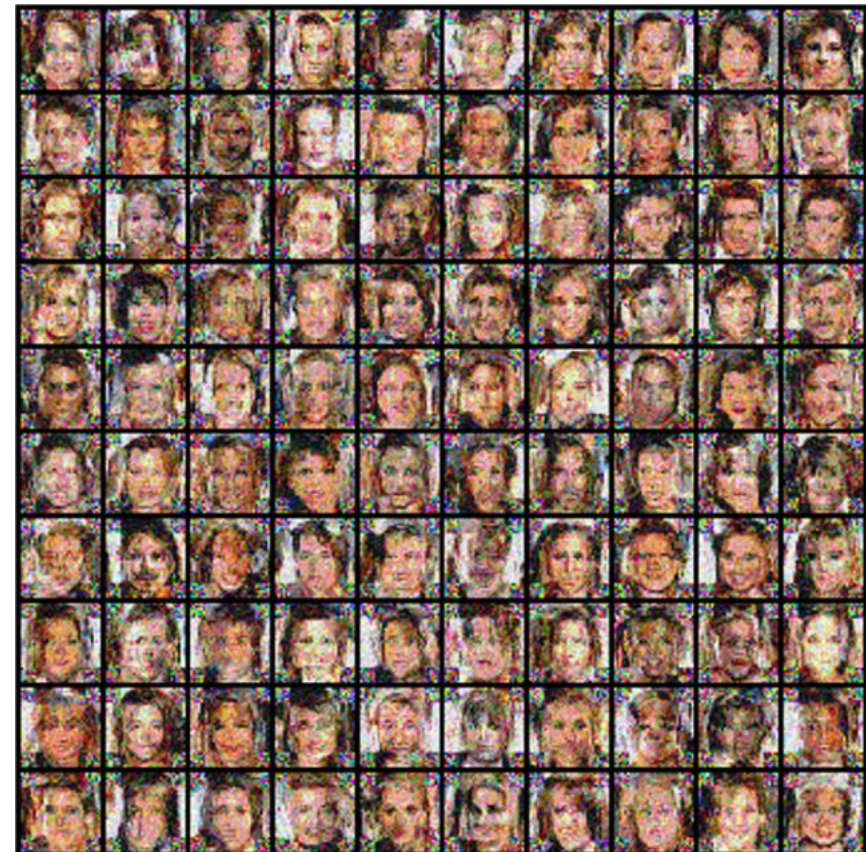


Good vs Bad Partitioning

Checkerboard $\times 4$; channel squeeze;
channel $\times 3$; channel unsqueeze;
checkerboard $\times 3$



(Mask top half; mask bottom
half; mask left half; mask right
half) $\times 2$



Lecture overview

- Foundations of Flows (1-D)
- 2-D Flows
- **N-D Flows**
 - Autoregressive Flows and Inverse Autoregressive Flows
 - RealNVP (like) architectures
 - **Glow, Flow++, FFJORD**
- Dequantization

Choice of coupling transformation

- A Bayes net defines coupling dependency, but what invertible transformation f to use is a design question

$$\mathbf{x}_i = f_{\theta}(\mathbf{z}_i; \text{parent}(\mathbf{x}_i))$$

- Affine transformation is the most commonly used one (NICE, RealNVP, IAF-VAE, ...)

$$\mathbf{x}_i = \mathbf{z}_i \cdot \mathbf{a}_{\theta}(\text{parent}(\mathbf{x}_i)) + \mathbf{b}_{\theta}(\text{parent}(\mathbf{x}_i))$$

- More complex, nonlinear transformations -> better performance
 - CDFs and inverse CDFs for Mixture of Gaussians or Logistics (Flow++)
 - Piecewise linear/quadratic functions (Neural Importance Sampling)

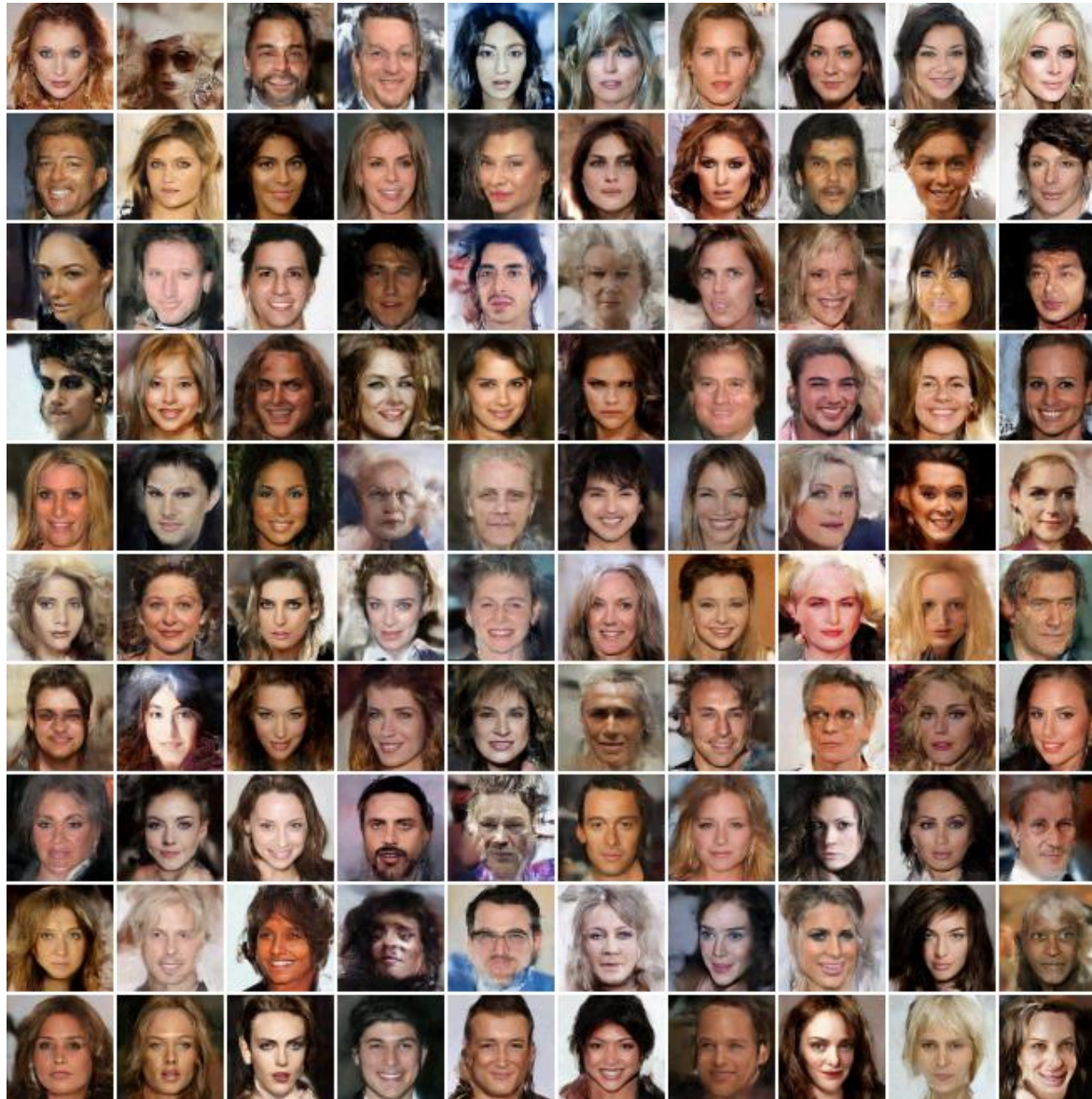
NN architecture also matters

- Flow++ = MoL transformation + self-attention in NN
 - Bayes net (coupling dependency), transformation function class, NN architecture all play a role in a flow's performance.

Table 2. CIFAR10 ablation results after 400 epochs of training.
Models not converged for the purposes of ablation study.

Ablation	bits/dim	parameters
uniform dequantization	3.292	32.3M
affine coupling	3.200	32.0M
no self-attention	3.193	31.4M
Flow++ (not converged for ablation)	3.165	31.4M

Flow++



Samples from Flow++
trained on 64x64 CelebA

Other classes of flows

- Glow ([link](#))
 - Replacing permutation with 1×1 convolution (soft permutation)
 - Large-scale training
- Continuous time flows (FFJORD)
 - Allows for unrestricted architectures. Invertibility and fast log probability computation guaranteed.



Architectural Taxonomy

Sparse connection

$$f(\boldsymbol{x})_t = g(\boldsymbol{x}_{1:t})$$

1. Autoregressive

IAF/MAF/NAF
SOS polynomial
UMNN

2. Block coupling

NICE/RealNVP/Glow
Cubic Spline Flow
Neural Spline Flow

Residual Connection

$$f(\boldsymbol{x}) = \boldsymbol{x} + g(\boldsymbol{x})$$

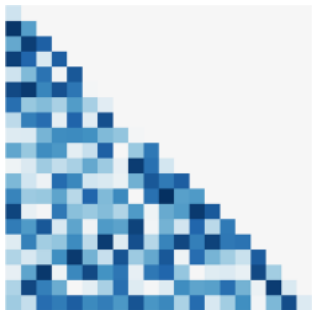
3. Det identity

Planar/Sylvester
flows
Radial flow

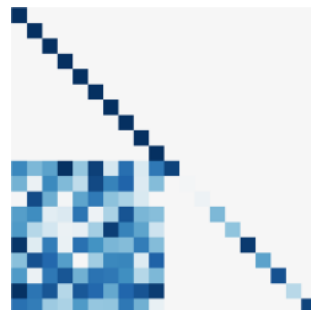
4. Stochastic estimation

Residual
Flow
FFJORD

Jacobian



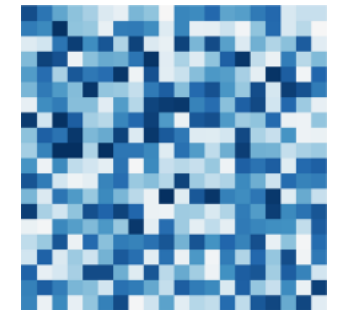
(Lower triangular)



(Lower triangular +
structured)



(Low rank)

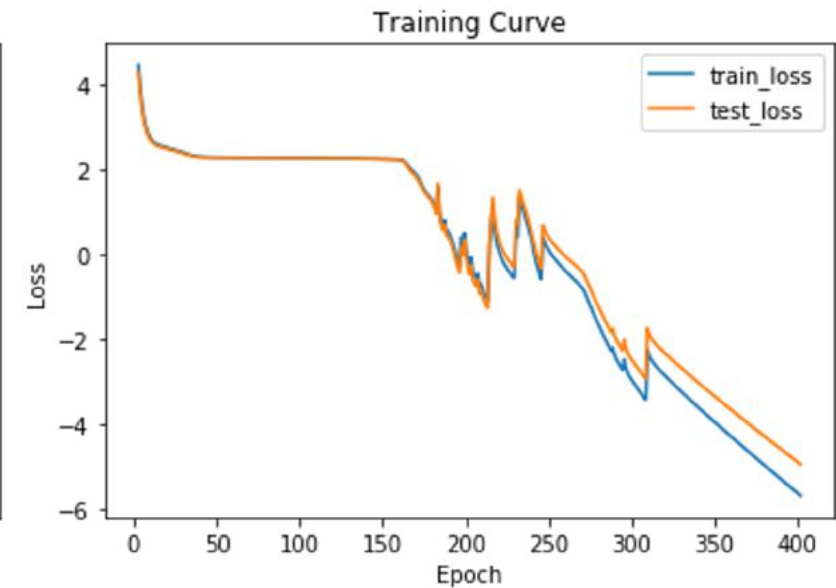
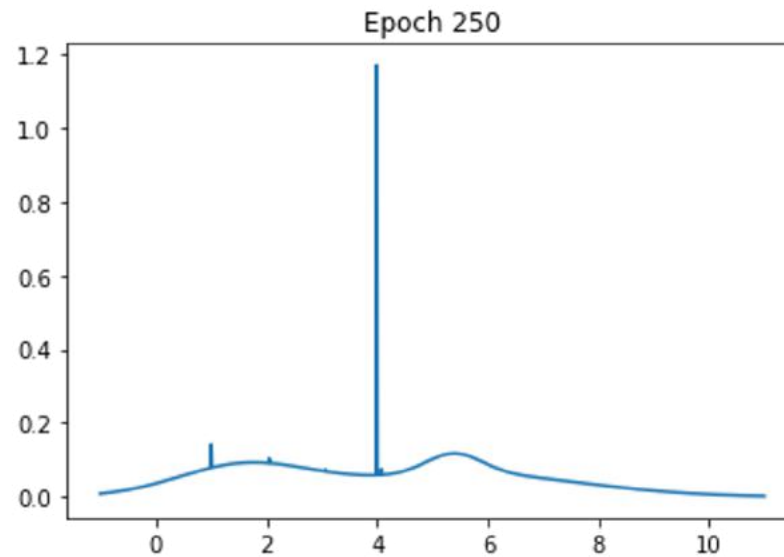
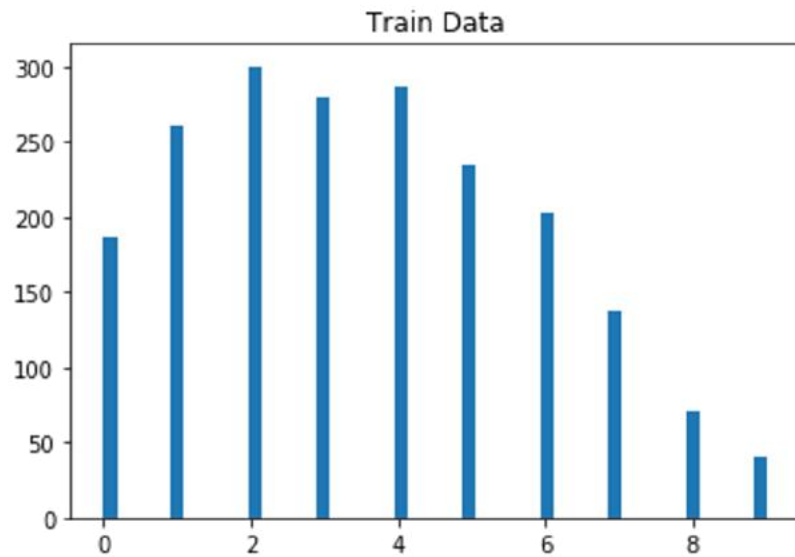


(Arbitrary)

Lecture overview

- Foundations of Flows (1-D)
- 2-D Flows
- N-D Flows
- Dequantization

Flow on Discrete Data Without Dequantization...



Continuous flows for discrete data

- A problem arises when fitting continuous density models to discrete data: degeneracy
 - When the data are 3-bit pixel values, $\mathbf{x} \in \{0, 1, 2, \dots, 255\}$
 - What density does a model assign to values between bins like 0.4, 0.42...?
- Correct semantics: we want the integral of probability density within a discrete interval to approximate discrete probability mass

$$P_{\text{model}}(\mathbf{x}) := \int_{[0,1)^D} p_{\text{model}}(\mathbf{x} + \mathbf{u}) d\mathbf{u}$$

Continuous flows for discrete data

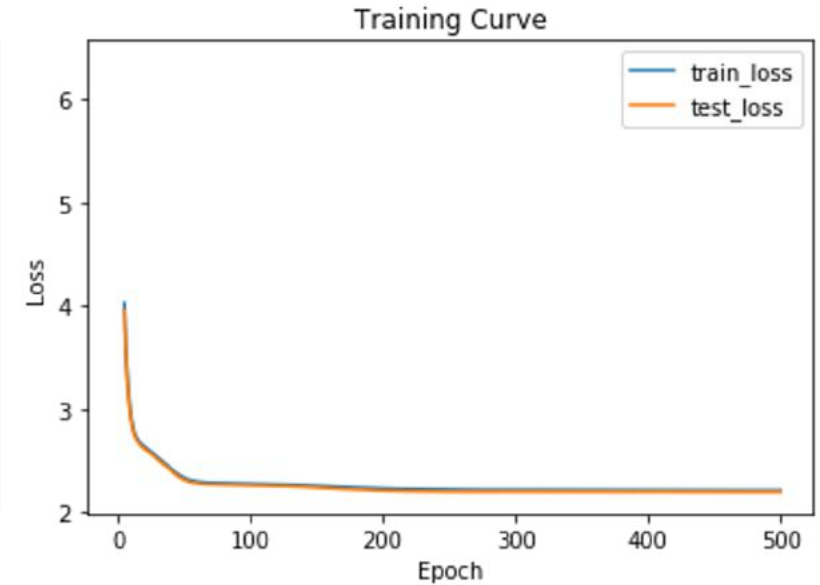
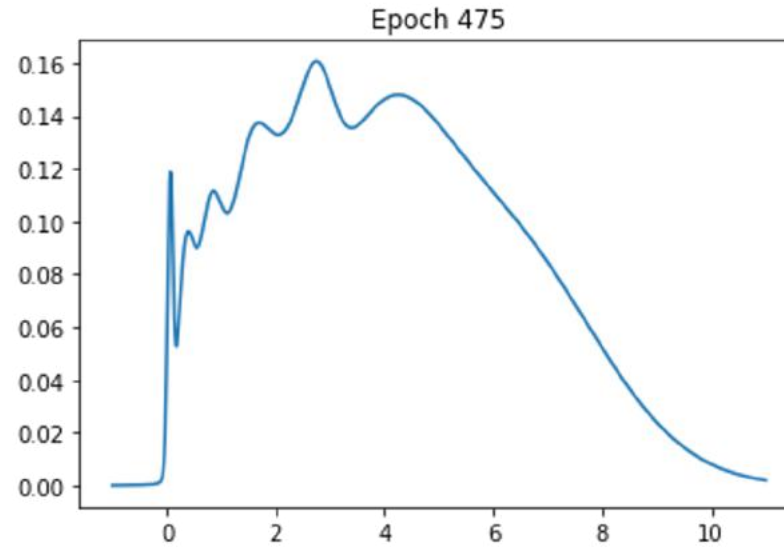
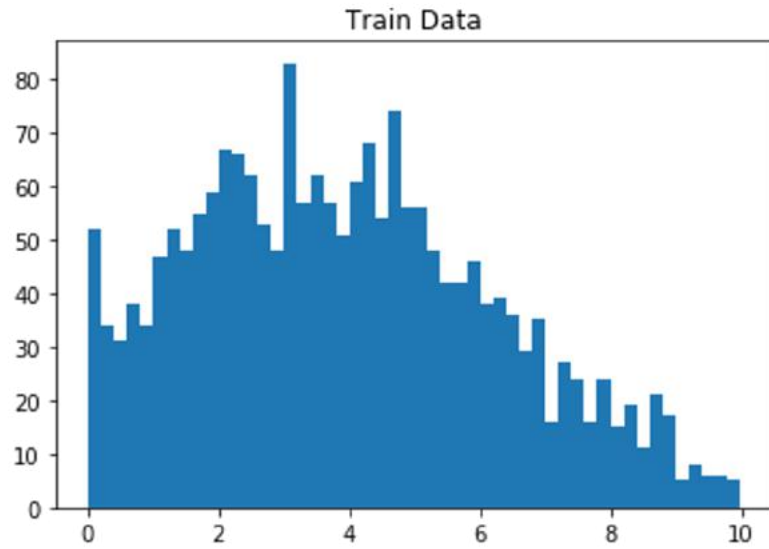
- Solution: **Dequantization**. Add noise to data.

$$\mathbf{x} \in \{0, 1, 2, \dots, 255\}$$

- We draw noise \mathbf{u} uniformly from $[0, 1)^D$

$$\begin{aligned}\mathbb{E}_{\mathbf{y} \sim p_{\text{data}}} [\log p_{\text{model}}(\mathbf{y})] &= \sum_{\mathbf{x}} P_{\text{data}}(\mathbf{x}) \int_{[0,1)^D} \log p_{\text{model}}(\mathbf{x} + \mathbf{u}) d\mathbf{u} \\ &\leq \sum_{\mathbf{x}} P_{\text{data}}(\mathbf{x}) \log \int_{[0,1)^D} p_{\text{model}}(\mathbf{x} + \mathbf{u}) d\mathbf{u} \\ &= \mathbb{E}_{\mathbf{x} \sim P_{\text{data}}} [\log P_{\text{model}}(\mathbf{x})]\end{aligned}$$

Flow on Discrete Data With Dequantization



Applications

- FloWaveNet
 - A flow-based generative model for raw audio synthesis
 - Efficiently samples raw audio in real-time

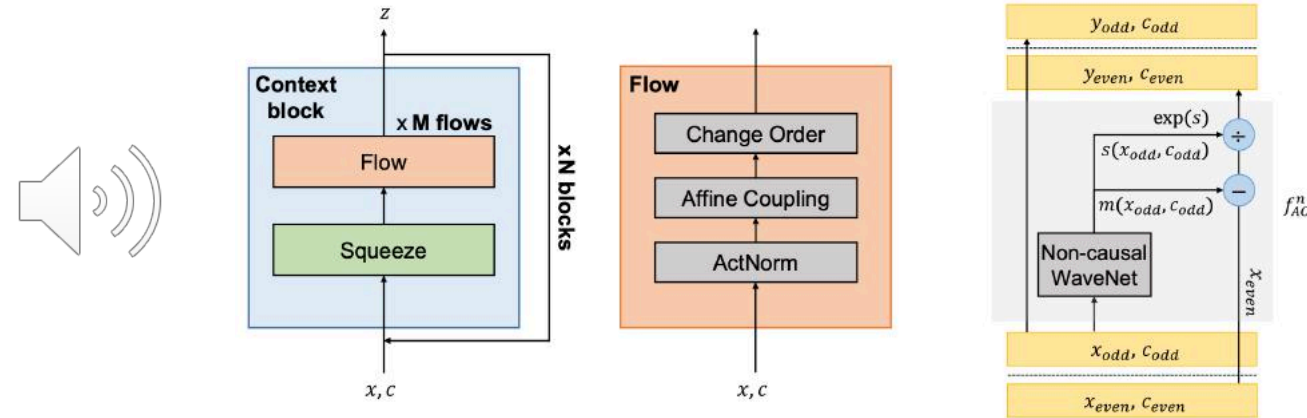


Figure 1. Schematic diagram of FloWaveNet. Left: an entire forward pass of the FloWaveNet consisting of N context blocks. Middle: an abstract diagram of the flow operation. Right: a detailed version of the affine coupling operation.

- SRFlow
 - A normalizing flow based super-resolution method, allowing diversity
 - Outperforms state-of-the-art GAN-based approaches



LR Input

SRFlow



Output: SR Image Distribution

Future directions

- The ultimate goal: a likelihood-based model with
 - fast sampling
 - fast inference
 - fast training
 - good samples
 - good compression
- Flows seem to let us achieve some of these criteria.
- But how exactly do we design and compose flows for great performance? That's an open question.

Next lecture: Variational Autoencoders