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Measuring the quality of discrete representations of efficient sets in multiple objective mathematical programming

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Abstract. One way of solving multiple objective mathematical programming problems is finding discrete representations of the efficient set. A modified goal of finding good discrete representations of the efficient set would contribute to the practicality of vector maximization algorithms. We define coverage, uniformity and cardinality as the three attributes of quality of discrete representations and introduce a framework that includes these attributes in which discrete representations can be evaluated, compared to each other, and judged satisfactory or unsatisfactory by a Decision Maker. We provide simple mathematical programming formulations that can be used to compute the coverage error of a given discrete representation. Our formulations are practically implementable when the problem under study is a multiobjective linear programming problem. We believe that the interactive algorithms along with the vector maximization methods can make use of our framework and its tools.

Key words. multiple objective mathematical programming – vector optimization – efficient set – discrete representation

1. Introduction

Multiple Objective Mathematical Programming (MOMP), which allows for simultaneous optimization of more than one objective function, provides a flexible modeling framework. Mathematically, the MOMP problem can be expressed as:

$$(MOMP) \qquad Maximize \quad f(x), \quad subject\ to\ x \in X,$$

where $X \subseteq \mathbf{R^n}$ is the set of feasible alternatives and $f = (f_1, \ldots, f_k) : \mathbf{R^n} \to \mathbf{R^k}$, $k \geq 2$, is a vector–valued function. Note that X can be any set, continuous or discrete, and the objective function f can be of any form.

Solving the MOMP problem is about studying the inherent trade–offs among conflicting objectives. *Efficient solutions* are the ones that possess the relevant trade–off information.

Definition 1. *$x^o \in \mathbf{R^n}$ is an* efficient solution *for the MOMP problem if $x^o \in X$ and there exists no $x \in X$ such that $f(x) \geq f(x^o)$ with $f_j(x) > f_j(x^o)$ for some $j \in \{1, \ldots, k\}$.*

The set of all efficient solutions of the MOMP problem will be denoted by X_E. The most–preferred solution of the Decision Maker (DM) should belong to X_E, as solutions

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that are not efficient can be improved upon in at least one objective without worsening the others. In this study, we will assume that X_E is a bounded set.

As X_E is usually a difficult set to deal with, very different solution approaches that ask for the DM's input at different stages of the solution procedure have emerged. *A priori methods*, methods that use prior articulation of preferences, such as multiattribute utility theory technique [9], goal programming [14], compromise programming [25], reduce the MOMP problem to a single objective problem by imposing an implicit or explicit utility function that is assumed to model the DM's preferences, thus they aspire to find a single solution to the problem. Despite their restrictive assumptions, the interest in some of the a priori methods may be due to their similarity to the more familiar Mathematical Programming construct. *The interactive methods* are among the most practical ways of solving MOMP problems. Today, some of the interactive MOMP algorithms have evolved into decision support systems ([12]) that provide a friendly environment for modeling as well as problem solving. However, some of these algorithms are sometimes criticized for relying too much on the information provided by the DM, and confining their search to only local portions of the X_E.

Vector maximization methods, also referred to as posterior methods, are based on the sole assumption that the DM prefers more to less in each objective function in MOMP, hence they propose identifying all of the efficient solutions of MOMP and presenting them to the DM for the identification of the most–preferred solution. In spite of the significant amount of research effort directed towards the vector maximization approach and the proposed methods, failure in practical utilization of the methods is mostly due to their heavy computational requirements and the near–impossible assignment of studying the overwhelming set of efficient solutions to identify the most–preferred solution attributed to the DM. The reader is referred to [4] for a recent critical overview of the approaches to MOMP, and to [7], [10], and references therein for a more detailed and thorough discussion of solution approaches to MOMP.

When MOMP has linear objective functions and a polyhedral feasible set, the resulting problem that is called a Multiple Objective Linear Programming (MOLP) problem has mathematical features that make it easier to characterize and obtain the efficient set compared to the more general case. However, it is still not an easy task to find all of the efficient solutions and present them to the DM. Thus most of the vector maximization methods have concentrated on finding special subsets of the efficient set. The most popular such subset is the set of efficient extreme points of the multiple objective linear programming problem. It has been noticed that the set of efficient extreme points may still contain too many points to be studied by the DM, and it lacks uniformity in the sense that some portions of the efficient set may end up being overemphasized whereas some regions are highly missed. Moreover, there is no reason why a DM would be solely interested in extreme point efficient solutions. The attractiveness of efficient extreme points mostly lies in their mathematical properties.

The idea of generating a finite set of discrete points different than the set of efficient extreme points to represent the efficient set is not recent. Intra–set point generation suggested by Steuer and Harris [23] that refers to generating certain convex combinations of the efficient extreme points, cone shift method proposed by Armann [1] that aims to generate a discrete subset of the efficient set are among the ideas proposed towards improving the quality of solutions presented to the DM. Clustering proposed by Morse [15], and filtering proposed by Steuer and Harris [23] are based on the idea of eliminating some of the efficient points obtained to attain a better quality representation. Methods that manage to concentrate on particular portions of the efficient set and build a subset of efficient extreme points [22] are also proposed.

The goal of finding truly globally–representative subsets of the efficient set has been proposed by Benson and Sayin [4], whose "Global Shooting Procedure" generates global representations of efficient sets which are not limited to extreme points. Theirs is a procedure that applies to a general class of multiple objective mathematical programs, takes a global approach, works in the outcome space and has the potential to be computationally–insensitive to problem size.

Although the ingredients that make up the quality of a representation have been discussed([1], [21], [4]), the attributes of the quality of a discrete representation have not been fully materialized. In cases where pictures of the efficient set and the representation can be drawn, a subjective visual evaluation can be performed to decide if the representation is satisfactory, or to compare representations with each other.

The goal of this study is to help redefine the vector maximization approach. Ultimately, we envision that the DM who has constructed a MOMP problem with or without assistance, and who is also informed about the meaning of an efficient solution, receives a discrete representation of the efficient set as a solution to his/her problem. This representation conforms to the standards imposed by the DM implicitly or explicitly. He/She is then free to identify the most–preferred solution in this discrete representation, again with or without assistance. To realize this goal, a mathematical definition of a discrete representation, and practically–measurable quantities to assess the quality of a given discrete representation are needed. Then vector maximization methods may aspire to find discrete representations of a prescribed quality.

This study provides the framework for assessing the quality of a given discrete representation through the introduction of a δ–uniform d_ϵ–representation. This is a definition of a discrete representation given a *coverage error* ϵ and a *uniformity level* δ measured in metric d. In Sect. 2, we discuss the properties of good discrete representations, and define our three quality attributes. In Sect. 3, we propose tools for computing the values of the quality attributes of given discrete representations. In Sect. 4, we discuss the use of our tools when the problem under study is a multiple objective linear programming problem. Section 5 contains a discussion of issues towards employing our framework within the vector optimization approach. Section 6 contains our concluding remarks.

2. Representations of the efficient set

There are three dimensions of interest in evaluating a discrete representation of the efficient set.

1. **Coverage.** All of the efficient solutions of the problem i.e. all of the elements of X_E must be well–represented. A *globally–representative* subset of X_E should contain points from every portion of the efficient set without missing any region.
2. **Uniformity.** The representation should be *uniform* in the sense that it does not include any redundancies. Points in clusters are not desired in the representation

because they do not contribute as much to the information being presented to the DM. In the name of uniformity, an ideal representation may contain, for instance, points that are exactly at the same distance to each other.

3. **Cardinality.** The representation should consist of a *reasonable number of elements*. Too many points to study pose a threat to consume the limited information processing stamina of the DM. As a result, some portions of the efficient set may end up being unexplored by the DM at all although representative points from those portions do exist.

Note that the above properties may be interrelated. As the representative quality increases, so should the number of points in the representation. As uniformity increases, the number of points in the representation could potentially decrease.

In our definition of a discrete representation, we use two parameters, ϵ which is a parameter that signifies how precisely the efficient set is being approximated, and δ, which implies that points within a representation do not get closer to each other more than a δ amount. Another factor that is implicitly present in our definition of discrete representations is a metric $d(x, y)$ defined on $\mathbf{R^n}$ which is used in measuring the quantities ϵ and δ.

Definition 2. *Let $\epsilon > 0$ be a real number. Let $D \subseteq Z$ be a discrete set. D is called a d_ϵ–representation of Z if for any $z \in Z$, there exists $y \in D$ such that $d(z, y) \leq \epsilon$.*

In the definition above, $\epsilon > 0$ is the coverage error of D in representing the set Z measured in metric d. Note that if D is a $d_{\bar{\epsilon}}$ –representation of Z, then D is a d_ϵ–representation of Z for all $\epsilon \geq \bar{\epsilon}$. Hence, when we refer to the coverage error ϵ of a d_ϵ–representation D of Z, we imply the smallest ϵ that meets the definition.

The existence of a discrete representation of finite cardinality when Z is bounded is shown in the Appendix. For our purposes, representing a set, even an unbounded one, with infinitely many points would not be acceptable. We believe that an unbounded efficient set may be appropriately truncated before a discrete representation is sought.

For some of our later results to go through, it is necessary that $d(x, y)$ constitutes a norm on $\mathbf{R^n}$ when considered as the norm of $(x - y)$ (see, for instance, [13]). For the reader who is not familiar with the metrics, the following group of functions that we will use later are widely–utilized metrics on $\mathbf{R^n}$ that also constitute norms.

Definition 3. *For $x, y \in \mathbf{R^n}$,*

$$l^p(x, y) = \begin{cases} (|x_1 - y_1|^p + \ldots + |x_n - y_n|^p)^{1/p} & \text{for } p = 1, 2, \ldots \\ \max_{i=1\ldots n} |x_i - y_i| & \text{for } p = \infty, \end{cases}$$

is referred to as the l^p metric.

In Fig. 1, the $\bar{\epsilon}$ marks *a worst–represented element of Z with respect to the Euclidean metric l^2*. Thus we refer to the D in Fig. 1 as an $l^2_{\bar{\epsilon}}$–representation of Z. It is clear that D is an l^2_ϵ–representation of Z for all $\epsilon \geq \bar{\epsilon}$.

Definitions similar to the definition of a d_ϵ–representation above can be seen in a number of contexts. For instance, an equivalent definition is given by Statnikov and Matusov [20] in a related context of representing a continuous set with a discrete one.

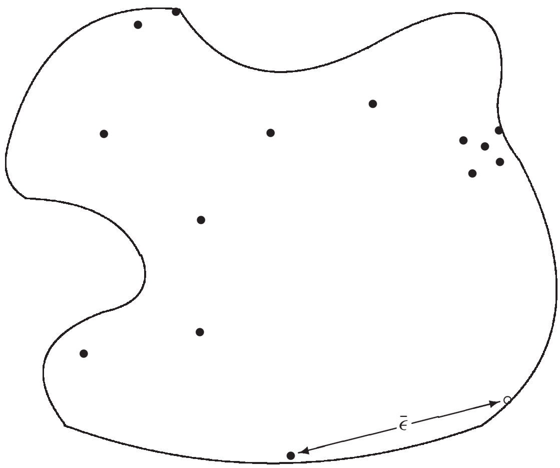


Fig. 1. Representing a set

To measure the uniformity level of a representation, we incorporate another parameter, δ, which constitutes a lower bound on the distance allowed between pairs of points in the representation.

Definition 4. *Let $Z \subseteq \mathbf{R^n}$ be any set and let D be a d_ϵ–representation of Z. Then D is called a δ–uniform d_ϵ–representation if*

$$\min_{x,y\in D, x\neq y} \{d(x, y)\} \geq \delta.$$

A δ–uniform d_ϵ–representation then does not contain any pair of points that are at a distance less than δ to each other. Observe that if a d_ϵ–representation D is $\bar{\delta}$–uniform, it is δ–uniform for all $\delta \leq \bar{\delta}$. Therefore hereafter when we mention a δ–uniform d_ϵ–representation, we imply the largest δ that satisfies δ–uniformity for D.

The three dimensions we have attributed to a discrete representation D of a set Z form a multicriteria framework to assess the quality of a discrete representation. In a d_ϵ–representation D of a set Z, a smaller ϵ, i.e. smaller coverage error or higher precision, a larger δ, i.e. greater uniformity, and a smaller cardinality (denoted $|D|$ hereafter) are desirable. For example, the four l^2_ϵ–representations that appear in Fig. 2 can be compared to each other within this multicriteria framework. The set Z to be represented is the square with a side of length a. The set D in each case consists of the discrete points labeled by bullets. In each instance, ϵ again marks a worst–represented element of Z. In Fig. 2 (b) and (c), there are other worst–represented elements of Z that are not marked to maintain visual clarity. The ϵ, δ and $|D|$ given for each case easily point out that the representation in Fig. 2(b) would be preferred to the one in Fig. 2(c) or (d), and the representation in Fig. 2(a) is preferred to the one in Fig. 2(d). These quantitative implications are expected to be in line with the reader's judgment by visual inspection since the l^2 metric, the Euclidean distance, is a visually familiar measure on the plane.

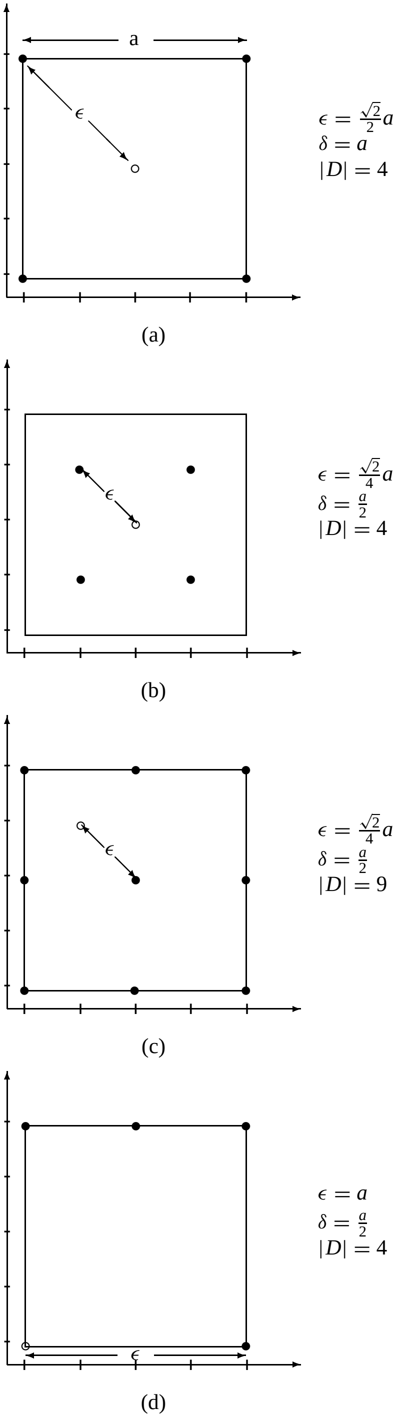


Fig. 2. Alternative l_{ϵ}^{2} representations

3. Computing the values of the quality attributes

Measuring the coverage error ϵ and the uniformity level δ will require a translation of Definitions 2 and 4 respectively into mathematical expressions. We have mentioned that, as we attribute quantities as dimensions of quality to a representative set, the coverage error ϵ is the smallest, and the uniformity level δ is the largest quantity that meets the definition. Thus we obtain the following expressions to compute ϵ and δ. Given a d_ϵ–representation D of a set Z, the coverage error is given by the quantity

$$\epsilon = \max_{z \in Z} \min_{x \in D} d(z, x). \tag{1}$$

For a fixed element z of Z, how well it is covered is determined by the closest point to z in the representation D. How well the entire set Z is covered depends on how well an arbitrary element of Z is covered, and thus the coverage error ϵ is equal to the maximum of coverage error quantities for individual points in Z.

Similarly, the uniformity level δ is determined by the quantity

$$\delta = \min_{x, y \in D, x \neq y} d(x, y). \tag{2}$$

From Equation (2) and the fact that D is of finite cardinality, computing the uniformity level δ is simple as long as the metric d is computable.

From Equation (1), computing the coverage error ϵ is not trivial unless Z itself is a finite discrete set. As our efforts are directed towards the case in which Z is the (continuous) efficient set of a MOMP problem, we will have to elaborate more on computing ϵ.

3.1. Computing the coverage error ϵ

Suppose that $D = \{x^1, \ldots, x^N\}$. From Equation (1), the coverage error ϵ can be computed by solving the following mathematical program $MP(d)$:

$$\begin{array}{lll} MP(d) & & \max\ \epsilon \\ & \text{s.t.} & \epsilon \leq d(z, x^i), \quad i = 1, \ldots, N, \\ & & z \in Z. \end{array}$$

The computational difficulty of computing ϵ depends on two factors: the characteristics of the metric d, and the properties of the set Z. For instance, if Z is a polyhedral set and d is a linear function, than ϵ can be computed at the expense of solving a linear programming problem. In practice, the most intuitive instances of the metric d are given by l^p for the cases $p = 1, 2$ or ∞. Unfortunately, none of these leads to a simple linear function. Moreover, even in the case of a MOLP problem, the efficient set is hardly ever a simple polyhedral set. Given a d_ϵ–representation D of Z, computing ϵ practically will require further transformation. Below we identify the cases in which ϵ can be computed by solving a relatively simple mathematical program.

When $p = \infty$, the l^p metric, also referred to as the Tchebycheff distance, labels the maximum coordinatewise distance among two points x and $y \in \mathbf{R^n}$ as the overall distance among them (Definition 3). The Tchebycheff distance has been widely used within the area of Multiple Criteria Decision Making, probably due to its intuitive and mathematical appeal. Consider a d_ϵ–representation D of a set $Z \subseteq \mathbf{R^n}$. Take an element z of Z. The point x in D that represents z is the one that is closest to z. If the Tchebycheff distance between x and z is $l^\infty(x, z)$, we know that all coordinatewise distances ($|x_j - z_j|$, $j = 1, \ldots, n$) between x and z are less than or equal to $l^\infty(x, z)$. When each of the coordinates is physically meaningful as in the case of optimization, and the measured distance is some form of an *error*, as in our case it is the *error in coverage of* z, the above property becomes an attractive feature that does not exist in other instances of the l^p metric.

We now present the following mixed integer programming formulation $MP(l^\infty)$ that is the Tchebycheff version of the mathematical programming formulation $MP(d)$. Let M denote a sufficiently large positive number. Define $e_n \in \mathbf{R^n}$ and $e_N \in \mathbf{R^N}$ as vectors whose entries are all 1. Let $d = (d_1, \ldots, d_N) \in \mathbf{R^N}, z, u^i, o^i, t^i, s^i \in \mathbf{R^n}$, $i = 1, \ldots, N$ denote the variables.

$$
\begin{aligned}
MP(l^\infty) \qquad & \max \quad \epsilon \\
\text{s.t.} \qquad & e_N\epsilon - d \le 0, && (3)\\
& -e_n d_i + z + u^i = x^i, \quad i = 1, \ldots, N, && (4)\\
& e_n d_i + z - o^i = x^i, \quad i = 1, \ldots, N, && (5)\\
& u^i - Mt^i \le 0, \quad i = 1, \ldots, N, && (6)\\
& o^i - Ms^i \le 0, \quad i = 1, \ldots, N, && (7)\\
& e_n^T t^i + e_n^T s^i \le 2n - 1, \quad i = 1, \ldots, N, && (8)\\
& z \in Z,\\
& u^i, o^i \ge 0, \quad i = 1, \ldots, N,\\
& t^i, s^i \in \{0, 1\}, \quad i = 1, \ldots, N.
\end{aligned}
$$

To see how the formulation works, first recall that, with $d = l^\infty$, by Equation (1),

$$\epsilon = \max_{z \in Z} \min_{i=1..N} \max_{j=1..n} |z_j - x_j^i|.$$

For a given $z \in Z$, the d_i in the formulation stands for the Tchebycheff distance between z and x^i. To see this, first note that by constraints (4) and (5),

$$d_i \ge z_j - x_j^i, \quad j = 1, .., n, \tag{9}$$

and

$$d_i \ge x_j^i - z_j, \quad j = 1, .., n. \tag{10}$$

Moreover, the slack and surplus variables u^i and o^i in constraints (4) and (5) are introduced to ensure that at least one inequality among the $2n$ in (9) and (10) holds as an equality. This is provided through the related integer variables defined by constraints (6) and (7). Observe that the binary variables t^i, s^i are defined as indicator variables whose components identify if the associated component of a u^i, o^i is positive or not. Constraint (8), allowing at most $2n - 1$ positive components in a pair of

u^i and o^i, forces one among the $2n$ inequalities (9) and (10) to hold as an equality. Thus $d_i = \max_{j=1..n} |z_j - x^i_j|$. On the other hand, constraint (3) helps establish ϵ as the minimum of the $d_i, i = 1, .., N$. The objective function maximizes ϵ, ensuring the definition.

Observe that $MP(l^\infty)$ is a mixed integer programming problem with $2nN$ binary and $1 + n + (2n + 1)N$ continuous variables, and $(4n + 2)N$ constraints in addition to those that define Z. Although the size of problem $MP(l^\infty)$ becomes quite large as N grows, we do not expect this to be a problem for two reasons. First, the majority of the constraints in the formulation do not express physical limitations, but exist in order to compute various quantities. Second, for our purposes of representing the efficient set, N should be reasonably small. Computational experiments would demonstrate if our anticipation is valid.

When Z is a polyhedral set, the problem is solvable with widely available mixed integer linear programming solvers. We will utilize the fact that the efficient set of a MOLP problem is a union of its efficient faces in the following section.

The rectilinear distance, or the l^1 metric, constitutes another case for which a mixed integer programming formulation ($MP(l^1)$) can express the problem of computing the coverage error ϵ. Building on the previous formulation, define an additional set of variables $a^i \in \mathbf{R^n}$, $i = 1, \ldots, N$. Then $MP(l^1)$ replaces constraints (4) and (5) with

$$-a^i + z + u^i = x^i, \quad i = 1, \ldots, N, \tag{11}$$

$$a^i + z - o^i = x^i, \quad i = 1, \ldots, N, \tag{12}$$

and constraint (8) with

$$t^i_j + s^i_j \leq 1, \quad i = 1, \ldots, N, \quad j = 1, \ldots, n, \tag{13}$$

and appends the constraint

$$d_i - e_n^T a^i = 0, \quad i = 1, \ldots, N, \tag{14}$$

to the formulation so that the $d_i, i = 1 \ldots N$ again measure the distance of $z \in Z$ to $x^i \in D$. The reader is referred to [18] for a detailed discussion of the formulation.

3.2. *Computing an upper bound on the coverage error ϵ*

The cases $p = 1$ and $p = \infty$ constitute the two extremes for the measure l^p. Given two points x, y, the $l^1(x, y)$ will be the largest and $l^\infty(x, y)$ will be the smallest measure among all values of p. The only remaining geometrically interpretable case of the l^p metric with $p = 2$ leads to a case that is not easily linearizable even with the help of binary variables. It is possible to attack the formulation $MP(l^2)$ as a nonlinear programming problem. However, we find it beyond the scope of this paper to explore solution methods for nonlinear problems. Instead, we will provide below a way of computing an upper bound on the coverage error ϵ when the feasible set Z is a polytope with all of its extreme points explicitly known. Let $V = \{v^1, \ldots, v^K\}$ denote the set of extreme points of Z. Let $\lambda \in \mathbf{R^K}$ be the set of decision variables in addition to $\bar{\epsilon}$. The following simple linear programming formulation with $K + 1$ continuous variables and

$N + n + 1$ constraints in addition to those that define Z, applies to any metric d that constitutes a norm.

$$\begin{aligned}
LPUB(d) \qquad & \bar{\epsilon}^* = \max \quad \bar{\epsilon} \\
\text{s.t.} \qquad & \bar{\epsilon} - \sum_{j=1}^{K} \lambda_j d(v^j, x^i) \le 0, \quad i = 1, \dots, N, \\
& \sum_{j=1}^{K} \lambda_j v^j - z = 0, \\
& z \in Z, \\
& \sum_{j=1}^{K} \lambda_j = 1, \\
& \lambda_j \ge 0, \quad j = 1 \dots, K.
\end{aligned}$$

Theorem 1. *The optimal value $\bar{\epsilon}^*$ of LPUB(d) constitutes an upper bound on the coverage error of D over Z with respect to the norm d.*

After we see the following two simple facts, the above result will become obvious.

Lemma 1. *Let $a_1 \le b_1, \dots, a_N \le b_N$ be a set of $2N$ real numbers. Then min $\{a_1, \dots, a_N\} \le$ min $\{b_1, \dots, b_N\}$.*

Proof. Obvious. □

Lemma 2. *Let $\Lambda = \{\lambda \in \mathbf{R^K} | \sum_{j=1}^{K} \lambda_j v^j \in Z, \sum_{j=1}^{K} \lambda_j = 1, \lambda_j \ge 0, j = 1, \dots, K\}$. For $z \in Z$, let $\epsilon(z) = \min_{i=1,\dots,N} d(z, x^i)$, and for $\lambda \in \Lambda$, let $\bar{\epsilon}(\lambda) = \min_{i=1,\dots,N} \sum_{j=1}^{K} \lambda_j d(v^j, x^i)$. Then*

$$\max_{z \in Z} \epsilon(z) \le \max_{\lambda \in \Lambda} \bar{\epsilon}(\lambda).$$

Proof. First, observe that $Z = \{z \in \mathbf{R^n} | z = \sum_{j=1}^{K} \lambda_j v^j$ *for some* $\lambda \in \Lambda\}$. Let $\hat{z} \in Z$ be such that $\epsilon(\hat{z}) = \max_{z \in Z} \epsilon(z)$; $\bar{\lambda} \in \Lambda$ be such that $\bar{\epsilon}(\bar{\lambda}) = \max_{\lambda \in \Lambda} \bar{\epsilon}(\lambda)$. Define the associated $\hat{\lambda}$ and $\bar{z}$ as $\hat{\lambda} \in \Lambda$ such that $\hat{z} = \sum_{j=1}^{K} \hat{\lambda}_j v^j$, and $\bar{z} \in Z$ such that $\bar{z} = \sum_{j=1}^{K} \bar{\lambda}_j v^j$.

Since $\hat{z} = \sum_{j=1}^{K} \hat{\lambda}_j v^j$, $d(\hat{z}, x^i) = d(\sum_{j=1}^{K} \hat{\lambda}_j v^j, \sum_{j=1}^{K} \hat{\lambda}_j x^i)$ as $\sum_{j=1}^{K} \hat{\lambda_j} = 1$. Then $d(\hat{z}, x^i) = d(\sum_{j=1}^{K} \hat{\lambda}_j (v^j - x^i), 0) \le \sum_{j=1}^{K} d(\hat{\lambda}_j (v^j - x^i), 0) = \sum_{j=1}^{K} \hat{\lambda}_j d(v^j - x^i, 0) = \sum_{j=1}^{K} \hat{\lambda}_j d(v^j, x^i)$ since d defines a norm.

As for $i = 1 \dots, N$, $d(\hat{z}, x^i) \le \sum_{j=1}^{K} \hat{\lambda}_j d(v^j, x^i)$, by Lemma 1, $\epsilon(\hat{z}) \le \bar{\epsilon}(\hat{\lambda})$ follows. By definition of $\bar{\lambda}, \bar{\epsilon}(\hat{\lambda}) \le \bar{\epsilon}(\bar{\lambda})$. Then $\epsilon(\hat{z}) \le \bar{\epsilon}(\hat{\lambda}) \le \bar{\epsilon}(\bar{\lambda})$, which implies $\epsilon(\hat{z}) \le \bar{\epsilon}(\bar{\lambda})$. By definition of $\hat{z}$ and $\bar{\lambda}$, it follows that $\max_{z \in Z} \epsilon(z) \le \max_{\lambda \in \Lambda} \bar{\epsilon}(\lambda)$. □

To see why Theorem 1 works, we just need to observe that the optimal value of problem $LPUB(d)$, $\bar{\epsilon}^*$, is equal to $\max_{\lambda \in \Lambda} \bar{\epsilon}(\lambda)$ as defined in Lemma 2. Obviously, the coverage error of D over Z is given by $\max_{z \in Z} \epsilon(z)$ again following the notation in Lemma 2. Thus the result follows.

As ϵ is the coverage error of the representative set D, an upper bound on ϵ establishes a cap on the error present in the representation. That is, the coverage error ϵ of D can be no worse than the optimal value of $LPUB(d)$. If $\bar{\epsilon}^*$ is an acceptable quantity already, the tightness of the bound does not become an issue. If $\bar{\epsilon}^*$ is too large a quantity to be judged an acceptable error in coverage, it is still possible that the true coverage error ϵ of D is acceptable depending on how tight a bound is provided by $LPUB(d)$. In the example of Sect. 4, and in a few similar small examples, the upper bound $\bar{\epsilon}^*$ was about 10% higher than the coverage error ϵ both for l^∞ and l^1. However, the tightness of the bound provided by $LPUB(d)$ can only be realistically explored by extensive numerical experimentation, which we will not stop to do in order not to lose our primary focus.

4. Measuring the quality of discrete representations of the efficient set

Given a (continuous) set $Z \subseteq \mathbf{R^n}$ and its discrete representation D, the previous section contains the straight–forward mathematical programming formulations to compute the coverage error ϵ with respect to the l^∞ and l^1 metric. For the formulations to be practically implementable, the set Z needs to be available. When $Z = X_E$, the efficient set of a MOMP problem, it is generally not possible to obtain and express X_E in a simple way. We will not explore the nonlinear case further as we see it beyond the scope of our study. If practical ways of finding and expressing the efficient set in the nonlinear case are developed, our formulations for measuring the coverage error could possibly become tractable.

The MOLP problem constitutes a special case of the MOMP problem. It is well–known that the efficient set of a MOLP problem can be expressed as a union of its efficient faces. The computational experiments in [17] indicate that the number of (maximally) efficient faces of a MOLP problem may not be too large. Each efficient face being a polyhedron, the formulations $MP(l^\infty)$ and $MP(l^1)$ can then be used to measure the coverage error over individual faces. In particular, the following procedure can be employed to compute the coverage error ϵ of D.

Obtain the efficient set X_E of the MOLP. Let $X_E = \cup_{i=1}^{F} X_i$ where each X_i denotes an efficient face.

Solve F subproblems $MP(l^\infty)$ or $MP(l^1)$ with $Z = X_i$ and D in each to compute ϵ_i, $i = 1, \dots, F$.

The coverage error ϵ of D is given by $\epsilon = \max_{i=1}^{F} \epsilon_i$.

When solving the individual subproblems, all that is needed is the explicit expression of the efficient face X_i in the formulation. If efficient faces are expressed by the constraints that hold as equality ([24]), the X_i are easily represented in the formulation by converting the types of the associated original constraints to equality. If the efficient set is expressed as a collection of the optimal solution sets to a number of parametric optimization problems ([8]), then each X_i can be expressed as $Z \cap \{z \in \mathbf{R^n} | \beta^T z = \beta_o\}$, for some $\beta \in \mathbf{R^n}$ and $\beta_o \in \mathbf{R}$. If the efficient faces are available as the convex hulls of groups of efficient extreme points, however, extra work is needed to express the efficient faces within the mathematical programming formulation. In that case, one might opt to solve $LPUB(d)$ so as to find an upper bound for the coverage error.

Example. Consider the following MOLP example:

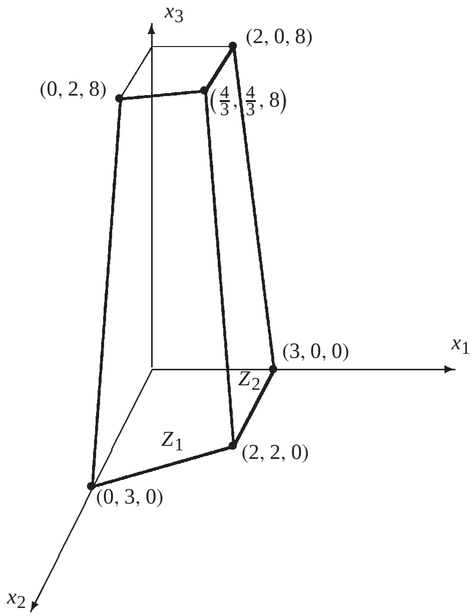
$$\begin{aligned} \text{vmax} \quad & x_1, \quad x_2, \quad x_3 \\ \text{s.t.} \quad & 4x_1 + 8x_2 + x_3 \leq 24, \\ & 8x_1 + 4x_2 + x_3 \leq 24, \\ & x_3 \leq 8, \\ & x_1, x_2, x_3 \geq 0. \end{aligned}$$

The efficient set in this example consists of the two two–dimensional faces labeled as Z_1 and Z_2 in Fig. 3(a). There are six efficient extreme points of the problem. We will look at various discrete representations of the efficient set. As we decompose the efficient set into Z_1 and Z_2, observe that the expression of the X_i to be used in the associated mathematical programming formulation can be obtained by respectively forcing the first or the second constraint to an equality.

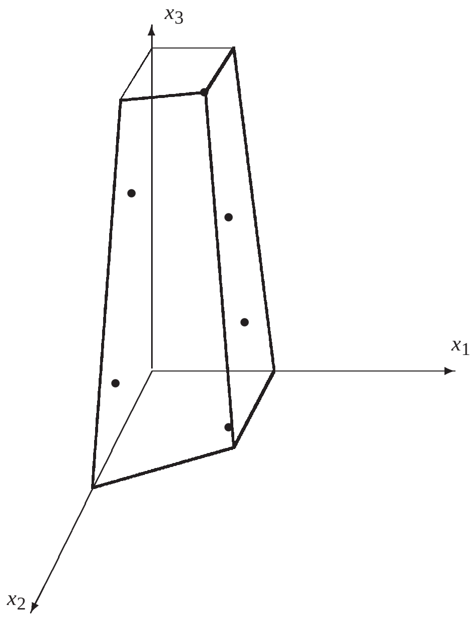
Take the set of efficient extreme points as the representative set D, i.e. $D = \{(0, 2, 8), (1.33, 1.33, 8), (2, 0, 8), (0, 3, 0), (2, 2, 0), (3, 0, 0)\}$. When we solve $MP(l^\infty)$ with $Z = Z_1$ and D, we obtain $\epsilon_1^\infty = 4$ with $z^1 = (0, 2.5, 4) \in Z_1$ as a worst–represented point, and with $Z = Z_2$ and D, $\epsilon_2^\infty = 4$ with $z^2 = (2.5, 0, 4) \in Z$ as a worst–represented point. Note the symmetry present in the two faces Z_1 and Z_2 and the existence of many alternative worst–represented points that have an x_3 value of 4. Thus the coverage error of the set of the efficient extreme points equals 4, with a uniformity level $\delta^\infty = 1.33$ determined by the two points $(0, 2, 8)$ (or $(2, 0, 8)$) and $(1.33, 1.33, 8)$ in the representation, and $|D| = 6$.

As a different representation, consider the one displayed in Fig. 3(b). The representation consists of points $D = \{(2.23, 0.56, 3.9), (1.93, 1.93, 0.77), (0.67, 2, 5.33), (0.375, 2.625, 1.5), (2.625, 0.375, 1.5), (1.33, 1.33, 8)\}$ and is obtained using an instance of the Global Shooting Procedure [4]. In Z_1, we find that a worst–represented point is $(0.31, 2.41, 3.41)$ with $\epsilon_1^\infty = 1.92$. In Z_2, a worst represented point is $(2.25, 0, 6)$, with $\epsilon_2^\infty = 2$. The coverage error of $\epsilon^\infty = 2$, uniformity level $\delta^\infty = 1.56$ and $|D| = 6$ indicates that this is a better l_ϵ^∞–representation compared to the set of efficient extreme points.

When the representation that consists of efficient extreme points is evaluated using the l^1 metric, $\epsilon^1 = 5.19$, $\delta^1 = 2$ with a worst–represented point $x = (0.69, 2.19, 3.69)$ or $x = (2.19, 0.69, 3.69)$ is obtained. The second representation yields $\epsilon^1 = 3.44$, $\delta^1 = 2.88$ with a worst–represented point $x = (2.16, 0, 6.71)$. In other words, the sum of the coordinatewise errors for the worst–represented point is 3.44. Note that the l^1 metric does not give us a simple interpretation as the l^∞ metric does.



(a) Efficient extreme points as a discrete representation



(b) A discrete representation by the global shooting procedure

Fig. 3. Example: two discrete representations

5. Guidelines for evaluating discrete representations of the efficient set

Measuring in the outcome space. Efficient solutions in the outcome space, also referred to as efficient outcomes or nondominated solutions, are the images of the decision space solutions. That is, if we let $Y_E = f(X_E) = \{y \in \mathbf{R^k} | y = f(x) \ \textit{for some}\ x \in X_E\}$, it is recommended to work with Y_E instead of X_E due to a number of reasons. Studies on the geometrical properties of the set of efficient outcomes, such as that of Dauer's [6] and Benson's [2] indicate that the structure of the efficient set in the outcome space is expected to be simpler than the structure of the efficient set in the decision space. As the number of objectives is usually much smaller than the number of decision variables, the dimensionality of the two spaces suggest that studying Y_E is easier than studying X_E for the DM. Furthermore, the DM is primarily interested in outcomes rather than values of decision variables as long as problem constraints are met. Therefore we recommend that all computations be carried out in the outcome space as we will explain below.

If Y_E and its representation are explicitly known, then the formulations of the previous section can be used to compute ϵ, which then will be the coverage error in the outcome space. If what we have is an explicit statement of X_E, the formulations can be easily modified. The discrete representation D can be easily mapped into the outcome space if it is originally stated in the decision space, hence we will assume that the discrete representation D of the efficient set is given in the outcome space, i.e. $D = \{y^1 = f(x^1), \dots, y^N = f(x^N)\} \subseteq \mathbf{R^k}$. Without loss of generality, assume that all the $y^i, i = 1, \dots, N$ are distinct. Then the uniformity level δ can be measured as $\delta = \min_{i,j \in \{1,..,N\}, i \neq j} d(y^i, y^j)$.

Following the previously used notation, letting e_k be a vector of size k with all entries equal to 1, we can rewrite formulation $MP(l^\infty)$ in the outcome space by replacing x^i's by y^i's, $i = 1, \dots, N$, introducing the variable $y \in \mathbf{R^k}$ to replace the variable z, and adding the constraint

$$y = f(z) \tag{15}$$

to enforce the fact that y is the image of z in the outcome space. The variables u^i, o^i, t^i, s^i, y would then belong to $\mathbf{R^k}$, which is the result of the transformation into the outcome space. If we consider k being significantly smaller than n, the formulation in the outcome space would also be significantly smaller than in the decision space.

Choosing the metric. We favor the use of the Tchebycheff norm, l^∞, within our framework. If used to measure ϵ and δ in the outcome space, the l^∞ metric provides us with a guarantee that the maximum possible error of the solution in any of the criteria is at most ϵ, and all of the points in the representation are at least δ apart in the outcome space with respect to each criterion. As all of the other instances of the l^p metric tend to add up coordinatewise distances, this is usually inappropriate when coordinates refer to criteria of the MOMP problem, which for instance, may be measuring cost in one criterion and environmental hazard in another. Therefore using the l^∞ metric is the safest unless there is evidence that any other metric is more appropriate for a particular application and the consequences of using such a function is fully understood. If the ranges of the criteria are significantly different or the DM attaches different importance to different criteria, using the weighted l^∞ metric may be more appropriate to provide a balance among the criteria.

Definition 5. *Given $w_i > 0, i = 1, \dots, n$, the weighted l^p metric for $x, y \in \mathbf{R}^\mathbf{n}$, is given by*

$$l_w^p(x, y) = \begin{cases} (w_1^p|x_1 - y_1|^p + \ldots + w_n^p|x_n - y_n|^p)^{1/p} & \text{for } p = 1, 2, \ldots \\ \max_{i=1\ldots n} w_i|x_i - y_i| & \text{for } p = \infty. \end{cases}$$

Given $\bar{w} = (1/w_1, \dots, 1/w_n)^T$, the formulation $MP(l^\infty)$ can easily accommodate weights by rewriting constraints (4) and (5) as

$$-\bar{w}d_i + z + u^i = x^i, \quad i = 1, \dots, N, \tag{4'}$$

and

$$\bar{w}d_i + z - o^i = x^i, \quad i = 1, \dots, N. \tag{5'}$$

Another way of dealing with this situation is to let the DM assign not one, but k coverage errors $\epsilon_1, \dots, \epsilon_k$ separately for each criterion. In that case, our framework can accommodate such a generalization at the expense of solving multiple mathematical programs $MP(l^\infty)$ for each coordinate. As presented in constraint (15), if one redefines $y = f_i(z)$ for criterion i and lets the other associated variables of size k to become accordingly of size 1, $MP(l^\infty)$ in the outcome space would measure the coverage error only in that criterion. Specifying different coverage error levels would be meaningful only if the l^∞ metric is in use.

Specifying the coverage error. Specifying absolute quantities for the coverage error and the uniformity level would not be appropriate as these quantities should be a function of the problem data and the DM's expectations. The error the DM is willing to tolerate in the solution of a vector optimization problem may be most appropriately specified by the DM. Yet the DM needs some assistance prior to specifying an acceptable error. We believe that the range information of each criterion over the efficient set would be very valuable for a DM in order to decide on an acceptable error. In the example of the previous section, if the DM is informed that the first and the second objectives run in the range [0, 3] and the third objective runs in the range [0, 8] over the efficient set, she may find an error of $\epsilon^\infty = 2$ too large, and for instance, $\epsilon^\infty = 0.5$ acceptable. To find the minimum values taken by the criteria over the efficient set, any of the available exact or heuristic approaches could be used ([19], [3], [11]).

Specifying the uniformity level and the cardinality. The uniformity level δ is of interest to the DM in a somewhat indirect fashion. What directly interests the DM is the number of points that exist in the representation. Moreover, the error the DM is willing to tolerate in coverage hints that representative points do not need to get closer to each other more than an ϵ amount. Therefore, unless otherwise specified, we recommend using the value of the coverage error ϵ as a benchmark to evaluate the uniformity level of the representation. We also remark that using elimination techniques ([23], [15]) can help improve the uniformity level of a discrete representation.

On the other hand, the DM might quite easily specify a maximum number of points that she is willing to study as a solution to a MOMP problem. The only critical issue here is to note the possible conflict between the cardinality of the representative set and the coverage error that can be tolerated. As the cardinality of the set decreases, the coverage error is expected to increase.

6. Future directions

We proposed a framework for measuring the quality of discrete representations of efficient sets. We then provided means of computing the quality attributes when the l^∞ metric is used. A similar formulation is possible for the l^1 metric. Moreover, the computations can be carried out in the outcome space. When the problem is a MOLP problem, our formulations can be practically implemented using a mixed integer programming solver as the efficient set decomposes into efficient faces. An informal computational experiment with a problem that has a feasible set of a polyhedral $Z \subseteq \mathbf{R^{50}}$ defined via 40 constraints, $k = 3$ objectives and with $N = 50$ representative points shows that solving $MP(l^\infty)$ in the outcome space using the routines of Cplex Callable library [5] takes around 50 CPU seconds on a Sun Ultra Model 140 workstation (with a 143 mHz CPU). When the number of points $N = 30$, the same problem takes only 15 CPU seconds. However, more experimentation is needed to assess if our procedure is computationally tractable for reasonable problem sizes of the MOLP case.

The formulations we proposed to measure the coverage error could constitute the core of a method that has the goal of finding discrete representations of prescribed quality in the MOLP case since the efficient set decomposes into efficient faces. Given an efficient face, and an initial efficient point that serves as the seed of a discrete representation, the formulation $MP(l^\infty)$ (preferably in the outcome space) can be applied to find a worst–represented point that lies in the face. Adding the resulting point itself to the representation, the formulation can be applied iteratively until a prescribed cardinality or a coverage error is achieved. The assessment of the computational behavior of such an approach requires further research.

Finally, variations of the framework proposed here, for instance one that incorporates statistical measures rather than deterministic ones, can be considered. Experimentation within our framework in actual decision making settings and consideration of the behavioral issues could provide the guidelines along which the approach should evolve into a practical and useful tool.

7. Appendix

We provide the mathematical definitions that build towards the existence of discrete representations. In the following definitions (see, for instance, Rudin [16]), $\mathbf{R^n}$ is regarded as a metric space with the metric $d(x, y)$ on $\mathbf{R^n}$.

Definition 6. *Let $x \in \mathbf{R^n}$ be a point. Let $\epsilon > 0$. Then the ϵ–neighborhood of x, denoted $B_\epsilon(x)$ is defined as $B_\epsilon(x) = \{y \in \mathbf{R^n} | d(x, y) < \epsilon\}$.*

Definition 7. *Let $Z \subseteq \mathbf{R^n}$. A point x is a* limit point *of Z if, for any $\epsilon > 0$, $B_\epsilon(x)$ contains a point $y \neq x$ such that $y \in Z$. If $x \in Z$ is not a limit point of Z, then x is called an* isolated point *of Z.*

Definition 8. *$Z \subseteq \mathbf{R^n}$ is called a* discrete set *if every element $x \in Z$ is an isolated point of Z.*

Definition 9. *$Z \subseteq \mathbf{R^n}$ is called a* compact set *if every open cover of Z contains a finite subcover, i.e. for any collection of open sets $\{C_\alpha\} \subseteq \mathbf{R^n}$ such that $Z \subseteq \cup_\alpha \{C_\alpha\}$ there are finitely many indices $\alpha_1, \ldots \alpha_N$ such that $Z \subseteq \cup_{i=1}^N C_{\alpha_i}$.*

For any set $Z \subseteq \mathbf{R^n}$, and any $\epsilon > 0$, a set $D \subseteq Z$ that satisfies $Z \subseteq \cup_{x \in D} B_\epsilon(x)$ and $d(x, y) > \frac{\epsilon}{2}$ for any $x, y \in D$ is a d_ϵ–representation of Z since such a set D is discrete. However, depending on the properties of the set Z and the metric d, the set D may not be finite. The following result ensures that when Z is bounded, d_ϵ–representations that are of finite cardinality exist for any $\epsilon > 0$.

Proposition 1. *Let $Z \subseteq \mathbf{R^n}$ be a nonempty bounded set. Let $\epsilon > 0$. Then there exists $D \subseteq Z$ such that D is a d_ϵ–representation of Z and D is of finite cardinality.*

Proof. Let $Z \subseteq \mathbf{R^n}$ be any nonempty bounded set. Let $\epsilon > 0$. Let $\bar{Z}$ denote the closure of Z. Since $\bar{Z}$ is compact, and $\bar{Z} \subseteq \cup_{z \in \bar{Z}} B_{\frac{\epsilon}{2}}(z)$, there exists $z^1, \ldots, z^N$ such that

$$\bar{Z} \subseteq \cup_{i=1}^N B_{\frac{\epsilon}{2}}(z^i). \tag{16}$$

Now, construct $D = \{x^1, \ldots, x^N\} \subseteq Z$ such that $x^i = z^i$ if $z^i \in Z$, and x^i satisfies $d(x^i, z^i) < \frac{\epsilon}{2}$ if $z^i \notin Z$. Note that for any point $\bar{z} \in \bar{Z}$ that does not belong to Z, there exists a point $x \in Z$ that is arbitrarily close to $\bar{z}$. Now, take any $x \in Z$. Since $Z \subseteq \bar{Z}$, by (16), $d(x, z^i) < \frac{\epsilon}{2}$ for some $i = 1, \ldots, N$. By the way D is constructed, $d(x, x^i) \leq d(x, z^i) + d(z^i, x^i) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$. Since D is of finite cardinality, it cannot contain any limit points(by Corollary to Theorem 2.20 in [16]) and therefore is a discrete set. Thus D is a d_ϵ–representation of Z.

□

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