Q 6:

Function: $f: \{0,1\}^n \rightarrow \{0,1\}$

$$f: \begin{cases} 0 & \text{for } 0^{\frac{N}{2}} 1^{\frac{N}{2}} \\ 1 & \text{for } \#1 \text{ in output}^{0 \rightarrow \frac{N}{2}} + \#0 \text{ in output}^{\frac{N}{2}+1 \rightarrow N} = \frac{N}{2} \end{cases}$$

Algorithem:

1. Start $|0\rangle^{\otimes n}$

2. Apply Hadamard to each qubit:

$$H^{\otimes n}|0\rangle^{\otimes n} = \frac{1}{\sqrt{2^n}} \sum_{x \in \{0,1\}^n} |x\rangle$$

3. Apply U_f on all qubits:

$$U_f = (-1)^{f(x)}|x\rangle = (-1)^{b[\text{index}]}|x\rangle$$

$$\frac{1}{\sqrt{2^n}} \sum_{x \in \{0,1\}^n} |x\rangle \xrightarrow{U_f} \frac{1}{\sqrt{2^n}} \sum_{x \in \{0,1\}^n} (-1)^{b[\text{index of } x]}|x\rangle$$

$b[\text{index of } x]$: bit value at position x in binary output

for 0: function is constant
→ $(-1)^0$ for all inputs

for 1: function is balanced
→ $(-1)^1$ for half and $(-1)^0$ for the other half of inputs

4. Apply Hadamard on all qubits:

$$\frac{1}{\sqrt{2^n}} \sum_{x \in \{0,1\}^n} (-1)^{b[\text{index}]}|x\rangle$$

$$\xrightarrow{H^{\otimes n}} \frac{1}{2^n} \sum_{x \in \{0,1\}^n} (-1)^{b[\text{index}]} \sum_{y \in \{0,1\}^n} (-1)^{x \cdot y}|y\rangle$$

5. Measure output:

For $0^{\frac{N}{2}} 1^{\frac{N}{2}} \xrightarrow{\text{prob.}} |0^n\rangle$

otherwise not $|0^n\rangle$