Yildiz Technical University

Department of Control and Automation Engineering

Vehicle System Dynamics and Control

Modeling and Control of Vehicle Dynamics: A Study on Magic Tire Formula and LQR-Based Direct Yaw Moment Control

Asst. Prof. Dr. M. Tolga Emirler

Name Surname: Aylin Öztürk

Student Number: 20016928

Signature:

Istanbul, Turkey

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ABSTRACT

This study investigates two fundamental aspects of vehicle dynamics: the nonlinear tire behavior modeled by the Magic Tire Formula, and the design of a Direct Yaw Moment Control (DYC) system using Linear Quadratic Regulator (LQR) control strategy. Simulations were performed in MATLAB to analyze the tire forces and self-aligning moment under varying vertical loads, followed by a state-space modeling and control design to stabilize vehicle yaw dynamics.

1. INTRODUCTION

Modern vehicle safety and handling systems heavily rely on accurate modeling of tire dynamics and advanced control strategies. Traditional linear tire models fail to capture the nonlinear behavior observed under high slip conditions, motivating the use of more sophisticated models such as the Magic Tire Formula (Pacejka Model). This model enables realistic simulation of tire forces including longitudinal force, lateral force, and self-aligning moment based on slip ratio and slip angle.

In addition to modeling, controlling the vehicle's yaw behavior is crucial for stability, especially during aggressive maneuvers. To this end, Direct Yaw Moment Control (DYC) provides an effective approach to manipulate the yaw moment by applying differential braking or torque distribution. In this study, an LQR-based control algorithm was developed to achieve desired yaw and sideslip angle performance using a linear single-track vehicle model.

2. METHOD

2.1. Magic Tire Formula Implementation

Normally, linear tire models are used for the simplicity of simulations. However, when dealing with large slip angles and ratios, a more accurate representation of tire behavior is needed. The Magic Tire Formula (Pacejka Model) is a non-linear tire model that captures the complex relationship between tire forces and slip conditions, especially under extreme driving conditions. This model includes large slip angles and ratios and provides mathematical expressions for three key forces or moments acting on a tire:

• Longitudinal Force Fx(s) :The force acting in the direction in which the wheel is moving.. It depends on the slip ratio (s), which represents the difference between the wheel's rotational speed and the vehicle's actual ground speed. As the slip ratio increases, the longitudinal force increases until it reaches a peak and then decreases.

• Lateral Force Fy(α) : The force acting sideways, perpendicular to the wheel's direction. It affects the vehicle's steering and handling. It depends on the slip angle (α\alphaα), which is the angle between the wheel's heading and the direction of travel.

• Self-Aligning Moment Mz(α) : This is a torque (rotational force) that tries to return the tire to its original direction. It also depends on the slip angle (α\alphaα) and is important for understanding the tire's role in vehicle stability and handling.

The general formula:

Y(s)=D·sin[(C·arctan(B(s+Sh)−E·(B(s+Sh)−arctan(B(s+Sh)))))] +Sv

- The longitudinal tire force

$$F_x(s) = D\sin\left[C\arctan\left(B(s+S_h) - E\left(B(s+S_h) - \arctan\left(B(s+S_h)\right)\right)\right)\right] + S_v$$

- The lateral tire force

$$F_y(\alpha) = D\sin\left[C\arctan\left(B(\alpha+S_h) - E\left(B(\alpha+S_h) - \arctan\left(B(\alpha+S_h)\right)\right)\right)\right] + S_v$$

- For the self- aligning moment the cosine version of the formula is given by

$$M_{sa}(\alpha) = D\sin\left[C\arctan\left(B(\alpha+S_h) - E\left(B(\alpha+S_h) - \arctan\left(B(\alpha+S_h)\right)\right)\right)\right] + S_v$$

α (Slip angle): This is the angle between the direction of the tire and the direction of motion of the vehicle. It affects the lateral force on the tire.

2.1.1. Code Implementation in MATLAB

	Load, F_z, kN	B	C	D	E	S_h	S_v	BCD
F_y, N	2	0.244	1.50	1936	−0.132	−0.280	−118	780.6
	4	0.239	1.19	3650	−0.678	−0.049	−156	1038
	6	0.164	1.27	5237	−1.61	−0.126	−181	1091
	8	0.112	1.36	6677	−2.16	0.125	−240	1017
M_z, N · m	2	0.247	2.56	−15.53	−3.92	−0.464	−12.5	−9.820
	4	0.234	2.68	−48.56	−0.46	−0.082	−11.7	−30.45
	6	0.164	2.46	−112.5	−2.04	−0.125	−6.00	−45.39
	8	0.127	2.41	−191.3	−3.21	−0.009	−4.22	−58.55
F_x, N	2	0.178	1.55	2193	0.432	0.000	25.0	605.0
	4	0.171	1.69	4236	0.619	0.000	70.6	1224
	6	0.210	1.67	6090	0.686	0.000	80.1	2136
	8	0.214	1.78	7711	0.783	0.000	104	2937

Source: Bakker et al. (1987).

Figure 1: Values of the coefficients in the Magic Tire Formula (tire slip angle in degrees and slip ratio in %. Reference: J. Y. Wong, Theory of Ground Vehicles, Fifth Edition, Wiley, 2022.

```
% Fx vs alip ratio plotting
figure;
hold on; %keeping the current plot and adding new plot
for i = 1:length(Fz_values)
    Fz = Fz_values(i); %assign corresponding Fz value from table
    %assigning B,C,D,E,Sh,Sv,BCD values from the data set
    B = fx_values(i,2); C = fx_values(i,3); D = fx_values(i,4); E = fx_values(i,5);
Sh=fx_values(i,6);
    Sv=fx_values(i,7);BCD= fx_values(i,8);

    % Pacejka (Magic Tire) Formula for Fx calculation
    Fx = D * sin(C * atan(B *((slip_ratio +Sh) - E * (B * (slip_ratio +Sh) - atan(B
* (slip_ratio +Sh)))))) +Sv;

    plot(slip_ratio , Fx, 'DisplayName', sprintf('Fz = %d kN', Fz));
end
xlabel('slip ratio (in %) '); ylabel('Fx (N) '); title('Longitudinal force -- Slip
ratio'); grid on;% labeling axis
legend; %adding legends on the plot
hold off; % clear old plots
```

```
% Fy vs slip angle plottinh
figure;
hold on;
for i = 1:length(Fz_values)
    Fz = Fz_values(i);%assign corresponding Fz vertical force acting on the tire
    %assigning B,C,D,E,Sh,Sv,BCD values from the table
    B = fy_values(i,2); C = fy_values(i,3); D = fy_values(i,4); E = fy_values(i,5);
Sh=fy_values(i,6);
    Sv=fy_values(i,7);BCD= fy_values(i,8);

     % Pacejka (Magic Tire) Formula for Fy calculation
    Fy = D * sin(C * atan(B * (slip_angle +Sh) - E * (B * (slip_angle +Sh) - atan(B
* (slip_angle +Sh))))) +Sv;

    plot(slip_angle, Fy, 'DisplayName', sprintf('Fz = %d kN', Fz));
end
xlabel('slip angle (in deg) '); ylabel('Fy (N) '); title('Lateral Force -- Slip
Angle'); grid on; %labeling the axis
legend;% adding legends on the plot
hold off; % clear old plots
```


```
% Fy vs slip angle plottinh
figure;
hold on;
for i = 1:length(Fz_values)
    Fz = Fz_values(i);%assign corresponding Fz vertical force acting on the tire
    %assigning B,C,D,E,Sh,Sv,BCD values from the table
    B = fy_values(i,2); C = fy_values(i,3); D = fy_values(i,4); E = fy_values(i,5);
Sh=fy_values(i,6);
    Sv=fy_values(i,7);BCD= fy_values(i,8);

     % Pacejka (Magic Tire) Formula for Fy calculation
    Fy = D * sin(C * atan(B * (slip_angle +Sh) - E * (B * (slip_angle +Sh) - atan(B
* (slip_angle +Sh))))) +Sv;

    plot(slip_angle, Fy, 'DisplayName', sprintf('Fz = %d kN', Fz));
end
xlabel('slip angle (in deg) '); ylabel('Fy (N) '); title('Lateral Force -- Slip
Angle'); grid on; %labeling the axis
legend;% adding legends on the plot
hold off; % clear old plots
```

```
% Msa vs slip angle plotting
figure;
hold on;
for i = 1:length(Fz_values)
    Fz = Fz_values(i); %assign corresponding Fz value
    %assigning B,C,D,E,Sh,Sv,BCD values
    B = Msa_values(i,2); C = Msa_values(i,3); D = Msa_values(i,4); E =
Msa_values(i,5); Sh=Msa_values(i,6);
    Sv=Msa_values(i,7);BCD= Msa_values(i,8);

    %pacejka (Magic Tire) Formula for Msa calculation
    Msa = D * sin(C * atan(B * (slip_angle +Sh) - E * (B * (slip_angle +Sh) -
atan(B * (slip_angle +Sh))))) +Sv;

    plot(slip_angle, Msa, 'DisplayName', sprintf('Fz = %d kN', Fz));
end
xlabel('slip angle (in deg) '); ylabel('Msa(N) '); title('Self-Aligning Moment --
Slip Angle'); grid on; %labeling the axis
legend; %adding legedns on the polt
hold off;
```

2.2. LQR-Based Direct Yaw Moment Control Design

For front Wheel steering we can write the linearized single track model in state-spaceform as follows:

$$\dot{x} = Ax + Bu \qquad x = \begin{bmatrix} \beta & r \end{bmatrix}^T$$

$$y = Cx \qquad u = \begin{bmatrix} \delta_f & M_z \end{bmatrix}^T$$

$$A = \begin{bmatrix} \dfrac{-\mu C_f - \mu C_r}{mv} & -1 + \left(\dfrac{\mu C_r l_r - \mu C_f l_f}{mv^2} \right) \\ \dfrac{\mu C_r l_r - \mu C_f l_f}{I_z} & \dfrac{-\mu C_f l_f^{\,2} - \mu C_r l_r^{\,2}}{I_z v} \end{bmatrix}, \quad B = \begin{bmatrix} \dfrac{\mu C_f}{mv} & 0 \\ \dfrac{\mu C_f l_f}{I_z} & \dfrac{1}{I_z} \end{bmatrix}$$

The desired vehicle motion is represented as follows in state space form:

$$\dot{x}_d = Ax_d + \begin{bmatrix} b_{11} \\ b_{21} \end{bmatrix} \delta_f$$

Define an error state between the actual state xand the desired state xd as e=x-xd.

Differentiating the error yields the tracking error dynamics:

$$\dot{e} = \dot{x} - \dot{x}_d = Ae + \begin{bmatrix} b_{12} \\ b_{22} \end{bmatrix} M_z$$

After the determination of the tracking error dynamics, the optimal corrective yaw moment is calculated as follows:

$$M_z^* = -k_1(\beta - \beta_d) - k_2(r - r_d)$$

where the gains k1and k2are determined based on LQR optimal control theory by minimizing the cost function whish is represented as follow

$$J = \int_0^\infty \left[w_\beta (\beta - \beta_d)^2 + w_r (r - r_d)^2 + w_{M_z} M_z^{\,2} \right] dt$$

Our Q and R matrices are:

$$Q = \begin{bmatrix} w_\beta & 0 \\ 0 & w_r \end{bmatrix},\ R = \begin{bmatrix} w_{M_z} \end{bmatrix}$$

2.2.2. Code and Simulink Implementation in MATLAB

```
clear all
clc
close all
```

INTRODUCTION INFORMATIONS

```
%these values are given to us in the homework
x=2+0+0+1+6+9+2+8;
mu = 0.7;
Cf = 145000;   % N/rad
Cr = 185000;   % N/rad
m = 1500+x;      % kg (x = 8)
Iz = 2400+x;     % kg*m^2 (x = 8)
lf = 1.1;      % m
lr = 1.5;      % m
v = 20;        % m/s

% State-space A matrix elements
A11 = (-mu * (Cf + Cr)) / (m * v);
A12 = -1 + (mu * (Cr * lr - Cf * lf)) / (m * v^2);
A21 = (mu * (Cr * lr - Cf * lf)) / Iz;
A22 = (-mu * (Cf * lf^2 + Cr * lr^2)) / (Iz * v);

B11= (mu*Cf)/(m*v);
B12=0;
B21= (mu*Cf*lf)/Iz;
B22= 1/ Iz;
```

PART A

```
% Define A and B matrices
A = [A11, A12; A21, A22];
B=[B11,B12; B21,B22];
disp('A matrix:');
```

```
A matrix:
```

```
disp(A);
```

```
   -7.5589   -0.8649
   34.0198   -8.5294
```

```
Cont=[B A*B]; %our controllbility matrix
```

```
if rank(Cont) == 2 % checking whether system is controllable or not
    disp("system is  controllable");
end
```

```
system is  controllable
```

```
trace_A = A11 + A22;
det_A = A11*A22 - A12*A21;

% Compute eigenvalues
lambda1 = (trace_A + sqrt(trace_A^2 - 4*det_A)) / 2;
lambda2 = (trace_A - sqrt(trace_A^2 - 4*det_A)) / 2;
disp('eigenvalues of A:')
```

```
eigenvalues of A:
```

```
disp(lambda1); disp(lambda2);
```

```
  -8.0442 + 5.4025i

  -8.0442 - 5.4025i
```

```
% Eigenvalues must be on the left half plane to be stable
if real(lambda1) < 0 && real(lambda2) < 0
    disp('The open-loop system is STABLE');
else
    disp('The open-loop system is UNSTABLE');
end
```

```
The open-loop system is STABLE
```

PART B

Differentiating the error yields the tracking error dynamics:

$$\dot{e} = \dot{x} - \dot{x}_d = Ae + \begin{bmatrix} b_{12} \\ b_{22} \end{bmatrix} M_z$$

```
% Error Dynamics are given as e_dot = A * e + B_error * Mz
B_error = B(:,2);  % sadece Mz etkisi
C=eye(2);
D=zeros(2,1);
sys_error = ss(A, B_error,C , D);
disp('Error dynamics state-space model when input is Mz only:');
```

```
Error dynamics state-space model when input is Mz only:
```

```
sys_error
```

```
sys_error =

  A = 
             x1       x2
```

```
  x1   -7.559  -0.8649
  x2    34.02   -8.529

B =
          u1
  x1       0
  x2  0.0004119

C =
      x1  x2
  y1   1   0
  y2   0   1

D =
      u1
  y1   0
  y2   0

Continuous-time state-space model.
Model Properties
```

```
% Compute desired yaw rate r_d
numerator = Cf * Cr * (lf + lr) * v;
denominator = (Cr * Cf * (lf + lr)^2) + (Cr * lr - Cf * lf) * m * v^2;
```

The gains k1 and k2 are determined based on LQR optimal control theory by minimizing the cost function which is given as follows

$$J = \int_0^\infty \left[w_\beta \left(\beta - \beta_d\right)^2 + w_r \left(r - r_d\right)^2 + w_{M_z} {M_z}^2 \right] dt$$

```
Q = [100000, 0;
     0, 100000];

R = [0.000001];

% LQR K gain matrix
K= lqr(A, B_error, Q, R)
```

```
K = 1×2
10^5 ×
    0.5485    2.9583
```

```
fprintf("LQR kazanç matrisi: K = [%.4f  %.4f]\n", K(1), K(2));
```

```
LQR kazanç matrisi: K = [54849.1690  295832.0156]
```

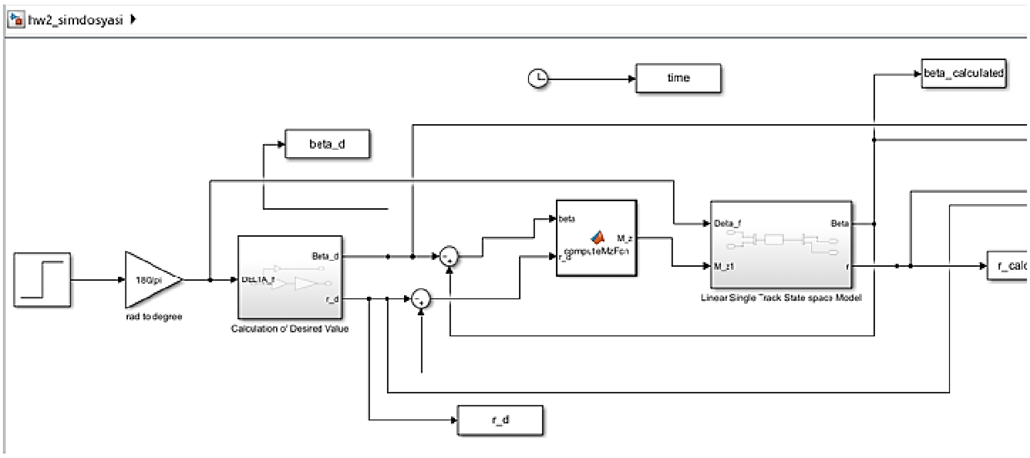
```
k1=K(1)
```

```
k1 =
5.4849e+04
```

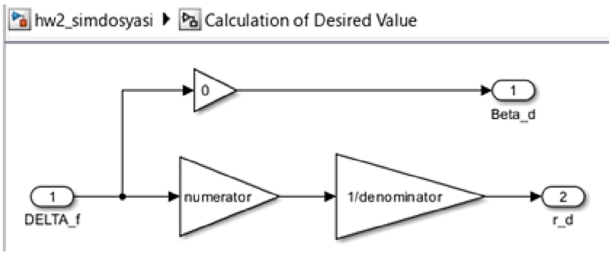
```
k2=K(2)
```

```
k2 =
2.9583e+05
```

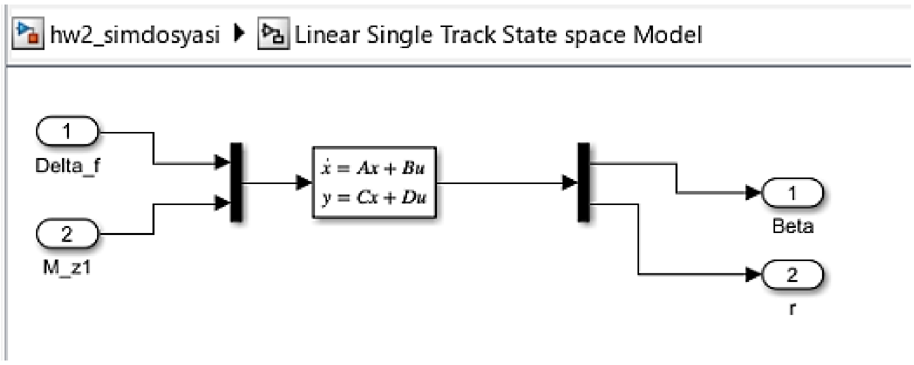
Simulink models are represented below



Inside of the calculation of desired values where we calculated our desired beta and r values



Inside of the Linear Single Track State Space Model where we model our systems dynamics in to state-space model



Inside of the Matlab function where we calculate our M_z with the help of beta , r and K values from the calculation of LQR

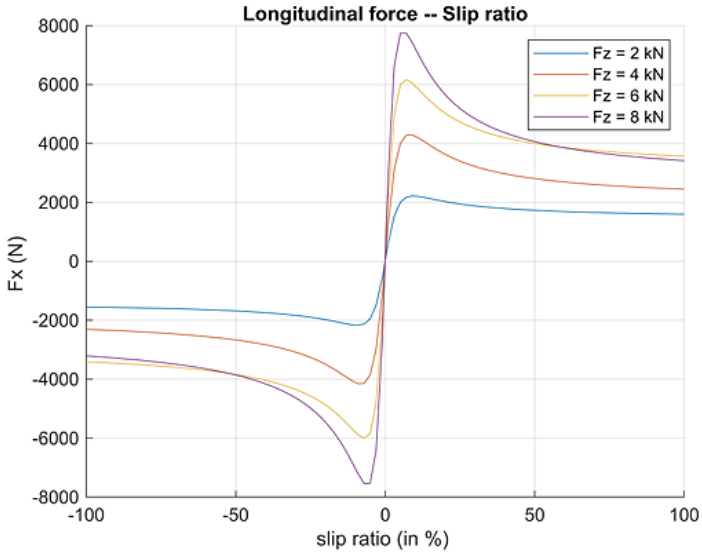
MATLAB Function

hw2_simdosyasi ▸ MATLAB Function

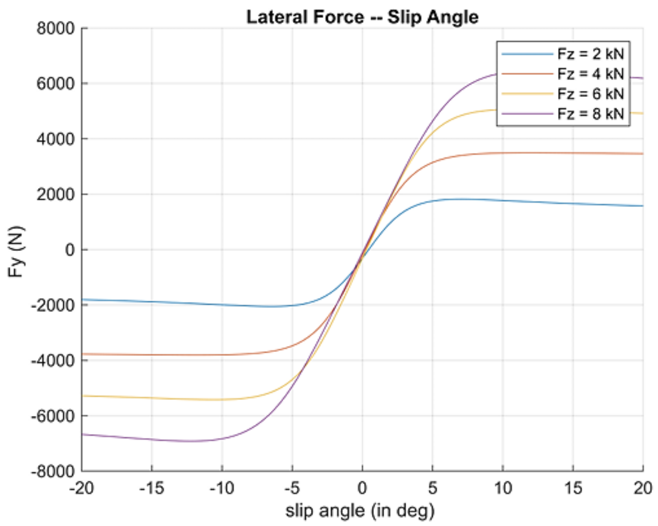
```
function M_z = computeMzFcn(beta,r_d)

    K1 =5.4849e+04  ;
    K2 =2.9583e+05  ;
    M_z = (-K1 * beta) + (-K2 * r_d);
end
```

3. RESULTS AND DISCUSSION

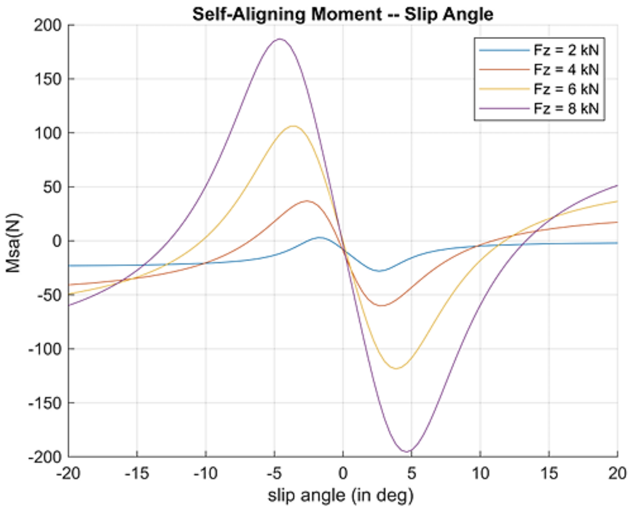
3.1. Magic Tire Results



The longitudinal force (Fx) increases with slip ratio, reaches a peak, and then decreases—this is typical tire behavior due to the transition from adhesion to sliding. The peak force becomes higher with increasing vertical load (Fz). This confirms that tires generate more traction under higher vertical loads, which is crucial for performance during acceleration and braking.



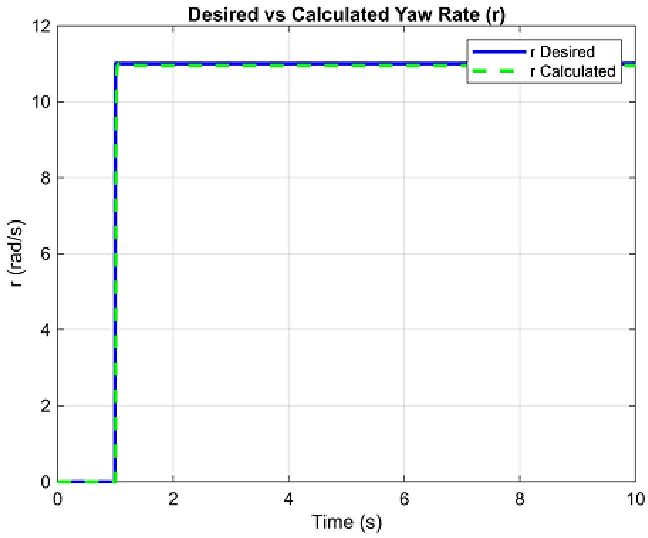
The lateral force (Fy) increases with slip angle up to a peak, then saturates and slightly declines. This behavior reflects the tire's ability to generate cornering force until it reaches its grip limit. This shows that heavier loading improves cornering capability, but also introduces nonlinearity earlier, which may affect stability during aggressive maneuvers.



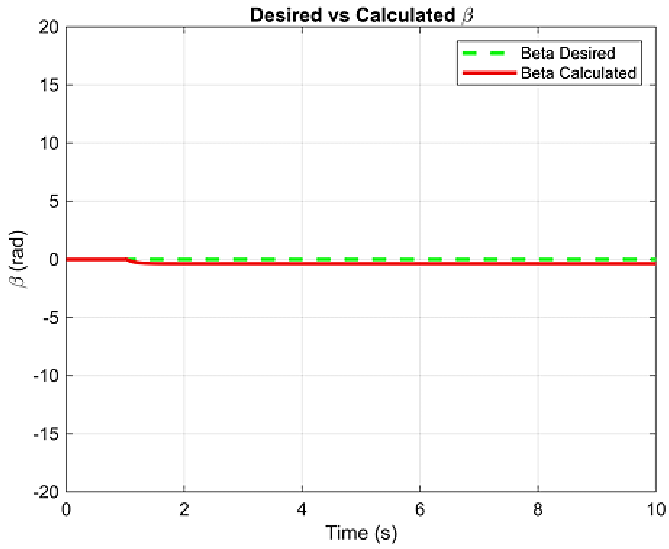
The self-aligning moment (Mz) does not start from zero, due to non-zero vertical shift (Sv) in the Magic Tire Formula. The moment increases with slip angle, reaches a positive peak, and then becomes negative at high slip angles. Among the tested values, Fz = 6 kN showed a strong and stable peak, suggesting an optimal balance. At Fz = 8 kN, although the moment is strong, it drops off faster, possibly reducing stability at high slip angles.

These observations collectively highlight the load sensitivity of tire behavior. As vertical force increases, so does the tire's ability to generate forces—but it also becomes more nonlinear and sensitive to slip, especially in lateral and aligning characteristics.

3.2. LQR-Based Direct Yaw Moment Control Results



The blue line represents the target yaw rate the control system aims to achieve and represents the actual yaw rate produced by the vehicle dynamics. Both desired and calculated yaw rates rise sharply and match closely indicate that there is almost zero steady-state error , stabilizing around 11 rad/s.The controller effectively tracks the desired yaw rate with minimal error and very fast response.



The green dashed line represents the reference sideslip angle, and the red line represents the actual sideslip angle of the vehicle. The calculated sideslip angle (β) does not settle exactly to zero. There is a steady-state error of approximately −0.2 degrees. The close match between desired and actual β indicates that the vehicle dynamics are well-managed, and undesired lateral movement is well-controlled.

As a result, yhe system performs very well in both yaw rate and sideslip angle control. The vehicle behaves as expected under control, and the results validate the effectiveness and robustness of the controller design.

4. CONCLUSION

This study illustrates the importance of nonlinear tire modeling for accurate vehicle simulation and the effectiveness of optimal control techniques in maintaining vehicle stability. The Magic Tire Formula provided realistic behavior under high slip conditions, and the LQR-based DYC system successfully stabilized the yaw dynamics. Together, they represent powerful tools for advanced vehicle dynamics analysis and control system design.

5. REFERENCES

- L. Güvenç, Vehicle Control Systems PhD Course, Lecture Notes, Istanbul Technical University.
- R. Rajamani, VehicleDynamics andControl, 2nd Edition, Springer, 2012.
- B. AksunGüvenç, Vehicle System Dynamics BScCourse, Lecture Notes, Istanbul Technical University